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IMPLICIT-EXPLICIT PREDICTOR-CORRECTOR METHODS COMBINED WITH IMPROVED SPECTRAL METHODS FOR PRICING EUROPEAN STYLE VANILLA AND EXOTIC OPTIONS*

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Abstract. In this paper we present a robust numerical method to solve several types of European style option pricing problems. The governing equations are described by variants of Black-Scholes partial differential equations (BS-PDEs) of the reaction-diffusion-advection type. To discretise these BS-PDEs numerically, we use the spectral methods in the asset (spatial) direction and couple them with a third-order implicit-explicit predictor-corrector (IMEX-PC) method for the discretisation in the time direction. The use of this high-order time integration scheme sustains the better accuracy of the spectral methods for which they are well-known. Our spectral method consists of a pseudospectral formulation of the BS-PDEs by means of an improved Lagrange formula. On the other hand, in the IMEX-PC methods, we integrate the diffusion terms implicitly whereas the reaction and advection terms are integrated explicitly. Using this combined approach, we first solve the equations for standard European options and then extend this approach to digital options, butterfly spread options, and European calls in the Heston model. Numerical experiments illustrate that our approach is highly accurate and very efficient for pricing financial options such as those described above.

Key words. European options, butterfly spread options, digital options, Black-Scholes equation, barycentric interpolation, implicit-explicit predictor-corrector methods

AMS subject classifications. 39A05, 65M06, 65M12, 91G60

1. Introduction.

In this paper we consider a class of European style options described by Black-Scholes equations [7]. In general, closed-form analytical solutions of some of these Black-Scholes PDEs do not exist and therefore one has to resort to numerical methods in order to solve them. In the literature, the following four main families of methods have been developed and extensively used for Black-Scholes PDEs: lattice methods [10, 21, 32], Monte Carlo simulations [5, 13, 41, 45], finite difference (FD) methods [11, 42, 59], and analytical approximations [20, 27, 35]. The first two are classified as stochastic simulation methods since they approximate the underlying process directly. The other two methods are usually performed on the Black-Scholes PDEs with appropriate approximate boundary conditions.

Popular techniques such as lattice methods can be very efficient for valuing simple calls and puts, however, they become less efficient when valuing more complicated options. FD methods are more desirable over binomial (or trinomial) trees because the transition from a differential equation to a difference equation is easier when the grid/mesh is simple and regular. This offers more flexibility as compared to the lattice methods. However, it is well known that the kink at the strike price in the payoff function causes lower-order convergence when higher-order FD schemes are applied to solve these option pricing PDEs.

Numerous ideas have been proposed to enhance the convergence of FD methods. Clarke and Parrott [19] used a coordinate transformation, stretched the region around the strike price where there is a discontinuity in the first derivative of the final condition, and found that the accuracy of their implicit FD method was improved. Another way of obtaining more grid points around the discontinuity is to use adaptive grid points as in Persson and von Sydow [44]. Recently, Oosterlee et al. [43] obtained a fourth-order accurate solution

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for European options using the grid stretching transformation [52] in combination with the fourth-order spatial discretisation based on a five-point stencil and the fourth-order backward differencing formula (BDF4) for time discretisation. More recently, Tangman et al. [50] considered the higher-order compact (HOC) schemes and used a grid stretching that concentrates the grid nodes at the strike price for the European options.

In this paper we will explore spectral methods to discretise the option pricing problems in the asset (spatial) direction. Spectral methods are a class of approximation methods that are well known for the task of solving partial differential equations [17]. For smooth enough solutions, they are exponentially convergent in the number of degrees of freedom [16, 24, 49]. Although widely used in fields such as fluid mechanics, their use in option pricing have been rare. The main drawback for their direct application to option pricing is that the payoff functions for typical options or the initial conditions in the governing PDEs are nonsmooth. Thus, the collocation approximations are reduced to low-order accuracy, making them not competitive with existing finite difference methods. The literature is rich in ideas for overcoming this problem. One approach is to regularise the initial condition as proposed by Greenberg [28]. Suh [47, 48] used the Broadie-Detemple [12] approach and obtained a significant improvement of the pseudospectral method over the finite difference methods (FDM) while solving PDEs and PIDEs (partial integro-differential equations) in finance. Tangman et al. [51] presented a new approach which consists in dividing the set of Chebyshev points into two at the strike price E. To this end, the new set of points will cluster the grid nodes not only at the boundaries but also at the singularity located at the strike price for a European option. Using such a strategy, the Chebyshev collocation method achieved fourth-order accuracy. Zhu [60] proposed a spectral element method based on the regularisation approach of Greenberg [28] to price European options with and without jumps in one and two dimensions. He successfully recovered the exponential accuracy of spectral methods.

To discretise the problem in time direction, we use a class of implicit-explicit (IMEX) methods. These methods have been used in conjunction with spectral methods [16] to solve problems involving different types of PDEs. Ascher et al. [4] constructed families of first-, second-, third-, and fourth-order IMEX multistep methods to solve convection-diffusion equations. Ruuth [46] used IMEX multistep methods and efficiently solved reaction-diffusion problems in pattern formation. Recently, Hundsdorfer and Ruuth [34] extended the construction of IMEX multistep methods with general monotonicity and boundedness properties to hyperbolic systems with stiff source or relaxation terms. IMEX multistep methods also appear in the field of option pricing. In particular, for jump-diffusion PIDE, Almendral and Oostelee [2] proposed a second-order backward differentiation formula (BDF). Feng and Linetsky [22] proposed an extrapolation approach in combination with the first-order accurate IMEX-Euler scheme. Their experiments show that the extrapolation method improved significantly over the first-order IMEX-Euler scheme in solving the jump-diffusion PIDE. Another family of IMEX schemes is based on Runge-Kutta methods. Ascher et al. [3] constructed IMEX Runge-Kutta methods for solving convection-diffusion-reaction problems. De Frutos [25, 26] introduced IMEX-RK methods as an alternative to other existing time integration methods for pricing options. We refer the interested readers to [3, 8, 14, 15, 25, 36] for recent developments on IMEX-RK methods.

The class of IMEX methods that we will be using belongs to the family of IMEX-PC schemes. These are successfully applied to solve stiff PDEs. The main idea is to split the basic multistep IMEX into predictor-corrector (PC) schemes. Cash [18] used this idea to construct a new class of multistep methods. By splitting the BDF, he obtained a new BDF which has considerably better stability than the standard BDF while maintaining the same accuracy. Voss and Casper [55] used a split version of the Adams-Moulton formulae as a

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novel family of PC schemes for stiff ODEs. Voss and Khaliq [56] considered the θ-methods in a linearly implicit form as the predictor and derived an implicit second-order PC scheme for reaction-diffusion problems. Recently, Li et al. [40] adopted the strategy found in [4] to construct a family of higher-order IMEX-PC schemes for nonlinear parabolic differential equations. Their numerical results show that these IMEX-PC methods have a significant better stability than those found in [4]. More recently, Grooms and Julien [29] derived a fourth-order IMEX-PC scheme. Their method used the fourth-order total variation IMEX scheme found in [34] as a predictor and the fourth-order BDF scheme as a corrector. To the best of our knowledge, IMEX-PC methods have not been used to price financing options, except in [37] where a second-order IMEX-PC scheme is used to price American options.

In this paper we present a spectral method based on the improved Lagrange formula to compute European, digital, and butterfly spread options. Our method is coupled with a third-order IMEX-PC for time integration. The reason for using higher-order IMEX-PC is that we expect our spectral method to provide exponential accuracy, which is usually affected by lower-order temporal schemes. We then extend this approach to solve a two-dimensional option pricing problem described by the Heston model.

The rest of this paper is organised as follows: in Section 2, we describe the formulation of the option pricing problem in the Black-Scholes framework. In Section 3, the spatial approximations of the pricing equations using spectral methods are considered. In Section 4, we review the IMEX-PC methods for solving the semi-discrete system resulting from the spatial discretisation. The overall method is analysed in Section 5. Numerical experiments are conducted in Section 6. The extension of the proposed approach to a two-dimensional case is given in Section 7. Finally, in Section 8 we present some concluding remarks and scope for future research.

2. The mathematical model. Consider the financial market model given by the following tuple $\mathcal{M} = \left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_\tau)_{\tau\geq 0}, (S_\tau)_{\tau\geq 0}\right)$ where Ω is the set of all possible outcomes of the experiment known as the sample space, $\mathcal{F}$ is the set of all events, i.e., permissible combinations of outcomes, $\mathbb{P}$ is a map $\mathcal{F} \to [0,1]$ which assigns a probability to each event, $\mathcal{F}_\tau$ is a natural filtration, and S_τ is a risky underlying asset price process. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as a probability space. Let Z_τ be a $\mathbb{P}$-Brownian motion, $\sigma > 0$ the volatility of the underlying asset, $\mu > 0$ the expected rate of return, $r > 0$ the interest rate, and $\delta > 0$ the continuous dividend yield. Without loss of generality, μ, σ, r, and δ are assumed to be constant. Then under the equivalent martingale measure $\mathbb{Q}$, the stochastic process of the asset price S_τ is assumed to follow the geometric Brownian motion

$$\frac{dS_\tau}{S_\tau} = \mu dt + \sigma dZ_\tau. \tag{2.1}$$

Now, consider a portfolio that involves short selling of one unit of a European call option and long holding of Δ_τ units of the underlying asset. The portfolio value $\Pi(S_\tau, \tau)$ at time τ is then given by

$$\Pi = -V + \Delta_\tau S_\tau, \tag{2.2}$$

where $V = V(S_\tau, \tau)$ denotes the value of the option. The jump in the value of the portfolio in one time step is

$$d\Pi = -dV + \Delta_\tau dS_\tau.$$

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Note that Δ_τ changes with time τ, reflecting the dynamic nature of hedging. Since V is a stochastic function of S_τ, we apply Ito's lemma to compute its differential, which gives

$$dV = \frac{\partial V}{\partial \tau} d\tau + \frac{\partial V}{\partial S_\tau} dS_\tau + \frac{\sigma^2 S_\tau^2}{2} \frac{\partial^2 V}{\partial S_\tau^2} d\tau. \tag{2.3}$$

Substituting (2.1) and (2.3) into (2.2) and simplifying, we obtain

$$d\Pi = \left[-\frac{\partial V}{\partial \tau} - \frac{\sigma^2 S_\tau^2}{2} \frac{\partial^2 V}{\partial S_\tau^2} + \left(\Delta_\tau - \frac{\partial V}{\partial S_\tau} \right) \mu S_\tau \right] d\tau + \left(\Delta_\tau - \frac{\partial V}{\partial S_\tau} \right) \sigma S_\tau dZ_\tau.$$

The cumulative financial gain on the portfolio at time τ is given by

$$\begin{aligned} G(\Pi(S_\tau, \tau)) &= \int_0^\tau -dV + \int_0^\tau \Delta_\tau dS_\tau \\ &= \int_0^\tau \left[-\frac{\partial V}{\partial u} - \frac{\sigma^2 S_u^2}{2} \frac{\partial^2 V}{\partial S_u^2} + \left(\Delta_u - \frac{\partial V}{\partial S_u} \right) \mu S_u \right] du \\ &\quad + \int_0^u \left(\Delta_u - \frac{\partial V}{\partial S_u} \right) \sigma S_u dZ_u. \end{aligned} \tag{2.4}$$

The stochastic component of the portfolio gain stems from the second term of (2.4).

Suppose we adopt the dynamic hedging strategy by choosing $\Delta_u = \frac{\partial V}{\partial S_u}$ at all times $u < \tau$. Then the financial gain becomes deterministic at all times. By virtue of no arbitrage, the financial gain should be the same as the gain from investing on the risk free asset with a dynamic position whose value equals $-V + S_u \frac{\partial V}{\partial S_u}$. The deterministic gain from this dynamic position of the riskless asset is given by

$$\widetilde{G}_\tau = \int_0^u \left(-rV + (r-\delta) S_u \frac{\partial V}{\partial S_u} \right) du.$$

By equating these two deterministic gains $G(\Pi(S_\tau, \tau))$ and $\widetilde{G}_\tau$, we have

$$-\frac{\partial V}{\partial u} - \frac{\sigma^2 S_u^2}{2} \frac{\partial^2 V}{\partial S_u^2} = \left(-rV + (r-\delta) S_u \frac{\partial V}{\partial S_u} \right), \quad 0 < u < \tau,$$

which is satisfied for any asset price S if $V(S, \tau)$ satisfies the equation

$$\frac{\partial V}{\partial \tau} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0, \quad 0 < \tau < T.$$

The above partial differential equation is called the Black-Scholes equation [7].

Now, by a change of variables $t = T - \tau$ (T is the time of expiration), we can rewrite the above equation as

$$\frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV. \tag{2.5}$$

The boundary and the final conditions make the difference between American and European style options as well as between puts and calls and other types of options. In this article, we consider European vanilla, binary, and spread options, whose final and boundary conditions are given in Section 6, where we provide numerical results. We then, in Section 7, extend this approach to solve a two-dimensional option pricing problem described by the Heston model.

3. Spectral method for the discretisation in space. In our spectral discretisation in space, we will be using a class of Lagrange interpolation formulae. This interpolation is theoretically very powerful and deplored mainly for numerical practice as reported in many textbooks of numerical analysis [1]. With slight modifications, the Lagrange formula is indeed of great practical use. This has been noted by several authors, including Henrici [30] and Werner [58]. Berrut and Trefethen [6] modified the Lagrange polynomial through the formula of barycentric interpolation and proposed an improved Lagrange formula. In this section, we review the improved Lagrange formula and propose a spatial dicretisation of the option pricing problems discussed in earlier sections.

3.1. Lagrange interpolation. We would like to find the polynomial $p_N(x)$ from the vector space of all polynomials of degree at most N that interpolates the data f_j at distinct interpolation points x_j, $j = 0, \ldots, N$, i.e.,

$$p_N(x_j) = f_j, \quad j = 0, \ldots, N.$$

Recall that the Lagrange form of $p_N(x)$ is ([39])

$$p_N(x) = \sum_{j=0}^{N} f_j \ell_j(x), \quad \ell_j(x) = \prod_{k=0,k\neq j}^{N} \frac{x - x_k}{x_j - x_k}, \tag{3.1}$$

where the Lagrange polynomial ℓ_j corresponding to the node x_j has the property

$$\ell_j(x_k) = \begin{cases} 1 & j = k, \\ 0 & \text{otherwise.} \end{cases}$$

The drawbacks of the Lagrange formula (3.1) are

1. It takes $\mathcal{O}(N^2)$ additions and multiplications for each evaluation of $p_N(x)$.
2. A new computation from scratch has to be performed if we add a new pair of data (x_{N+1}, f_{N+1}).
3. Instability may be present in numerical computation.

It would be advantageous to modify the formula (3.1) in order to overcome the above shortcomings.

3.2. A modified Lagrange formula. Following [6], the Lagrange formula (3.1) can be rewritten in such a way that $p_N(x)$ is computed in $\mathcal{O}(N)$ operations. We define $\ell(x)$, the numerator of ℓ_j in (3.1), as

$$\ell(x) = \frac{1}{x - x_j} \prod_{k=0}^{N} (x - x_k).$$

In addition, if we define the barycentric weight by

$$w_j = \frac{1}{\prod_{k=0,k\neq j}^{N} (x_j - x_k)}, \quad j = 0, \ldots, N,$$

i.e., $w_j = 1/\ell'(x_j)$, then ℓ_j in (3.1) becomes

$$\ell_j(x) = \ell(x) \frac{w_j}{x - x_j}.$$

Consequently, the Lagrange formula (3.1) becomes

$$p_N(x) = \ell(x) \sum_{j=0}^{N} \frac{w_j}{x - x_j} f_j. \tag{3.2}$$

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3.3. Barycentric formula. The formula (3.2) can be written in a more elegant way. If we represent the constant function $f(x) = 1$, we obtain

$$1 = \sum_{j=0}^{N} \ell_j(x) = \ell(x) \sum_{j=0}^{N} \frac{w_j}{x - x_j}. \tag{3.3}$$

Dividing (3.2) by (3.3), we get the barycentric formula for p_N

$$p_N(x) = \frac{\sum_{j=0}^{N} \frac{w_j}{x-x_j} f_j}{\sum_{j=0}^{N} \frac{w_j}{x-x_j}}. \tag{3.4}$$

This is the most used form of Lagrange interpolation in practice. We see that the formula (3.4) is special case of (3.2).

A significant advantage of the spectral collocation method based on the modified barycentric Lagrange interpolation is that after the transformation, the derivatives in the underlying differential equation do not have to be transformed correspondingly as it is usual in other spectral collocation methods. More details regarding the convergence and stability properties of the modified Lagrange formula are extensively discussed in [6, 33, 57].

3.4. Calculation of the component matrices. Suppose that the solution u of the semi-discrete version of the PDE (2.5) is represented in the Lagrange form

$$u(x) = \sum_{j=0}^{N} u_j \ell_j(x). \tag{3.5}$$

Then the first and the second derivatives of u are given by

$$u'(x) = \sum_{j=0}^{N} u_j \ell_j'(x), \quad u''(x) = \sum_{j=0}^{N} u_j \ell_j''(x). \tag{3.6}$$

The barycentric formula of ℓ_j is given by

$$\ell_j(x) = \frac{\frac{w_j}{x-x_j}}{\sum_{k=0}^{N} \frac{w_k}{x-x_k}}. \tag{3.7}$$

Multiplying both sides of (3.7) by $x - x_i$ and simplifying, we get

$$\ell_j(x) \sum_{k=0}^{N} w_k \frac{x - x_i}{x - x_k} = w_j \frac{x - x_i}{x - x_j}. \tag{3.8}$$

Let

$$s(x) = \sum_{k=0}^{N} w_k \frac{x - x_i}{x - x_k}.$$

Then the first and the second derivatives of (3.8) yield the following equations

$$\ell_j'(x)s(x) + \ell_j(x)s'(x) = w_j \left(\frac{x - x_i}{x - x_j} \right)' \tag{3.9}$$

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and

$$\ell_j''(x)s(x) + 2\ell_j'(x)s'(x) + \ell_j(x)s''(x) = w_j \left(\frac{x - x_i}{x - x_j}\right)''. \tag{3.10}$$

To find the entries of the first and second differentiation matrices, we solve (3.9) and (3.10) at $x = x_i$. This gives

$$s(x_i) = w_i, \quad s'(x_i) = \sum_{k=0,k\neq i}^{N} w_k/(x_i - x_k), \quad s''(x_i) = -2\sum_{k=0,k\neq i}^{N} w_k/(x_i - x_k)^2.$$

When $i \neq j$ we obtain

$$\ell_j(x_i) = 0, \quad \ell_j'(x_i) = \frac{w_j/w_i}{x_i - x_j}, \quad \ell_j''(x_i) = -2\frac{w_j/w_i}{x_i - x_j}\left[\sum_{k=0,k\neq i}^{N} \frac{w_k/w_i}{x_i - x_k} - \frac{1}{x_i - x_j}\right].$$

When $i = j$ we obtain

$$\ell_j'(x_j) = -\sum_{i\neq j}^{N} \ell_j'(x_i), \quad \ell_j''(x_j) = -\sum_{i\neq j}^{N} \ell_j''(x_i).$$

The above can be used for the entries of the first- and second-order differentiation matrices $D^{(1)}$ and $D^{(2)}$ which are given by

$$D_{ij}^{(1)} = \ell_j'(x_i), \quad D_{ij}^{(2)} = \ell_j''(x_i).$$

3.5. Chebyshev grid transformations. Spectral methods are exponentially accurate for smooth problems but in option pricing problems the initial condition is typically not differentiable and may be discontinuous. It is known (see, e.g., [53]) that local grid refinements may improve the accuracy near a region of singularity and hence improve the overall accuracy of the numerical method. Therefore, a local grid refinement near the non-differentiable or discontinuous payoff condition seems to be a logical choice to retain a satisfactory accuracy. In this paper we use an analytic coordinate transformation to stretch grids around strike prices. Following [53], we use the transformation

$$x = g(z) = \alpha + \beta \sinh\left[\sinh^{-1}\left(\frac{1+\alpha}{\beta}\right)\frac{1+z}{2} - \sinh^{-1}\left(\frac{1-\alpha}{\beta}\right)\frac{1-z}{2}\right], \tag{3.11}$$

where α is the point of singularity in the Chebyshev domain $[-1, 1]$, β is a parameter that determines the stretching rate around α, and $z_k = \cos(\pi k/N)$ are the Chebyshev-Gauss-Lobatto (CGL) collocation points.

In the case of multiple regions of singularity, it is possible to combine maps with a single point of singularity in order to concentrate points around these regions. Suppose that we have a collection of maps $h_k(z)$, $k = 1, \ldots, n$, which cluster points around regions of rapid change δ_k with distribution parameters β_k. We define such maps by

$$G(z) = H^{-1}(z), \tag{3.12}$$

where

$$H(z) = \sum_{k=1}^{n} a_k h_k^{-1}(z), \quad \sum_{k=1}^{n} a_k = 1,\ a_k > 0.$$

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In the case of butterfly spread options, we have three singularities and therefore we will have

$$H(z) = a_1 h_1^{-1}(z) + a_2 h_2^{-1}(z) + a_3 h_3^{-1}(z), \quad a_1 + a_2 + a_3 = 1,\ a_1, a_2, a_3 > 0. \tag{3.13}$$

Maps such as (3.12) are nonlinear and have to be solved numerically using generic nonlinear equation solvers.

3.6. Application to the Black-Scholes PDE. The Black-Scholes PDE (2.5) is discretized in the asset (space) direction by means of a modified barycentric Lagrange collocation (BLC) approach. Let $x = g(z_j)$ be the transformed Chebyshev points. Then the first step is to transform $x \in [-1, 1]$ into $S \in [S_m, S_M]$ that better suits the option at hand. We do this through $x = (2S - (S_M - S_m))/(S_M + S_m)$ where S_m and S_M are the minimal and the maximal values of the underlying asset. Now writing $V(S, t) = u(x, t)$, the PDE (2.5) together with its initial and boundary conditions yield

$$\begin{aligned}
&u_t = p(x)u_{xx} + q(x)u_x + ru, && \\
&u(x, 0) = u^0, && -1 \le x \le 1, S_m \le S \le S_M, \\
&u(-1, t) = u_0,\ \ u(1, t) = u_N, && 0 \le t \le T,
\end{aligned}$$

where

$$p(x) = \frac{1}{2}\sigma^2 S^2 \left(\frac{2}{S_M - S_m}\right)^2,\ \ q(x) = (r - \delta)S\left(\frac{2}{S_M - S_m}\right).$$

Substituting (3.5) and (3.6) yields the following system of nonlinear ODEs

$$\begin{aligned}
&u_t(x, t) = p(x)\sum_{k=0}^{N} u_k(t)\ell_k''(x) + q(x)\sum_{j=0}^{N} u_k(t)\ell_k'(x) + r\sum_{k=0}^{N} u_k(t)\ell_k(x), \\
&u_0 = u(-1, t), \quad u_N = u(1, t).
\end{aligned} \tag{3.14}$$

In order to write (3.14) in matrix form, we introduce the following matrix and vector notation

$$\begin{aligned}
&\mathbf{u} = [u_1, u_2, \ldots, u_{N-1}]^T, && & \\
&D^{(1)} = \left(D_{ij}^{(1)}\right), && D_{ij}^{(1)} = \ell_j'(x_i), && i, j = 1, \ldots, N-1, \\
&D^{(2)} = \left(D_{ij}^{(2)}\right), && D_{ij}^{(2)} = \ell_j''(x_i), && i, j = 1, \ldots, N-1, \\
&P = \mathrm{diag}(p(x_i)), && Q = \mathrm{diag}(q(x_i)), && i = 1, \ldots, N-1,
\end{aligned}$$

moreover I denotes an $(N-1)\times(N-1)$ identity matrix. P and Q are diagonal matrices whose entries are $p(x_i)$ and $q(x_i), i = 1, 2, \ldots, N-1$, respectively. Consequently, (3.14) can be expressed as an initial value problem of the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = A\mathbf{u} + \mathrm{g}(t, \mathbf{u}), \quad u(0) = u_0, \tag{3.15}$$

where

$$\begin{aligned}
A &= PD^{(2)} \\
\mathrm{g}(t, \mathbf{u}) &= \Big[QD^{(1)}\mathbf{u} - rI\mathbf{u} + \left(p(x_i)D_{i0}^{(2)} + q(x_i)D_{i0}^{(1)} + rI_{i0}\right)u_0 \\
&\qquad + \left(p(x_i)D_{iN}^{(2)} + q(x_i)D_{iN}^{(1)} + rI_{iN}\right)u_N\Big]^T.
\end{aligned}$$

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4. Implicit-explicit predictor-corrector method for the discretisation in time. The system of ODEs (3.15) can be solved by means of standard ODE time integrators. The main challenge when dealing with this type of problems is that explicit time integrators are inadequate because the diffusion term is typically stiff and necessitates excessively small time steps. On the other hand, the use of stiffly accurate implicit time integrators which are unconditionally stable is practically time consuming. In order to avoid these problems, it could be interesting to separate non-stiff and stiff terms. The non-stiff term has to be solved explicitly whereas the stiff term has to be integrated implicitly. Such time integrators are known as implicit-explicit (IMEX) time integrators and have been used for the time integration of spatially discretised PDEs of reaction-diffusion type [46]. In this article, we use IMEX-PC methods to integrate the system of ODEs obtained after a spatial discretisation of the PDE (2.5) mentioned above.

Let us consider the system of ODEs (3.15)

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathrm{g}(t,\mathbf{u}), \quad \mathbf{u}(t_0) = \mathbf{u}_0,$$

and let k be the time step-size and u_n the approximation of the solution at $t_n = kn$. Following the strategy of [4], we may write the general s-step IMEX method when applied to the system of ODEs (3.15) as

$$\sum_{j=0}^{s} a_j u_{n+j} = k\sum_{j=0}^{s} b_j A u_{n+j} + k\sum_{j=0}^{s-1} c_j \mathrm{g}(t_{n+j}, u_{n+j}), \tag{4.1}$$

where $a_s \neq 0$. Following [40], the split form of (4.1) yields the following IMEX-PC

$$(a_s I - k b_s A)\tilde{u}_{n+s} = \sum_{j=0}^{s-1}(-a_j u_{n+j} + k b_j A u_{n+j} + k c_j \mathrm{g}(t_{n+j}, u_{n+j})), \quad \text{Predictor} \tag{4.2}$$

$$\begin{aligned}(a_s I - k b_s A)u_{n+s} = \sum_{j=0}^{s-1}&(-a_j u_{n+j} + k b_j A u_{n+j} + k b_j \mathrm{g}(t_{n+j}, u_{n+j})) \\ &+ k b_s \mathrm{g}(t_{n+s}, \tilde{u}_{n+s}).\end{aligned} \quad \text{Corrector} \tag{4.3}$$

The above IMEX-PC uses the IMEX of [4] as the predictor and implicit schemes as the corrector. Only the non-stiff term is corrected; the corrector treats the stiff term implicitly. This significantly reduces the computational cost compared with general implicit methods. As compared to the PC used in [37, 55], the present strategy does not require the use of iterative solvers such as Newton's method.

We denote by IMEX-PC(s,m) the s-step implicit-explicit predictor-corrector of order m.

IMEX-PC$(1,m)$: the IMEX-PC$(1,m)$ is a family of 1-step, one-parameter (γ) IMEX-PC schemes of order m and can be written as follows:

$$\begin{aligned}(I-\gamma kA)\,\tilde{u}_{n+1} &= [I + (1-\gamma)kA]\,u_n + k\mathrm{g}(t_n,u_n), && \text{Predictor}\\ (I-\gamma kA)\,u_{n+1} &= [I + (1-\gamma)kA]\,u_n + (1-\gamma)k\mathrm{g}(t_n,u_n) && \\ &\quad + \gamma k\mathrm{g}(t_{n+1},\tilde{u}_{n+1}), && \text{Corrector}\end{aligned}$$

where the parameter $0 \leq \gamma \leq 1$ prevents large truncation errors. The choice $\gamma = 1$ yields an IMEX-PC(1,1) scheme.

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IMEX-PC$(2,m)$: the IMEX-PC$(2,m)$ is a family of 2-step, two-parameter (γ and c) IMEX-PC schemes of order m and can be written as follows:

$$\begin{aligned}
&\left[\left(\gamma+\tfrac{1}{2}\right)I-\left(\gamma+\tfrac{c}{2}\right)kA\right]\tilde{u}_{n+1}\\
&\quad=[2\gamma I+(1-\gamma-c)kA]u_n+\left[\left(\tfrac{1}{2}-\gamma\right)I+\tfrac{c}{2}kA\right]u_{n-1} &&\text{Predictor}\\
&\qquad+(\gamma+1)k\mathrm{g}(t_n,u_n)-\gamma k\mathrm{g}(t_{n-1},u_{n-1}),
\end{aligned}$$

$$\begin{aligned}
&\left[\left(\gamma+\tfrac{1}{2}\right)I-\left(\gamma+\tfrac{c}{2}\right)kA\right]\tilde{u}_{n+1}\\
&\quad=[2\gamma I+(1-\gamma-c)kA]u_n+\left[\left(\tfrac{1}{2}-\gamma\right)I+\tfrac{c}{2}kA\right]u_{n-1}\\
&\qquad+(\gamma+\tfrac{c}{2})k\mathrm{g}(t_{n+1},\tilde{u}_{n+1})+(1-\gamma-c)k\mathrm{g}(t_n,u_n) &&\text{Corrector}\\
&\qquad+\tfrac{c}{2}k\mathrm{g}(t_{n-1},u_{n-1}).
\end{aligned}$$

Choosing $(\gamma,c)=(0,1)$ we obtain an IMEX-PC(2,2) scheme.

IMEX-PC$(3,m)$: the IMEX-PC$(3,m)$ is a family of 3-step, three-parameter (γ, θ, and c) IMEX-PC schemes of order m and can be written as follows:

$$\begin{aligned}
&\left[\left(\tfrac{1}{2}\gamma^2+\gamma+\theta\right)I-\left(\tfrac{\gamma^2+\gamma}{2}+c\right)kA\right]\tilde{u}_{n+1}\\
&=\left[\left(\tfrac{3}{2}\gamma^2+2\gamma-\tfrac{1}{2}+\theta\right)I+\left(1-\gamma^2-3c+\tfrac{23}{12}\theta\right)kA\right]u_n\\
&\quad+\left[\left(-\tfrac{3}{2}\gamma^2-\gamma+1\right)I+\left(\tfrac{\gamma^2-\gamma}{2}+3c-\tfrac{4}{3}\theta\right)kA\right]u_{n-1}\\
&\quad+\left[\left(\tfrac{1}{2}\gamma^2-\tfrac{1}{6}\right)I+\left(\tfrac{5}{12}\theta-c\right)kA\right]u_{n-2} &&\text{Predictor}\\
&\quad+\left(\tfrac{\gamma^2+3\gamma}{2}+1+\tfrac{23}{12}\theta\right)k\mathrm{g}(t_n,u_n)-\left(\gamma^2+2\gamma+\tfrac{4}{3}\theta\right)k\mathrm{g}(t_{n-1},u_{n-1})\\
&\quad+\left(\tfrac{\gamma^2+\gamma}{2}+\tfrac{5}{12}\theta\right)k\mathrm{g}(t_{n-2},u_{n-2}),
\end{aligned}$$

$$\begin{aligned}
&\left[\left(\tfrac{1}{2}\gamma^2+\gamma+\theta\right)I-\tfrac{k}{6}A\right]\tilde{u}_{n+1}\\
&=\left[\left(\tfrac{3}{2}\gamma^2+2\gamma-\tfrac{1}{2}+\theta\right)I+\left(1-\gamma^2-3c+\tfrac{23}{12}\theta\right)kA\right]u_n\\
&\quad+\left[\left(-\tfrac{3}{2}\gamma^2-\gamma+1\right)I+\left(\tfrac{\gamma^2-\gamma}{2}+3c-\tfrac{4}{3}\theta\right)kA\right]u_{n-1}\\
&\quad+\left[\left(\tfrac{1}{2}\gamma^2-\tfrac{1}{6}\right)I+\left(\tfrac{5}{12}\theta-c\right)kA\right]u_{n-2} &&\text{Corrector}\\
&\quad+\left(\tfrac{\gamma^2+\gamma}{2}+c\right)k\mathrm{g}(t_{n+1},\tilde{u}_{n+1}+\left(1-\gamma^2-3c+\tfrac{23}{12}\theta\right)k\mathrm{g}(t_n,u_n))\\
&\quad-\left(\tfrac{\gamma^2-\gamma}{2}+3c-\tfrac{4}{3}\theta\right)k\mathrm{g}(t_{n-1},u_{n-1})\\
&\quad+\left(\tfrac{5}{12}\theta-c\right)k\mathrm{g}(t_{n-2},u_{n-2}).
\end{aligned}$$

The choice $(\gamma,\theta,c)=(1,0,0)$ yields an IMEX-PC(3,3) scheme.

5. Analysis of the method. In [40], Li et al. gave stability and convergence results for IMEX-PC methods for solving stiff problems. We briefly recall some of them and associate these with our option pricing problems. Then we compare the stability regions of

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these IMEX-PC methods to those of the existing IMEX methods [4]. The order of accuracy of the present IMEX-PC is given by the following theorem.

THEOREM 5.1 ([40]). *Let us suppose that the s-step IMEX predictor schemes (4.2) are of order p and that the corrector schemes (4.3) have order q. Then the resulting IMEX-PC is of order* $\min\{p+1, q\}$.

We would like to analyse the stability of the IMEX-PC schemes (4.2) and (4.3) when applied to the PDE problem (2.5). It is beneficial to transform this PDE into one with constant coefficients by considering the transformation $x = \log(S/E)$, where E is the strike price. Therefore the problem (2.5) becomes

$$\frac{\partial V}{\partial t} = b\frac{\partial^2 V}{\partial x^2} - a\frac{\partial V}{\partial x} - cV, \quad -\infty \le x \le \infty,\ 0 \le t \le T, \tag{5.1}$$

where $b = \frac{1}{2}\sigma^2$, $a = -(r - \delta - \frac{1}{2}\sigma^2)$, $c = r$, V denotes the value of the European options, $t = T - \tau$ is the time to expiry, and T is the expiration (maturity) time.

The first step is to find a spectral representation of this problem. To this end, we consider the following change of variables

$$V(x,t) = e^{i\xi x}u(t). \tag{5.2}$$

The substitution of (5.2) into (5.1) yields the scalar test equation

$$u' = H(\xi)u(t) + G(\xi)u(t) \tag{5.3}$$

where $H(\xi) = -b\xi^2$ and $G(\xi) = -ia\xi - c$. By applying the IMEX-PC methods (4.2) and (4.3) to the scalar test equation (5.3) with step size k, we obtain

$$(a_s - kH(\xi)b_s)\tilde{u}_{n+s} = \sum_{j=0}^{s-1}[-a_j + kH(\xi)b_j + kG(\xi)c_j]u_{s+j}, \tag{5.4}$$

and

$$(a_s - kH(\xi)b_s)u_{n+s} = \sum_{j=0}^{s-1}[-a_j + kH(\xi)b_j + kG(\xi)b_j]u_{s+j} + kG(\xi)b_s\tilde{u}_{n+s}\,. \tag{5.5}$$

Substituting the variables $z = kH(\xi)$, $w = kG(\xi)$, and $R^n = u_n$, into the Equations (5.4) and (5.5) and plugging in (5.4) into (5.5) yields the following characteristic equation

$$\varphi(R; z, w) = R^s - \sum_{j=0}^{s-1}\left[\frac{-a_j + zb_j + wb_j}{(a_s - zb_s)} + \frac{wb_s}{(a_s - zb_s)^2}(-a_j + zb_j + wc_j)\right]R^j. \tag{5.6}$$

Note that the IMEX-PC is linearly stable when all the roots of the characteristic polynomial (5.6) have modulus less than or equal to one. In other words, let $R_i(z,w)$ be the roots of the characteristic polynomial for $i = 1, 2, \ldots, s$. Then we define the stability region S of the method as

$$S = \left\{(z,w) \in \mathbb{C}^2 : |R_i(z,w)| \le 1, \forall i\right\}.$$

The root of the characteristic polynomial of the IMEX-PC(1,2) method is given by

$$R(z,w) = \frac{1 - 2\gamma z + z + \gamma^2 z^2 - \gamma z^2 + w + \gamma w^2}{(1-\gamma z)^2},$$

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whereas the root of the characteristic polynomial of the first-order IMEX method [4] is given by

$$R(z,w) = \frac{1+z-\gamma z+w}{1-\gamma z}.$$

For higher-order PC methods we do not provide general explicit expressions of their characteristic polynomials. We rather confine our study to special cases. The choice $(\gamma, c) = (1,0)$ gives the following characteristic polynomial

$$\left(z-\frac{3}{2}\right)R^2 + \left(2+\frac{4w+4w^2}{3-2z}\right)R - \left(\frac{1}{2}+\frac{w+2w^2}{3-2z}\right) = 0,$$

whereas the root of the characteristic polynomial of the second-order IMEX method [4] is given by

$$\left(\frac{3}{2}-z\right)R^2 - (2+2w)R + \left(\frac{1}{2}+w\right) = 0.$$

Similarly, the choice $(\gamma, \theta, c) = (1,0,0)$ for the 3-step PC gives

$$\begin{aligned}\left(\frac{11}{6}-z\right)R^3 &- \left(3+\frac{6w(3+3w)}{11-6z}\right)R^2 \\ &- \left(-\frac{3}{2}+\frac{6w(-3-6w)}{22-12z}\right)R - \left(\frac{1}{3}+\frac{2w(1+3w)}{11-6z}\right) = 0,\end{aligned}$$

whereas the root of the characteristic polynomial of the third-order IMEX method [4] is given by

$$\left(\frac{11}{6}-z\right)R^3 - (3+3w)R^2 + \left(\frac{3}{2}+w\right)R - \left(\frac{1}{3}+w\right) = 0.$$

Figure 5.1 shows the stability region of the IMEX scheme (4.1) and the IMEX-PC schemes (4.2) and (4.3) in the (z,w)-plane. Figure 5.1 (top) represents the region of stability of the IMEX(1,2) and IMEX-PC(1,2) schemes with $\gamma = \frac{1}{2}$. Figure 5.1 (left bottom) shows the stability region of the IMEX(2,2) and IMEX-PC(2,2) methods with $(\gamma, c) = (1,0)$, and Figure 5.1 (right bottom) shows the stability region of the IMEX(3,3) and IMEX-PC(3,3) methods with $(\gamma, c, \theta) = (1,0,0)$. Clearly, we observe that in all cases the stability region of the IMEX scheme [4] is included in the stability region of the proposed IMEX-PC scheme. This show that the proposed IMEX-PC methods have larger stability regions and therefore are more stable than the IMEX methods suggested in [4].

6. Numerical experiments. In this section, we present some numerical results that we obtained using the proposed approach. We consider European call, put, digital call, and butterfly spread options. Further extensions will be discussed in Section 7.

6.1. European call options. A European call option gives the holder the right to exercise the option at maturity time T. To buy the underlying asset at maturity time T makes sense if the asset price is higher than the exercise price $(S > E)$ because one can buy the asset for E and sell it immediately on the market for S. If this is not the case, then the option is worthless. The value of a European call option can be determined by solving equation (2.5) subject to the initial condition

$$V(S,0) = \max(S-E,0), \tag{6.1}$$

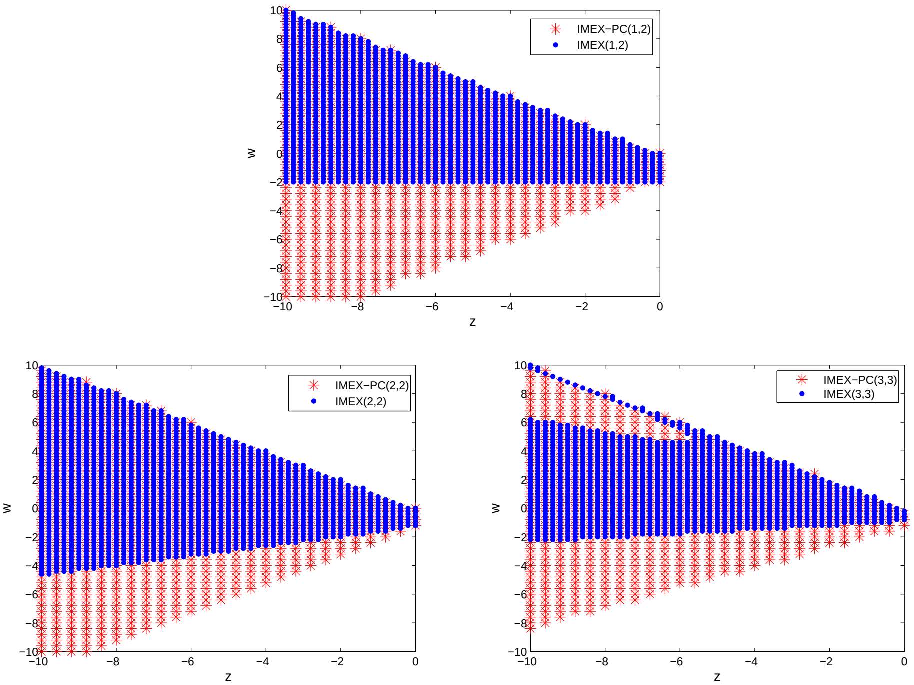

FIG. 5.1. *Absolute stability regions of the IMEX (4.1) and IMEX-PC (4.2)–(4.3): IMEX(1,2) and IMEX-PC(1,2) with* $\gamma = \frac{1}{2}$ *(top), IMEX(2,2) and IMEX-PC(2,2) with* $(\gamma, c) = (1, 0)$ *(bottom left), and IMEX(3,3) and IMEX-PC(3,3) with* $(\gamma, c, \theta) = (1, 0, 0)$ *(bottom right).*

where E is the strike price of the option V. The boundary conditions are

$$V(0,t) = 0, \qquad V(S,t) = S\mathrm{e}^{-\delta t} - E\mathrm{e}^{-rt}, \quad \text{as } S \to \infty. \tag{6.2}$$

The analytic solution of the Black-Scholes equation (2.5) for European call options is known [7, 59] and expressed as

$$V(S,t) = S\mathrm{e}^{-\delta t} N(d_1) - E\mathrm{e}^{-rt} N(d_2), \tag{6.3}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}, \quad d_2 = d_1 - \sigma\sqrt{t}, \tag{6.4}$$

and $N(\cdot)$ is the cumulative probability distribution function for a standardised normal variable

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x. \tag{6.5}$$

Numerical results are obtained with $T = 0.5$, 1, and 2 years as maturity times with $S_{\min} = 0$ and $S_{\max} = 200$ with strike price $E = 45$. The number of space mesh points is $N = 80$, and the other parameters are as indicated in the Tables 6.1–6.5. The accuracy of the present method was measured by means of the maximum error

$$L_\infty = \max_{i=1,\ldots,N} |u_i - V_i|$$

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TABLE 6.1
Comparison of European call option valuation using barycentric Lagrange collocation (BLC) with Chebyshev-Gauss-Lobatto (CGL) points and the finite difference method (FD) with uniform grid points.

Schemes	$T = 0.5$		$T = 1$		$T = 2$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	Parameters: $r = 0.05,\ \sigma = 0.2,\ \delta = 0.00$					
FD	1.9265(-3)	7.5269(-3)	1.7084(-3)	6.2297(-3)	1.5913(-3)	5.4520(-3)
BLC	1.8753(-3)	9.3260(-3)	1.5087(-3)	6.3006(-3)	1.1892(-3)	4.1491(-3)
	Parameters: $r = 0.07,\ \sigma = 0.04,\ \delta = 0.03$					
FD	7.2924(-3)	6.1838(-2)	1.1283(-2)	7.2901(-2)	1.3508(-2)	8.5026(-2)
BLC	8.2255(-3)	6.1964(-2)	4.8242(-3)	4.4339(-2)	3.0501(-3)	2.2628(-2)
	Parameters: $r = 0.1,\ \sigma = 0.3,\ \delta = 0.05$					
FD	1.4858(-3)	4.5463(-3)	1.2127(-3)	3.2754(-3)	9.7763(-4)	2.3725(-3)
BLC	1.4758(-3)	5.8705(-3)	1.1751(-3)	3.9176(-3)	9.3080(-4)	2.4952(-3)

TABLE 6.2
Comparison of European call option valuation using barycentric Lagrange collocation (BLC) with transformed Chebyshev-Gauss-Lobatto (CGL) points and the finite difference method (FD) with non-uniform grid points.

Schemes	$T = 0.5$		$T = 1$		$T = 2$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	Parameters: $r = 0.05,\ \sigma = 0.2,\ \delta = 0.00$					
FD	6.8107(-4)	1.4774(-3)	1.2556(-3)	2.6516(-3)	2.3119(-3)	4.5751(-3)
BLC	3.0696(-9)	8.8089(-9)	4.6061(-9)	1.3591(-8)	1.0053(-8)	3.6679(-8)
	Parameters: $r = 0.07,\ \sigma = 0.04,\ \delta = 0.03$					
FD	1.9240(-3)	8.2068(-3)	2.3690(-3)	7.6678(-3)	3.3822(-3)	7.1610(-3)
BLC	4.0924(-8)	1.7177(-7)	6.9650(-8)	3.2579(-7)	1.5729(-7)	6.5362(-7))
	Parameters: $r = 0.1,\ \sigma = 0.3,\ \delta = 0.05$					
FD	4.2991(-4)	8.1881(-4)	7.2360(-4)	1.3072(-3)	1.1133(-3)	1.8128(-3)
BLC	2.8688(-9)	1.1267(-8)	6.1628(-9)	2.5323(-8)	8.1796(-9)	3.8962(-8)

and the root mean square error

$$L_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(u_i - V_i)^2},$$

where N is the number of points used in the discretisation in one particular direction, V_i is the exact solution of the Black-Scholes equation given by (6.3), and u_i is the numerical approximation to the exact solution of the Black-Scholes equation. For comparison purposes, we present the absolute, maximum, and root mean square errors. However, we also add the relative errors to get a better idea of the performance of our method. We evaluate the value of a European option by finite differences (FD) using uniform grids, and barycentric Lagrange collocation (BLC) using the Chebyshev-Gauss-Lobatto (CGL) points for various option parameters. The results are displayed in Table 6.1.

Although in theory and for a range of practical problems, the higher accuracy of general spectral methods over finite difference methods [9, 23, 24] has been shown and demonstrated, one can observe from Table 6.1 that the BLC has a moderately smaller error than that of the FD. Numerically, higher-order methods, in particular spectral methods, have difficulties in accurately approximating the solution in the region of singularity, i.e., the region of dramatic change. In fact, spectral collocation methods are adequate for problems involving

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TABLE 6.3
Comparison of European put option valuation using barycentric Lagrange collocation (BLC) with transformed Chebyshev-Gauss-Lobatto (CGL) points and the finite difference method (FD) with uniform grid points.

Schemes	$T=0.5$		$T=1$		$T=2$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	Parameters: $r=0.05,\ \ \sigma=0.2,\ \ \delta=0.00$					
FD	1.8853(-4)	3.9139(-4)	3.6984(-4)	9.5309(-4)	7.8517(-4)	2.1599(-3)
BLC	3.4402(-9)	9.4734(-9)	2.8691(-9)	1.0938(-8)	6.7411(-9)	3.6676(-8)
	Parameters: $r=0.07,\ \ \sigma=0.04,\ \ \delta=0.03$					
FD	1.7707(-3)	7.7987(-3)	1.8909(-3)	7.4343(-3)	2.0736(-3)	7.4714(-3)
BLC	4.0791(-8)	1.7180(-7)	6.9573(-8)	3.2570(-7)	1.5726(-7)	6.5359(-7)
	Parameters: $r=0.1,\ \ \sigma=0.3,\ \ \delta=0.05$					
FD	2.8525(-4)	5.9174(-4))	5.0501(-4)	1.3007(-3)	8.1141(-4)	2.2758(-3)
BLC	2.6238(-9)	8.3153(-9)	6.0125(-9)	1.1213(-8)	6.1796(-9)	1.4678(-8)

TABLE 6.4
Comparison of European digital call option valuation using barycentric Lagrange collocation (BLC) with transformed Chebyshev-Gauss-Lobatto (CGL) points and the finite difference method (FD) with uniform grid points.

Schemes	$T=0.5$		$T=1$		$T=2$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	Parameters: $r=0.05,\ \ \sigma=0.2,\ \ \delta=0.00$					
FD	6.6648(-3)	2.9135(-2)	5.4136(-3)	1.9930(-2)	4.2782(-3)	1.3269(-2)
BLC	9.6328(-6)	1.3466(-5)	8.0988(-6)	1.1035(-5)	4.9873(-6)	6.6544(-6)
	Parameters: $r=0.07,\ \ \sigma=0.04,\ \ \delta=0.03$					
FD	2.5120(-2)	2.4646(-1)	1.9574(-2)	1.5775(-1)	1.4467(-2)	1.1473(-1)
BLC	3.2479(-5)	6.2697(-5)	1.9218(-5)	4.2730(-5)	9.6899(-6)	2.7942(-5)
	Parameters: $r=0.1,\ \ \sigma=0.3,\ \ \delta=0.05$					
FD	5.2655(-3)	1.8233(-2)	4.2214(-3)	1.2180(-2)	3.2368(-3)	7.7694(-3)
BLC	6.0569(-6)	8.1388(-6)	1.9365(-6)	2.5231(-6)	3.2007(-6)	1.6415(-5)

smooth initial conditions. In the present case, the first derivative of the initial condition is discontinuous at the strike price E. As a result, the BLC method cannot be significantly superior to FD as far as the accuracy is concerned.

In order to improve the accuracy of the BLC method, we use resolution grids in the region of dramatic change. We utilise the transformation (3.11) to increase the number of points in the region around the strike price $S = E$. Therefore, from Table 6.2, we observe a significant improvement of the BLC method when concentrating more grid points near the strike price, while with the FD method the improvement is moderate. This is because high resolution grids in the region of singularity at E allow the BLC to capture the rapid change in the option price, while in the region of low change, the BLC method gives very accurate results with a small number of grid points.

In Figure 6.1, we illustrate the trade-off between computational time and the accuracy as the time step is refined for the IMEX-PC(1,1) and IMEX(1,1) methods with the choice $\gamma = 1$, for the IMEX-PC(2,2) and IMEX(2,2) methods with the choice $(\gamma, c) = (1, 0)$, and for the IMEX-PC(3,3) and IMEX(3,3) methods with the choice $(\gamma, \theta, c) = (1, 0, 0)$ at time $T = 0.5$. The following parameters are used: $S_{\min} = 0$, $S_{\max} = 200$, $r = 0.2$, $\sigma = 0.3$, $\delta = 0.0$, $E = 45$, $N = 100$, and $\beta = 0.5 \times 10^{-4}$. In all cases for two methods of the same order, the IMEX-PC schemes show better results as compared to the IMEX schemes. One observes

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TABLE 6.5
Comparison of European butterfly spread option valuation using barycentric Lagrange collocation (BLC) with transformed Chebyshev-Gauss-Lobatto (CGL) points and the finite difference method (FD) with non-uniform grid points.

Schemes	$T = 0.5$		$T = 1$		$T = 2$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	Parameters: $r = 0.05,\ \ \sigma = 0.2,\ \ \delta = 0.00$					
FD	1.0574(-2)	4.3091(-2)	8.6844(-3)	3.1513(-2))	6.4097(-3)	1.9811(-2)
BLC	2.3652(-6)	5.2302(-6)	2.0773(-6)	3.8961(-6)	2.5712(-5)	1.3077(-4)
	Parameters: $r = 0.07,\ \ \sigma = 0.04,\ \ \delta = 0.03$					
FD	3.2409(-2)	2.8985(-1)	3.1689(-2)	2.1213(-1)	3.3502(-2)	1.7927(-1)
BLC	1.0341(-6)	3.4514(-6)	7.1643(-6)	3.9076(-6)	6.2304(-5)	2.4926(-4)
	Parameters: $r = 0.1,\ \ \sigma = 0.3,\ \ \delta = 0.05$					
FD	8.0303(-3)	2.8515(-2)	6.0627(-3)	1.7545(-2)	2.5792(-2)	9.2620(-2)
BLC	4.6265(-6)	2.4136(-5)	1.1158(-6)	5.6118(-6)	2.1265(-5)	9.2620(-5)

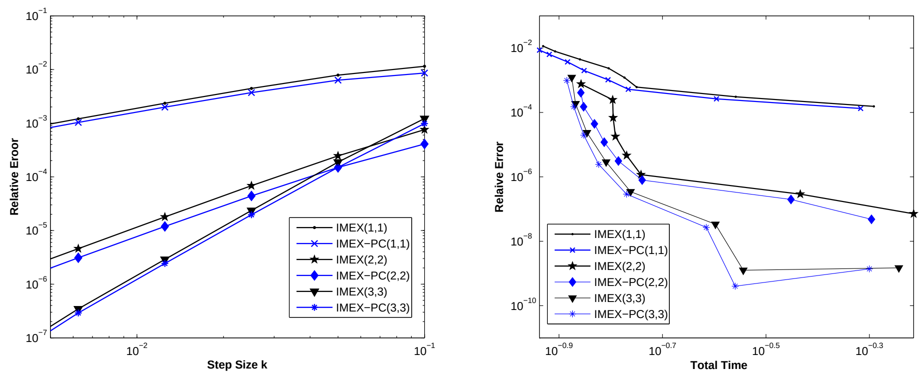

FIG. 6.1. *Performance of different IMEX-PC against IMEX methods for pricing European call options with $N = 100$, $r = 0.1$, $\sigma = 0.2$, $\delta = 0.0$, $E = 45$, and $\beta = 0.5 \times 10^{-5}$ at $T = 0.5$.*

that IMEX-PC(3,3) has the best convergence compared to other methods. Therefore in the remainder of this paper, we use IMEX-PC(3,3) as time integrating method.

Figure 6.2 illustrates the convergence of the mapped BLC method for different values of β. It can be observed that the mapped BLC converges much better than the FD method. Different values of the parameter β leads to different accuracy. The choice $\beta = 0.5 \times 10^{-1}$ shows the worst accuracy but is still very satisfactory compared to the FD method. The smaller the value of β, the better is the accuracy because then more points are clustered near the strike price E. However, we find that $\beta = 0.5 \times 10^{-4}$ gives better accuracy than $\beta = 0.5 \times 10^{-5}$. The main reason is that there are not enough points left away from the region of regularity and therefore $\beta = 0.5 \times 10^{-4}$ seems to be the optimal choice for valuing European call and put options. In the experiments below, we therefore chose $\beta = 0.5 \times 10^{-4}$. In addition, we investigate the tradeoff between computational time and the accuracy as the asset grid space is refined. Clearly the BLC method is faster than the FD method and achieves spectral convergence as expected.

Figure 6.3 represents the numerical solution for a European call option together with its Delta (Δ), Gamma (Γ), and the numerical error. All these results are very satisfactory and free of oscillations.

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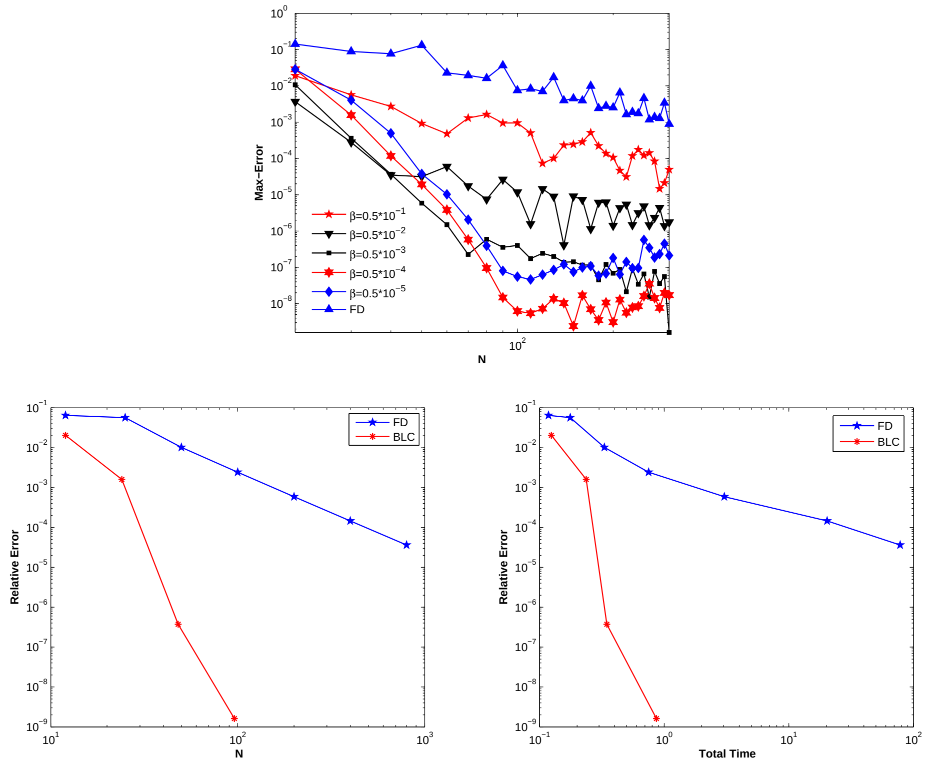

FIG. 6.2. *Convergence of the mapped BLC method for European call options with* $k = 5.10^{-4}$, $S_{\min} = 0$, $S_{\max} = 200$, $r = 0.1$, $\sigma = 0.2$, $\delta = 0.0$, $E = 45$, $T = 0.5$.

6.2. European put options. Given the value of a call option, it is possible to compute the value of the corresponding put option via the put-call-parity [38]. However, puts and calls do not always share the same properties. Therefore, we also evaluate European put options by our approach.

The value of a European put can be computed numerically by solving the PDE (2.5) subject to the initial condition

$$V(S,0) = \max(E - S, 0),$$

and the boundary conditions

$$\begin{aligned} V(0,t) &= Ee^{-rt}, \\ V(S,t) &= 0 \quad \text{as} \;\; S \to \infty. \end{aligned}$$

The benchmark used to validate our numerical scheme is the analytic solution of the Black-Scholes equation (2.5) given by

$$Ee^{-rt}N(-d_2) - Se^{-\delta t}N(d_1),$$

where d_1, d_2 are defined in (6.4) and N is the cumulative normal distribution defined in (6.5).

We use the same set of parameters as in the valuation of European call options. The results are presented in Table 6.3. It can be seen that the conclusions are similar to those for the European call options. Therefore, our approach is consistent. Hence, the approach using the

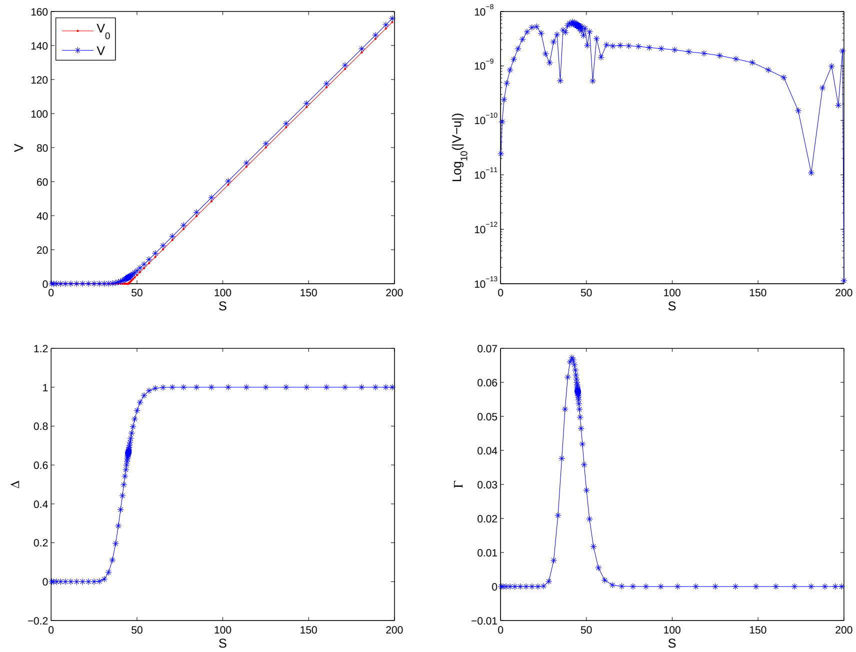

FIG. 6.3. *Valuation of the European call options (top left), its error (top right), Δ (bottom left), and Γ using the barycentric Lagrange collocation (BLC) method with $N = 80$, $k = 0.001$, $S_m = 0$, $S_M = 200$, $r = 0.1$, $\sigma = 0.2$, $\delta = 0.0$, $E = 45$, $T = 0.5$.*

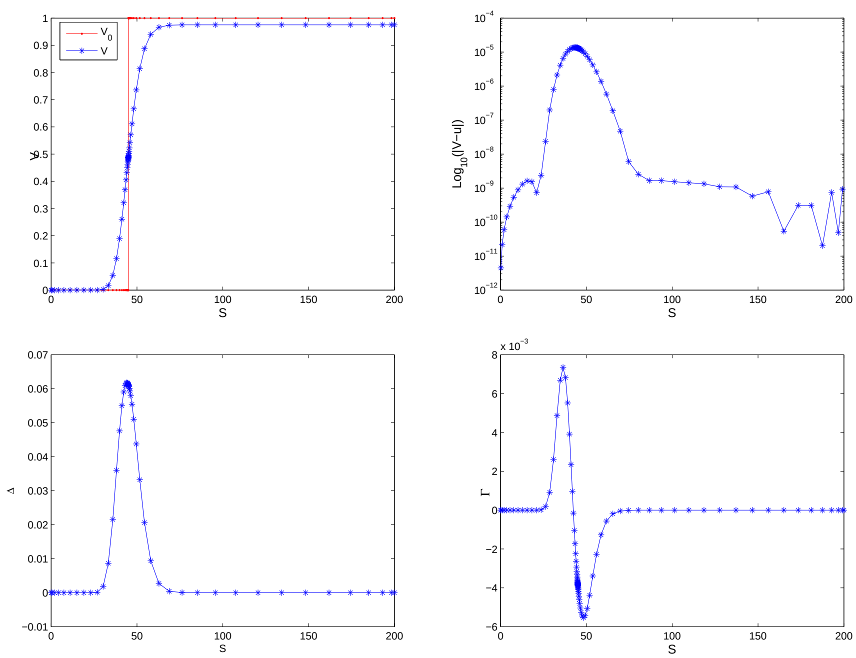

FIG. 6.4. *Valuation of the European digital call options (top left), its error (top right), Δ (bottom left), and Γ using the barycentric Lagrange collocation (BLC) method with $N = 80$, $k = 0.001$, $S_m = 0$, $S_M = 200$, $r = 0.1$, $\sigma = 0.2$, $\delta = 0.0$, $E = 45$, $T = 0.5$.*

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grid refinement at the strike price is found to perform significantly better than the FD method in terms of accuracy for valuating European option pricing problems.

Now, we investigate the utility of our approach to price two types of exotic options, namely European digital call options and butterfly spread options.

6.3. European digital call options. Another type of option that we are dealing within this paper is the digital call option. This option belongs to the class of exotic options. Such contracts are traded between a financial institution (e.g., a bank) and a customer and not at exchanges. A digital call option, also known as cash-or-nothing call or binary option, is an option with payoff zero before the strike price and one (or any fixed amount) after the strike price. As an example of these options, we solve the Black-Scholes PDE model (2.5) with the payoff function given by

$$V(S,0) = \begin{cases} 1 & \text{for } S \geq E, \\ 0 & \text{for } S \leq E, \end{cases}$$

with the following boundary conditions

$$\begin{aligned} &V(0,t) = 0, \\ &V(S,t) = e^{-rt} \quad \text{as } S \to \infty. \end{aligned}$$

The analytic solution for the digital option is

$$V(S,t) = e^{-rt} N(d_2),$$

where d_2 is in defined in (6.4). The discontinuous initial conditions for digital options are susceptible to cause numerical oscillations of the Greeks when time integrators such as the Crank-Nicolson method are used. However, our approach produces a non-oscillatory behaviour of the Greeks. Figure 6.4 represents the numerical solution for the digital call option together with its Delta (Δ), Gamma (Γ), and the numerical error. All these results are very satisfactory and free of oscillations. We also investigate the maximum error and the root mean square error for different maturity times and different parameters as chosen in the previous experiments. The results are presented in Table 6.4. Our approach (BLC) using the grid refinement at strike price is found to perform significantly better than the FD method in terms of accuracy for valuating European digital call option pricing problems, although the results are less accurate than in the case of European calls and puts. The main reason resides in the smoothness of the initial conditions. While the European call and put has a discontinuity in the first derivative of the payoff, the digital options have discontinuities in the payoff itself, i.e., the digital options, which are less smooth than the European vanilla options, produce less accurate results compared to those of the European vanilla options for the same grid stretching parameter. This is consistent with the convergence of spectral methods, which relies on the smoothness of the initial conditions.

6.4. Butterfly spread options. The butterfly spread is a combination of four options. Two long position calls with exercise price E_1 and E_3 and two short position calls with exercise price $E_2 = (E_1 + E_3)/2$. The value of a European butterfly spread call option can be determined by solving Equation (2.5) subject to the initial condition

$$V(S,0) = \max(S - E_1) - 2\max(S - E_2) + \max(S - E_3),$$

and the boundary conditions

$$V(S,t) = 0 \quad \text{as } S \to 0, \qquad V(S,t) = 0 \quad \text{as } S \to \infty.$$

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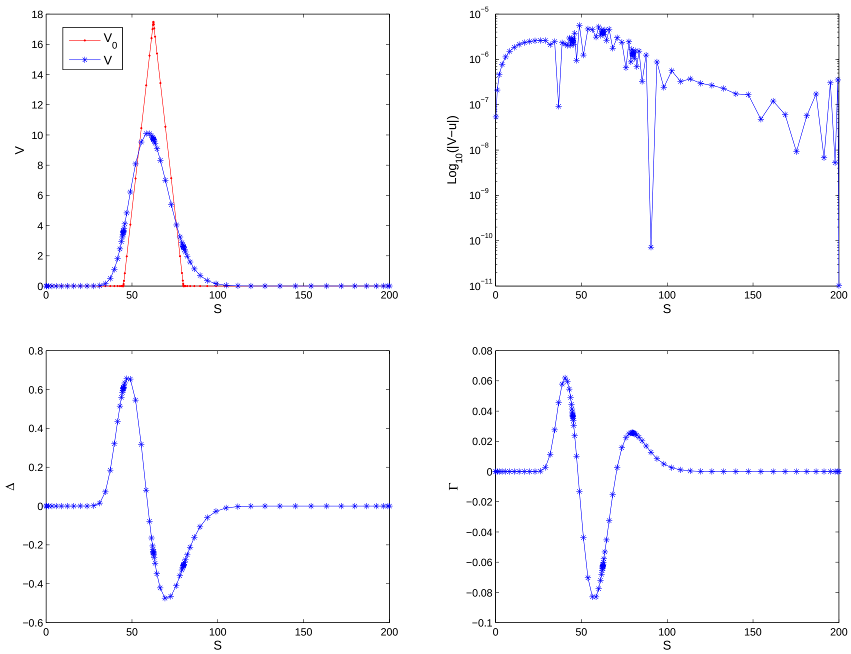

FIG. 6.5. *Valuation of butterfly spread options by the barycentric Lagrange collocation (BLC) method with* $N = 80$, $k = 0.001$, $S_m = 0$, $S_M = 200$, $r = 0.1$, $\sigma = 0.2$, $\delta = 0.0$, $E_1 = 45$, $E_3 = 80$, $T = 0.5$.

In this particular case, we need to stretch the grid points at three different strike prices in order to improve the accuracy of the BLC method. The suitable map is chosen from (3.13) with $a_1 = a_2 = a_3 = 1/3$, and the grid stretching parameters are $\beta_1 = \beta_2 = \beta_3 = 0.5 \times 10^{-3}$.

Figure 6.5 displays the numerical values of the butterfly spread option together with its Δ, Γ, and its error with $N = 80$, $S_{\min} = 0$, $S_{\max} = 200$, $r = 0.1$, $\sigma = 0.2$, $\delta = 0.0$, $E_1 = 45$, $E_3 = 80$ at $T = 0.5$. To ensure that the error is dominated by the spatial discretisation, we choose the time step $k = 0.001$. All the results are satisfactory and free of oscillations. To further investigate the accuracy of the mapped BLC method for pricing butterfly spread options, we compare the results with those obtained by using the FD method. The results are presented in Table 6.5. We observe that the results obtained with the mapped BLC method are more accurate than those of the FD method. Very accurate results are obtained for different values of option parameters for different expiry times.

7. Extension of the proposed approach to solve the Heston model. The stochastic volatility model of Heston [31] is one of the most popular equity option pricing models. This model is an extension of the Black-Scholes PDE to two-dimensional form. Before we explain the extension of the proposed approach, we describe this model.

Let $V(S, \nu, t)$ denotes the value of the option if at time $T - t$ the underlying asset price equals S and its variance equals ν. Heston's stochastic volatility model [31] implies that V satisfies the two-dimensional parabolic PDE

$$V_t = \frac{1}{2}S^2\nu V_{SS} + \frac{1}{2}\sigma^2\nu V_{\nu\nu} + \rho\sigma S\nu V_{S\nu} + rSV_S + \kappa(\eta - \nu)V_\nu - rV,$$

for $0 \leq t \leq T$, $S > 0$, $\nu > 0$. The parameter $\kappa > 0$ is the volatility mean-reversion rate, $\eta > 0$ is the long-term mean, σ is the volatility of the variance, $\rho \in [-1, 1]$ is the correlation between the underlying asset and the variance, and r is the interest rate.

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The initial condition for a call option is

$$V(S,\nu,0) = \max(S-E,0), \quad 0 \le S \le S_M,\ 0 \le \nu \le \nu_M, \tag{7.1}$$

where E is the strike price of the option. Boundary conditions are given by

$$\begin{aligned}
V(0,\nu,t) &= 0, && 0 \le t \le T,\ 0 \le \nu \le \nu_M,\\
V(S_M,\nu,t) &= S_M - Ee^{-rt}, && 0 \le t \le T,\ 0 \le \nu \le \nu_M,\\
V_\nu(S,0,t) &= 0, && 0 \le t \le T,\ 0 \le S \le S_M,\\
V_\nu(S,\nu_M,t) &= 0, && 0 \le t \le T,\ 0 \le S \le S_M.
\end{aligned}$$

Let $V(S,\nu,T) = Y(S,t) + C(S,\nu,t)$, where Y satisfies the Black-Scholes equation (2.5) for a call option (6.1)–(6.2). Then the Heston model can be written in terms of C as

$$C_t = \frac{1}{2}S^2\nu C_{SS} + \frac{1}{2}\sigma^2\nu C_{\nu\nu} + \rho\sigma S\nu C_{S\nu} + rSC_S + \kappa(\eta-\nu)C_\nu - rC + F,$$

where

$$F(S,\nu,t) = \rho\sigma S\nu Y_{S\nu} + \frac{1}{2}\sigma^2\nu Y_{\nu\nu} + \kappa(\eta-\nu)Y_\nu.$$

The change of variables

$$S = Ee^{L_1 x},\ \ \nu = \eta e^{L_2\omega},\ \ \text{and}\ \ c(x,\omega,t) = C(S,\nu,t)\ \ \text{on}\ \ (x,\omega) = [-1,1]\times[-1,1]$$

yields the Heston PDE of the form

$$\begin{aligned}
c_t = {}& \frac{1}{2}\nu L_1^{-2}c_{xx} + \frac{1}{2}\sigma^2\nu^{-1}L_2^{-2}c_{\omega\omega} + \rho\sigma L_1^{-1}L_2^{-1}c_{x\omega} + \left(r - \frac{1}{2}\nu\right)L_1^{-1}c_x\\
&+ \left[\left(\frac{1}{2}\sigma^2 - \kappa\eta\right) + \kappa\right]L_2^{-1}c_\omega - rc + E^{-1}F(Ee^{L_1x},\eta e^{L_2\omega},t).
\end{aligned} \tag{7.2}$$

The initial condition is

$$c(x,\omega,0) = 0. \tag{7.3}$$

Boundary conditions are given by

$$\begin{aligned}
c(-1,\omega,t) &= 0, && 0 \le t \le T,\ \ -1 \le \omega \le 1,\\
c(1,\omega,t) &= 0, && 0 \le t \le T,\ \ -1 \le \omega \le 1,\\
c_\nu(x,-1,t) &= 0, && 0 \le t \le T,\ \ -1 \le x \le 1,\\
c_\nu(x,1,t) &= 0, && 0 \le t \le T,\ \ -1 \le x \le 1.
\end{aligned} \tag{7.4}$$

In order to discretise the two-dimensional problem (7.2)–(7.4), we introduce the two-dimensional version of the approximation (3.4), viz.,

$$u(x,\omega) = \frac{\sum_{j=0}^{N_x}\sum_{k=0}^{N_\omega}\frac{w_j w_k}{(x-x_j)(\omega-\omega_k)}u(x_k,\omega_k)}{\sum_{j=0}^{N_x}\sum_{k=0}^{N_\omega}\frac{w_j w_k}{(x-x_j)(\omega-\omega_k)}},$$

where w_j, for $j = 0,\dots,N_x$, and w_k, for $k = 0,\dots,N_\omega$, are the barycentric weights defined by $w_0 = 1/2$, $w_N = (-1)^{N_x}/2$, and $w_j = (-1)^j, j = 1,\dots,N_x-1$.

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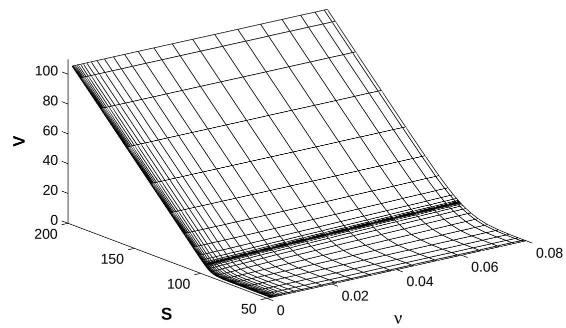

FIG. 7.1. *Values of European call options in the Heston model using the BLC method with* $N = 30$, $E = 100$, $T = 0.5$, $\kappa = 2$, $\eta = 0.01$, $\rho = 0.5$, $r = 0.1$, $L_1 = \ln(2), \ln(8)$, $\beta_S = 0.5 \times 10^{-3}$, *and* $\beta_\nu = 10^{-2}$ *(grid stretching parameters in* S-, ν- *directions, respectively) at* $T = 0.5$.

In this article, our extension of the BLC to two dimensions depends on the utilisation of the Kronecker product for matrices denoted by "$\otimes$". We explain the notation as per below.

Let **A** be an $m \times n$ matrix and **B** a $p \times q$ matrix. The Kronecker or tensor product of **A** and **B** is the matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

The interested reader can find a review of the properties of the Kronecker product in [54].

We utilise the Kronecker product notation because it provides for a clear separation of operators in multiple dimensions. For instance, we consider the discretisation of the first- and second-order derivative operators in two dimensions as follows

$$\begin{aligned} c_x(x,\omega) &\to \left(D_x^{(1)} \otimes I_\omega\right) c, \\ c_{xx}(x,\omega) &\to \left(D_x^{(2)} \otimes I_\omega\right) c, \\ c_\omega(x,\omega) &\to \left(I_x \otimes D_\omega^{(1)}\right) c, \\ c_{\omega\omega}(x,\omega) &\to \left(I_x \otimes D_\omega^{(2)}\right) c, \\ c_{x\omega}(x,\omega) &\to \left(D_x^{(1)} \otimes D_\omega^{(1)}\right) c, \end{aligned} \tag{7.5}$$

where I_x and I_ω are the identity matrices in x and ω directions, respectively, and $D_x^{(1,2)}$ and $D_\omega^{(1,2)}$ are the first- and second-order differentiation matrices in x and ω directions, respectively. Denoting $\mathbf{X} = \mathbf{x} \otimes \mathbf{1}_{N_\omega}^T$, $\mathbf{\Omega} = \mathbf{1}_{N_x} \otimes \omega^{\mathbf{T}}$, $\mathbf{C} = c(\mathbf{X}, \mathbf{\Omega}, t)$, $\mathbf{V} = \mathrm{diag}(\eta e^{L_2 \mathbf{\Omega}})$

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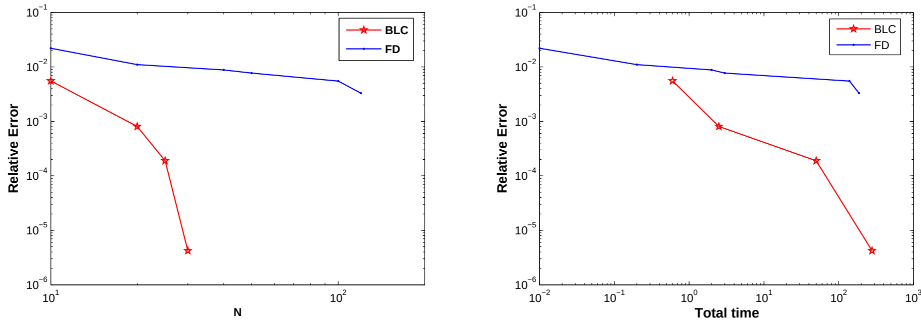

FIG. 7.2. *Performance of the BLC against the FD method for pricing European call options in the Heston model with* $E = 100$, $T = 0.5$, $\kappa = 2$, $\eta = 0.01$, $\rho = 0.5$, $r = 0.1$, $L_1 = \ln(2)$, *and* $\ln(8)$.

and substituting (7.5) into (7.2) yields

$$
\begin{aligned}
\dot{\mathbf{C}} &= \frac{1}{2}\mathbf{V}L_1^{-2}\left(D_x^{(2)} \otimes I_\omega\right)\mathbf{C} + \frac{1}{2}\sigma^2\mathbf{V}^{-1}L_2^{-2}\left(I_x \otimes D_\omega^{(2)}\right)\mathbf{C} \\
&\quad + \rho\sigma L_1^{-1}L_2^{-1}\left(D_x^{(1)} \otimes D_\omega^{(1)}\right)\mathbf{C} + \left(r - \frac{1}{2}\mathbf{V}\right)L_1^{-1}\left(D_x^{(1)} \otimes I_\omega\right)\mathbf{C} \\
&\quad + \left[\left(\frac{1}{2}\sigma^2 - \kappa\eta\right) + \kappa\right]L_2^{-1}\left(I_x \otimes D_\omega^{(1)}\right)\mathbf{C} \\
&\quad - r\left(I_x \otimes I_\omega\right)\mathbf{C} + E^{-1}F(Ee^{L_1\mathbf{X}}, \eta e^{L_2\mathbf{\Omega}}, t).
\end{aligned} \tag{7.6}
$$

Equation (7.6) can be written in the form of a global matrix as

$$\dot{\mathbf{C}} = A\mathbf{C} + g(\mathbf{C}, t), \tag{7.7}$$

where

$$A = \frac{1}{2}\mathbf{V}L_1^{-2}\left(D_x^{(2)} \otimes I_\omega\right)\mathbf{C} + \frac{1}{2}\sigma^2\mathbf{V}^{-1}L_2^{-2}\left(I_x \otimes D_\omega^{(2)}\right)\mathbf{C},$$

is the stiff part of the PDE (7.2) and

$$
\begin{aligned}
g(\mathbf{C}, t) &= \rho\sigma L_1^{-1}L_2^{-1}\left(D_x^{(1)} \otimes D_\omega^{(1)}\right)\mathbf{C} + \left(r - \frac{1}{2}\mathbf{V}\right)L_1^{-1}\left(D_x^{(1)} \otimes I_\omega\right)\mathbf{C} \\
&\quad + \left[\left(\frac{1}{2}\sigma^2 - \kappa\eta\right) + \kappa\right]L_2^{-1}\left(I_x \otimes D_\omega^{(1)}\right)\mathbf{C} - r\left(I_x \otimes I_\omega\right)\mathbf{C} \\
&\quad + E^{-1}F(Ee^{L_1\mathbf{X}}, \eta e^{L_2\mathbf{\Omega}}, t),
\end{aligned}
$$

is the non-stiff part. We next apply the IMEX-PC(3,3) defined in Section 4 to solve the system of ODEs (7.7).

We compare the performance of the BLC method against that of the FD method to compute the European call option prices under the Heston model. The parameter values used in the simulation are $E = 100$, $T = 0.5$, $\kappa = 2$, $\eta = 0.01$, $\rho = 0.5$, $r = 0.1$, $L_1 = \ln(2)$, and $\ln(8)$. Figure 7.1 represents the value of the option plotted at the final time $T = 0.5$ using the BLC method coupled with IMEX-PC(3,3) with time step $k = 0.001$. Here a non-uniform grid is applied in both directions S and ν such that many points lie in the neighborhood of $S = K$ and $\nu = 0$, respectively. This is motivated by the fact that the initial condition (7.1) possesses a discontinuity in its first derivative at $S = E$ and that for $\nu \approx 0$, the

Heston PDE is advection-dominated. The results obtained here are in good agreement with the analytical solution proposed in [31]. In Figure 7.2 (left), we plot the relative error against the number of spatial grids in the asset direction. In Figure 7.2 (right), the relative error is plotted against the computational time. For this problem, the BLC method is faster than the FD method and achieves a spectral convergence as expected.

8. Concluding remarks and scope for future research. In this paper, we have considered a spectral approach based on a barycentric Lagrange discretisation in space and combined it with a third-order IMEX-PC time marching method for pricing European vanilla, digital, and butterfly spread options. The method was first designed for one-dimensional problems and then extended to two-dimensional problems. The proposed method is also analysed for stability. Extensive comparisons are carried out and presented in form of tables and figures. It can be seen from these comparative results that we achieve high-order accuracy using coordinate transformations that stretch the points around the strike price. These results show that our method is very accurate and reliable in pricing the class of options indicated in this paper. Currently, we are exploring the utility of this approach to solve other classes of option pricing problems.

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