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REMARKS ON THE CIARLET-RAVIART MIXED FINITE ELEMENT*

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Abstract. This paper derives a new scheme for the mixed finite element method for the biharmonic equation in which the flow function is approximated by piecewise quadratic polynomial and vortex function by piecewise linear polynomials. Assuming that the partition, with triangles as elements, is quasi-uniform, then the proposed scheme can achieve the approximation order that is observed by the Ciarlet-Raviart mixed finite element when approximating the flow function and the vortex functions by piecewise quadratic polynomials.

Key words. Ciarlet-Raviart mixed finite element, biharmonic problem.

AMS subject classification. 65L60.

1. Review of the Ciarlet-Raviart mixed element scheme. Consider the biharmonic problem (with clamped boundary conditions)

$$\begin{cases} \Delta^2\phi = f, & \text{in } \Omega, \\ \phi = \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the domain Ω is a convex polygon in R^2.

Let by $H^s(\Omega)$ denote the Sobolev space, let $\|\cdot\|_s$ denote its norm, and let $\|\cdot\|_0$ denote the norm of the space $L^2(\Omega)$. Let $H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$ with $\|\cdot\|_{-1}$ as its norm. It is well known that for $f \in H^{-1}(\Omega)$, (1.1) admits only one solution ϕ satisfying

$$\phi \in H^3(\Omega), \text{ and } \|\phi\|_3 \le C \cdot \|f\|_{-1}. \tag{1.2}$$

The Ciarlet-Raviart mixed finite element method is used to simultaneously approximate the flow function ϕ and the vortex $-\Delta\phi$:
With $u := -\Delta\phi$, consider the following variational problem corresponding to (1.1):

$$\begin{cases} \text{Find } (u,\phi) \in H^1(\Omega) \times H_0^1(\Omega), \text{ such that} \\ \displaystyle\int_\Omega uv\,dxdy - \int_\Omega \nabla v \nabla\phi\,dxdy = 0, \quad \forall v \in H^1(\Omega); \\ \displaystyle\int_\Omega \nabla u \nabla\psi\,dxdy = -\int_\Omega f\psi\,dxdy, \quad \forall \psi \in H_0^1(\Omega). \end{cases} \tag{1.3}$$

Let $\mathcal{T}_h = \{K\}$ be a quasi-uniform partition of Ω with h the maximum diameter of the partition. Set

$$\begin{aligned} X_h &:= \{v \in C^0(\overline{\Omega}) : \ v|_K \in P_m, \ \forall\, K \in \mathcal{T}_h\}, \\ M_h &:= X_h \cap H_0^1(\Omega). \end{aligned}$$

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Consider the discrete variational problem used to approximate (1.3):

$$\left\{\begin{array}{l}\text{Find } (u_h,\phi_h)\in X_h\times M_h \text{ such that}\\ \displaystyle\int_\Omega u_h v dxdy-\int_\Omega \nabla v\nabla\phi_h dxdy=0,\quad \nabla v\in X_h;\\ \displaystyle\int_\Omega \nabla u_h\nabla\psi dxdy=-\int_\Omega f\cdot\psi dxdy,\quad \forall\psi\in M_h.\end{array}\right. \tag{1.4}$$

Equation (1.4) is called the Ciarlet-Raviart scheme for biharmonic problem. From this it can be seen that subspaces of X_h, namely X_h and M_h, are used for the approximation of both spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. The assumption $M_h = X_h\cap H_0^1(\Omega)$ yields a significant simplification in the proof of error estimates.

In [1] and [2], it was shown, using different approaches, that the following error estimates for the Ciarlet-Raviart scheme hold:

$$\left\{\begin{array}{l}\|\phi-\phi_h\|_1\le C\cdot h^{s-1}\|\phi\|_s\,;\\ \|u-u_h\|_\delta\le C\cdot h^{s-2-\delta},\quad \delta=0,1.\end{array}\right.$$

where $1\le s\le\min\{k+1,r\}$, $u\in H^r(\Omega)$. These estimates depend upon the order k of the polynomials and the smoothness r of generalized solution u. However, in general case, the solution of (1.1) statisfies $\phi\in H^3(\Omega)$ and $u=-\Delta\phi\in H^1(\Omega)$, so that the approximation order can not be increased by increasing the order k of the piecewise polynomials in the spaces M_h and X_h. Hence under the natural smoothness assumptions, to achieve a higher approximation order, a reasonable choice would be to take the degree of polynomial to be 2 for M_h and a lower degree than 2 for X_h. The aim of this paper is to look for such spaces M_h and X_h.

2. Main results and proofs. Let $\mathcal{T}_{2h}$ be a quasi-uniform triagulation of Ω. $\mathcal{T}_h$ is a triangulation obtained by connecting all middle points of edges for each triangle in $\mathcal{T}_{2h}$. Define V_i $(i=1,2)$ to be the order of the associated piecewise polynomial spaces defined on $\mathcal{T}_{ih}$. It is obvious that $V_i\in H^1(\Omega)$. Taking $X_h=V_1$ and $M_h=V_2\cap H_0^1(\Omega)$ in the Ciarlet-Raviart mixed element model (1.3), then we will have shown that the same conclusions hold for error estimates as the case in which X_h and M_h are taken as the quadratic piecewise polynomial spaces. In our case, as $M_h\subset X_h$ is not valid, the error estimates cannot be proved with the approach used in [1].

Let H, M and X be three real Banach spaces with norms $\|\cdot\|_H$, $\|\cdot\|_M$ and $\|\cdot\|_X$, respectively, and let X be continuously embedded into H, denote by $X\hookrightarrow H$. Assume that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, bounded bilinear forms defined on $H\times H$ and $H\times M$, respectively, satisfy

$$|a(u,v)|\le C\cdot\|u\|_H\|v\|_H\,,\forall u\in X,\quad \forall v\in X, \tag{2.1}$$

$$|b(u,\psi)|\le C\cdot\|u\|_X\|v\|_M\,,\forall u\in X,\quad \forall\psi\in M. \tag{2.2}$$

Consider the following abstract problem:
For any $f\in X'$ and any $g\in M'$, find $(u,\phi)\in X\times M$ such that

$$\left\{\begin{array}{l}a(u,v)-b(v,\phi)=\langle f,v\rangle,\quad \forall v\in X,\\ b(u,\psi)=\langle g,\psi\rangle,\quad \forall\psi\in M,\end{array}\right. \tag{2.3}$$

where X' and M' are the dual spaces of X and M, and where $\langle\cdot,\cdot\rangle$ represents the dual inner product between X' and X or M' and M, respectively. The discrete variational

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approximation of (2.3) is:

$$\text{(2.4)} \qquad \begin{cases} \text{Find } (u_h, \phi_h) \in X_h \times M_h \text{ such that} \\ a(u_h, v) - b(v, \phi_h) = \langle f, v \rangle, \quad \forall v \in X_h, \\ b(u_h, \psi) = \langle g, \psi \rangle, \quad \forall \psi \in M_h, \end{cases}$$

where $X_h \subset X$ and $M_h \subset M$ are finite dimensional spaces. The following lemma is proved in [2]:

LEMMA 2.1. *Assume that the following hypotheses are satisfied:*

H1 *For any $(f, g) \in D$, the problem (2.3) has only one solution, where D is the subspace of $X' \times M'$.*

H2 *If G is a Banach space and $M \hookrightarrow G$, then $\forall d \in G'$, the following problem has only one solution:*

$$\text{(2.5)} \qquad \begin{array}{l} Find \ (y_d, z_d) \in X \times M \ such \ that \\ a(v, y_d) - b(v, z_d) = 0, \quad \forall v \in X, \\ b(y_d, \psi) = \langle d, \psi \rangle, \quad \forall \psi \in M, \end{array}$$

H3 *There exists a constant $\alpha > 0$, independent of h, such that*

$$a(v, v) \geq \alpha \|v\|_H^2, \quad \forall v \in X_h.$$

H4 *There exists a constant $S(h)$ satisfying*

$$\|v\|_X \leq S(h) \|v\|_H, \quad \forall v \in X_h.$$

H5 *There exists an operator $P : Y \longrightarrow X_h$, such that*

$$b(y - Py, \psi) = 0, \quad \forall y \in Y, \ \forall \psi \in M_h,$$

where $Y := \operatorname{span}\{\{y_d\}_{d \in G'}, u\}, (u, \varphi)$ is the solution of (2.3), and (y_d, z_d) is the solution of (2.5) corresponding to $d \in G'$.

Then (2.4) admits only one solution (u_h, φ_h) which satisfies the error estimates

$$\text{(2.6)} \qquad \|u - u_h\|_H \leq C \cdot \left(\|u - Pu\|_H + S(h) \|\varphi - \psi\|_M\right), \forall \psi \in M_h,$$

$$\text{(2.7)} \quad \|\varphi - \varphi_h\|_G \leq \sup_{d \in G'} \frac{b(y_d - Py_d, \varphi - \psi) + a(u - u_h, Py_d - y_d) + b(u - u_h, z_d - v)}{\|d\|_{G'}},$$

$\forall \psi, v \in M_h$.

Now, we introduce another lemma proven in [5].

LEMMA 2.2. $\forall v \in C(\Omega)$,

$$\text{(2.8)} \qquad \|I_2 v - I_1 v\|_a \leq \sqrt{\frac{2}{3}} \|I_1 v\|_a,$$

where $\|w\|_a^2 = (\nabla w, \nabla w) = |w|_1$ and $I_i : C^0 \longrightarrow V_i$, i.e., I_i is the piecewise interpolation operator of order i on all vertices of triangles of $\mathcal{T}_{ih}$.

Now we are in a position to state a main result of this paper.

THEOREM 2.3. *When $X_h = V_1$, and $M_h = V_2 \cap H_0^1(\Omega)$, there exists only one solution (u_h, φ_h) for (1.4) which satisfies the error estimates*

$$\text{(2.9)} \qquad \|u - u_h\| \leq C \cdot h, \text{ and}$$

$$\|\varphi-\varphi_h\|_\delta \le C\cdot h^2, \quad \delta=0,1. \tag{2.10}$$

Proof. Take $X=H^1(\Omega)$, $M=H_0^1(\Omega)$, $H=L^2(\Omega)$, $a(u,v)=\int_\Omega uvdxdy$, $b(v,\psi)=\int_\Omega \nabla v\nabla\psi dxdy$, $D=0\times H^{-1}(\Omega)$, $G=H_0^1(\Omega)$ and $G'=H^{-1}(\Omega)$. Then we will use Lemma 1 to prove this theorem. It is obvious that (2.1), (2.2) and hypotheses H1–H3 are valid. As the partition is quasi-uniform, H4 is valid for $S(h)=C\cdot h^{-1}$. Hence it is necessary to construct an operator P such that H5 be satisfied, and then to estimate $\|u-Pu\|_0$, $\|y_d-Py_d\|_0$ and $|y_d-Py_d|_1$.

For a given $v\in H^1(\Omega)$, consider the auxiliary problem:

Find $w\in V_1$ such that

$$\int_\Omega \nabla w\nabla\psi dxdy=\int_\Omega \nabla v\nabla\psi dxdy, \forall\psi\in V_2; \tag{2.11}$$

$$\int_\Omega wdxdy=\int_\Omega vdxdy. \tag{2.12}$$

Equation (2.11) is equivalent to

$$\int_\Omega \nabla w\nabla(I_2\psi)dxdy=\int_\Omega \nabla v\nabla(I_2\psi)dxdy, \quad \forall\psi\in V_1. \tag{2.13}$$

Define the quotient space $H^1(\Omega)/P_0$, where P_0 is the polynomial space of order 0. From [1], this space is a Banach space with its norm defined as

$$\mathring{v}\in H^1(\Omega)/P_0 \longrightarrow \left\|\mathring{v}\right\|_1=\inf_{p\in P_0}\|v+p\|_1,$$

where v is any element of the equivalence class $\mathring{v}$. For any $\mathring{v}$, one has

$$\mathring{v}\in H^1(\Omega)/P_0 \longrightarrow \left|\mathring{v}\right|_1=|v|_1, \tag{2.14}$$

and

$$\left\|\mathring{v}\right\|_1\le C\cdot\left|\mathring{v}\right|_1. \tag{2.15}$$

In the quotient space $H^1(\Omega)/P_0$, define

$$(\mathring{u},\mathring{v})_0=\int_\Omega \nabla u\nabla vdxdy. \tag{2.16}$$

Then

$$(\mathring{u},\mathring{v})_0=\int_\Omega \nabla u\nabla vdxdy=\int_\Omega \nabla v\nabla udxdy=(\mathring{v},\mathring{u})_0; \tag{2.17}$$

$$\begin{aligned}(\mathring{u}+\mathring{w},\mathring{v})_0 &= \textstyle\int_\Omega \nabla(u+w)\nabla vdxdy\\ &= \textstyle\int_\Omega \nabla u\nabla wdxdy+\int_\Omega \nabla w\nabla vdxdy\\ &= (\mathring{u},\mathring{v})_0+(\mathring{w},\mathring{v})_0;\end{aligned} \tag{2.18}$$

and

$$(\mathring{u},\mathring{u})_0=\int_\Omega |\nabla u|^2\,dxdy=|u|_1^2\ge c\cdot\left\|\mathring{u}\right\|_1^2. \tag{2.19}$$

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Hence, $(\overset{\circ}{u}, \overset{\circ}{v})_0$ is an inner product in $H^1(\Omega)/P_0$.

By the definition of $\left\|\overset{\circ}{v}\right\|_1$, one has

$$\left|\overset{\circ}{v}\right|_1 = \inf_{p\in P_0} |v+p|_1 \leq \inf_{p\in P_0} \|v+p\|_1 = \left\|\overset{\circ}{v}\right\|_1. \tag{2.20}$$

It is thus realized from (2.14), (2.15) and (2.20) that the norm $\sqrt{(\overset{\circ}{u}, \overset{\circ}{u})_0}$, derived from the above inner product, is equivalent to $\left\|\overset{\circ}{u}\right\|_1$. Hence, $H^1(\Omega)/P_0$ is a Hilbert space with respect to the inner product $(\overset{\circ}{u}, \overset{\circ}{v})_0$.

Let V_1/P_0 be the subspace of $H^1(\Omega)/P_0$, and define

$$I(\overset{\circ}{w}, \overset{\circ}{u}) := \int_\Omega \nabla w \nabla (I_2 u) dxdy. \tag{2.21}$$

Then in terms of Lemma 2, for any $w \in V_1$ it can be seen that

$$\begin{array}{l} \int_\Omega \nabla w \nabla (I_2 w) dxdy = \int_\Omega \nabla w \nabla (I_2 w - I_1 w + I_1 w) dxdy \\ = \int_\Omega |\nabla w|^2 dxdy - \int_\Omega \nabla w \nabla (I_1 w - I_2 w) dxdy \\ \geq |w|_1^2 - \sqrt{\frac{2}{3}} |w|_1^2 = (1 - \sqrt{\frac{2}{3}}) |w|_1^2. \end{array} \tag{2.22}$$

Combining (2.21), (2.22), (2.13) and (2.14) results in

$$I(\overset{\circ}{w}, \overset{\circ}{w}) \geq C \cdot \left\|\overset{\circ}{w}\right\|_1, \quad \forall \overset{\circ}{w} \in V_1/P_0. \tag{2.23}$$

From (2.20) and (2.21),

$$\begin{array}{rcl} I(\overset{\circ}{w}, \overset{\circ}{v}) & = & \int_\Omega \nabla w \nabla (I_2 v) dxdy \\ & \leq & |w|_1 \cdot |I_2 v|_1 \leq C \cdot |w|_1 \cdot |v|_1 \\ & \leq & C \cdot \left\|\overset{\circ}{w}\right\|_1 \cdot \left\|\overset{\circ}{v}\right\|_1, \quad \forall \overset{\circ}{w}, \overset{\circ}{v} \in V_1/P_0. \end{array} \tag{2.24}$$

By the definition (2.21) and $\overset{\circ}{w} + \overset{\circ}{u} = \overset{\circ}{w + u}$, one has

$$I(\overset{\circ}{w} + \overset{\circ}{u}, \overset{\circ}{v}) = I(\overset{\circ}{w}, \overset{\circ}{v}) + I(\overset{\circ}{u}, \overset{\circ}{v}) \tag{2.25}$$

and also

$$I(\overset{\circ}{w}, \overset{\circ}{u} + \overset{\circ}{v}) = I(\overset{\circ}{w}, \overset{\circ}{v}) + I(\overset{\circ}{u}, \overset{\circ}{v}). \tag{2.26}$$

Hence, $I(\overset{\circ}{u}, \overset{\circ}{v})$ is a continuous positive definite bilinear form. For a fixed v define the functional

$$g(\overset{\circ}{\psi}) := \int_\Omega v \nabla (I_2 \psi) dxdy, \tag{2.27}$$

so that $g(\overset{\circ}{\psi})$ is a continuous on V_1/P_0. By the Lax-Milgram Lemma, the following problem has only one solution $\overset{\circ}{w}$:

$$\left\{ \begin{array}{l} \text{Find } \overset{\circ}{w} \in V_1/P_0 \text{ such that} \\ I(\overset{\circ}{w}, \overset{\circ}{v}) = g(\overset{\circ}{v}), \quad \forall \overset{\circ}{v} \in V_1/P_0. \end{array} \right. \tag{2.28}$$

Consequently, there exists a class of solutions for (2.13). All of solutions are same except for a constant. Thus for any $v \in H^1(\Omega)$, the solution of (2.11), satisfying (2.12), is unique. Denote by $w \in V_1$ this solution. An operator P satisfying H5 can be defined as $P : H^1(\Omega) \to V_1, w = Pv$. Finally from Lemma 1 there is only one solution (u_h, φ_h) for (1.4).

In order to establish the estimates (2.9) and (2.10), one has to estimate $|Py_d - y_d|_1$ and $\|Py_d - y_d\|_0$. As $G' = H^{-1}(\Omega)$, by (1.2) the solution of (2.4) obeys $z_d \in H^3(\Omega) \cap H_0^2(\Omega)$, $y_d \in H^1(\Omega)$ and

$$\|z_d\|_3 \leq C \cdot \|d\|_{-1} \text{ and } \|y_d\|_1 \leq C \cdot \|d\|_{-1}. \tag{2.29}$$

From the property of P, $Py_d \in V_1$ and

$$\int_\Omega \nabla(Py_d - y_d)\nabla\psi dxdy = 0, \quad \forall \psi \in V_2.$$

Hence,

$$\begin{aligned} |Py_d|_1^2 &\leq C \cdot a(Py_d, I_2(Py_d)) = C \cdot a(y_d, I_2(Py_d)) \\ &\leq C \cdot |y_d|_1 \cdot |I_2(Py_d)|_1 \\ &\leq C \cdot |y_d|_1 \cdot |Py_d|_1 . \end{aligned}$$

That is,

$$|Py_d|_1 \leq C \cdot |y_d|_1 .$$

Consequently,

$$|Py_d - y_d|_1 \leq |Py_d|_1 + |y_d|_1 \leq C \cdot |y_d|_1 \leq C \cdot \|d\|_{-1} . \tag{2.30}$$

Now we will use Nitsche's technique to estimate $\|Py_d - y_d\|_0$. Let z be the solution of the variational problem

$$\begin{cases} z \in H^1(\Omega), \\ b(v, z) = (v, y_d - Py_d), \quad \forall v \in H^1(\Omega). \end{cases} \tag{2.31}$$

From [1], $z \in H^2(\Omega)$, and

$$\|z\|_2 \leq C \cdot \|Py_d - y_d\|_0 . \tag{2.32}$$

Taking $v := y_d - Py_d$ in (2.31) results in, with (2.11) and (2.32),

$$\begin{aligned} \|Py_d - y_d\|_0^2 &= b(y_d - Py_d, z) = b(y_d - Py_d, z - I_2 z) \\ &\leq \|Py_d - y_d\|_1 \cdot \|z - I_2 z\|_1 \leq C \cdot \|d\|_{-1} \cdot h \cdot \|z\|_2 \\ &\leq C \cdot h \cdot \|d\|_{-1} \cdot \|Py_d - y_d\|_0 . \end{aligned}$$

That is,

$$\|Py_d - y_d\|_0 \leq C \cdot h \|d\|_{-1} , \tag{2.33}$$

and by (2.6)

$$\|u - u_h\|_0 \leq C \cdot \left(\|u - Pu\|_0 + C \cdot h^{-1} \|\varphi - I_2\varphi\|_1\right) \leq C \cdot h.$$

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This completes the proof for (2.9). On the other hand, take $v = I_2 z_d$ and $\psi = I_2\varphi$ in (2.7). Then it can be derived in terms of (2.30), (2.33) and (2.29) that

$$
\begin{aligned}
\|\varphi - \varphi_h\|_1 &\leq \sup_{d \in H^{-1}(\Omega)} \frac{b(y_d - Py_d, \varphi - \psi) + a(u - u_h, Py_d - y_d) + b(u - u_h, z_d - I_2 z_d)}{\|d\|_{-1}} \\
&\leq \sup_{d \in H^{-1}(\Omega)} \frac{|y_d - Py_d|_1 \cdot |\varphi - I_2\varphi|_1 + \|u - u_h\|_0 \cdot \|Py_d - y_d\|_0 + |u - u_h|_1 \cdot |z_d - I_2 z_d|_1}{\|d\|_{-1}} \\
&\leq \sup_{d \in H^{-1}(\Omega)} \frac{C \cdot \|d\|_{-1} \cdot h^2 \|\varphi\|_3 + C \cdot h^2 \|d\|_{-1} + C \cdot h^2 \|d\|_{-1}}{\|d\|_{-1}} \\
&\leq C \cdot h^2
\end{aligned}
$$

That is, (2.10) and the theorem is proven. □

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