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ETNA
Kent State University
etna@mcs.kent.edu

MATRIX CONTINUED FRACTIONS RELATED TO FIRST-ORDER LINEAR RECURRENCE SYSTEMS*

P. LEVRIE[†‡] AND A. BULTHEEL[†]

Abstract. We introduce a matrix continued fraction associated with the first-order linear recurrence system $Y_k = \theta_k Y_{k-1}$. A Pincherle type convergence theorem is proved. We show that the n-th order linear recurrence relation and previous generalizations of ordinary continued fractions form a special case. We give an application for the numerical computation of a non-dominant solution and discuss special cases where θ_k is constant for all k and the limiting case where $\lim_{k\to+\infty}\theta_k$ is constant. Finally the notion of adjoint fraction is introduced which generalizes the notion of the adjoint of a recurrence relation of order n.

Key words. recurrence systems, recurrence relations, matrix continued fractions, non-dominant solutions.

AMS subject classifications. 40A15, 65Q05.

1. Introduction. Continued fractions are closely related to linear recurrence relations (see [11], [14]) and can be defined using the composition of linear fractional transformations. In this paper we look at linear first-order recurrence systems, and we associate matrix continued fractions with them. These matrix continued fractions (MCF's) are generalisations of ordinary continued fractions, of generalized continued fractions (or n-fractions, see [3]), and of the general n-fractions introduced in [13].

In section 2 we given the definition of an (r, s)-matrix continued fraction associated with a first-order recurrence system of the form

$$Y_k = \theta_k Y_{k-1}\ , \quad k = 0, 1, ...,$$

with $\theta_k \in \mathbb{C}^{n\times n}$ and $Y_k \in \mathbb{C}^{n\times 1}$. We prove a Pincherle type convergence theorem for these MCF's and we show that they can be generated using linear fractional transformations with matrix elements.

In section 3 and 4 we show that these MCF's are generalizations of the generalized continued fractions that are associated with linear recurrence relations.

In section 5 we give some references to the case $r = s$.

In section 6 an application is given: we show how MCF's can be used to calculate non-dominant solutions of the recurrence system in a stable manner. Other algorithms to solve this problem can be found in [4], [10], [16], [17] and [26].

In the next two sections we consider some special cases: the case that the matrix of the recurrence system does not depend on k, i.e.,

$$\theta_k = \theta \quad \text{for all} \quad k$$

and the case that the recurrence system is of Poincaré-type, i.e.,

$$\lim_{k\to+\infty} \theta_k = \theta.$$

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† Department of Computing Science, K.U.Leuven, Celestijnenlaan 200A, B-3001 Heverlee, Belgium.
‡ Departement IWT, Karel de Grote-Hogeschool, Campus KIHA, Salesianenlaan 30, B-2660 Hoboken, Belgium (paul@kiha.be).

In each of these cases we prove that if the eigenvalues of θ are all different in modulus, then the associated MCF's converge. Furthermore we look at the convergence of some sequences associated with these MCF's.

In the final section we look at the adjoint recurrence system and discuss duality.

2. Matrix Continued Fractions: Definitions. We consider the first-order recurrence system

$$Y_k = \theta_k Y_{k-1} \quad , \qquad k = 0, 1, \ldots, \tag{2.1}$$

with $Y_k \in \mathbb{C}^{n\times 1}$ and $\theta_k \in \mathbb{C}^{n\times n}$, where we assume that all θ_k are nonsingular. The matrices θ_k are divided into four blocks

$$\theta_k = \begin{pmatrix} c_k & d_k \\ a_k & b_k \end{pmatrix}$$

with $c_k \in \mathbb{C}^{r\times r}$, $d_k \in \mathbb{C}^{r\times s}$, $a_k \in \mathbb{C}^{s\times r}$, $b_k \in \mathbb{C}^{s\times s}$ and with $r + s = n$.

This leads to a splitting up of the vectors Y_k into two parts:

$$Y_k = \begin{pmatrix} Y_k^{(1)} \\ Y_k^{(2)} \end{pmatrix}$$

with $Y_k^{(1)} \in \mathbb{C}^{r\times 1}$ and $Y_k^{(2)} \in \mathbb{C}^{s\times 1}$.

A solution Z_k of this system is completely determined by the initial value Z_{-1}:

$$Z_k = \Theta_k Z_{-1} \quad , \quad k = 0, 1, \ldots,$$

with

$$\Theta_k = \theta_k \theta_{k-1} \ldots \theta_1 \theta_0.$$

We use the following notation for the blocks of Θ_k:

$$\Theta_k = \begin{pmatrix} C_k & D_k \\ A_k & B_k \end{pmatrix} = \begin{pmatrix} c_k & d_k \\ a_k & b_k \end{pmatrix} \cdot \begin{pmatrix} c_{k-1} & d_{k-1} \\ a_{k-1} & b_{k-1} \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} c_0 & d_0 \\ a_0 & b_0 \end{pmatrix}.$$

If $X_k \in \mathbb{C}^{n\times n}$ satisfies

$$X_k = \theta_k X_{k-1} \quad , \quad k = 0, 1, \ldots,$$

with X_{-1} regular, then the columns of X_k constitute n linearly independent solutions of (2.1). Such a sequence X_k is called a fundamental system of solutions of (2.1).

We define the (r, s)-matrix continued fraction (MCF) associated with the first-order recurrence system (2.1) by its sequence of approximants

$$\frac{A_k}{B_k} \ , \quad k = 0, 1, 2, \ldots,$$

where the division of matrices should be interpreted as a multiplication from the left with the inverse

$$\frac{P}{Q} = Q^{-1} P.$$

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etna@mcs.kent.edu

The matrix continued fraction is said to converge if

$$\lim_{k\to+\infty} \frac{A_k}{B_k} \in \mathbb{C}^{s\times r}.$$

The tail of the MCF for the m-th approximant is defined as the MCF associated with the system

$$Y_k = \theta_{k+m} Y_{k-1} \quad , \qquad k = 0, 1, \ldots,$$

We have the following generalization of a result by Pincherle - Van der Cruyssen [23]:

THEOREM 2.1. *The MCF associated with the system (2.1) converges if and only if the recurrence system (2.1) has a fundamental system of solutions* $X_k \in \mathbb{C}^{n\times n}$*:*

$$X_k = \Theta_k X_{-1} \quad , \qquad k = 0, 1, \ldots, \quad \textit{with} \quad X_{-1} \quad \textit{regular},$$

satisfying

$$(\alpha) \qquad X^c_{-1} \quad \textit{is regular;}$$

$$(\beta) \qquad \lim_{k\to+\infty} \frac{X^a_k}{X^b_k} = 0,$$

where

$$X_k = \begin{pmatrix} X^c_k & X^d_k \\ X^a_k & X^b_k \end{pmatrix}, \qquad X^c_k \in \mathbb{C}^{r\times r}.$$

Proof. Let us first assume that (α) and (β) are satisfied. We set $\Theta_{-1} = I_n$. Since

$$X_k = \begin{pmatrix} X^c_k & X^d_k \\ X^a_k & X^b_k \end{pmatrix} = \Theta_k X_{-1} = \begin{pmatrix} C_k & D_k \\ A_k & B_k \end{pmatrix} X_{-1},$$

we get, by setting

$$F = \begin{pmatrix} F^c & F^d \\ F^a & F^b \end{pmatrix}, = (X_{-1})^{-1}$$

that

$$\Theta_k = X_k F, \quad \text{i.e.,} \quad \begin{pmatrix} C_k & D_k \\ A_k & B_k \end{pmatrix} = \begin{pmatrix} X^c_k & X^d_k \\ X^a_k & X^b_k \end{pmatrix} \cdot \begin{pmatrix} F^c & F^d \\ F^a & F^b \end{pmatrix}. \tag{2.2}$$

Multiplying we get

$$A_k = X^a_k F^c + X^b_k F^a$$

and

$$B_k = X^a_k F^d + X^b_k F^b.$$

Hence

$$\frac{A_k}{B_k} = \frac{X^a_k F^c + X^b_k F^a}{X^a_k F^d + X^b_k F^b} = \frac{\frac{X^a_k}{X^b_k} \cdot F^c + F^a}{\frac{X^a_k}{X^b_k} \cdot F^d + F^b},$$

and we get immediately from (β) that

$$\lim_{k\to+\infty} \frac{A_k}{B_k} = \frac{F^a}{F^b},$$

if F^b is regular. To prove that F^b is regular, we observe that

$$X_{-1}F = I_n,$$

hence

$$0 = X^c_{-1}F^d + X^d_{-1}F^b.$$

If F^b is singular, we can find a vector $V \in \mathbb{C}^{s\times 1}$ for which $F^b V = 0$. From the previous equation we then get

$$0 = X^c_{-1}F^d V,$$

or, since X^c_{-1} is assumed to be regular:

$$F^d V = 0.$$

Together with $F^b V = 0$ this would imply that F is singular, a contradiction.

Let us now assume that the matrix continued fraction associated with (2.1) converges and that

$$\lim_{k\to+\infty} \frac{A_k}{B_k} = T_0.$$

The sequence of matrices

$$\begin{pmatrix} C_k - D_k \cdot T_0 & D_k \\ A_k - B_k \cdot T_0 & B_k \end{pmatrix} = \begin{pmatrix} C_k & D_k \\ A_k & B_k \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -T_0 & I_s \end{pmatrix} \tag{2.3}$$

is a fundamental system of solutions of (2.1) satisfying (β) and (α) since the rightmost matrix is obviously regular and

$$\lim_{k\to+\infty} \frac{A_k - B_k \cdot T_0}{B_k} = \lim_{k\to+\infty} \frac{A_k}{B_k} - T_0 = 0.$$

□ A similar result for the case $r = s$ was proved in [2].

For of a second-order linear homogeneous recurrence relation

$$y_{k+1} = b_k y_k + a_k y_{k-1} \quad , \quad k = 0, 1, \ldots \tag{2.4}$$

with $a_k, b_k \in \mathbb{C}$ (corresponding to

$$\theta_k = \begin{pmatrix} 0 & 1 \\ a_k & b_k \end{pmatrix}$$

in our notation) the previous theorem is given in [6]: the ordinary continued fraction

$$\frac{a_0|}{|b_0} + \frac{a_1|}{|b_1} + \ldots + \frac{a_k|}{|b_k} + \ldots$$

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Kent State University
etna@mcs.kent.edu

converges if and only if the recurrence relation (2.4) has a solution f_k with $f_{-1} \neq 0$ satisfying

$$\lim_{k \to +\infty} \frac{f_k}{g_k} = 0$$

with g_k a solution of (2.4) linearly independent of f_k. The solution f_k is called a non-dominant (or minimal) solution of (2.4). The solution g_k is called dominant. It is well–known that the computation of non-dominant solutions using forward recurrence is numerically unstable.

The condition (β) of the theorem expresses that the solutions spanned by the first r columns of X_k are dominated by the solutions spanned by the last $s = n - r$ columns.

Let the MCF related to the system (2.1) converge to T_0. It follows from the proof of the previous theorem that a non-dominant solution Z_k of (2.1) is in the subspace spanned by the columns of the matrix

$$\begin{pmatrix} C_k - D_k \cdot T_0 \\ A_k - B_k \cdot T_0 \end{pmatrix}. \tag{2.5}$$

Thus its initial conditions Z_{-1} satisfy

$$Z_{-1}^{(2)} = -T_0 \cdot Z_{-1}^{(1)}.$$

Furthermore we have

$$Z_0^{(1)} = (c_0 - d_0 \cdot T_0) \cdot Z_{-1}^{(1)}.$$

If we assume that the m-th tail converges, i.e., the MCF associated with the system

$$Y_k = \theta_{k+m} Y_{k-1} \quad , \quad k = 0, 1, \ldots$$

converges for all m to the matrix T_m, then the solution Z_k of the system (2.1) which is in the column space of (2.5) satisfies:

$$Z_{k-1}^{(2)} = -T_k \cdot Z_{k-1}^{(1)} \ , \tag{2.6}$$

$$Z_k^{(1)} = (c_k - d_k \cdot T_k) \cdot Z_{k-1}^{(1)} \ , \tag{2.7}$$

and it is easy to prove that

$$T_k = \frac{a_k + T_{k+1} \cdot c_k}{b_k + T_{k+1} \cdot d_k}. \tag{2.8}$$

We note that the subspace spanned by the columns of (2.5) is equal to the subspace spanned by the columns of

$$\begin{pmatrix} X_k^c \\ X_k^a \end{pmatrix},$$

since it is a consequence of (2.2) and (2.3) that

$$\begin{pmatrix} C_k - D_k \cdot T_0 & D_k \\ A_k - B_k \cdot T_0 & B_k \end{pmatrix} = \begin{pmatrix} X_k^c & X_k^d \\ X_k^a & X_k^b \end{pmatrix} \cdot \begin{pmatrix} F^c & F^d \\ F^a & F^b \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -T_0 & I_s \end{pmatrix},$$

ETNA
Kent State University
etna@mcs.kent.edu

and hence

$$\begin{pmatrix} C_k - D_k \cdot T_0 \\ A_k - B_k \cdot T_0 \end{pmatrix} = \begin{pmatrix} X_k^c \\ X_k^a \end{pmatrix} \cdot (F^c - F^d T_0)$$

with $(F^c - F^d T_0)$ regular ($F^a - F^b T_0 = 0$ from the proof of theorem 1).

This construction will be used in section 4 to find a numerically stable method to compute an non-dominant solution of the recurrence (2.1).

Note that the approximants of the matrix continued fraction associated with the system (2.1) may be calculated from the composition of linear fractional transformations

$$\begin{aligned} & s_k(W) = \frac{a_k + W c_k}{b_k + W d_k} && (k = 0, 1, \ldots) \\ & S_0(W) = s_0(W) \qquad \text{and} \quad S_k(W) = S_{k-1}(s_k(W)) && (k = 1, 2, \ldots) \end{aligned} \tag{2.9}$$

with $W \in \mathbb{C}^{s \times r}$.

We have the following theorem :

THEOREM 2.2.

$$S_k(W) = \frac{A_k + W C_k}{B_k + W D_k}.$$

Proof. By induction on k, using simple algebra. □

Hence

$$S_k(0) = \frac{A_k}{B_k},$$

the k-th approximant of the MCF.

3. Example 1: Linear recurrence relations.

We show that a classical recurrence relation of order n fits in the framework of (r, s)-MCF's.

Let

$$\theta_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_k^{(n)} & \alpha_k^{(n-1)} & \alpha_k^{(n-2)} & \cdots & \alpha_k^{(1)} \end{pmatrix}. \tag{3.1}$$

with $\alpha_k^{(n)} \neq 0$ for all k. If we put

$$Y_k = \begin{pmatrix} z_{k+1} \\ z_{k+2} \\ \vdots \\ z_{k+n} \end{pmatrix}, \quad k = -1, 0, 1, \ldots.$$

in (2.1), then this first–order system is equivalent with the n-th-order linear recurrence relation

$$z_{k+n} = \alpha_k^{(1)} z_{k+n-1} + \alpha_k^{(2)} z_{k+n-2} + \cdots + \alpha_k^{(n)} z_k\ , \quad k = 0, 1, 2, \ldots \tag{3.2}$$

If we denote by $z_k^{(1)}, z_k^{(2)}, \ldots, z_k^{(n)}$, $(k = 0, 1, 2, \ldots)$, the solutions of (3.2) with initial values given by

$$\begin{pmatrix} z_0^{(1)} & z_0^{(2)} & \cdots & z_0^{(n)} \\ z_1^{(1)} & z_1^{(2)} & \cdots & z_1^{(n)} \\ \vdots & \vdots & & \vdots \\ z_{n-1}^{(1)} & z_{n-1}^{(2)} & \cdots & z_{n-1}^{(n)} \end{pmatrix} = I_n,$$

then it is easy to see that

$$\Theta_k = \begin{pmatrix} z_k^{(1)} & z_k^{(2)} & \cdots & z_k^{(n)} \\ z_{k+1}^{(1)} & z_{k+1}^{(2)} & \cdots & z_{k+1}^{(n)} \\ \vdots & \vdots & & \vdots \\ z_{k+n-1}^{(1)} & z_{k+n-1}^{(2)} & \cdots & z_{k+n-1}^{(n)} \end{pmatrix},$$

and the k-th approximant of the (r, s)-matrix continued fraction associated with (2.1, 3.1) is given by

$$S_k(0) = \begin{pmatrix} z_{k+r}^{(r+1)} & \cdots & z_{k+r}^{(n)} \\ \vdots & & \vdots \\ z_{k+n-1}^{(r+1)} & \cdots & z_{k+n-1}^{(n)} \end{pmatrix}^{-1} \cdot \begin{pmatrix} z_{k+r}^{(1)} & \cdots & z_{k+r}^{(r)} \\ \vdots & & \vdots \\ z_{k+n-1}^{(1)} & \cdots & z_{k+n-1}^{(r)} \end{pmatrix}. \tag{3.3}$$

Let us assume that the matrix continued fraction associated with the system $Y_k = \theta_{k+m} Y_{k-1}$ converges for $m = 0, 1, \ldots$ to T_m. In this case (2.7) reduces to

$$Z_k^{(1)} = \left(\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \end{pmatrix} \cdot T_k \right) \cdot Z_{k-1}^{(1)}$$

or, with $T_k = (t_k^{(i,j)})$,

$$Z_k^{(1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ t_k^{(1,1)} & t_k^{(1,2)} & t_k^{(1,3)} & \cdots & t_k^{(1,r)} \end{pmatrix} \cdot Z_{k-1}^{(1)}.$$

This equation is of the same form as (3.1). Hence the recurrence relation (3.2) reduces to

$$z_{k+r} = t_k^{(1,r)} z_{k+r-1} + t_k^{(1,r-1)} z_{k+r-2} + \cdots + t_k^{(1,1)} z_k \ , \quad k = 0, 1, 2, \ldots \ ,$$

a linear recurrence relation of order r. We note that only the first row of T_k is needed, and that the calculation of this first row of T_k from (2.8) can be done without the use of the other rows (see [12]).

With (3.3) we can prove that this method is equivalent to using the generalized continued fractions (n-fractions) of de Bruin [3]- Van der Cruyssen [23] in the case $r = n - 1$, the generalized continued fraction of Zahar [24] in the case $n = 1$, or the generalized n-fractions in [13] for the general case.

4. Example 2: Vector recurrence relations. Let $n = m \cdot p$ and

$$\theta_k = \begin{pmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ \alpha_k^{(p)} & \alpha_k^{(p-1)} & \alpha_k^{(p-2)} & \cdots & \alpha_k^{(1)} \end{pmatrix}$$

with $\alpha_k^{(i)} \in \mathbb{C}^{m\times m}$, and $\alpha_k^{(p)}$ regular for all k.

If we put

$$Y_k = \begin{pmatrix} z_{k+1} \\ z_{k+2} \\ \vdots \\ z_{k+p} \end{pmatrix} , \quad k = -1, 0, 1, \ldots, \quad \text{with } z_k \in \mathbb{C}^{m\times 1}$$

in (2.1), then this first-order system is equivalent with

$$z_{k+p} = \alpha_k^{(1)} z_{k+p-1} + \alpha_k^{(2)} z_{k+p-2} + \cdots + \alpha_k^{(p)} z_k \ , \quad k = 0, 1, 2, \ldots .$$

A set of equations of this form is called a vector recurrence relation (see e.g. [21]). As the previous example is a special case of this one ($m = 1$), the results from the previous section can easily be adapted to this type of recurrence relation.

5. Example 3: (r, r)-matrix continued fractions. The general case of an MCF with $r = s$ was studied e.g. in [20] and [22] (see also [5], [2]).

In all these references the division of matrices is interpreted as a multiplication from the right with the inverse (see also section 9).

6. Application: Numerical calculation of non-dominant solutions of a recurrence system. We use theorem 1 to calculate solutions of the recurrence system (2.1) which in a certain sense are non-dominant (condition (β) of the theorem), and cannot be calculated numerically from (2.1) using forward recurrence. We take an example from [10]. Let

$$\theta_k = \frac{1}{4} \begin{pmatrix} \frac{2k+5}{\sqrt{2}} - 2k - 4 + 2\sqrt{2} & \sqrt{2} - 4\sqrt{2} & -\frac{2k+3}{\sqrt{2}} + 2k + 4 + 2\sqrt{2} \\ \frac{2k+5}{\sqrt{2}} - 2k - 6 - 2\sqrt{2} & \sqrt{2} + 4\sqrt{2} & -\frac{2k+3}{\sqrt{2}} + 2k + 6 - 2\sqrt{2} \\ \frac{2k+5}{\sqrt{2}} - 2k - 8 + 2\sqrt{2} & \sqrt{2} - 4\sqrt{2} & -\frac{2k+3}{\sqrt{2}} + 2k + 8 + 2\sqrt{2} \end{pmatrix} .$$

A fundamental system X_k of solutions is given by

$$X_k = \begin{pmatrix} (1/\sqrt{2})^{k+1} & k+2 & (\sqrt{8})^{k+1} \\ (1/\sqrt{2})^{k+1} & k+3 & -(\sqrt{8})^{k+1} \\ (1/\sqrt{2})^{k+1} & k+4 & -(\sqrt{8})^{k+1} \end{pmatrix} , \quad k = -1, 0, 1, \ldots .$$

Hence the conditions of theorem 1 are satisfied for $r = 1, s = 2$, and the $(1, 2)$-matrix continued fraction associated with the recurrence system

$$Y_k = \theta_{k+m} Y_{k-1} \ , \quad k = 0, 1, \ldots$$

converges for all m to T_m, say. It is easy to see that we cannot calculate the solution

$$(2/(\sqrt{2})^k, 2/(\sqrt{2})^k, 2/(\sqrt{2})^k)^\tau, \quad k = 0, 1, \ldots ,$$

ETNA
Kent State University
etna@mcs.kent.edu

Table 6.1

Absolute errors in the calculation of a non-dominant solution Z_k of the recurrence system of section 6 using forward recurrence and a $(1,2)$-MCF with $N = 39$.

k	forward	MCF
-1	0	8.9E-8
9	3.1E-13	5.8E-07
19	9.9E-09	1.0E-06
29	3.2E-04	1.5E-06
39	1.0E+01	1.9E-06

Table 6.2

Absolute errors in the calculation of a non-dominant solution Z_k of the recurrence system of section 6 using forward recurrence and a $(2,1)$-MCF with $N = 49$.

k	forward	MCF
-1	0	0
9	4.7E-13	1.4E-13
19	1.4E-08	2.7E-12
29	4.5E-04	4.9E-08
39	1.5E+01	1.6E-03
49	4.9E+05	5.3E+01

in a stable manner using forward recurrence. The conditions of theorem 1 are satisfied with $r = 1$ and $s = 2$. We use (2.6) and (2.7) to calculate approximations to Z_k: with (2.8) we calculate for some index N

$$T_N = 0, \qquad T_k = \frac{a_k + T_{k+1} \cdot c_k}{b_k + T_{k+1} \cdot d_k}, \quad k = N-1, N-2, \ldots, 1, 0,$$

and then we use (2.6) and (2.7) to get approximations to the solution we want, with $Z_{-1}^{(1)} = 2$. For $N = 39$ the results are given in the tables 6.1 and 6.2. We have also calculated the solution Z_k using forward recurrence. In table 6.1 the maximum of the absolute errors in the three components of Z_k is given for some values of k.

The solution

$$(k+1, k+2, k+3)^\tau, \quad k = 0, 1, \ldots,$$

is also non-dominant. The conditions of theorem 1 are satisfied with $r = 2$ and $s = 1$. In table 6.2 we use the same methods as before, with $N = 49$.

This method is related to the method described in [26] in the same way as Gautschi's method [6] to calculate minimal solutions of linear second-order recurrence relations is related to Olver's method [18].

We note that the theoretical method behind this algorithm is known in the literature as *method of embedding* (see [1]).

7. Special cases I - Periodic MCF. Let us assume that the matrix of the recurrence system is constant:

$$\theta_k = \theta = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Then we have $\Theta_k = \theta^{k+1}$. Let us also assume that θ has eigenvalues which are all different in modulus:

$$|\lambda_1| < |\lambda_2| < \ldots < |\lambda_n|.$$

Let $\Lambda \in \mathbb{C}^{n\times n}$ be defined by $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ ($\mathrm{diag}(\alpha, \ldots, \gamma)$ is a diagonal matrix with the given arguments as diagonal elements in the given order), and $P \in \mathbb{C}^{n\times n}$ is the matrix whose columns are the corresponding eigenvectors $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$:

$$\theta P = P\Lambda.$$

Set

$$P = \begin{pmatrix} P^c & P^d \\ P^a & P^b \end{pmatrix} \quad \text{with} \quad P^c \in \mathbb{C}^{r\times r}.$$

Thus if

$$Y_k = \theta Y_{k-1} \quad , \quad k = 0, 1, \ldots \tag{7.1}$$

then Y_k is in the column space of

$$P_k = \theta^{k+1} P = P\Lambda^{k+1}.$$

Assume P^c is regular. Then it follows from theorem 1 that the MCF associated with (7.1) converges to some T_0 say, which is given by

$$T_0 = \frac{Q^a}{Q^b},$$

where

$$Q = \begin{pmatrix} Q^c & Q^d \\ Q^a & Q^b \end{pmatrix} = P^{-1}.$$

Note that if P are the right eigenvectors of θ, $\theta\, P = P\, \Lambda$, then $Q = P^{-1}$ are the left eigenvectors of θ, $Q\, \theta = \Lambda\, Q$. Moreover because

$$Q\, P = I_n,$$

we have

$$Q^a P^c + Q^b P^a = 0 \quad \text{or} \quad (Q^b)^{-1} Q^a = -P^a (P^c)^{-1}.$$

In terms of the recurrence (2.8) we have the following result: *the sequence U_k generated by*

$$U_{-1} = 0, \quad U_{k+1} = \frac{a + U_k \cdot c}{b + U_k \cdot d}$$

is the sequence of approximants $U_k = T_k$; hence it converges to

$$T_0 = -P^a \cdot (P^c)^{-1} = (Q^b)^{-1} \cdot Q^a.$$

Note that T_0 is constructed from the eigenvectors associated with the smallest eigenvalues of θ, thus it is associated with non-dominant solutions of the recurrence (7.1).

ETNA
Kent State University
etna@mcs.kent.edu

If we use the matrix continued fraction (2.8) in the forward direction, i.e., if we set

$$U_k = \frac{a + U_{k+1} \cdot c}{b + U_{k+1} \cdot d} \quad \Rightarrow \quad U_{k+1} = -(a - b \cdot U_k) \cdot (c - d \cdot U_k)^{-1}.$$

then, defining $V_k = -(U_k)^\tau$, we find that it satisfies

$$V_{k+1} = \frac{a^\tau + V_k \cdot b^\tau}{c^\tau + V_k \cdot d^\tau}.$$

To apply the previous results to the recurrence system with matrix

$$\mu = \begin{pmatrix} b^\tau & d^\tau \\ a^\tau & c^\tau \end{pmatrix} = J\,\theta^\tau\,J^\tau \ ; \quad J = \begin{pmatrix} 0 & I_s \\ I_r & 0 \end{pmatrix},$$

we need the eigenvalue decomposition of μ. We use $Q\,\theta = \Lambda\,Q$ to get

$$J\,\theta^\tau\,J^\tau\,J\,Q^\tau = J\,Q^\tau\,\Lambda^\tau \ \text{ or } \ \mu\,\tilde{Q} = \tilde{Q}\,\Lambda\ , \quad \tilde{Q} = J\,Q^\tau.$$

Subdividing Q as follows

$$Q = \begin{pmatrix} Q^{c'} & Q^{d'} \\ Q^{a'} & Q^{b'} \end{pmatrix} \ \text{ with } \ Q^{d'} \in \mathbb{C}^{s\times s},$$

we get

$$J\,Q^\tau = \begin{pmatrix} (Q^{d'})^\tau & (Q^{b'})^\tau \\ (Q^{c'})^\tau & (Q^{a'})^\tau \end{pmatrix},$$

and hence, if $Q^{d'}$ is regular, the sequence V_k will converge to $-(Q^{c'})^\tau(Q^{d'})^\tau$.

Subdividing P as

$$P = \begin{pmatrix} P^{c'} & P^{d'} \\ P^{a'} & P^{b'} \end{pmatrix} \ \text{ with } \ P^{d'} \in \mathbb{C}^{r\times r},$$

we obtain from $Q\,P = I_n$ that

$$Q^{c'}\,P^{d'} + Q^{d'}\,P^{b'} = 0.$$

Thus we have that *the sequence* $U_k = -V_k^\tau = -A_k\,C_k^{-1}$ *generated by*

$$U_{-1} = 0, \quad U_{k+1} = -(a - b \cdot U_k) \cdot (c - d \cdot U_k)^{-1}$$

converges to

$$(Q^{d'})^{-1} \cdot Q^{c'} = -P^{b'} \cdot (P^{d'})^{-1},$$

if $Q^{d'}$ *is regular.* Note that this MCF is associated with the eigenvectors for the largest eigenvalues of θ. It is associated with dominant solutions of the recurrence (7.1).

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8. Special cases II - Limit periodic MCF. Let us assume that the matrix of the recurrence system satisfies:

$$\lim_{k\to+\infty} \theta_k = \theta = \begin{pmatrix} c & d \\ a & b \end{pmatrix}. \tag{8.1}$$

In [19] Perron proved the following theorem:

THEOREM 8.1. *If the recurrence system (2.1) has the property (8.1) with for all k $\det\theta_k \neq 0$, and if the eigenvalues of θ are all different in modulus :*

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|,$$

then for each $j \in \{1, 2, \ldots, n\}$ the recurrence system (2.1) has a solution

$$X_k^{(j)} = (x_k^{(1,j)} \; ... \; x_k^{(n,j)})^\tau,$$

where

$$\lim_{k\to\infty} \frac{x_{k+1}^{(i,j)}}{x_k^{(i,j)}} = \lambda_j$$

for all $i \in \{1, 2, \ldots, n\}$ for which the eigenvector $(p_1^{(j)}, \ldots, p_n^{(j)})^\tau$ corresponding to the eigenvalue λ_j has i-th component different from zero, i.e., $p_i^{(j)} \neq 0$. Furthermore, if $p_i^{(j)} \neq 0$, then

$$\lim_{k\to+\infty} \frac{x_k^{(m,j)}}{x_k^{(i,j)}} = \frac{p_m^{(j)}}{p_i^{(j)}}$$

for all $m \neq i$. We combine this theorem with a result by Máté and Nevai [15]:

THEOREM 8.2. *If the recurrence system (2.1) has the property (8.1), and if the eigenvalues of θ are all different in modulus, then for every solution Z_k of (2.1) either $Z_k = 0$ for all large enough k, or $Z_k \neq 0$ for all large enough k, and in this case there is a j with $1 \leq j \leq n$ and a sequence of complexe numbers γ_k such that*

$$\lim_{k\to+\infty} \frac{Z_k}{\gamma_k} = p^{(j)}.$$

This leads to

THEOREM 8.3. *If the recurrence system (2.1) has the property (8.1) with $\det\theta_k \neq 0$ for all k and if*

$$\theta\, P = P\,\Lambda \;, \quad \Lambda = diag(\lambda_1, \ldots, \lambda_n),$$

where

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|,$$

is the eigenvalue decomposition of θ with all eigenvalues different in modulus, then there exists a fundamental system of solutions X_k for the recurrence (2.1) and complex diagonal matrices $\Gamma_k = \mathrm{diag}(\gamma_k^{(1)}, \ldots, \gamma_k^{(n)})$ such that

$$\lim_{k\to+\infty} X_k \Gamma_k^{-1} = P \quad \textit{and} \quad \lim_{k\to+\infty} \Gamma_{k+1}\Gamma_k^{-1} = \Lambda.$$

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Moreover,

$$\lim_{k\to+\infty}\frac{\gamma_k^{(i)}}{\gamma_k^{(j)}}=0 \quad \textit{for all} \quad j>i. \tag{8.2}$$

Let us assume that the conditions of theorem 5 are satisfied, and let X_k be the matrix which has the solutions $X_k^{(1)},\ldots,X_k^{(n)}$ of the system (2.1) as columns in the given order, with

$$X_k^{(j)}=(x_k^{(1,j)}\ldots x_k^{(n,j)})^\tau$$

as in theorem 3. Then we can write

$$X_k=(P+\Phi_k)\,\Gamma_k$$

with $\Phi_k\in\mathbb{C}^{n\times n}$ and

$$\lim_{k\to+\infty}\Phi_k=0.$$

Set

$$\Phi=\begin{pmatrix}\Phi^c & \Phi^d\\ \Phi^a & \Phi^b\end{pmatrix} \quad \text{with} \quad \Phi^c\in\mathbb{C}^{r\times r}.$$

If P^b and P^c are regular, then the conditions of theorem 1 are satisfied from some $k=k_0$ on, i.e.,

$$X_k^c \;\text{ is }\; \text{regular}$$

(this follows from the regularity of P^c) and

$$\lim_{k\to+\infty}\frac{X_k^a}{X_k^b}=0.$$

To prove this we note that

$$\frac{X_k^a}{X_k^b}=\frac{(P^a+\Phi_k^a)\operatorname{diag}(\gamma_k^{(1)},\ldots,\gamma_k^{(r)})}{(P^b+\Phi_k^b)\operatorname{diag}(\gamma_k^{(r+1)},\ldots,\gamma_k^{(n)})}$$

or

$$(X_k^b)^{-1}X_k^a=\operatorname{diag}\left(\frac{1}{\gamma_k^{(r+1)}},\ldots,\frac{1}{\gamma_k^{(n)}}\right)(P^b+\Phi_k^b)^{-1}(P^a+\Phi_k^a)\operatorname{diag}(\gamma_k^{(1)},\ldots,\gamma_k^{(r)}).$$

The element in the i-th row and j-th column of the matrix on the right is of the form

$$\frac{\gamma_k^{(j)}}{\gamma_k^{(r+i)}}\,u_k \quad \text{with} \quad \lim_{k\to+\infty}u_k\in\mathbb{C}.$$

It now follows immediately from (8.2) that all elements of $(X_k^b)^{-1}X_k^a$ tend to zero if $k\to+\infty$.

Hence the tail, i.e., the MCF associated with the system

$$Y_k = \theta_{k+m} Y_{k-1} \quad , \quad k = 0, 1, \ldots$$

converges for all $m \geq k_0$. Let us assume from now on, for the sake of simplicity, that $k_0 = 0$. Using (2.6) we then get

$$T_k = -X^a_{k-1} (X^c_{k-1})^{-1},$$

and taking the limit for $k \to +\infty$ using theorem 5, we find

$$\lim_{k\to+\infty} T_k = - \lim_{k\to+\infty} (P^a + \Phi^a_{k-1})(P^c + \Phi^c_{k-1})^{-1} = -P^a (P^c)^{-1},$$

Hence the tails of the continued fraction (2.8) converge to a matrix built from the eigenvectors of θ corresponding to the r smallest eigenvalues. If we use the matrix continued fraction (2.8) in the forward direction, we get

$$U_k = \frac{a_k + U_{k+1} \cdot c_k}{b_k + U_{k+1} \cdot d_k} \quad \Rightarrow \quad U_{k+1} = -(a_k - b_k \cdot U_k) \cdot (c_k - d_k \cdot U_k)^{-1},$$

and using induction it is easy to prove that

$$U_{k+1} = -(A_k - B_k \cdot U_0)(C_k - D_k \cdot U_0)^{-1}.$$

Taking $U_0 = 0$ we find

$$U_{k+1} = -A_k (C_k)^{-1}.$$

We set

$$X_k = \begin{pmatrix} X_k^{c'} & X_k^{d'} \\ X_k^{a'} & X_k^{b'} \end{pmatrix} \text{ with } X_k^{a'} \in \mathbb{C}^{s\times s},$$

and

$$F = \begin{pmatrix} F^{c''} & F^{d''} \\ F^{a''} & F^{b''} \end{pmatrix} \text{ with } F^{a''} \in \mathbb{C}^{r\times r},$$

and then we get from $\Theta_k = X_k F$ that

$$A_k = X_k^{a'} F^{c''} + X_k^{b'} F^{a''} \text{ and } C_k = X_k^{c'} F^{c''} + X_k^{d'} F^{a''}.$$

Using the notation

$$\Gamma_k^{(s)} = \mathrm{diag}(\gamma_k^{(1)}, \ldots, \gamma_k^{(s)}) \text{ and } \Gamma_k^{(r)} = \mathrm{diag}(\gamma_k^{(s+1)}, \ldots, \gamma_k^{(n)}),$$

as a consequence of theorem 5 we now have

$$X_k^{a'} = (P^{a'} + \Phi_k^{a'})\, \Gamma_k^{(s)} \text{ and } X_k^{c'} = (P^{c'} + \Phi_k^{c'})\, \Gamma_k^{(s)},$$

$$X_k^{b'} = (P^{b'} + \Phi_k^{b'})\, \Gamma_k^{(r)} \text{ and } X_k^{d'} = (P^{d'} + \Phi_k^{d'})\, \Gamma_k^{(r)},$$

where the division of P and Φ_k into blocks is the same as that for X_k.

If the matrix $F^{a''}$ is regular, we can write

$$U_{k+1} = -A_k(C_k)^{-1} = -A_k(F^{a''})^{-1}(\Gamma_k^{(r)})^{-1}\left[C_k(F^{a''})^{-1}(\Gamma_k^{(r)})^{-1}\right]^{-1}$$

with

$$A_k(F^{a''})^{-1}(\Gamma_k^{(r)})^{-1} = (P^{a'} + \Phi_k^{a'})\Gamma_k^{(s)}F^{c''}(F^{a''})^{-1}(\Gamma_k^{(r)})^{-1} + P^{b'} + \Phi_k^{b'} \text{ and}$$

$$C_k(F^{a''})^{-1}(\Gamma_k^{(r)})^{-1} = (P^{c'} + \Phi_k^{c'})\Gamma_k^{(s)}F^{c''}(F^{a''})^{-1}(\Gamma_k^{(r)})^{-1} + P^{d'} + \Phi_k^{d'}.$$

Since

$$\lim_{k\to+\infty} \Gamma_k^{(s)}F^{c''}(F^{a''})^{-1}(\Gamma_k^{(r)})^{-1} = 0,$$

we obtain

$$\lim_{k\to+\infty} U_{k+1} = -P^{b'}(P^{d'})^{-1},$$

if the matrix $P^{d'}$ is regular. Hence *the sequence U_k generated by*

$$U_0 = 0, \quad U_{k+1} = -(a_k - b_k \cdot U_k)\cdot(c_k - d_k\cdot U_k)^{-1},$$

and obtained by using (2.8) *in the forward direction, converges to*

$$-\begin{pmatrix} p_{r+1}^{(s+1)} & \dots & p_{r+1}^{(n)} \\ \vdots & & \vdots \\ p_n^{(s+1)} & \dots & p_n^{(n)} \end{pmatrix}\begin{pmatrix} p_1^{(s+1)} & \dots & p_1^{(n)} \\ \vdots & & \vdots \\ p_r^{(s+1)} & \dots & p_r^{(n)} \end{pmatrix}^{-1},$$

a matrix built from the eigenvectors of θ corresponding to the r largest eigenvalues.

We now apply these results to the first-order recurrence system

$$Y_k = \theta_k^{-1}Y_{k-1},$$

and we write

$$\theta_k^{-1} = \begin{pmatrix} \hat{c}_k & \hat{d}_k \\ \hat{a}_k & \hat{b}_k \end{pmatrix}.$$

Then the sequence U_k generated by

$$U_0 = 0, \quad U_{k+1} = -(\hat{a}_k - \hat{b}_k\cdot U_k)\cdot(\hat{c}_k - \hat{d}_k\cdot U_k)^{-1} \tag{8.3}$$

will converge to

$$-\begin{pmatrix} p_{r+1}^{(r)} & \dots & p_{r+1}^{(1)} \\ \vdots & & \vdots \\ p_n^{(r)} & \dots & p_n^{(1)} \end{pmatrix}\begin{pmatrix} p_1^{(r)} & \dots & p_1^{(1)} \\ \vdots & & \vdots \\ p_r^{(r)} & \dots & p_r^{(1)} \end{pmatrix}^{-1},$$

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since $p^{(r)}, \ldots, p^{(1)}$ are the eigenvectors of θ_k^{-1} corresponding to the r largest eigenvalues. The matrix above is equal to $-P_a \cdot (P_c)^{-1}$. Now the recurrence (8.3) can be rewritten as

$$U_{k+1} = \frac{a_k + U_k \cdot c_k}{b_k + U_k \cdot d_k}$$

since

$$(b_k + U_k \cdot d_k)^{-1} \cdot (a_k + U_k \cdot c_k) = -(\hat{a}_k - \hat{b}_k \cdot U_k) \cdot (\hat{c}_k - \hat{d}_k \cdot U_k)^{-1}$$

which follows easily from the identity

$$\begin{pmatrix} c_k & d_k \\ a_k & b_k \end{pmatrix} \cdot \begin{pmatrix} \hat{c}_k & \hat{d}_k \\ \hat{a}_k & \hat{b}_k \end{pmatrix} = I_n.$$

Hence we reach the following conclusion *the sequence U_k generated by*

$$U_0 = 0, \quad U_{k+1} = \frac{a_k + U_k \cdot c_k}{b_k + U_k \cdot d_k}$$

converges to

$$-P^a \cdot (P^c)^{-1} = \lim_{k \to +\infty} T_k.$$

If $r = s = 1$, $c_k = 0$ and $d_k = 1$ for all k, this sequence is called the reverse continued fraction associated with (2.8) and it was studied by J. Gill in the papers [7], [8], and [9]. The sequence U_k may be used to accelerate the convergence of the given MCF : with (2.8) and theorem 2 it is easy to prove that

$$T_0 = S_k(T_{k+1}).$$

Hence it follows from the previous result that instead of using $S_k(0)$ as an approximation to T_0 for some large value of k, it is better to use

$$S_k(U_{k+1})$$

with U_k defined and calculated as above.

9. Duality. In section 2 we defined the division of matrices as a left division:

$$\frac{P}{Q} = Q^{-1}P.$$

Using the right division instead would give a completely dual development. As we saw in the previous sections, such right divisions appear if we invert a linear fractional transformation such as (2.9): if $Z = t(W)$ is defined by

$$Z = t(W) = (b + W \cdot d)^{-1}(a + W \cdot c),$$

then we have that

$$W = t^{-1}(Z) = (-a + b \cdot Z)(c - d \cdot Z)^{-1}.$$

Note that the coefficients b and d appear on the left of Z, while d and c are on the right of W in the definition of t. Therefore it is obvious that the θ-multiplication will shift to the other side. Thus if $Z = Z_1^{-1} Z_2$ and $W = W_2 W_1^{-1}$, then

$$(Z_2\ Z_1) = (W\ I_r)\theta \quad \Rightarrow \quad \begin{pmatrix} W_2 \\ W_1 \end{pmatrix} = \hat{\theta} \begin{pmatrix} Z \\ I_s \end{pmatrix}, \quad \text{with } \hat{\theta} = \begin{pmatrix} b & -a \\ -d & c \end{pmatrix}.$$

Note that even though t and t^{-1} are each others inverse, θ and $\hat{\theta}$ are not inverses. Only in the real scalar case ($r = s = 1$) we have $\hat{\theta}\theta^{\tau} = (\det\theta) I_2$. So if we replace $\hat{\theta}$ by $\tilde{\theta} = \hat{\theta}/\det\theta$ in the scalar case, we would have $\tilde{\theta} = \theta^{-\tau}$.

The recurrence system

$$Y_k = \theta_k^{-\tau} Y_{k-1} \tag{9.1}$$

is called the adjoint system of (2.1) (see [1]). If X_k is a fundamental system of solutions of (2.1), then $X_k^{-\tau}$ is a fundamental system of solutions of the adjoint system (9.1).

If we use the following notation for the blocks of $\Theta_k^{-\tau}$:

$$\Theta_k^{-\tau} = \begin{pmatrix} B_k' & -A_k' \\ -D_k' & C_k' \end{pmatrix} \quad \text{with} \quad B_k' \in \mathbb{C}^{r \times r};$$

then it is easy to prove that the k-th approximant of the MCF associated with (2.1) can be recovered from the blocks of $\Theta_k^{-\tau}$ in the following way:

$$\frac{A_k}{B_k} = \left(\frac{A_k'}{B_k'} \right)^{\tau}.$$

We note that for the n-th-order linear recurrence relation (3.2) the adjoint equation is given by

$$z_k = \alpha_{k+1}^{(n)} z_{k+1} + \alpha_{k+2}^{(n-1)} z_{k+2} + \ldots + \alpha_{k+n}^{(1)} z_{k+n}. \tag{9.2}$$

This is related to our definition of adjoint because it can be shown [25] that if $(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)})^{\tau}$ is a solution of (9.1) with θ_k as in (3.1), then $z_k = x_k^{(n)}/\alpha_k^{(1)}$ satisfies (9.2).

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