Electronic Transactions on Numerical Analysis.
Volume 4, pp. 89-105, September 1996.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

AN ANALYSIS OF THE POLE PLACEMENT PROBLEM. I. THE SINGLE-INPUT CASE *

VOLKER MEHRMANN† AND HONGGUO XU‡

Abstract. For the solution of the single-input pole placement problem we derive explicit expressions for the feedback gain matrix as well as the eigenvector matrix of the closed-loop system. Based on these formulas we study the conditioning of the pole-placement problem in terms of perturbations in the data and show how the conditioning depends on the condition number of the closed loop eigenvector matrix, which is a similar to a generalized Cauchy matrix, the norm of the feedback vector and the distance to uncontrollability.

Key words. pole placement, condition number, perturbation theory, Jordan form, explicit formulas, Cauchy matrix, stabilization, feedback gain, distance to uncontrollability.

AMS subject classifications. 65F15, 65F35, 65G05, 93B05, 93B55.

1. Introduction. We study single-input time-invariant linear systems

$$\dot{x} = dx(t)/dt = Ax(t) + bu(t),\ x(0) = x_0, \tag{1.1}$$

where $A \in \mathcal{C}^{n,n}(\mathcal{R}^{n,n})$, $b, x(t) \in \mathcal{C}^n(\mathcal{R}^n)$, and $u(t) \in \mathcal{C}(\mathcal{R})$. One of the most studied problems for such systems is the pole placement problem. Here we discuss the single-input case:

Problem 1. *Single-input pole placement (SIPP):*Given a set of n complex numbers $\mathcal{P} = \{\lambda_1, \ldots, \lambda_n\} \subset \mathcal{C}$, find a vector $f \in \mathcal{C}^n$, such that the set of eigenvalues of $\lambda(A - bf^T)$ is equal to $\mathcal{P}$. (Here we assume in the real case that the set $\mathcal{P}$ is closed under complex conjugation.)

The vector f is called feedback gain vector, since if $u(t) = -f^T x(t)$, then (1.1) is the closed-loop system and the solution is $x(t) = e^{(A-bf^T)t}x_0$. It is well known [14, 25] that the feedback gain vector f exists for all sets $\mathcal{P} \subset \mathcal{C}$ if and only if (A, b) is controllable, i.e.,

$$\operatorname{rank}[b, A - \lambda I] = n, \forall \lambda \in \mathcal{C}, \tag{1.2}$$

or equivalently

$$\text{if } w^H A = \lambda w^H \text{ and } w^H b = 0 \text{ then } w = 0. \tag{1.3}$$

For the solution of the SIPP problem explicit formulas were introduced in [1, 3] and numerous numerical algorithms for computing f have been devised; see for example [15, 18, 17, 21, 24].

There are several papers that study the perturbation theory of this problem [2, 19, 22, 20] and the stability of the numerical algorithms [4, 5].

Note that it is important to distinguish between two aspects of the pole placement problem, the computation of the feedback f and the computation of the closed loop

* Received February 8, 1996. Accepted for publication August 9, 1996. Communicated by P. M. VanDooren

† Fakultät für Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG This research was support by DFG project: Singuläre Steuerungsprobleme, Me 790/7-1

‡ Department of Mathematics, Fudan University, Shanghai 200433, P. R. China Current address: Fakultät für Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, This research was upported by Alexander von Humboldt Foundation and Chinese National Natural Foundation.

ETNA
Kent State University
etna@mcs.kent.edu

matrix $A - bf^T$ or its spectrum, repectively. This leads to some confusion in the literature. In our opinion the most important goal of the pole placement is that the implemented poles of the closed loop system are close to the desired ones. If the desired poles of the exact closed loop system are very sensitive to perturbations then this ultimate goal cannot be guaranteed. And this may happen even if the computation of f is reliable or even exact. See Example 2 below, where although the exact f is used the desired stabilization could not be achieved, although the desired poles were far away from the imaginary axis.

To analyze these problems and to get first order perturbation results we derive several explicit formulas for f and for the Jordan canonical form of $A - bf^T$ in terms of A, b, and $\mathcal{P}$.

The results support the statements made in [12] that in general one cannot expect that the closed loop system has a spectrum close to the desired one, since it is very likely that at least one of three contributing factors in the perturbation result, the norm of the feedback vector, the spectral condition number of the closed loop matrix, or the distance to uncontrollability, is large. Clearly there are pole placement problems, which are well-conditioned, for example, when all the poles are moved only slightly or not at all, and the original eigenvalues of A are insensitive to perturbations. But in many practical problems, where pole placement is used, for example in stabilization, moving the poles just slightly will not be enough and in this case it is very likely that the problem is ill-conditioned.

Throughout this paper, A^T, A^H represent the transpose and conjugate transpose of the matrix A, respectively; $\mathcal{N}(A)$ denotes the nullspace and $\Lambda(A)$ the set of eigenvalues of A. By $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0$ we denote the singular values of A in decreasing order, e_i is the $i-th$ unit vector and $e := \sum_{i=1}^n e_i$. The norms used in this paper are the Euclidian vector norm and the associated operator norm.

2. Explicit formulas for f and the Jordan canonical form of $A - bf^T$. We will now derive explicit formulas for the solution of the SIPP problem with closed loop eigenvalues $\lambda_1, \ldots, \lambda_s$ having algebraic multiplicities $r_1, \ldots, r_s$. It is well known, that in the single input case the geometric multiplicity of the assigned closed loop eigenvalues can be at most one. This is easily seen from the fact that if $A - bf^T$ has two different eigenvectors corresponding to the same eigenvalue λ then $\mathrm{rank}(A - bf^T - \lambda I) \leq n-2$ and thus $\mathrm{rank}[b, A - bf^T - \lambda I] \leq n-1$ which contradicts the controllability of (A, b). Furthermore it is also well known [1, 25] that the feedback gain vector f is uniquely determined. The Jordan canonical form of the closed-loop matrix is

$$A - bf^T = G\mathrm{diag}(J_{r_1}(\lambda_1), \ldots, J_{r_s}(\lambda_s))G^{-1}, \tag{2.1}$$

where $J_{r_i}(\lambda_i)$ is a Jordan block to the eigenvalue λ_i of size r_i. The explicit relationship between the data $A, b, \mathcal{P}$ and the solution f has been first observed in [1], and different formulas were obtained in [3]. Here we derive new formulas as well as a formula for the closed-loop eigenvector matrix G.

THEOREM 2.1. *Let a controllable pair (A, b) be given, and let $\mathcal{P} := \{\lambda_1, \ldots, \lambda_s\}$ be a set of pairwise different complex numbers, and let $r_1, \ldots, r_s > 0$ with $\sum_{i=1}^s r_i = n$ be the associated multiplicities of the λ_i. Let vectors $w_j^i \in \mathcal{C}^{n+1}$, $j = 1, \ldots, r_i$, $i = 1, \ldots, s$ be nonzero solutions of*

$$[b, A - \lambda_i I]w_j^i = u_{j-1}^i,\ j = 1, \ldots, r_i,\ i = 1, \ldots, s, \tag{2.2}$$

where $u_0^i = 0$*, for* $i = 1, \ldots, s$ *and* $u_j^i \in \mathcal{C}^n$ *is the lower part in the partitioning* $w_j^i = \begin{bmatrix} -\alpha_j^i \\ u_j^i \end{bmatrix}$*. Then the eigenvector matrix of the closed loop system is*

$$G = [u_1^1, \ldots, u_{r_1}^1, \ldots, u_1^s, \ldots, u_{r_s}^s], \tag{2.3}$$

i.e., we have the Jordan canonical form of the closed loop matrix

$$A - bf^T = G\,\mathrm{diag}(J_{r_1}(\lambda_1), \ldots, J_{r_s}(\lambda_s))G^{-1} =: GJG^{-1}. \tag{2.4}$$

Furthermore the feedback gain vector is

$$f^T = \alpha^T G^{-1}, \tag{2.5}$$

where $\alpha^T := \begin{bmatrix} \alpha_1^1 & \ldots & \alpha_{r_1}^1 & \ldots & \alpha_1^s & \ldots & \alpha_{r_s}^s \end{bmatrix}$*.*

Proof. Solutions w_j^i of (2.2) always exist, since the system is underdetermined and even if the right hand side is zero, they can be chosen nonzero. Note that (2.2) can be rewritten as

$$(A - \lambda_i I)u_j^i = \alpha_j^i b + u_{j-1}^i, \tag{2.6}$$

and thus we have

$$A \begin{bmatrix} u_1^i & \ldots & u_{r_i}^i \end{bmatrix} = b \begin{bmatrix} \alpha_1^i & \ldots & \alpha_{r_i}^i \end{bmatrix} + \begin{bmatrix} u_1^i & \ldots & u_{r_i}^i \end{bmatrix} J_{r_i}(\lambda_i), \tag{2.7}$$

for $i = 1, \ldots, s$ and

$$AG = b\alpha^T + GJ. \tag{2.8}$$

If we can show that G is nonsingular, then it is obvious that the feedback gain vector f is as in (2.5), and the Jordan canonical form of $A - bf^T$ is as in (2.4).

So it remains to show the nonsingularity of G. Suppose that G is singular, then the nullspaces of G and G^H are nonempty. From (2.8) we obtain that

$$v^H b\alpha^T u = 0, \tag{2.9}$$

for all $u \in \mathcal{N}(G)$ and $v \in \mathcal{N}(G^H)$, i.e., either $v^H b = 0$ for all $v \in \mathcal{N}(G^H)$ or $\alpha^T u = 0$ for all $u \in \mathcal{N}(G)$. If $\alpha^T u = 0$ for all $u \in \mathcal{N}(G)$, then from (2.8) we get that $GJu = 0$ for all $u \in \mathcal{N}(G)$. Thus $Ju \in \mathcal{N}(G)$ for all $u \in \mathcal{N}(G)$, i.e., $\mathcal{N}(G)$ is an invariant subspace of J. So there exists an eigenvector u_0 of J in $\mathcal{N}(G)$. Since J is in Jordan canonical form, we have that $u_0 = \beta\, e_k$ with $\beta \neq 0$ and $k = 1 + \sum_{j=0}^{i-1} r_j$ for some integer i (with $r_0 = 0$). Inserting this in $\alpha^T u_0 = 0$ and $Gu_0 = 0$ we get that $\alpha_1^i = 0$, $u_1^i = 0$, which means that $w_1^i = 0$. This contradicts the assumption that all $w_1^i \neq 0$. So there is at least one vector $\tilde{u} \in \mathcal{N}(G)$ such that $\alpha^T \tilde{u} \neq 0$. But then by (2.9) we have $v^H b = 0$ for all $v \in \mathcal{N}(G^H)$. Using the same argument as before, we obtain that $A^H v \in \mathcal{N}(G^H)$ and hence there exists a left eigenvector v_0 of A in $\mathcal{N}(G^H)$ such that $v_0^H b = 0$. But this is a contradiction to the controllability of (A, b). □

This result displays the exact relationship between the solutions of the SIPP problem f, $A - bf^T$ and the data $A, b, \mathcal{P}$. We see that the matrix G plays a central role in the SIPP problem. It is not only the eigenvector matrix of the closed-loop system but the coefficient matrix of the linear system for f as well. We will make use of this fact several times below.

Note also that Theorem 2.1 gives a different proof of Wonham's original result that the pole placement problem is solvable for every pole set if and only if the system is controllable [25].

If A and b are real and the set of poles is closed under conjugation then we have the following result.

THEOREM 2.2. *Let a real controllable pair (A, b) be given and let $\mathcal{P} := \{\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{s_1}, \bar{\lambda}_{s_1}, \lambda_{2s_1+1}, \ldots, \lambda_s\}$ be a set of pairwise different complex numbers, where $\{\lambda_{2s_1+1}, \ldots, \lambda_s\}$ are real and the others are nonreal. Let $r_1, r_1, \ldots, r_{s_1}, r_{s_1}, r_{2s_1+1}, \ldots, r_s > 0$ with $2\sum_{i=1}^{s_1} r_i + \sum_{i=2s_1+1}^{s} r_i = n$ be the associated multiplicities. If λ_i is non-real, then let $w_j^i \in \mathcal{R}^{n+1,2}$, $i = 1, \ldots, s$ be nonzero solutions of the system*

$$[b, A]w_j^i - w_j^i \begin{bmatrix} Re(\lambda_i) & Im(\lambda_i) \\ -Im(\lambda_i) & Re(\lambda_i) \end{bmatrix} = u_{j-1}^i,\ i = 1, \ldots, s_1,\ j = 1, \ldots, r_i, \tag{2.10}$$

where $u_0^i = 0$, for $i = 1, \ldots, s_1$ and $u_j^i \in \mathcal{R}^{n,2}$ is the lower part in the partitioning $w_j^i = \begin{bmatrix} [-\alpha_j^i, -\beta_j^i] \\ u_j^i \end{bmatrix}$. If λ_i is real then let w_j^i, u_j^i be as in (2.2). Then the transformation matrix to real Jordan form for the closed loop system is

$$G = [u_1^1, \ldots, u_{r_1}^1, \ldots, u_1^s, \ldots, u_{r_s}^s], \tag{2.11}$$

i.e., we have the real Jordan canonical form of the closed loop matrix

$$A - bf^T = G\,\mathrm{diag}(\hat{J}_{r_1}(\lambda_1), \ldots, \hat{J}_{r_{s_1}}(\lambda_{s_1}), \ldots, J_{r_s}(\lambda_s))G^{-1} =: GJG^{-1}, \tag{2.12}$$

where

$$\hat{J}_{r_i}(\lambda_i) = \begin{bmatrix} Re(\lambda_i) & Im(\lambda_i) & 1 & 0 & & & & \\ -Im(\lambda_i) & Re(\lambda_i) & 0 & 1 & & & & \\ & & Re(\lambda_i) & Im(\lambda_i) & \ddots & 0 & & \\ & & -Im(\lambda_i) & Re(\lambda_i) & 0 & \ddots & & \\ & & & & \ddots & \ddots & 1 & 0 \\ & & & & & \ddots & 0 & 1 \\ & & & & & & Re(\lambda_i) & Im(\lambda_i) \\ & & & & & & -Im(\lambda_i) & Re(\lambda_i) \end{bmatrix}.$$

Furthermore the feedback gain vector is

$$f^T = \alpha^T G^{-1}, \tag{2.13}$$

where $\alpha^T := \begin{bmatrix} \alpha_1^1 & \beta_1^1 & \ldots & \alpha_{r_1}^1 & \beta_{r_1}^1 & \ldots & \alpha_1^s & \beta_1^s & \ldots & \alpha_{r_s}^s & \beta_{r_s}^s \end{bmatrix}$.

Proof. The proof is analogous to the proof in the complex case. ☐

The exact formulas for the SIPP problem become simpler if none of the eigenvalues of A is reassigned as a pole.

THEOREM 2.3. *Let a controllable pair (A, b) be given, let $\mathcal{P} := \{\lambda_1, \ldots, \lambda_s\}$ be a set of pairwise different complex numbers and let $r_1, \ldots, r_s$ be the associated multiplicities. Assume furthermore that*

$$\{\lambda_1, \ldots, \lambda_s\} \cap \Lambda(A) = \emptyset. \tag{2.14}$$

Let $\bar e = \sum_{i=1}^{s} e_{k_i}$*, where* $k_i = 1 + \sum_{j=0}^{i-1} r_j,\ i = 1, \dots, s,$ *and* $r_0 = 0$*. Then we have that*

$$\tilde G := [(A-\lambda_1 I)^{-1}b, \dots, (A-\lambda_1 I)^{-r_1}b, \dots, (A-\lambda_s I)^{-1}b, \dots, (A-\lambda_s I)^{-r_s}b] \tag{2.15}$$

is nonsingular,

$$f^T = \bar e^T \tilde G^{-1}, \tag{2.16}$$

and

$$A - bf^T = \tilde G J \tilde G^{-1}. \tag{2.17}$$

Proof. By assumption we have that $A - \lambda_i I$ is nonsingular for all i, and thus by (2.2) or (2.6) we have

$$u_1^i = \alpha_1^i (A - \lambda_i I)^{-1} b$$

and

$$u_j^i = (A - \lambda_i I)^{-1} u_{j-1}^i + \alpha_j^i (A - \lambda_i I)^{-1} b.$$

Hence

$$[u_1^i, \dots, u_{r_i}^i] = [(A-\lambda_i I)^{-1} b, \dots, (A - \lambda_i I)^{-r_i} b] T_i,$$

where

$$T_i = \begin{pmatrix} \alpha_1^i & \alpha_2^i & \dots & \alpha_{r_i}^i \\ & \alpha_1^i & \ddots & \vdots \\ & & \ddots & \alpha_2^i \\ & & & \alpha_1^i \end{pmatrix}$$

is a triangular Toeplitz matrix. We furthermore have

$$G = \tilde G \operatorname{diag}(T_1, \dots, T_s) =: \tilde G T.$$

Since $A - \lambda_i I$ is nonsingular, the assumption that $w_1^i \neq 0$ implies that $\alpha_1^i \neq 0$. Thus, we have that $T_1, \dots, T_s$ and hence also T are nonsingular. The nonsingularity of G then implies the nonsingularity of $\tilde G$.

From (2.5) we directly obtain

$$\begin{aligned} f^T &= \alpha^T G^{-1} = \alpha^T T^{-1} \tilde G^{-1} \\ &= [1, 0, \dots, 0, 1, 0, \dots, 0, \dots, 1, 0, \dots, 0] \tilde G^{-1} = \bar e^T \tilde G^{-1}. \end{aligned}$$

By (2.4) and the fact that T_i and J_{r_i} commute, we obtain

$$\begin{aligned} A - bf^T &= GJG^{-1} = \tilde G T J T^{-1} \tilde G^{-1} \\ &= \tilde G \operatorname{diag}(T_1 J_{r_1} T_1^{-1}, \dots, T_s J_{r_s} T_s^{-1}) \tilde G^{-1} = \tilde G J \tilde G^{-1}. \end{aligned}$$

□

Again there is a real version of this result, which we omit here.

If the n poles to be assigned are pairwise different, then the formulas simplify even further. We have the following obvious corollaries.

COROLLARY 2.4. *Let a controllable pair (A, b) be given and let $\mathcal{P} := \{\lambda_1, \ldots, \lambda_n\}$ be a set of pairwise different complex numbers. Let vectors $w_i \in \mathcal{C}^{n+1}$ be nonzero solutions of*

$$[b, A - \lambda_i I]w_i = 0 \tag{2.18}$$

for $i = 1, \ldots, n$. Partition $w_i = \begin{bmatrix} -\alpha_i \\ u_i \end{bmatrix}$ with $u_i \in \mathcal{C}^n$, and define

$$\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n),\ G_1 := [u_1, \ldots, u_n].$$

Then

$$f^T = [\alpha_1, \ldots, \alpha_n]G_1^{-1}, \tag{2.19}$$

and

$$A - bf^T \ = G_1 \Lambda G_1^{-1}. \tag{2.20}$$

Proof. Clear from Theorem 2.1. □

COROLLARY 2.5. *Let a controllable pair (A, b) be given and let $\mathcal{P} := \{\lambda_1, \ldots, \lambda_n\}$ be a set of pairwise different complex numbers such that $\mathcal{P} \cap \Lambda(A) = \emptyset$. Let*

$$\tilde{G}_1 := [(A - \lambda_1 I)^{-1}b, (A - \lambda_2 I)^{-1}b, \ldots, (A - \lambda_n I)^{-1}b]. \tag{2.21}$$

Then

$$f^T = e^T \tilde{G}_1^{-1}, \tag{2.22}$$

and

$$A - bf^T \ = \tilde{G}_1 \ \Lambda \tilde{G}_1^{-1}. \tag{2.23}$$

Proof. Clear from Theorem 2.3. □

Both Corollaries have obvious real versions which we omit here.

The results in Theorems 2.1, 2.3 and Corollaries 2.4, 2.5 give a concrete relationship between the data $A, b, \mathcal{P}$ and the solutions f, $A - bf^T$ of the SIPP problem. This relationship is closely related to that described in [3] but different from the formula in [1]. But since f is unique, all these formulas must be equivalent. For completeness we therefore derive Ackermann's formula from Theorem 2.1.

COROLLARY 2.6. *Let a controllable pair (A, b) be given, let $\mathcal{P} := \{\lambda_1, \ldots, \lambda_s\}$ be a set of pairwise different complex numbers and let $r_1, \ldots, r_s$ be the associated multiplicities. Let $\Phi(\lambda) := \prod_{j=1}^{s}(\lambda - \lambda_j)^{r_j}$ and $\Gamma := [b, Ab, \ldots, A^{n-1}b]$. Then*

$$f^T = e_n^T \Gamma^{-1} \Phi(A). \tag{2.24}$$

Proof. By (2.6) we immediately obtain that

$$(A - \lambda_i I)^{r_i}[u_1^i, \ldots, u_{r_i}^i] = [b, (A - \lambda_i I)b, \ldots, (A - \lambda_i I)^{r_i - 1}b]\hat{T}_i,$$

with the Hankel matrix $\hat{T}_i := \begin{bmatrix} 0 & \dots & 0 & \alpha_1^i \\ \vdots & 0 & \dots & \vdots \\ 0 & \dots & & \\ \alpha_1^i & \dots & \dots & \alpha_{r_i}^i \end{bmatrix}$.

With $\Phi_i(\lambda) := \prod_{j=1 j\neq i}^{s}(\lambda-\lambda_j)^{r_j}$ we have

$$\Phi(A)[u_1^i, \dots, u_{r_i}^i] = \Phi_i(A)[b, (A-\lambda_i I)b, \dots, (A-\lambda_i I)^{r_i-1}b]\hat{T}_i.$$

Since the polynomials $\Phi_i(\lambda)(\lambda-\lambda_i)^t$, for $t = 0, \dots, r_i - 1$ are monic, there exists an $n \times r_i$ matrix $R_i = \begin{bmatrix} \hat{R}_i \\ S_i \end{bmatrix}$ with S_i unit upper triangular such that $\Phi_i(A)[b, (A - \lambda_i)b, \dots, (A-\lambda_i)^{r_i-1}] = \Gamma R_i$. Therefore $\Phi(A)[u_1^i, \dots, u_{r_i}^i] = \Gamma R_i \hat{T}_i$, and hence $\Phi(A)G = \Gamma[R_1\hat{T}_1, \dots, R_s\hat{T}_s]$. The result then follows immediately from (2.5) and the special form of R_i and $\hat{T}_i$. $\square$

Remark 1. An immediate consequence of Ackermann's formula [1] is that the feedback gain vector f is a continuous function of the data, i.e., the elements of A, b and the chosen poles $\lambda_1, \dots, \lambda_n$. This is very important from the perturbation theory point of view, since it allows first order perturbation theory as it is carried out in [19, 22]. An advantage of the new formulas that we have given is that we also explicitly obtain the eigenvector matrix, which is a key in the analysis of the accuracy of the solution of the SIPP problem.

Remark 2. The above results can also be used to design new algorithms for solving the SIPP problem. We may first solve the n underdetermined systems (2.2) to form the eigenvector matrix G. These are independent linear systems which can even be solved in parallel. After this we can compute the feedback gain f by solving an additional linear system. This is not the best way to solve the pole placement problem as will be demonstrated below. But by splitting the computation in these two parts, we see explicitly the freedom that we have in designing a numerical method. It lies in the choice of the solutions of the n independent systems, which corresponds to the non-uniqueness of the eigenvector matrix. Any matrix that commutes with the Jordan matrix of the closed loop matrix can be multiplied on the right to G. As an example consider the situation of Corollary 2.5. By multiplying nonsingular matrices from the right to the system $f^T\tilde{G}_1 = e^T$ we easily obtain the equivalent system: $f^T[(A-\lambda_1 I)^{-1}b, (A-\lambda_2 I)^{-1}(A-\lambda_1 I)^{-1}b, \dots, (\prod_{i=1}^{n}(A-\lambda_i I)^{-1})b] = e_1^T$. The freedom amounts to the construction of different methods for the computation of the feedback, i.e., different linear systems can be obtained in the final step, when we use different solutions of (2.2). It is not clear, which of the different possibilities leads to the best procedure. One possibility would be to choose those solutions of (2.2) that lead to the best conditioned system (2.5). Another possibility is that we first orthogonalize the matrix G (for example using the QR-decomposition). Then we would obtain the Schur-form instead of the Jordan-form. This is the theoretical basis of many pole assignment methods like [17, 18, 21, 24], see also [2]. But this may not be the optimal way in general, since we obtain one well conditioned subproblem, i.e., the inversion of the Schur vector matrix, while at the same time the computation of its columns before orthogonalization, which is implicitely included, may be ill-conditioned. In principle we could even try to balance the conditioning of the two parts of the solution, i.e., the final linear system (2.5) and the computation of the columns of G, by making the condition numbers of the subproblems roughly equal.

Remark 3. The advantage of the given explicit formulas for f and the eigenvector

ETNA
Kent State University
etna@mcs.kent.edu

matrix is that we immediately have a possibility to check the conditioning of the eigenvector matrix, while in the methods based on the Schur vector matrix, this information is not available. We have seen in this section that we can give explicit formulas for the solution of the SIPP problem and the eigenvector matrix of the closed loop system. Using these formulas we can now analyze the perturbation theory of the pole placement problem in more detail than it has been done previously.

3. Perturbation Theory. In two recent papers [19, 22] first-order perturbation results for the feedback gain f and the closed loop spectrum were given in terms of the information of the eigenvector matrix of $A - bf^T$. In this section we will use our explicit solution formulas to rewrite these perturbation results directly in terms of the original data $A, b, \mathcal{P}$. It is not difficult to obtain the following perturbation theorem by applying the results given in Section 2 to the perturbation theorem in [22], but for completeness and simplicity we will derive the results directly. In the following we discuss the special case that is characterized by the following assumptions.

We assume that A is diagonalizable, and we denote by δA, δb and $\delta\lambda$ small perturbations of A, b and $\lambda := [\lambda_1, \ldots, \lambda_n]^T$, respectively. We furthermore assume that the elements of λ and $\lambda + \delta\lambda$ are each pairwise different. Also we often use the vector $\delta a = [\delta a_1^T, \ldots, \delta a_n^T]^T$, where δa_i is the $i - th$ column of δA. Furthermore let A have the Jordan canonical form

$$A = X \Gamma X^{-1} \tag{3.1}$$

with $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, and let $\hat{b} := X^{-1}b = [\hat{b}_1, \ldots, \hat{b}_n]^T$. Introduce the following $n \times n$ matrices

$$B := \mathrm{diag}(\hat{b}_1, \ldots, \hat{b}_n), \tag{3.2}$$

$$C := [c_{i,j}] := [\frac{1}{\gamma_i - \lambda_j}], \tag{3.3}$$

$$Q := \mathrm{diag}(q_1, \ldots, q_n), \tag{3.4}$$

where $q^T := [q_1, \ldots, q_n] := e^T C^{-1}$.

We have the following variation of the perturbation result in [22].

THEOREM 3.1. *Consider the SIPP problem with data A, b, λ and consider a perturbed problem with data $\hat{A} := A + \delta A, \hat{b} := b + \delta b, \hat{\lambda} := \lambda + \delta\lambda$, where A is diagonalizable, the elements of λ, $\hat{\lambda}$ are each pairwise different and furthermore no pole is reassigned, neither in the perturbed nor in the unperturbed problem. Suppose furthermore that (A, b) and $(\hat{A}, \hat{b})$ are controllable and that $\|\delta A\|, \|\delta b\|, \|\delta\lambda\| \leq \epsilon$ for sufficiently small ϵ. Let $\tilde{G}_1 =: [g_{i,j}]$ be the eigenvector matrix as in (2.21). If f is the feedback gain of the unperturbed problem and $\hat{f} := f + \delta f$ is the feedback gain of the perturbated system, then we have $f^T = q^T B^{-1} X^{-1}$, $\tilde{G}_1 = XBC$ and*

$$\begin{aligned} \|\delta f\| &= \|\delta a^T K_A - \delta b^T K_b - \delta\lambda^T K_\lambda\| + O(\epsilon^2) && (3.5)\\ &\leq \epsilon(\|K_A\| + \|K_b\| + \|K_\lambda\|) + O(\epsilon^2), && (3.6) \end{aligned}$$

where

$$\begin{aligned} K_\lambda &= C^T Q B^{-1} X^{-1}, && (3.7)\\ K_b &= X^{-T} B^{-1} Q B^{-1} X^{-1}, && (3.8)\\ K_A &= \begin{bmatrix} X^{-T} B^{-1} Q C \,\mathrm{diag}(g_{11}, \ldots, g_{1n}) \\ \vdots \\ X^{-T} B^{-1} Q C \,\mathrm{diag}(g_{n1}, \ldots, g_{nn}) \end{bmatrix} C^{-1} B^{-1} X^{-1}. && (3.9) \end{aligned}$$

ETNA
Kent State University
etna@mcs.kent.edu

Proof. Suppose that $\tilde{G}_1$ and $\tilde{G}_1 + \delta\tilde{G}_1$ are the eigenvector matrices of the exact and the perturbed closed loop system, respectively. If we partition $\tilde{G}_1 = [g_1, \ldots, g_n]$ and $\delta\tilde{G}_1 = [\delta g_1, \ldots, \delta g_n]$, then from (2.21) we obtain

$$g_i = (A - \lambda_i I)^{-1} b, \;\; g_i + \delta g_i = (\hat{A} - \hat{\lambda}_i I)^{-1} \hat{b}.$$

This implies

$$\delta g_i = -(A - \lambda_i I)^{-1}((\delta A - \delta\lambda_i I) g_i - \delta b) + O(\epsilon^2).$$

Since $(\tilde{G}_1 + \delta\tilde{G}_1)^{-1} = \tilde{G}_1^{-1}(I - \delta\tilde{G}_1 \tilde{G}_1^{-1}) + O(\epsilon^2)$, we have

$$\begin{aligned} \delta f^T &= e^T((\tilde{G}_1 + \delta\tilde{G}_1)^{-1} - \tilde{G}_1^{-1}) \\ &= -e^T \tilde{G}_1^{-1} \delta\tilde{G}_1 \tilde{G}_1^{-1} + O(\epsilon^2) = -f^T \delta\tilde{G}_1 \tilde{G}_1^{-1} + O(\epsilon^2). \end{aligned}$$

Let $Z = [z_1, \ldots, z_n]$, where $z_i = (A - \lambda_i I)^{-T} f$ then

$$\begin{aligned} f^T \delta g_i &= f^T (A - \lambda_i I)^{-1} (\delta b + \delta\lambda_i g_i - \delta A g_i) \\ &= z_i^T \delta b + \delta\lambda_i z_i^T g_i - z_i^T \delta A g_i \\ &= \delta b^T z_i + \delta\lambda_i z_i^T g_i - \sum_{j=1}^{n} g_{ji} z_i^T \delta a_j \\ &= \delta b^T z_i + \delta\lambda_i z_i^T g_i - \delta a^T \begin{bmatrix} g_{1,i} z_i \\ \vdots \\ g_{n,i} z_i \end{bmatrix}, \end{aligned}$$

and thus

$$\begin{aligned} f^T \delta\tilde{G}_1 &= \delta b^T Z + \delta\lambda^T \operatorname{diag}(z_1^T g_1, \ldots, z_n^T g_n) \\ &- \delta a^T \begin{bmatrix} Z \operatorname{diag}(g_{11}, \ldots, g_{1n}) \\ \vdots \\ Z \operatorname{diag}(g_{n1}, \ldots, g_{nn}) \end{bmatrix}. \end{aligned}$$

From this we obtain

$$\begin{aligned} -\delta f^T &= \delta b^T Z \tilde{G}_1^{-1} + \delta\lambda^T \operatorname{diag}(z_1^T g_1, \ldots, z_n^T g_n) \tilde{G}_1^{-1} - \\ & \quad \delta a^T \begin{bmatrix} Z \operatorname{diag}(g_{11}, \ldots, g_{1n}) \\ \vdots \\ Z \operatorname{diag}(g_{n1}, \ldots, g_{nn}) \end{bmatrix} \tilde{G}_1^{-1} + O(\epsilon^2) \\ &=: \delta b^T K_b + \delta\lambda^T K_\lambda - \delta a^T K_A + O(\epsilon^2). \end{aligned} \tag{3.10}$$

Since the diagonal elements of $\Gamma = X^{-1} A X$ and the elements λ are each pairwise different, considering the form of B, we derive

$$g_i = X(\Gamma - \lambda_i I)^{-1} \hat{b} = XB[\frac{1}{\gamma_1 - \lambda_i}, \ldots, \frac{1}{\gamma_n - \lambda_i}]^T =: XBc_i,$$

so $\tilde{G}_1 = XBC$. Furthermore from $f = \tilde{G}_1^{-T} e = X^{-T} B^{-1} q$, we obtain

$$z_i = (A - \lambda_i I)^{-T} f = X^{-T} (\Gamma - \lambda_i I)^{-1} X^T f = X^{-T} B^{-1} Q c_i,$$

and thus $Z = X^{-T}B^{-1}QC$. It is well known, e.g. [8], that $C^{-1} = [\tilde{c}_{ij}]$, with

$$\tilde{c}_{ij} = -\frac{\prod_{k=1}^{n}(\lambda_i - \gamma_k)\ \prod_{k=1}^{n}(\gamma_j - \lambda_k)}{(\gamma_j - \lambda_i)\prod_{k=1 k\neq i}^{n}(\lambda_i - \lambda_k)\prod_{k=1 k\neq j}^{n}(\gamma_j - \gamma_k)},$$

and hence

$$q_i = \frac{\prod_{k=1}^{n}(\gamma_i - \lambda_k)}{\prod_{k=1,k\neq i}^{n}(\gamma_i - \gamma_k)},\ i = 1,\ldots,n.$$

Thus, we obtain

$$Z^T\tilde{G}_1 \ = C^TQB^{-1}X^{-1}XBC = C^TQC = \mathrm{diag}(c_1^TQc_1,\ldots,c_n^TQc_n).$$

Since $z_i^Tg_i = c_i^TQB^{-1}X^{-1}XBc_i = c_i^TQc_i$, we have

$$\mathrm{diag}(z_1^Tg_1,\ldots,z_n^Tg_n) = Z^T\tilde{G}_1.$$

Inserting these results in (3.10) we have finished the proof. □

Remark 4. The previous theorem can be easily modified also to the case of multiple eigenvalues or pole placement problems where a number of poles stays fixed. The formulas in Section 2 allow this.

The major difference will be that we do not have such nice explicit formulas for K_λ, K_b and K_A. We would rather have to do a perturbation analysis for the linear systems (2.2) and would obtain rather messy formulas.

On the other hand, from a perturbation theory point of view it makes no sense to consider multiple poles with higher order Jordan blocks in the closed loop system, since it is well known, e.g. [23], that arbitrary small perturbations destroy the Jordan structure drastically and even the multiplicity. For non-diagonalizable matrices A the result is easily extended.

Also the case that certain poles are kept fix in the pole placement problem can be avoided, since via a reduction to Schur-form (as it is used in most known pole placement algorithms) we can split the problem into two subproblems: one which is not altered so that its poles are fixed and one where every pole is changed. For the latter subsystem we are again in the situation of our theorem.

From Theorem 3.1 we see, and this has already been observed in [12, 22], that several factors contribute to the conditioning of the SIPP problem. These are essentially $\|(XB)^{-1}\|$, $\|C^{-1}\|$ and $\|q\|$, where the latter is again directly related to $\|C^{-1}\|$. While $\|C^{-1}\|$ depends strongly on the choice of poles in relation to the eigenvalues of A, the first factor relates to the conditioning of the open-loop eigenvector matrix and the *distance to uncontrollability*, [11]. This distance is commonly defined, e.g. [7], as

$$d_{uc}(A,b) := \min_{\lambda\in\mathcal{C}}\sigma_n[b, A-\lambda I]. \tag{3.11}$$

Furthermore, when $A - bf^T$ is diagonalizable, as in [6], we define κ as the (scaled) spectral condition number of $A-bf^T$, i.e., $\kappa := \|G_0\|\|G_0^{-1}\|$ where G_0 is the eigenvector matrix of $A-bf^T$ with columns scaled to be of norm one. Analogously let $\hat{\kappa}$ be the scaled spectral condition number of the closed loop matrix of the perturbed problem. Then we have the following perturbation result in terms of κ, $\|f\|$ and $d_{uc}(A,b)$. Note that here the assumptions are less restrictive than in the previous theorem.

Theorem 3.2. *Consider the SIPP problem with data A, b, λ and consider a perturbed problem with data $\hat{A} := A+\delta A$, $\hat{b} := b+\delta b$, $\hat{\lambda} := \lambda+\delta\lambda$, where the components of*

ETNA
Kent State University
etna@mcs.kent.edu

λ and $\hat{\lambda}$ are each pairwise different. Suppose that $\|\delta A\|$, $\|\delta b\|$, $\|\delta\lambda\| \leq \epsilon$ for sufficiently small ϵ and that

$$\epsilon < d_{uc}(A,b)/4. \tag{3.12}$$

Let $G =: \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}$ be the eigenvector matrix of the unperturbed closed loop system, satisfying $f^T G = \alpha^T$, $\begin{bmatrix} -\alpha^T \\ G \end{bmatrix} = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix} =: W$, $\|w_i\| = 1$ and $[b, A - \lambda_i I]w_i = 0$. Let analogously $\hat{G} =: \begin{bmatrix} \hat{u}_1 & \dots & \hat{u}_n \end{bmatrix}$ be the eigenvector matrix of the perturbed system, i.e., $\hat{f}^T\hat{G} = \hat{\alpha}^T$, $\begin{bmatrix} -\hat{\alpha}^T \\ \hat{G} \end{bmatrix} = \begin{bmatrix} \hat{w}_1 & \dots & \hat{w}_n \end{bmatrix} =: \hat{W}$, $\|\hat{w}_i\| = 1$ and $[\hat{b}, \hat{A} - \hat{\lambda}_i I]\hat{w}_i = 0$. Then we have the following inequality for $\delta f := \hat{f} - f$:

$$\begin{aligned} \|\delta f\| &\leq \frac{15\sqrt{n}\epsilon}{4d_{uc}(A,b)}\sqrt{1+\|f\|^2}\|\hat{G}^{-1}\| \\ &\leq \frac{15\sqrt{n}\epsilon\hat{\kappa}}{4d_{uc}(A,b)}\left(\max_i \sqrt{\frac{\|\hat{A}-\hat{\lambda}_i I\|^2}{\|\hat{b}\|^2}+1}\right)\sqrt{1+\|f\|^2}, \end{aligned} \tag{3.13}$$

where $\hat{\kappa}$ is the scaled spectral condition number of $\hat{A} - \hat{b}\hat{f}^T$.

Furthermore if $15\sqrt{n}\epsilon\|G^{-1}\| \leq 15\sqrt{n}\epsilon\kappa \max_i \sqrt{\frac{\|A-\lambda_i I\|^2}{\|b\|^2}+1} < 4d_{uc}(A,b)$ then

$$\begin{aligned} \|\delta f\| &\leq \frac{15\sqrt{n}\epsilon\|G^{-1}\|\sqrt{1+\|f\|^2}}{4d_{uc}(A,b) - 15\sqrt{n}\epsilon\|G^{-1}\|} \\ &\leq \frac{15\sqrt{n}\epsilon\kappa \max_i \sqrt{(\frac{\|A-\lambda_i I\|^2}{\|b\|^2}+1)}\sqrt{1+\|f\|^2}}{4d_{uc}(A,b) - 15\sqrt{n}\epsilon\kappa\max_i\sqrt{(\frac{\|A-\lambda_i I\|^2}{\|b\|^2}+1)}}, \end{aligned} \tag{3.14}$$

where κ is the scaled spectral condition number of $A - bf^T$.

Proof. Inequality (3.12) implies that

$$d_{uc}(A+\delta A, b+\delta b) \geq d_{uc}(A,b) - 2\epsilon \geq \frac{1}{2}d_{uc}(A,b) > 0.$$

Thus $(\hat{A},\hat{b})$ is controllable and $\hat{G}^{-1}$ exists. By definition we have $[b, A-\lambda_i I]\hat{w}_i = -[\delta b, \delta A - \delta\lambda_i I]\hat{w}_i$. Let $U_i\Sigma_i V_i^H$ be the singular value decomposition of $[b, A-\lambda_i I]$. Then we have

$$V_i^H\hat{w}_i = -\Sigma_i^{-1}U_i^H[\delta b, \delta A - \delta\lambda_i I]\hat{w}_i,$$

and hence

$$\|V_i^H\hat{w}_i\| \leq \frac{3\epsilon}{\sigma_n(\Sigma_i)} \leq \frac{3\epsilon}{d_{uc}(A,b)}. \tag{3.15}$$

Since w_i spans the nullspace of $[b, A-\lambda_i I]$ and has norm one, it follows that $\begin{bmatrix} V_i^H \\ w_i^H \end{bmatrix}$ is unitary. Thus

$$|w_i^H\hat{w}_i|^2 = 1 - \|V_i^H\hat{w}_i\|^2 \geq 1 - (\frac{3\epsilon}{d_{uc}(A,b)})^2.$$

Multiplying the i-th column of $\hat{G}$ and $\hat{\alpha}^T$ by a scalar does not change $\hat{f}^T$, so we may assume, without loss of generality, that $w_i^H \hat{w}_i > 0$. Then

$$1 - w_i^H \hat{w}_i = \frac{1-(w_i^H \hat{w}_i)^2}{1+w_i^H \hat{w}_i} \le (\frac{3\epsilon}{d_{uc}(A,b)})^2.$$

By this inequality we obtain

$$\begin{aligned} \|w_i - \hat{w}_i\| &= \| \begin{bmatrix} V_i^H \\ w_i^H \end{bmatrix} (w_i - \hat{w}_i)\| \\ &= \|e_{n+1} - \begin{bmatrix} V_i^H \hat{w}_i \\ w_i^H \hat{w}_i \end{bmatrix} \| \\ &= \sqrt{\|V_i^H \hat{w}_i\|^2 + (1 - w_i^H \hat{w}_i)^2} \\ &\le \sqrt{(\frac{3\epsilon}{d_{uc}(A,b)})^2 + (\frac{3\epsilon}{d_{uc}(A,b)})^4} \\ &\le \frac{15\epsilon}{4d_{uc}(A,b)}, \end{aligned}$$

and thus

$$\|W - \hat{W}\| \le \frac{15\epsilon\sqrt{n}}{4d_{uc}(A,b)}.$$

Using the explicit formulas for f, $\hat{f}$ we obtain

$$\hat{f}^T - f^T = -\begin{bmatrix} 1 & f^T \end{bmatrix} \begin{bmatrix} -(\hat{\alpha}^T - \alpha^T) \\ \hat{G} - G \end{bmatrix} \hat{G}^{-1} = -\begin{bmatrix} 1 & f^T \end{bmatrix} (\hat{W} - W)\hat{G}^{-1}.$$

So $\|\delta f\| \le \|\hat{G}^{-1}\| \|W - \hat{W}\| \sqrt{1+\|f\|^2} \le \frac{15\epsilon\|\hat{G}^{-1}\|\sqrt{n}}{4d_{uc}(A,b)} \sqrt{1+\|f\|^2}$. Since $[\hat{b}, \hat{A} - \hat{\lambda}_i I] \begin{bmatrix} -\hat{\alpha}_i \\ \hat{u}_i \end{bmatrix} = 0$, it follows that

$$\|\hat{u}_i\| \ge \frac{\|\hat{b}\| |\hat{\alpha}_i|}{\|\hat{A} - \hat{\lambda}_i I\|}.$$

Using the assumption that $\|\hat{w}_i\|^2 = \|\hat{u}_i\|^2 + |\hat{\alpha}_i|^2 = 1$ we get

$$\|\hat{u}_i\| \ge \sqrt{\frac{\|\hat{b}\|^2}{\|\hat{A} - \hat{\lambda}_i I\|^2 + \|\hat{b}\|^2}}.$$

Let $\hat{U} := \mathrm{diag}(\frac{1}{\|\hat{u}_1\|}, \dots, \frac{1}{\|\hat{u}_n\|})$ and $\hat{G}_0 := \hat{G}\hat{U}$, then $\|\hat{G}_0\| \ge 1$ and hence

$$\begin{aligned} \|\hat{G}^{-1}\| &= \|\hat{U}(\hat{G}\hat{U})^{-1}\| \le \|\hat{U}\| \|\hat{G}_0^{-1}\| \\ &\le \|\hat{G}_0^{-1}\| \max_i \sqrt{\frac{\|\hat{A} - \hat{\lambda}_i I\|^2}{\|\hat{b}\|^2} + 1} \\ &\le \hat{\kappa} \max_i \sqrt{\frac{\|\hat{A} - \hat{\lambda}_i I\|^2}{\|\hat{b}\|^2} + 1}, \end{aligned}$$

which finishes the first part of the proof.

For the second part we use that $\|G-\hat{G}\| \leq \|W-\hat{W}\|$, and $\|G^{-1}\| \leq \kappa \max_i \sqrt{\frac{\|A-\lambda_i I\|^2}{\|b\|^2}+1}$, and hence

$$
\begin{aligned}
\|\hat{G}^{-1}\| &= \|G^{-1}(I+(\hat{G}-G)G^{-1})^{-1}\| \\
&\leq \frac{\|G^{-1}\|}{1-\|G^{-1}\|\|\hat{G}-G)\|} \\
&\leq \kappa \frac{\max_i \sqrt{\frac{\|A-\lambda_i I\|^2}{\|b\|^2}+1}}{1-\kappa\frac{15\epsilon\sqrt{n}}{4d_{uc}(A,b)}\max_i \sqrt{\frac{\|A-\lambda_i I\|^2}{\|b\|^2}+1}},
\end{aligned}
$$

and from this the second part follows. □

We see from this theorem, that a small distance to uncontrollability, a large feedback gain or an ill-conditioned eigenvector matrix of the perturbed closed loop system may cause a large error in f.

REMARK 5. Condition (3.12) guarantees that the perturbed system is controllable and thus that the solution exists and can be obtained via the explicit formula of Theorem 2.1. In principle using (3.12) we can cancel the term $\frac{\epsilon}{d_{uc}(A,b)}$ from the bounds, but then we would not have a perturbation results of the form (perturbation of the data) $\times$ (amplification factor). Note further that we could use (3.15) to remove $d_{uc}(A,b)$ from the bounds and replace it by $\min_i \sigma_n([b, A-\lambda_i I])$ which makes the bounds more sharp.

To illustrate the result consider the following example:

EXAMPLE 1. Consider $X = B = I$ in Theorem 3.1 , then $K_\lambda = C^T Q, K_b = Q$ and

$$
K_A = \begin{bmatrix} QC\,\mathrm{diag}(c_{11},\ldots,c_{1n}) \\ \vdots \\ QC\,\mathrm{diag}(c_{n1},\ldots,c_{nn}) \end{bmatrix} C^{-1}.
$$

Suppose that

$$
\beta_1 = \min_{\gamma_i\in\lambda(A),\lambda_j\in\mathcal{P}} |\gamma_i-\lambda_j|,\ \ \beta_2 = \max_{\gamma_i,\gamma_j\in\lambda(A);i\neq j} |\gamma_i-\gamma_j|.
$$

Using the formulas for q_i and C above we have in Frobenius norm

$$
\|K_\lambda\|_F \geq n(\frac{\beta_1}{\beta_2})^{n-1},\ \ \|K_b\|_F \geq n\beta_1(\frac{\beta_1}{\beta_2})^{n-1}.
$$

If we have for example $\beta_1/\beta_2 > 2$, which is not an unreasonable value, since in many applications the eigenvalues of A are in the right half plane and the chosen poles are in the left half plane, then $\|K_\lambda\|, \|K_b\|$ will increase at least as 2^{n-1}! Although we do not know a similar lower bound for $\|K_A\|$, we expect that it will usually be even larger, because $\|K_A\|$ is equivalent to $\|Q\|\|C^{-1}\|$.

Note that $\frac{\beta_1}{\beta_2}$ can be viewed as the relative distance between $\Lambda(A)$ and $\mathcal{P}$.

In most of the pole placement algorithms at first the feedback gain vector $\hat{f}$ is computed from $A, b, \mathcal{P}$, which is, if a backward stable method is used, the exact feedback gain vector for a slightly perturbed problem $\hat{A}, \hat{b}, \hat{\mathcal{P}}$. With this feedback gain vector $\hat{f}$ then the closed loop system $A - b\hat{f}^T$ is formed. So far we have mainly considered perturbation bounds for f. But usually it is more important in practice, how far the actual eigenvalues of the implemented closed loop system $A - b\hat{f}^T$ are away from the desired poles.

THEOREM 3.3. *Consider the SIPP problem with data* $A, b, \mathcal{P} = \{\lambda_1, \ldots, \lambda_n\}$ *and consider a perturbed problem with data* $\hat{A} := A + \delta A, \hat{b} := b + \delta b, \hat{\mathcal{P}} := \{\lambda_1 + \delta\lambda_1, \ldots, \lambda_n + \delta\lambda_n\}$. *Assume that the desired poles* $\lambda_j,\ j = 1, \ldots, n$ *and the perturbed poles* $\lambda_j + \delta\lambda_j,\ j = 1, \ldots, n$ *are each pairwise different.*

Suppose further that $\|\delta A\|, \|\delta b\|, \|\delta\lambda\| \le \epsilon$ *for sufficiently small* ϵ *and that*

$$\epsilon < d_{uc}(A, b)/4. \tag{3.16}$$

(Here as before λ *and* $\delta\lambda$ *are vectors formed from the elements of* $\mathcal{P}$ *and* $\hat{\mathcal{P}}$, *respectively.)*

Let f, $\hat{f}$ *be the feedback gains of the unperturbed and perturbed system, respectively and let* $\hat{\kappa}$ *be the spectral condition number of the perturbed closed loop system* $\hat{A} - \hat{b}\hat{f}^T$. *Then for each of the eigenvalues* μ_i *of* $A - b\hat{f}^T$ *there is a pole* λ_i *of the unperturbed closed loop system* $A - bf^T$ *such that*

$$\begin{aligned} |\lambda_i - \mu_i| &< \epsilon(1 + (1 + \|\hat{f}\|)\hat{\kappa}) && (3.17)\\ &\le \epsilon(1 + \hat{\kappa} + \|f\|\hat{\kappa} \\ &+ \epsilon\hat{\kappa}^2 \frac{15\sqrt{n}\max_i \sqrt{\|\hat{A} - \hat{\lambda}_i I\|^2 + \|\hat{b}\|^2}}{4d_{uc}(A, b)\|\hat{b}\|}\sqrt{1 + \|f\|^2}). && (3.18) \end{aligned}$$

Proof. Applying the Bauer-Fike Theorem, e.g. [9], p. 342, to $A - b\hat{f}^T = \hat{A} - \hat{b}\hat{f}^T - \delta A + \delta b\hat{f}^T$, we obtain that for each of the desired poles λ_i, there is a pole μ_i of $A - b\hat{f}^T$, such that

$$|\mu_i - \lambda_i| = |\mu_i - \hat{\lambda}_i + \delta\lambda_i| \le \hat{\kappa}\|\delta A - \delta b\hat{f}^T\| + |\delta\lambda_i| \le \epsilon\hat{\kappa}(1 + \|\hat{f}\|) + \epsilon.$$

This gives the first bound and the second follows from Theorem 3.2 using $\|\hat{f}\| \le \|\delta f\| + \|f\|$. □

REMARK 6. We see from this result that if the norm of $\hat{f}$ or the spectral condition number of the closed loop system are large or (A, b) is near to an uncontrollable pair, then we cannot expect that the eigenvalues of the closed loop system $A - b\hat{f}^T$ are close to the desired eigenvalues. And this means that the desired goal of the pole placement problem is not achieved.

Here it becomes clear that the question of conditioning for the pole placement problem is rather tricky, since different quantities can be considered to be **results**. This may be f, or the closed loop system $A - bf^T$. The ultimate goal of the pole placement problem (in our opinion) should be that the implemented closed loop poles are close to the desired poles and this is obviously not the case if the right hand side in (3.17) is large. **In this sense the pole placement problem is ill-conditioned** if either of the contributing factors is large. To illustrate these observations consider the following example which shows that computing the feedback gain very accurately may not be enough to guarantee that the closed loop system has poles close to the desired poles. Even if we compute f exactly, in presence of slight perturbations like for example noise or measurement errors, the closed loop system may have eigenvalues which are far away from the desired ones.

ETNA
Kent State University
etna@mcs.kent.edu

$\|f_a - f_{mp}\|_2$=9.1e-4	$\|f_a - f_s\|_2$=1.96e11
$\|f_a - f_{mp}\|_2/\|f_a\|_2$=4.6e-15	$\|f_a - f_s\|_2/\|f_a\|_2$ =9.9e-1

TABLE 3.1
Errors in feedback gains

EXAMPLE 2. Consider the SIPP problem with data

$$A = \mathrm{diag}(1, 2, \ldots, 15),\ b = e^T,\ \mathcal{P} = \{-1, \ldots, -15\}.$$

The numerical results in this example were performed in Matlab version 4.2a on an HP 715-33 workstation, with machine epsilon 2.22×10^{-16}.

Let f_a, f_{mp}, f_s denote the exact feedback gain, the feedback gain obtained with the Matlab code of Miminis and Paige [17] and the gain obtained with formulas (2.21), (2.22), respectively. (The exact gain can be obtained as in the proof of Theorem 3.1, since all the elements of f_a are integers.)

We obtain that $\|f_a\|_2 = 1.98\ e{+}11$ and we have the absolute and relative errors for the computed feedbacks given in Table 1. We see that the Miminis/Paige procedure computes f essentially to full relative accuracy, while the direct formulas give extremely inaccurate results.

If we now determine the eigenvalues of the corresponding closed loop systems and let $\lambda, \lambda_a, \lambda_{mp}, \lambda_s$ denote the vector of desired poles, computed poles of $A - bf_a^T$, poles $A - bf_{mp}^T$, and poles of $A - bf_s^T$, respectively, (where the eigenvalue computation was done using the Matlab function *eig*), then we obtain the following closed loop spectra as depicted in Table 2 and Figure 1.

λ	λ_a	λ_{mp}	λ_s
-1	$-7.11e+01$	$-7.87e+01+9.03e+01i$	$-2.82e+01$
-2	$-2.10e+01+3.97e+01i$	$-7.87e+01-9.03e+01i$	$-1.85e+01+1.28e+01i$
-3	$-2.10e+01-3.97e+01i$	$6.99e+00+3.60e+01i$	$-1.85e+01-1.28e+01i$
-4	$-1.42e+00+1.99e+01i$	$6.99e+00-3.60e+01i$	$-7.59e+00+1.11e+01i$
-5	$-1.43e+00-1.99e+01i$	$4.17e+00+1.58e+01i$	$-7.58e+00-1.11e+01i$
-6	$-9.07e+00$	$4.17e+00-1.58e+01i$	$1.52e+01$
-7	$4.37e-01+1.05e+01i$	$2.66e+00+8.87e+00i$	$-3.17e+00+6.89e+00i$
-8	$4.37e-01-1.05e+01i$	$2.66e+00-8.87e+00i$	$-3.17e+00-6.89e+00i$
-9	$6.50e-01+5.83e+00i$	$1.91e+00+5.14e+00i$	$-1.44e+00+3.86e+00i$
-10	$6.50e-01-5.83e+00i$	$1.91e+00-5.14e+00i$	$-1.44e+00-3.86e+00i$
-11	$6.30e-01+3.05e+00i$	$1.46e+00+2.74e+00i$	$-6.88e-01+1.86e+00i$
-12	$6.30e-01-3.05e+00i$	$1.46e+00-2.74e+00i$	$-6.88e-01-1.86e+00i$
-13	$4.11e-01$	$7.77e-01$	$-3.71e-01+5.15e-01i$
-14	$5.33e-01+1.21e+00i$	$1.13e+00+1.04e+00i$	$-3.71e-01-5.15e-01i$
-15	$5.33e-01-1.21e+00i$	$1.13e+00-1.04e+00i$	$1.17e+01$

Table 2: Closed loop eigenvalues

We see that the closed loop spectra are very far away from the desired ones, even if f is computed exactly. What is also striking is that even the stability of the closed loop system is lost in all three cases.

4. Conclusion. In this paper we have analyzed the single-input pole placement problem. We haven given new explicit formulas for the feedback gain f and have used

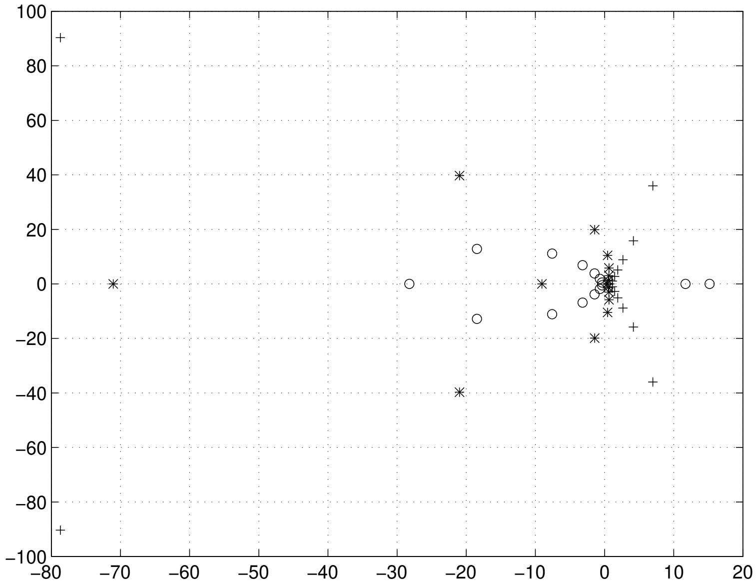

FIG. 3.1. *Closed loop spectra:* $* - \lambda_a$, $+ - \lambda_{mp}$, $o - \lambda_s$.

these to obtain perturbation results. From these results it follows that $\|f\|$, the spectral condition number of the closed loop matrix and the distance to uncontrollability are the governing terms in the perturbation bounds.

If the perturbation bound is large, then we cannot expect that the closed loop poles are close to the desired ones, and in this sense the pole placement problem can be viewed as ill-conditioned even if the computation of f is well-conditioned.

Unfortunately the spectral condition number of the closed loop system and the norm of f are proportional to the norm of the inverse of a Cauchy matrix, which is usually very large in particular for large system dimension. Thus in many circumstances we cannot expect that the closed loop poles are near to the desired ones. This partially verifies observations made in [12].

The analysis we have given leads to some open problems. Suppose that the system (A, b) is given, and that we only require that the poles lie in specified regions in the complex plane, see e.g. [12, 13].

1. *Can we choose the poles in these regions so that the pole placement problem is well-conditioned in the sense that the actual closed loop poles are close to the desired ones ?*

2. *Is the closed loop system obtained under such circumstances robust under perturbations ?*

These questions and related topics are currently under investigation, partial answers to these questions are given in a recent report [16].

5. Acknowledgment. We thank Dr. Chunyang He, Dr. Vasile Sima and Dr. Andras Varga for many valuable discussions and comments, which improved the paper substantially. We also thank the anonymous referees for valuable comments and criticism.

REFERENCES

[1] J. Ackermann, *Der Entwurf linearer Regelungssysteme im Zustandsraum*, Regelungstechnik und Prozessdatenverarbeitung, 7 (1972), pp. 297-300.

[2] M. Arnold, *Algorithms and Conditioning for Eigenvalue Assignment*, PhD thesis, Dept. of Mathematics, Northern Illinois University, De Kalb, 1993.

[3] W.L. Brogan, *Application of a determinant identity to pole-placement and observer problems.* IEEE Trans. Automat. Control, 19 (1974), pp. 612-614.

[4] C.L. Cox and W.F. Moss, *Backward error analysis for a pole assignment algorithm* SIAM J. Matrix Anal. Appl., 10 (1989), pp. 446-456.

[5] C.L. Cox and W.F. Moss *Backward error analysis for a pole assignment algorithm II: The complex case*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 1159-1171.

[6] J. Demmel, *The condition number of equivalence transformations that block diagonalize matrix pencils*, in Matrix Pencils, B. Kagström and A. Ruhe, eds., Springer Verlag, Berlin, Lecture Notes in Mathematies 973, 1982, pp. 2-16Mathematies 973, 1982, pp. 2-16. .

[7] R. Eising, Between controllable and uncontrollable, Systems and Control Letters, 4 (1984):263-264, 1984.

[8] T. Finck, G. Heinig, and K. Rost, *An inversion formula and fast algorithms for Cauchy-Vandermonde matrices*, Linear Algebra Appl., 183 (1993), pp. 179-191.

[9] G. H. Golub and C. F. Van Loan, *Matrix Computations.* Johns Hopkins Univ. Press, Baltimore Maryland, 2nd edition, 1989.

[10] R. T. Gregory and D. L. Karney, A Collection of Matrices for Testing Numerical Algorithms. Wiley, New York, 1969.

[11] C. He, *Estimating the distance to uncontrollability: A fast method and a slow one*, System and Control Lett,26 (1995), pp. 275-281.

[12] C. He, A. J. Laub and V. Mehrmann, *Placing plenty of poles is pretty preposterous.* Preprint $95-17$: DFG-Forschergruppe Scientific Parallel Computing, Fak. f. Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, May 1995.

[13] J. Hench, C. He, V. Kucera and V. Mehrmann, *Dampening controllers via a Riccati equation approach.* Preprint $95-18$: DFG-Forschergruppe Scientific Parallel Computing, Fak. f. Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, May 1995.

[14] T. Kailath, Linear Systems. Prentice Hall, Englewood Cliffs, New Jersey, 1980.

[15] J. Kautsky, N. K. Nichols and P. Van Dooren, Robust pole assignment in linear state feedback, Internat. J. Control, 41 (1985), pp. 1129-1155.

[16] V. Mehrmann and H. Xu, *Choosing poles so that the single-input pole placement problem is well-conditioned.* Preprint SFB393/96-01: Sonderforschungsbereich 393 "Numerische Simulation auf massiv parallelen Rechnern", Fak. f. Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, January 1996.

[17] G. S. Miminis and C. C. Paige, *A direct algorithm for pole assignment of time-invariant multi-input linear systems using state feedback. Automatica J. IFAC.*, 24 (1988), pp. 343-356.

[18] P.Hr. Petkov, N.D. Christov and M.M. Konstantinov, *A computational algorithm for pole assignment of linear multiinput systems* IEEE Trans. Automat. Control, 31 (1986), pp. 1044-1047.

[19] M.M. Konstantinov and P.Hr. Petkov, *Conditioning of linear state feedback.* Technical Report 93–61, Dept. of Engineering, Leicester University, 1993.

[20] M.M. Konstantinov, P.Hr. Petkov and N. Christov, *Conditioning of the output pole assignment*, Proceedings of the 3rd European Control Conference, Rome 1995, 3 (1995), pp. 2072-2077.

[21] V. Sima, *An efficient Schur Method to solve the stabilization problem.* IEEE Trans. Automat. Control, 26 (1981), pp. 724-725.

[22] J. G. Sun , *Perturbation analysis of the pole assignment problem*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 313-331.

[23] G.W. Stewart and J.G. Sun, *Matrix Perturbation Theory*, Academic Press, San Diego, 1990.

[24] A. Varga, *A Schur method for pole assignment*, IEEE Trans. Automat. Control, 26 (1981), pp. 517-519.

[25] W.M. Wonham, *On pole assignment in multi-input controllable linear systems*, IEEE Trans. Automat. Control, 12 (1967), pp. 660–665.