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DISCRETE POINCARÉ INEQUALITIES FOR ARBITRARY MESHES IN THE DISCRETE DUALITY FINITE VOLUME CONTEXT*

ANH HA LE†‡ AND PASCAL OMNES‡

Abstract. We establish discrete Poincaré type inequalities on a two-dimensional polygonal domain covered by arbitrary, possibly nonconforming meshes. On such meshes, discrete scalar fields are defined by their values both at the cell centers and vertices, while discrete gradients are associated with the edges of the mesh, like in the discrete duality finite volume scheme. We prove that the constants that appear in these inequalities depend only on the domain and on the angles between the diagonals of the diamond cells constructed by joining the two vertices of each mesh edge and the centers of the cells that share that edge.

Key words. Poincaré inequalities, finite volumes, discrete duality, arbitrary meshes.

AMS subject classifications. 65N08, 46E35.

1. Introduction. Let Ω be a two-dimensional polygonal domain. Let us introduce the following two Poincaré inequalities which will be mentioned throughout this article: the Friedrichs (also called Poincaré) inequality

$$\int_\Omega u^2(x)dx \le c_F \int_\Omega |\nabla u(x)|^2 dx, \quad \forall u \in H_0^1(\Omega), \tag{1.1}$$

and the Poincaré (also called mean Poincaré) inequality

$$\int_\Omega u^2(x)dx \le c_P \int_\Omega |\nabla u(x)|^2 dx, \quad \forall u \in H^1(\Omega) \text{ such that } \int_\Omega u(x)dx = 0, \tag{1.2}$$

where c_F and c_P are constants depending only on Ω. These two inequalities play an important role in the theory of partial differential equations. Here, $H^1(\Omega)$ is the Sobolev space of $L^2(\Omega)$ functions with generalized derivatives in $(L^2(\Omega))^2$, and $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with zero boundary values in the sense of traces on $\partial\Omega$. More details on the Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$ may be found, e.g., in [1].

This article considers discrete versions of Poincaré inequalities for the so-called discrete duality finite volume (DDFV) method with discretization on arbitrary meshes, as presented, e.g., in [11]. Originally developed for the discretization of (possibly heterogeneous, anisotropic, nonlinear) diffusion equations on arbitrary meshes [3, 6, 11, 15, 16, 20], this technique has found applications in other fields, like electromagnetics [17], div-curl problems [9] and Stokes flows [8, 18, 19], drift diffusion and energy transport models [4].

The originality of these schemes is that they work well on all kinds of meshes, including very distorted, degenerating, or highly nonconforming meshes; see the numerical tests in [11]. The name DDFV comes from the fact that these schemes are based on the definition of discrete gradient and divergence operators which verify a discrete Green formula.

Details about this method are recalled in Section 2. In this introduction, let us only mention that in the DDFV discretization scalar functions are discretized by their values both at the centers and at the vertices of a given mesh, and their gradients are evaluated on the

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†Institute for Computational Science and Technology Ho Chi Minh City, Quang Trung Software City, Tan Chanh Hiep Ward, District 12, Ho Chi Minh City, Vietnam (anh-ha.le@cea.fr).

‡CEA, DEN, DM2S-STMF, F-91191 Gif-sur-Yvette Cedex, France, and Université Paris 13, LAGA, CNRS UMR 7539, Institut Galilée, 99 Avenue J.-B. Clément, F-93430 Villetaneuse, France (pascal.omnes@cea.fr).

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so-called "diamond-cells" associated with the edges of the mesh. Each internal diamond-cell is a quadrilateral; its vertices are the two nodes of a given internal edge and the centers of the two cells which share this edge. Each boundary diamond cell is a degenerated quadrangle (i.e., a triangle); its vertices are the two nodes of a given boundary edge and the center of the corresponding cell and that of the boundary edge.

Then, the discrete version of the L^2 norm on the left-hand side of (1.1) and (1.2) is the half-sum of the L^2 norms of two piecewise constant functions, one defined by the discrete values given at the centers of the original ("primal" in what follows) cells, and the other defined by the discrete values given at the vertices of the primal mesh, to which we associate cells of a dual mesh. Moreover, the discrete version of the gradient L^2 norm on the right-hand side of (1.1) and (1.2) is the L^2 norm of the piecewise constant gradient vector field defined by its discrete values on the diamond-cells.

In the finite volume context, discrete Poincaré-Friedrichs inequalities have previously been proved in [12, Lemma 9.1, Lemma 10.2] and [14], respectively for so-called "admissible" meshes (roughly speaking, meshes such that each edge is orthogonal to the segment joining the centers of the two cells sharing that edge; see the precise definition in [12, Definition 9.1]) and for Voronoi meshes. Similar results on duals of general simplicial triangulations are proved in [21]. In the DDFV context, a discrete version of (1.1) is given for arbitrary meshes in [3]. However, the discrete constant c_F which appears in that paper depends on the mesh regularity in a rather intricate way; see [3, Formula (2.6) and Lemma 3.3].

The main result of our contribution is the proof of discrete versions of both (1.1) and (1.2) in the DDFV context, with constants c_F and c_P depending only on the domain and on the minimum angle in the diagonals of the diamond cells of the mesh.

Our proof of the discrete version of (1.1) is very similar to those given in [12] or [21]. We also prove a discrete version of (1.1) in a slightly more general situation when the domain is not simply connected and the discrete values of the function vanish only at the exterior boundary of the domain and are constant on each of the internal boundaries; this will have a subsequent application in the last section of the present work.

However, the task is more difficult for the mean-Poincaré inequality. Like in [12], it is divided into three steps. The first is the proof of this inequality on a convex subdomain; in the second, our proof differs from that in [12] because we actually do not need to prove a bound on the L^2 norm of the difference of discrete functions and their discrete mean value on the boundary of a convex subset, but rather an easier bound on the L^1 norm of this difference. The final step consists of dividing a general polygonal domain into several convex polygonal subdomains and in combining the first two steps to obtain the result.

As a consequence, we derive a discrete equivalent of the following statement (which is a particular case of a result given in [13]). Let us consider open, bounded, simply connected, convex polygonal domains $(\Omega_q)_{q\in[0,Q]}$ of $\mathbb{R}^2$, such that $\Omega_q \subset \Omega_0$ for all $q \in [1,Q]$, and $\bar\Omega_{q_1}\cap\bar\Omega_{q_2} = \emptyset$ for all $(q_1,q_2) \in [1,Q]^2$ with $q_1 \neq q_2$. Let Ω be defined by $\Omega = \Omega_0\backslash(\cup_{q=1}^{Q}\Omega_q)$. Let us denote by $\Gamma = \partial\Omega = \cup_{q=0}^{Q}\Gamma_q$, with $\Gamma_q = \partial\Omega_q$ for all $q \in [0,Q]$. Then, there exists a constant C, depending only on Ω, such that for all vector field $\mathbf{v}$ in $H(\mathrm{div},\Omega) \cap H(\mathrm{rot},\Omega)$, with $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ and $(\mathbf{v}\cdot\boldsymbol{\tau},1)_{\Gamma_q} = 0$ for all $q \in [1,Q]$, there holds

$$||\mathbf{v}||_{L^2(\Omega)} \leq C(||\nabla\cdot\mathbf{v}||_{L^2(\Omega)} + ||\nabla\times\mathbf{v}||_{L^2(\Omega)}).$$

The discrete equivalent has applications in the derivation of *a priori* error estimates for the DDFV method applied to the Stokes equations ([10]).

Let us mention that, although 3D extensions of the DDFV scheme have been published [2, 5, 6], the extension of our results to 3D is beyond the scope of this article.

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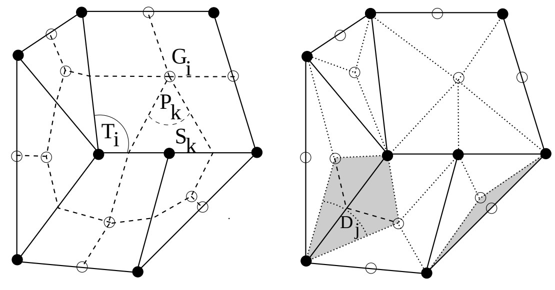

FIGURE 2.1. *A nonconforming primal mesh and its associated dual mesh (left) and diamond-mesh (right).*

The paper is organized as follows. Section 2 introduces some notations and definitions related to the meshes, to discrete differential operators and to discrete functions. In Section 3, discrete Poincaré inequalities are presented. First, we prove a discrete Poincaré inequality for discrete functions vanishing at the boundary of the polygonal domain, then we extend this result to the slightly more general case mentioned above, and we prove the discrete mean Poincaré inequality with the 3 steps described above. Finally, we present in Section 4 an application of the previous results to the derivation of another discrete inequality, relating the norm of discrete vector fields defined on the diamond cells and verifying special boundary conditions, to that of their divergence and curls defined on the primal and dual meshes. In Appendix A, we present the details of the proof of a lemma required for our main results.

2. Notations and Definitions. The following notations are summarized in Figure 2.1 and Figure 2.2. Let Ω be defined as above and be covered by a primal mesh with polygonal cells denoted by T_i, $i \in [1, I]$. With each T_i, we associate a point G_i located in the interior of T_i. Let us denote by S_k, with $k \in [1, K]$, the nodes of the cells. With any S_k, we associate a dual cell P_k by joining the points G_i associated with the primal cells surrounding S_k to the midpoints of the edges of which S_k is a node.

With any primal edge A_j with $j \in [1, J]$, we associate a so-called diamond-cell D_j obtained by joining the vertices $S_{k_1(j)}$ and $S_{k_2(j)}$ of A_j to the points $G_{i_1(j)}$ and $G_{i_2(j)}$ associated with the primal cells that share A_j as a part of their boundaries. When A_j is a boundary edge (there are J^Γ such edges), the associated diamond-cell is a flat quadrilateral (i.e., a triangle) and we denote by $G_{i_2(j)}$ the midpoint of A_j (thus, there are J^Γ such additional points G_i). The unit normal vector to A_j is $\mathbf{n}_j$ and points from $G_{i_1(j)}$ to $G_{i_2(j)}$. We denote by A'_{j1} (resp. A'_{j2}) the segment joining $G_{i_1(j)}$ (resp. $G_{i_2(j)}$) and the midpoint of A_j. Its associated unit normal vector, pointing from $S_{k_1(j)}$ to $S_{k_2(j)}$, is denoted by $\mathbf{n}'_{j1}$ (resp. $\mathbf{n}'_{j2}$). We also define vectors $\boldsymbol{\tau}_j$, $\boldsymbol{\tau}'_{j1}$ and $\boldsymbol{\tau}'_{j2}$ such that $(\mathbf{n}_j, \boldsymbol{\tau}_j)$, $(\mathbf{n}'_{j1}, \boldsymbol{\tau}'_{j1})$ and $(\mathbf{n}'_{j2}, \boldsymbol{\tau}'_{j2})$ are orthonormal, positively oriented basis of $\mathbb{R}^2$. In the case of a boundary diamond-cell, A'_{j2} reduces to $\{G_{i_2(j)}\}$ and does not play any role. Finally, for any diamond-cell D_j, we shall denote by $M_{i_\alpha k_\beta}$ the midpoint of $G_{i_\alpha(j)}S_{k_\beta(j)}$, with $(\alpha, \beta) \in \{1, 2\}^2$, M_j the midpoint of $S_{k_1(j)}S_{k_2(j)}$ and θ_{j_1} (resp θ_{j_2}) is defined to be the angle, smaller than $\pi/2$, between segment $S_{k_1(j)}S_{k_2(j)}$ and segment $G_{i_1(j)}M_j$ (resp $G_{i_2(j)}M_j$). We shall use the following definition.

DEFINITION 2.1. *We denote by $\theta^* > 0$ the largest angle in the mesh such that*

$$\theta_{j_1} \geq \theta^* \text{ and } \theta_{j_2} \geq \theta^*, \qquad \text{for all } j \in [1, J].$$

Now we shall associate discrete scalar values to the points G_i and S_k and discrete two-dimensional vector fields to the diamond-cells. This leads us to the following definitions.

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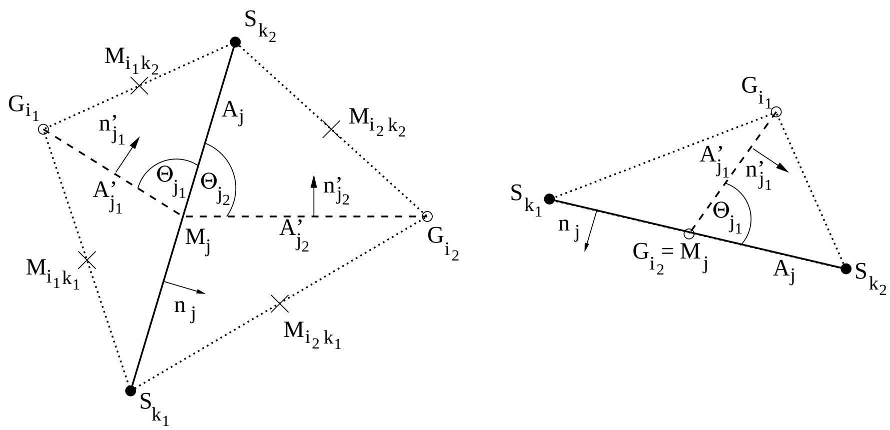

FIGURE 2.2. *Notations for the inner diamond-cell (left) and a boundary diamond mesh (right).*

DEFINITION 2.2. *Let* $\phi = (\phi_i^T, \phi_k^P)$ *and* $\psi = (\psi_i^T, \psi_k^P)$ *be in* $\mathbb{R}^I \times \mathbb{R}^K$. *Let* $\mathbf{v} = (\mathbf{v}_j)$ *and* $\mathbf{w} = (\mathbf{w}_j)$ *be in* $\left(\mathbb{R}^J\right)^2$. *We define the following scalar products and associated norms*

$$\begin{aligned}
(\phi, \psi)_{T,P} &:= \frac{1}{2}\left(\sum_{i\in[1,I]} |T_i|\phi_i^T\psi_i^T + \sum_{k\in[1,K]} |P_k|\phi_k^P\psi_k^P\right),\\
\|\phi\|_{T,P}^2 &:= (\phi,\phi)_{T,P},\\
(\mathbf{w},\mathbf{v})_D &:= \sum_{j\in[1,J]} |D_j|\mathbf{w}_j\cdot\mathbf{v}_j, \quad \|\mathbf{v}\|_D^2 := (\mathbf{v},\mathbf{v})_D.
\end{aligned}$$

DEFINITION 2.3. *Let* $\phi = (\phi_i^T, \phi_k^P)$ *be in* $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$. *We define the trace* $\tilde\phi$ *of* ϕ *on the boundary edges* $A_j \subset \Gamma$ *by* $\tilde\phi_j := \frac{1}{4}\left(\phi^P_{k_1(j)} + 2\phi^T_{i_2(j)} + \phi^P_{k_2(j)}\right)$. *We also define a discrete scalar product for the traces of* $\mathbf{v}\cdot\mathbf{n}$ *and* $\tilde\phi$ *on the boundaries* Γ_q

$$(\mathbf{v}\cdot\mathbf{n}, \tilde\phi)_{\Gamma_q,h} := \sum_{j\in\Gamma_q} |A_j|\,(\mathbf{v}_j\cdot\mathbf{n}_j)\,\tilde\phi_j$$

and on Γ

$$(\mathbf{v}\cdot\mathbf{n}, \tilde\phi)_{\Gamma,h} := \sum_{q\in[0,Q]} (\mathbf{v}\cdot\mathbf{n}, \tilde\phi)_{\Gamma_q,h}. \tag{2.1}$$

In the proof of discrete Poincaré inequalities, we often use the piecewise constant functions based on the discrete functions defined at the centers of each mesh; we make the following definitions

DEFINITION 2.4. *Let* $\phi \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$. *The piecewise constant functions* $\phi^T(x)$ *and* $\phi^P(x)$ *are defined as follows*

$$\begin{aligned}
\phi^T(x) &= \phi_i^T, \qquad \forall x \in T_i \text{ and } i \in [1, I];\\
\phi^P(x) &= \phi_k^P, \qquad \forall x \in P_k \text{ and } k \in [1, K].
\end{aligned}$$

We recall here the discrete gradient [7, 11] and (vector) curl operators [9] which have been constructed on the diamond cells.

DEFINITION 2.5. *Let $\phi = (\phi_i^T, \phi_k^P)$ be in $\mathbb{R}^{I+J^\Gamma}$. Its discrete gradient $\boldsymbol{\nabla}_h^D \phi$ and discrete curl $\boldsymbol{\nabla}_h^D \times \phi$ are defined by their values in the cells D_j through*

$$(\boldsymbol{\nabla}_h^D \phi)_j := \frac{1}{2|D_j|} \left\{ [\phi_{k_2}^P - \phi_{k_1}^P](|A'_{j1}|\mathbf{n'}_{j1} + |A'_{j2}|\mathbf{n'}_{j2}) + [\phi_{i_2}^T - \phi_{i_1}^T]|A_j|\mathbf{n}_j \right\},$$

$$(\boldsymbol{\nabla}_h^D \times \phi)_j := -\frac{1}{2|D_j|} \left\{ [\phi_{k_2}^P - \phi_{k_1}^P](|A'_{j1}|\boldsymbol{\tau'}_{j1} + |A'_{j2}|\boldsymbol{\tau'}_{j2}) + [\phi_{i_2}^T - \phi_{i_1}^T]|A_j|\boldsymbol{\tau}_j \right\}.$$

In the proof of our results, we shall use the following theorem [9, Theorem 4.7].

THEOREM 2.6 (Discrete Hodge decomposition). *Let $(\mathbf{v}_j)_{j\in[1,J]}$ be a discrete vector field defined by its values on the diamond-cells D_j. There exist unique*

$$\phi = (\phi_i^T, \phi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]},$$
$$\psi = (\psi_i^T, \psi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]},$$

and $(c_q^T, c_q^P)_{q\in[1,Q]}$ such that

$$\mathbf{v}_j = (\boldsymbol{\nabla}_h^D \phi)_j + (\boldsymbol{\nabla}_h^D \times \psi)_j, \quad \forall j \in [1, J], \tag{2.2}$$

$$\sum_{i\in[1,I]} |T_i|\phi_i^T = \sum_{k\in[1,K]} |P_k|\phi_k^P = 0,$$

$$\psi_i^T = 0, \quad \forall i \in \Gamma_0, \quad \psi_k^P = 0, \quad \forall k \in \Gamma_0, \tag{2.3}$$

and

$$\forall q \in [1, Q], \quad \psi_i^T = c_q^T, \quad \forall i \in \Gamma_q, \quad \psi_k^P = c_q^P, \quad \forall k \in \Gamma_q. \tag{2.4}$$

Moreover, the decomposition (2.2) is orthogonal. We shall also need the following construction of discrete divergence and (scalar) curl operators on both primal and dual cells.

DEFINITION 2.7. *Let $\mathbf{v} = (\mathbf{v}_j)$ be defined in $(\mathbb{R}^2)^J$ by its values on the diamond-cells. We define*

$$\left(\nabla_h^T \cdot \mathbf{v}\right)_i := \frac{1}{|T_i|} \sum_{j\in\partial T_i} |A_j|\mathbf{v}_j \cdot \mathbf{n}_{ji},$$

$$\left(\nabla_h^P \cdot \mathbf{v}\right)_k := \frac{1}{|P_k|} \left(\sum_{j\in\partial P_k} \left(|A'_{j1}|\mathbf{v}_j \cdot \mathbf{n}'_{j1k} + |A'_{j2}|\mathbf{v}_j \cdot \mathbf{n}'_{j2k}\right) + \sum_{j\in\partial P_k\cap\Gamma} \frac{|A_j|}{2}\mathbf{v}_j \cdot \mathbf{n}_j \right),$$

$$\left(\nabla_h^T \times \mathbf{v}\right)_i := \frac{1}{|T_i|} \sum_{j\in\partial T_i} |A_j|\mathbf{v}_j \cdot \boldsymbol{\tau}_{ji},$$

$$\left(\nabla_h^P \times \mathbf{v}\right)_k := \frac{1}{|P_k|} \left(\sum_{j\in\partial P_k} \left(|A'_{j1}|\mathbf{v}_j \cdot \boldsymbol{\tau}'_{j1k} + |A'_{j2}|\mathbf{v}_j \cdot \boldsymbol{\tau}'_{j2k}\right) + \sum_{j\in\partial P_k\cap\Gamma} \frac{|A_j|}{2}\mathbf{v}_j \cdot \boldsymbol{\tau}_j \right).$$

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The following result [9, Proposition 4.1], which consists of discrete Green formulas, has motivated the name "discrete duality".

THEOREM 2.8 (Discrete Green formulas). *For* $\phi = (\phi^T, \phi^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ *and* $\mathbf{v} \in (\mathbb{R}^2)^J$*, it holds that*

$$(\mathbf{v}, \boldsymbol{\nabla}_h^D \phi)_D = -(\nabla_h^{T,P} \cdot \mathbf{v}, \phi)_{T,P} + (\mathbf{v} \cdot \mathbf{n}, \tilde{\phi})_{\Gamma,h}, \tag{2.5}$$

$$(\mathbf{v}, \boldsymbol{\nabla}_h^D \times \phi)_D = (\nabla_h^{T,P} \times \mathbf{v}, \phi)_{T,P} - (\mathbf{v} \cdot \boldsymbol{\tau}, \tilde{\phi})_{\Gamma,h}. \tag{2.6}$$

3. Discrete Poincaré Inequalities. We first start with a discrete version of (1.1). Our result is a special case of that proved in [3, Lemma 3.3], but our expression of the discrete constant c_F is more accurate and simple, in that its dependence on the geometry of the cells occurs only through the angles between the diagonals of the diamond-cells. This is an important result in the DDFV context, since also *a priori* error estimation of the discrete solution of the Laplace equation obtained with this method only depends on the cell geometries through angles in the diamond-cells; see [11].

THEOREM 3.1 (Discrete Poincaré-Friedrichs inequality). *Let* Ω *be an open bounded polygonal domain; let us consider* $u = (u_i^T, u_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ *such that*

$$u_k^P = 0,\ \forall k \in \Gamma \ \text{ and } \ u_i^T = 0,\ \forall i \in \Gamma.$$

Let θ^* *be defined by Definition 2.1. Then, there exists a constant C only depending on* Ω *and* θ^* *such that*

$$\|u\|_{T,P} \leq C \|\boldsymbol{\nabla}_h^D u\|_D. \tag{3.1}$$

Proof. Let $u^T(\cdot)$ and $u^P(\cdot)$ be the piecewise constant functions defined in Definition 2.4. Then obviously $\|u\|_{T,P}^2 = (\|u^T\|^2_{L^2(\Omega)} + \|u^P\|^2_{L^2(\Omega)})/2$, so that, in order to prove (3.1), it suffices to prove

$$\|u^T\|_{L^2(\Omega)} \leq C \|\boldsymbol{\nabla}_h^D u\|_D, \tag{3.2}$$

$$\|u^P\|_{L^2(\Omega)} \leq C \|\boldsymbol{\nabla}_h^D u\|_D. \tag{3.3}$$

We shall first prove (3.2). Let $\mathbf{d}_1 = (0,1)^t$ and $\mathbf{d}_2 = (1,0)^t$; for $x \in \Omega$, let $\mathcal{D}_x^1$ and $\mathcal{D}_x^2$ be the straight lines going through x and parallel to the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$. For any edge $j \in [1, J]$ and any $x \in \Omega$, let us define $\chi_j^{T,1}(x)$ and $\chi_j^{T,2}(x)$ by

$$\chi_j^{T,\ell}(x) = \begin{cases} 1, & \text{if } A_j \cap \mathcal{D}_x^\ell \neq \emptyset, \\ 0, & \text{if } A_j \cap \mathcal{D}_x^\ell = \emptyset, \end{cases} \tag{3.4}$$

for $\ell = 1, 2$. For any $x = (x_1, x_2) \in \Omega$, we note that $\chi_j^1(x)$ only depends on x_1 and $\chi_j^2(x)$ only depends on x_2.

From the first formula of Definition 2.5 and simple geometry, it is easy to see that

$$(\boldsymbol{\nabla}_h^D u)_j \cdot \overrightarrow{G_{i_1(j)} G_{i_2(j)}} = u_{i_2(j)}^T - u_{i_1(j)}^T, \quad \forall j \in [1, J]. \tag{3.5}$$

Then, for any $i \in [1, I]$ and a.e. $x \in T_i$, let us follow the straight line $\mathcal{D}_x^\ell$ until it intersects the boundary Γ, and let us denote by $v_1(i) := i, v_2(i), \ldots, v_{n-1}(i)$, the indices of the primal cells that it intersects (in the order they are intersected), and by $v_n(i)$ the index in $[I+1, I+J^\Gamma]$ corresponding to the first boundary segment intersected by $\mathcal{D}_x^\ell$; see Figure 3.1. Then, since

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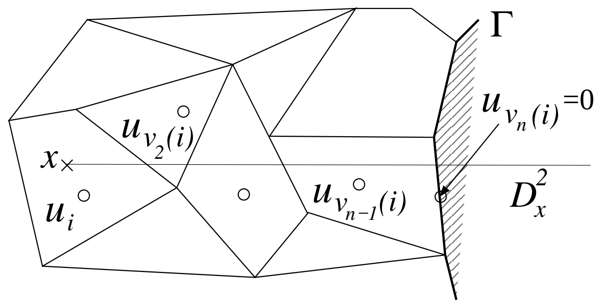

FIGURE 3.1. *Straight line $\mathcal{D}_x^2$ intersecting primal cells from point x to the boundary.*

$u^T_{v_n(i)} = 0$ because of the boundary conditions, we may write

$$\begin{aligned} u_i^T = u^T_{v_1(i)} &= (u^T_{v_1(i)} - u^T_{v_2(i)}) + (u^T_{v_2(i)} - u^T_{v_3(i)}) + \cdots + (u^T_{v_{n-1}(i)} - u^T_{v_n(i)}) \\ &= \sum_{m=1}^{n-1} (u^T_{v_m(i)} - u^T_{v_{m+1}(i)}), \end{aligned}$$

so that, since any couple $(u^T_{v_m(i)}, u^T_{v_{m+1}(i)})$ is a pair of neighboring values through an edge A_j intersected by $\mathcal{D}_x^\ell$, there holds, thanks to (3.5)

$$|u^T(x)| = |u_i^T| \leq \sum_{j=1}^{J} \left| (\boldsymbol{\nabla}_j^D u)_j \cdot \overrightarrow{G_{i_1(j)} G_{i_2(j)}} \right| \chi_j^{T,\ell}(x),$$

for $\ell = 1, 2$. Then, setting $v_j := \left| (\boldsymbol{\nabla}_j^D u)_j \cdot \overrightarrow{G_{i_1(j)} G_{i_2(j)}} \right|$, one has

$$(u^T(x))^2 \leq \left(\sum_{j=1}^{J} v_j \; \chi_j^{T,1}(x) \right) \left(\sum_{j=1}^{J} v_j \; \chi_j^{T,2}(x) \right).$$

Integrating the above inequality over T_i and summing over $i \in [1, I]$, yields

$$\|u^T\|^2_{L^2(\Omega)} \leq \int_\Omega \left[\left(\sum_{j=1}^{J} v_j \; \chi_j^{T,1}(x) \right) \left(\sum_{j=1}^{J} v_j \; \chi_j^{T,2}(x) \right) \right] dx. \tag{3.6}$$

Let $\alpha = \inf\{x_1 : (x_1, x_2) \in \Omega\}$ and $\beta = \sup\{x_1 : (x_1, x_2) \in \Omega\}$. For each $x_1 \in (\alpha, \beta)$, we denote by $H(x_1)$ the set of x_2 such that $x = (x_1, x_2) \in \Omega$. From the fact that $\int_{H(x_1)} \chi_j^{T,2}(x_2) dx_2 \leq |A_j|$ and $\int_\alpha^\beta \chi_j^{T,1}(x_1) dx_1 \leq |A_j|$, we infer that (3.6) may be written

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in the following way:

$$
\begin{aligned}
\|u^T\|^2_{L^2(\Omega)} &\leq \int_\alpha^\beta dx_1 \int_{H(x_1)} dx_2 \left[\sum_{j=1}^J v_j\ \chi_j^{T,1}(x_1) \sum_{j=1}^J v_j\ \chi_j^{T,2}(x_2) \right] \\
&\leq \int_\alpha^\beta \sum_{j=1}^J v_j\ \chi_j^{T,1}(x_1) \left(\int_{H(x_1)} \sum_{j=1}^J v_j\ \chi_j^{T,2}(x_2) dx_2 \right) dx_1 \\
&\leq \int_\alpha^\beta \sum_{j=1}^J v_j\ \chi_j^{T,1}(x_1) \left(\sum_{j=1}^J v_j \int_{H(x_1)} \chi_j^{T,2}(x_2) dx_2 \right) dx_1 \\
&\leq \int_\alpha^\beta \sum_{j=1}^J v_j\ \chi_j^{T,1}(x_1) \left(\sum_{j=1}^J v_j |A_j| \right) dx_1 \\
&\leq \left(\sum_{j=1}^J v_j |A_j| \right) \sum_{j=1}^J v_j \int_\alpha^\beta \chi_j^{T,1}(x_1) dx_1 \leq \left(\sum_{j=1}^J v_j |A_j| \right) \left(\sum_{j=1}^J v_j |A_j| \right).
\end{aligned}
$$

We thus obtain

$$
\|u^T\|^2_{L^2(\Omega)} \leq \left(\sum_{j=1}^J |(\boldsymbol{\nabla}_h^D u)_j . \overrightarrow{G_{i_1(j)} G_{i_2(j)}}|\, |A_j| \right)^2 . \tag{3.7}
$$

Finally, using the Cauchy-Schwarz inequality, we obtain

$$
\|u^T\|^2_{L^2(\Omega)} \leq \left(\sum_{j=1}^J |(\boldsymbol{\nabla}_h^D u)_j|^2 |G_{i_1(j)} G_{i_2(j)}|\, |A_j| \right) \left(\sum_{j=1}^J |G_{i_1(j)} G_{i_2(j)}|\, |A_j| \right).
$$

Since $|D_j| = \frac{1}{2}\left(|A_j|\, |G_{i_1} M_j| \sin\theta_{j_1} + |A_j|\, |G_{i_2} M_j| \sin\theta_{j_2}\right)$, we have that $|A_j|\, |G_{i_1} G_{i_2}| \leq \frac{2|D_j|}{\sin\theta^*}$, by Definition 2.1 and the triangle inequality. Moreover, since $\sum_{j=1}^J |D_j| = |\Omega|$, there holds

$$
\|u^T\|^2_{L^2(\Omega)} \leq \frac{4}{\sin^2\theta^*} |\Omega| \sum_{j=1}^J |(\boldsymbol{\nabla}_h^D u)_j|^2 |D_j|.
$$

We have completed the proof of inequality (3.2), with $C = \frac{2}{\sin\theta^*}|\Omega|^{1/2}$. We now turn to inequality (3.3). We shall use a very similar process to that employed in the proof of (3.2). A slight difference comes from the fact that dual cells may be non-convex, and that the straight lines $\mathcal{D}_x^\ell$ may thus intersect twice the boundary $A'_{j1} \cup A'_{j2}$ between two adjacent dual cells (see Figure 3.2), in which case it is not useful to introduce the difference $u^P_{k_2(j)} - u^P_{k_1(j)}$ in the calculation. We thus define $\chi_j^{P,1}(x)$ and $\chi_j^{P,2}(x)$ by

$$
\chi_j^{P,\ell}(x) = \begin{cases} 1, & \text{if either } A'_{j1} \cap \mathcal{D}_x^\ell \neq \emptyset \text{ or } A'_{j2} \cap \mathcal{D}_x^\ell \neq \emptyset\ , \\ 0, & \text{if } \left(A'_{j1} \cup A'_{j2}\right) \cap \mathcal{D}_x^\ell = \emptyset, \end{cases}
$$

for $\ell = 1, 2$. In the above definition, it is meant that the "either-or" is exclusive: if $\mathcal{D}_x^\ell$ intersects both A'_{j1} and A'_{j2}, then $\chi_j^{P,\ell}(x) = 0$. From the first formula of Definition 2.5, it is

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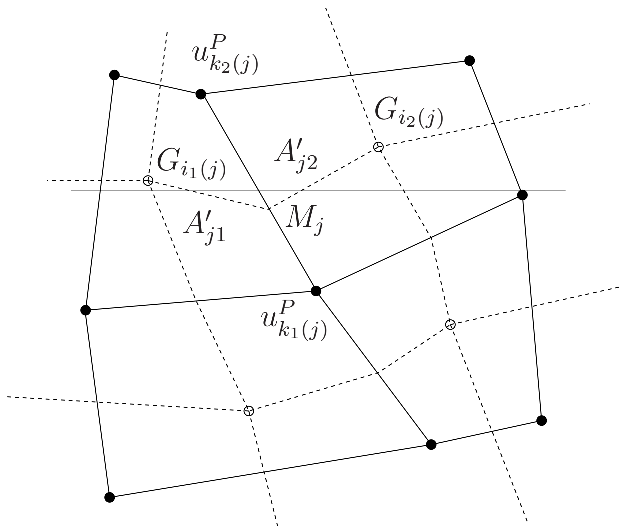

FIGURE 3.2. *The straight line $\mathcal{D}_x^2$ intersects twice the boundary $A'_{j1} \cup A'_{j2}$ of a non convex dual.*

easy to see that

$$(\boldsymbol{\nabla}_j^D u)_j \cdot \overrightarrow{S_{k_1(j)} S_{k_2(j)}} = u^P_{k_2(j)} - u^P_{k_1(j)}, \quad \forall j \in [1, J].$$

Thus, for any $k \in [1, K]$ and a.e. $x \in P_k$, one has

$$|u_k^P| \leq \sum_{j=1}^{J} |(\boldsymbol{\nabla}_h^D u)_j \cdot \overrightarrow{S_{k_1(j)} S_{k_2(j)}}|\; \chi_j^{P,\ell}(x), \quad \ell = 1, 2.$$

Using a similar process as in the proof of (3.2), and taking into account that

$$\int_\alpha^\beta \chi_j^{P,1}(x_1) dx_1 \leq |A'_{j_1}| + |A'_{j_2}| \quad \text{and} \quad \int_{H(x_1)} \chi_j^{P,2}(x_2) dx_2 \leq |A'_{j_1}| + |A'_{j_2}|,$$

we obtain

$$\|u^P\|^2_{L^2(\Omega)} \leq \left(\sum_{j=1}^{J} |(\boldsymbol{\nabla}_h^D u)_j| \, |A_j| (|A'_{j_1}| + |A'_{j_2}|) \right)^2 .$$

This allows to obtain, similarly as above,

$$\|u^P\|^2_{L^2(\Omega)} \leq \frac{4}{\sin^2 \theta^*} |\Omega| \sum_{j=1}^{J} |(\boldsymbol{\nabla}_h^D u)_j|^2 |D_j|,$$

which concludes the proof of inequality (3.3), with $C = \frac{2}{\sin \theta^*} |\Omega|^{1/2}$. □

We now turn to a generalization of Theorem 3.1, which will be useful in the last section of this work.

THEOREM 3.2 (Discrete Poincaré-Friedrichs inequality). *Let us consider open, bounded, simply connected, convex polygonal domains $(\Omega_q)_{q \in [0,Q]}$ of $\mathbb{R}^2$, such that $\Omega_q \subset \Omega_0$ for*

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all $q \in [1,Q]$, and $\bar{\Omega}_{q_1} \cap \bar{\Omega}_{q_2} = \emptyset$ for all $(q_1,q_2) \in [1,Q]^2$ with $q_1 \neq q_2$. Let Ω be defined by $\Omega = \Omega_0 \backslash (\cup_{q=1}^{Q} \Omega_q)$. Let us denote by $\Gamma = \partial\Omega = \cup_{q=0}^{Q} \Gamma_q$, with $\Gamma_q = \partial\Omega_q$ for all $q \in [0,Q]$.

Let $u = (u^T, u^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ be such that

$$\begin{aligned} u_k^P &= 0,\ \forall k \in \Gamma_0, \qquad & u_i^T &= 0,\ \forall i \in \Gamma_0, \\ u_k^P &= c_q^P,\ \forall k \in \Gamma_q, \qquad & u_i^T &= c_q^T,\ \forall i \in \Gamma_q,\ \forall q \in [1,Q]. \end{aligned} \tag{3.8}$$

For θ^* given by Definition 2.1, there exists a constant C depending only on Ω and θ^* such that (3.1) holds.

Proof. Like in Theorem 3.1, it suffices to prove both (3.2) and (3.3). We shall only prove (3.2), since the proof of (3.3) follows exactly the same lines.

The only difference in the proof of (3.2) in Theorem 3.2 with respect to Theorem 3.1 is that the straight line $\mathcal{D}_x^\ell$ may now intersect one or several internal boundaries Γ_q, with $q \in [1,Q]$, before intersecting the external boundary Γ_0; see Figure 3.3. For the sake of simplicity, we shall consider only one intersection with an internal boundary Γ_q (since the alternative may be treated exactly in the same way), and we denote by $v_{n_q}(i)$ and $v_{n_q+1}(i)$ the indices in $[I+1, I+J^\Gamma]$ corresponding to those intersected boundary edges of Γ_q. We may still write

$$u_i^T = \sum_{m=1}^{n-1} (u_{v_m(i)}^T - u_{v_{m+1}(i)}^T),$$

but now the couple $(u_{v_{n_q}(i)}^T, u_{v_{n_q+1}(i)}^T)$ *is not* a pair of neighboring values through an edge A_j intersected by $\mathcal{D}_x^\ell$. However, these two values are equal because of (3.8), so that

$$u_i^T = \sum_{\substack{m \in [1,n-1] \\ m \neq n_q}} (u_{v_m(i)}^T - u_{v_{m+1}(i)}^T).$$

Now, any couple $(u_{v_m(i)}^T, u_{v_{m+1}(i)}^T)$ in the above sum *is* a pair of neighboring values through an edge A_j of the mesh intersected by $\mathcal{D}_x^\ell$, so that there holds, thanks to (3.5),

$$|u_i^T| \leq \sum_{j=1}^{J} \left| (\boldsymbol{\nabla}_j^D u)_j \cdot \overrightarrow{G_{i_1(j)} G_{i_2(j)}} \right| \chi_j^{T,\ell}(x),$$

for $\ell = 1, 2$, and we finish the proof just like in the proof of (3.2). $\square$

Let us now turn to a discrete version of (1.2). As announced in the Introduction, the proof will be divided into three steps. The first step is to prove it in the case of a convex polygonal domain (Theorem 3.3), then we shall prove an inequality related to the mean value on the boundary of a convex polygonal domain (Theorem 3.5), and we shall conclude by the general case of a possibly non-convex polygonal domain (Theorem 3.6).

THEOREM 3.3 (Discrete mean Poincaré inequality for a convex polygonal domain). *Let Ω be an open bounded polygonal connected domain, and let ω be an open convex polygonal subset of Ω, with $\omega \neq \emptyset$. Let $u = (u_i^T, u_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$; the associated piecewise constant functions u^T, u^P are defined through Definition 2.4. Let θ^* be defined through Definition 2.1. Let us define the following mean-values:*

$$m_\omega^T(u) := \frac{1}{|\omega|} \int_\omega u^T(x)\,dx, \quad m_\omega^P(u) := \frac{1}{|\omega|} \int_\omega u^P(x)\,dx.$$

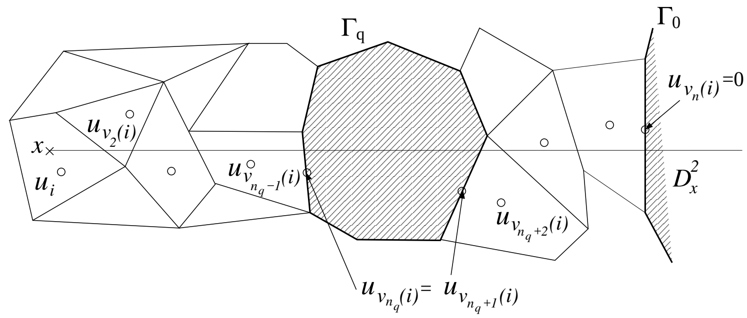

FIGURE 3.3. *Straight line $\mathcal{D}_x^2$ intersecting primal cells from point x to the boundary through internal boundary Γ_q.*

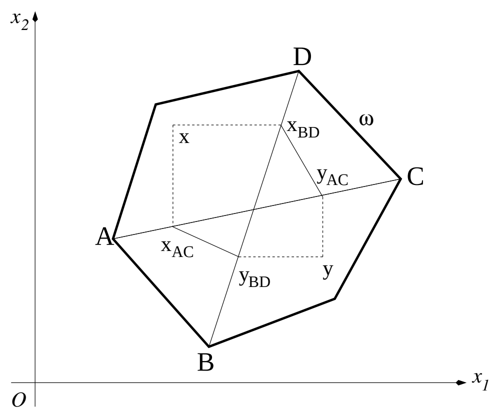

FIGURE 3.4. *Notation for points A, B, C, D and points x_{AC}, x_{BD}, y_{AC}, y_{BD}.*

Then, there exists a constant C only depending on Ω and θ^* such that

$$\|u^T - m_\omega^T(u)\|_{L^2(\omega)} \leq C \|\boldsymbol{\nabla}_h^D u\|_D, \tag{3.9}$$

and

$$\|u^P - m_\omega^P(u)\|_{L^2(\omega)} \leq C \|\boldsymbol{\nabla}_h^D u\|_D. \tag{3.10}$$

(Choosing $\omega = \Omega$ proves the discrete equivalent of (1.2) *if Ω is convex.)*

Proof. We only prove inequality (3.9). The proof of (3.10) may be adapted just like in the proof of Theorem 3.1. We first note that

$$\begin{aligned} \int_\omega |u^T(x) - m_\omega^T(u)|^2 dx &= \int_\omega \left| u^T(x) - \frac{1}{|\omega|} \int_\omega u^T(y) dy \right|^2 dx \\ &\leq \frac{1}{|\omega|} \int_\omega \int_\omega |u^T(x) - u^T(y)|^2 dy dx. \end{aligned} \tag{3.11}$$

We define points A, B, C, D belonging to $\overline{\omega}$ in the following way

$$\begin{aligned} x_A &= \inf\{x_1 : (x_1, x_2) \in \omega\}, \qquad & x_C &= \sup\{x_1 : (x_1, x_2) \in \omega\}, \\ y_B &= \inf\{y_2 : (y_1, y_2) \in \omega\}, \qquad & y_D &= \sup\{y_2 : (y_1, y_2) \in \omega\}. \end{aligned}$$

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Up to a rotation of ω, we may always suppose that those four points are distinct, except if ω is triangular; in that case, up to a rotation of ω, we may set $A = B$ and the proof is exactly the same as that below.

For any $x = (x_1, x_2) \in \omega$, we define $x_{AC} \in [AC]$ such that $(x_{AC})_1 = x_1$ and $x_{BD} \in [BD]$ such that $(x_{BD})_2 = x_2$. The notations are summarized in Figure 3.4. These points are used because, since x_{AC} does not depend on x_2, nor x_{BD} on x_1, they will help us simplify the quadruple integral in the right-hand side of (3.11) into double integrals. Moreover, since these points are all located on the two fixed straight lines $[AC]$ and $[BD]$, the evaluation of the remaining integrals may be treated in a systematic way, as it will be shown below.

Applying the triangle inequality leads to

$$\begin{aligned} |u^T(x) - u^T(y)| \le |u^T(x) - u^T(x_{BD})| + |u^T(x_{BD}) - u^T(y_{AC})| \\ + |u^T(y_{AC}) - u^T(y)|, \end{aligned} \tag{3.12}$$

and also to

$$\begin{aligned} |u^T(x) - u^T(y)| \le |u^T(x) - u^T(x_{AC})| + |u^T(x_{AC}) - u^T(y_{BD})| \\ + |u^T(y_{BD}) - u^T(y)|. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we have

$$\int_\omega \int_\omega |u^T(x) - u^T(y)|^2 dxdy \le \sum_{i=1}^{9} I_i, \tag{3.14}$$

where I_1–I_9 are defined and estimated in the following.

Treatment of I_1

$$I_1 = \int_\omega \int_\omega |u^T(x) - u^T(x_{BD})|\, |u^T(x) - u^T(x_{AC})|\, dxdy. \tag{3.15}$$

Using again (3.4) and (3.5), we may write

$$|u^T(x) - u^T(x_{AC})| \le \sum_{j=1}^{J} \chi_j^{T,1}(x) \left| (\nabla_h^D u)_j \cdot \overrightarrow{G_{i_1(j)} G_{i_2(j)}} \right| \tag{3.16}$$

and

$$|u^T(x) - u^T(x_{BD})| \le \sum_{j=1}^{J} \chi_j^{T,2}(x) \left| (\nabla_h^D u)_j \cdot \overrightarrow{G_{i_1(j)} G_{i_2(j)}} \right|. \tag{3.17}$$

Henceforth, we set for convenience $v_j = \left| (\nabla_h^D u)_j \cdot \overrightarrow{G_{i_1(j)} G_{i_2(j)}} \right|$. Recalling that $\chi_j^{T,1}(x)$ only depends on x_1 and $\chi_j^{T,2}(x)$ only depends on x_2, and noting that the integrand in (3.15) does not depend on y, there holds

$$\begin{aligned} I_1 &\le |\omega| \left(\int_{x_C}^{x_A} \sum_{j=1}^{J} \chi_j^{T,1}(x) v_j dx_1 \right) \left(\int_{y_B}^{y_D} \sum_{j=1}^{J} \chi_j^{T,2}(x) v_j dx_2 \right) \\ &\le |\omega| \left(\sum_{j=1}^{J} v_j \int_{x_C}^{x_A} \chi_j^{T,1}(x) dx_1 \right) \left(\sum_{j=1}^{J} v_j \int_{y_B}^{y_D} \chi_j^{T,2}(x) dx_2 \right). \end{aligned}$$

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We use that

$$\int_{x_A}^{x_C} \chi_j^{T,1}(x)dx_1 \leq |A_j|$$

and

$$\int_{y_B}^{y_D} \chi_j^{T,2}(x)dx_2 \leq |A_j|, \tag{3.18}$$

and obtain

$$I_1 \leq |\omega| \left(\sum_{j=1}^{J} |A_j| v_j \right)^2. \tag{3.19}$$

Treatment of I_2

$$I_2 = \int_\omega \int_\omega |u^T(x) - u^T(x_{BD})| \, |u^T(x_{AC}) - u^T(y_{BD})| \, dxdy.$$

Using inequality (3.17), we have

$$I_2 \leq \int_\omega \int_\omega \left(\sum_{j=1}^{J} \chi_j^2(x) \, v_j \right) |u^T(x_{AC}) - u^T(y_{BD})| \, dxdy.$$

By definition, $\chi_j^2(x)$ only depends on x_2 (which is in $[y_B, y_D]$), while x_{AC} only depends on x_1 (which is in $[x_A, x_C]$); of course, y_{BD} does not depend on x, so that

$$I_2 \leq \left(\sum_{j=1}^{J} v_j \int_{y_B}^{y_D} \chi_j^{T,2}(x)dx_2 \right) \int_\omega \int_{x_A}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})| \, dx_1 dy.$$

Thanks to (3.18), we thus have

$$I_2 \leq \left(\sum_{j=1}^{J} |A_j| v_j \right) \int_\omega \int_{x_A}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})| \, dx_1 dy.$$

Since y_{BD} only depends on y_2 and x_{AC} does not depend on y, the integration with respect to y_1 (which is in $[x_A, x_C]$) is straightforward and yields

$$I_2 \leq (x_C - x_A) \left(\sum_{j=1}^{J} |A_j| v_j \right) \int_{y_B}^{y_D} \int_{x_A}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})| \, dx_1 dy_2. \tag{3.20}$$

Treatment of I_3

$$I_3 = \int_\omega \int_\omega |u^T(x) - u^T(x_{BD})| \, |u^T(y_{BD}) - u^T(y)| \, dxdy.$$

This integral clearly decouples into two independent integrals

$$I_3 = \int_\omega |u^T(x) - u^T(x_{BD})| \, dx \int_\omega |u^T(y_{BD}) - u^T(y)| \, dy,$$

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which may be treated like in the estimation of I_1 thanks to (3.17), (3.18), and the fact that $\chi^{T,2}$ only depends on x_2. We obtain

$$I_3 \le (x_C - x_A)^2 \left(\sum_{j=1}^{J} |A_j| v_j \right)^2. \tag{3.21}$$

Treatment of I_4

$$I_4 = \int_\omega \int_\omega |u^T(x_{BD}) - u^T(y_{AC})| \, |u^T(x) - u^T(x_{AC})| \, dx dy.$$

We may proceed very similarly to the estimation of I_2 and we obtain that

$$I_4 \le (y_D - y_B) \left(\sum_{j=1}^{J} |A_j| v_j \right) \int_{x_A}^{x_C} \int_{y_B}^{y_D} |u^T(x_{BD}) - u^T(y_{AC})| \, dx_2 dy_1. \tag{3.22}$$

Treatment of I_5

$$I_5 = \int_\omega \int_\omega |u^T(x_{BD}) - u^T(y_{AC})| \, |u^T(x_{AC}) - u^T(y_{BD})| \, dx dy.$$

On the one hand, x_{BD} and y_{AC} do not depend on x_1; on the other hand, x_{AC} and y_{BD} do not depend on x_2, so that the integration with respect to x decouples into

$$I_5 \le \int_\omega \left(\int_{y_B}^{y_D} |u^T(x_{BD}) - u^T(y_{AC})| \, dx_2 \right) \left(\int_{x_A}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})| \, dx_1 \right) dy.$$

We also note that y_{BD} and x_{AC} do not depend on y_1 and that y_{AC} and x_{BD} do not depend on y_2, so that the integration with respect to y decouples into

$$I_5 \le \int_{x_A}^{x_C} \int_{y_B}^{y_D} |u^T(x_{BD}) - u^T(y_{AC})| \, dx_2 dy_1 \int_{y_B}^{y_D} \int_{x_A}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})| \, dx_1 dy_2. \tag{3.23}$$

Treatment of I_6

$$I_6 = \int_\omega \int_\omega |u^T(x_{BD}) - u^T(y_{AC})| \, |u^T(y_{BD}) - u^T(y)| \, dx dy.$$

We may proceed very similarly to the estimations of I_2 and I_4 and we obtain that

$$I_6 \le (x_C - x_A) \left(\sum_{j=1}^{J} |A_j| v_j \right) \int_{x_A}^{x_C} \int_{y_B}^{y_D} |u^T(x_{BD}) - u^T(y_{AC})| \, dx_2 dy_1. \tag{3.24}$$

Treatment of I_7

$$I_7 = \int_\omega \int_\omega |u^T(y_{AC}) - u^T(y)| \, |u^T(x) - u^T(x_{AC})| \, dx dy.$$

We may proceed very similarly to the estimation of I_3 and we obtain that

$$I_7 \le (y_D - y_B)^2 \left(\sum_{j=1}^{J} |A_j| v_j \right)^2. \tag{3.25}$$

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Treatment of I_8

$$I_8 = \int_\omega \int_\omega |u^T(y_{AC}) - u^T(y)|\, |u^T(x_{AC}) - u^T(y_{BD})|\, dxdy.$$

We may proceed very similarly to the estimations of I_2, I_4 and I_6 and we obtain that

$$I_8 \le (y_D - y_B) \left(\sum_{j=1}^{J} |A_j| v_j \right) \int_{y_B}^{y_D} \int_{x_A}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})|\, dx_1 dy_2. \tag{3.26}$$

Treatment of I_9

$$I_9 = \int_\omega \int_\omega |u^T(y_{AC}) - u^T(y)|\, |u^T(y_{BD}) - u^T(y)|\, dxdy.$$

We may proceed very similarly to the estimations of I_1 and we obtain that

$$I_9 \le |\omega| \left(\sum_{j=1}^{J} |A_j| v_j \right)^2. \tag{3.27}$$

In order to conclude the proof of Theorem 3.3, we need the following lemma, a proof of which is postponed to Appendix A.

LEMMA 3.4. *There exists a constant C_1, depending only on Ω, such that*

$$\int_{y_B}^{y_D} \int_{x_A}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})|\, dx_1 dy_2 \le C_1 \operatorname{diam}(\omega) \left(\sum_{j=1}^{J} |A_j| v_j \right),$$

$$\int_{x_A}^{x_C} \int_{y_B}^{y_D} |u^T(x_{BD}) - u^T(y_{AC})|\, dx_2 dy_1 \le C_1 \operatorname{diam}(\omega) \left(\sum_{j=1}^{J} |A_j| v_j \right).$$

Applying Lemma 3.4 and combining estimations (3.19) to (3.27) with the bound (3.14) results in

$$\int_\Omega \int_\Omega |u^T(x) - u^T(y)|^2 dxdy \le C_2^2 \left(\sum_{j=1}^{J} |A_j| v_j \right)^2,$$

where $C_2^2 = (4+4C_1+C_1^2) \operatorname{diam}^2(\omega)$. Now this inequality may be treated exactly like (3.7), and there holds

$$\int_\omega \int_\omega |u^T(x) - u^T(y)|^2 dxdy \le \frac{4C_2^2}{\sin^2 \theta^*} |\omega| \sum_{j=1}^{J} |(\boldsymbol{\nabla}_h^D u)_j|^2 |D_j|.$$

From (3.11), we have

$$\int_\omega (u^T(x) - m_\omega^T(u))^2 dx \le \frac{4C_2^2}{\sin^2 \theta^*} \sum_{j=1}^{J} |(\boldsymbol{\nabla}_h^D u)_j|^2 |D_j|,$$

which implies the desired result with $C = \frac{2C_2}{\sin \theta^*}$. $\square$

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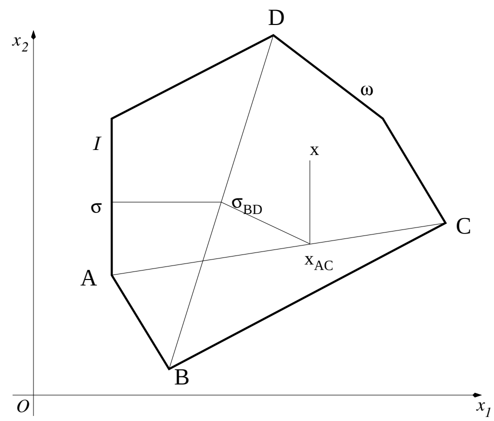

FIGURE 3.5. *Notation for points A, B, C, D and points x_{AC}, σ_{BD}.*

The second step in the proof of a discrete version of (1.2) is to establish an inequality related to the mean value on the boundary of a convex polygonal domain

THEOREM 3.5 (Mean boundary inequality). *Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^2$, and let ω be an open polygonal convex subset of Ω and $\mathcal{I} \subset \partial\omega$, with $|\mathcal{I}| > 0$; $|\mathcal{I}|$ is the one-dimensional Lebesgue measure of $\mathcal{I}$. Assume that $\mathcal{I}$ is included in a hyperplane of $\mathbb{R}^2$. Let $u = (u^T, u^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^J$ be given and the associated piecewise constant functions u^T and u^P be defined through Definition 2.4. Let $\gamma^T(u)(\sigma) = u_i^T$ for all $\sigma \in \overline{T}_i \cap \partial\omega$. (If $\sigma \in \overline{T}_i \cap \overline{T}_{i'}$, then the choice of u_i^T or $u_{i'}^T$ in the definition of γ^T does not matter.) Let $\gamma^P(u)(\sigma) = u_k^P$ for all $\sigma \in \overline{P}_k \cap \partial\omega$. (If $\sigma \in \overline{P}_k \cap \overline{P}_{k'}$, then the choice of u_k^P or $u_{k'}^P$ in the definition of γ^P does not matter.) Let $m_{\mathcal{I}}^T(u)$ (resp $m_{\mathcal{I}}^P(u)$) be the mean value of $\gamma^T(u)$ (resp $\gamma^P(u)$) on I. Let θ^* be defined through Definition 2.1. Then, there exists a constant C, only depending on Ω, ω, $\mathcal{I}$, and θ^*, such that*

$$\|u^T - m_{\mathcal{I}}^T(u)\|_{L^1(\omega)} \leq C\|\boldsymbol{\nabla}_h^D u\|_D, \tag{3.28}$$

$$\|u^P - m_{\mathcal{I}}^P(u)\|_{L^1(\omega)} \leq C\|\boldsymbol{\nabla}_h^D u\|_D. \tag{3.29}$$

Proof. Since $\mathcal{I}$ is included in a hyperplane, it may be assumed, without loss of generality, that $\mathcal{I} = \{0\} \times [a, b]$ and $\omega \subset \mathbb{R}_+ \times \mathbb{R}$; the convexity of ω is used here. We choose points A, B, C, and D, belonging to $\overline{\omega}$, such that

$$\begin{aligned} x_A &= \inf\{x_1 : (x_1, x_2) \in \omega\}, \qquad & x_C &= \sup\{x_1 : (x_1, x_2) \in \omega\}, \\ y_B &= \inf\{x_2 : (x_1, x_2) \in \omega\}, \qquad & y_D &= \sup\{x_2 : (x_1, x_2) \in \omega\}. \end{aligned}$$

It may happen, in particular cases, that those four points are not distinct, but this does not change the general idea of the proof. If $A = B$ and $\mathcal{I} = [BD]$, then it even simplifies the proof, since in that case we do not have to introduce the point σ_{BD} defined below.

For any $x = (x_1, x_2) \in \omega$ and $\sigma = (\sigma_1, \sigma_2) \in \mathcal{I}$, we define $x_{AC} \in AC$ such that $(x_{AC})_1 = x_1$, and $\sigma_{BD} \in BD$ such that $(\sigma_{BD})_2 = \sigma_2$. The notations are summarized in Figure 3.5. The following triangle inequality holds:

$$\begin{aligned} |u^T(x) - \gamma u^T(\sigma)| \leq |u^T(x) - u^T(x_{AC})| + |u^T(x_{AC}) - u^T(\sigma_{BD})| \\ + |\gamma u^T(\sigma) - u^T(\sigma_{BD})|. \end{aligned}$$

Moreover, there holds

$$\begin{aligned}\|u^T - m_{\mathcal{I}}^T(u)\|_{L^1(\omega)} &= \int_\omega \left| u^T(x) - \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} \gamma u^T(\sigma) d\sigma \right| dx \\ &= \int_\omega \left| \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} [u^T(x) - \gamma u^T(\sigma)] d\sigma \right| dx \\ &\le \frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} \left| u^T(x) - \gamma u^T(\sigma)\right| d\sigma dx,\end{aligned}$$

so that, taking into account the above triangle inequality, we obtain

$$\begin{aligned}\|u^T - m_{\mathcal{I}}^T(u)\|_{L^1(\omega)} \le &\frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} |u^T(x) - u^T(x_{AC})|\, d\sigma dx \\ &+ \frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} |u^T(x_{AC}) - u^T(\sigma_{BD})|\, d\sigma dx \\ &+ \frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} |\gamma u^T(\sigma) - u^T(\sigma_{BD})|\, d\sigma dx.\end{aligned}$$

We first observe that the function $|u^T(x) - u^T(x_{AC})|$ does not depend on the variable σ. Then, using similar techniques to those which led to (3.16) and the fact that $\int_{x_A}^{x_C} \chi_j^{T,1}(x) dx_1 \le |A_j|$, there holds

$$\frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} |u^T(x) - u^T(x_{AC})|\, d\sigma dx \le \operatorname{diam}(\omega)\left(\sum_{j=1}^J |A_j| v_j\right), \tag{3.30}$$

where we recall the notation $v_j = |(\boldsymbol{\nabla}_h^D u)_j \cdot \overrightarrow{G_{i_1(j)}G_{i_2(j)}}|$.

Then, we know that the function $|\gamma u^T(\sigma) - u^T(\sigma_{BD})|$ only depends on the variable σ, and, using similar techniques to those which led to (3.17) and the fact that $\int_{\mathcal{I}} \chi_j^{T,2}(\sigma) d\sigma \le |A_j|$, we have

$$\frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} |\gamma u^T(\sigma) - u^T(\sigma_{BD})|\, d\sigma dx \le \frac{|\omega|}{|\mathcal{I}|}\left(\sum_{j=1}^J |A_j| v_j\right). \tag{3.31}$$

Now, x_{AC} does not depend on the variable x_2, so that

$$\frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} |u^T(x_{AC}) - u^T(\sigma_{BD})|\, d\sigma dx \le \frac{\operatorname{diam}(\omega)}{|\mathcal{I}|}\int_{x_A}^{x_C}\int_{\mathcal{I}} |u^T(x_{AC}) - u^T(\sigma_{BD})|\, d\sigma dx_1.$$

Applying an inequality like in Lemma 3.4 leads to

$$\frac{1}{|\mathcal{I}|}\int_\omega\int_{\mathcal{I}} |u(x_{AC}) - u(\sigma_{BD})|\, d\sigma dx \le \frac{C_1 \operatorname{diam}^2(\omega)}{|\mathcal{I}|}\left(\sum_{j=1}^J |A_j| v_j\right). \tag{3.32}$$

Using (3.30), (3.31) and (3.32), we conclude that

$$\|u^T - m_{\mathcal{I}}^T(u)\|_{L^1(\omega)} \le \left[\operatorname{diam}(\omega) + \frac{|\omega|}{|\mathcal{I}|} + \frac{C^* \operatorname{diam}^2(\omega)}{|\mathcal{I}|}\right]\left(\sum_{j=1}^J |A_j| v_j\right).$$

Then, the Cauchy-Schwarz inequality yields (3.28). Similarly, we also obtain (3.29). ∎

Now, we come to the final step of our result.

THEOREM 3.6 (Mean Poincaré inequality). *Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^2$; let $u = (u^T, u^P)$ be in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, and $u^T(x)$, $u^P(x)$ be defined through Definition 2.4. Let θ^* be defined through Definition 2.1. Then, there exists a constant C only depending on Ω and θ^* such that*

$$\|u^T - m_\Omega^T(u)\|_{L^2(\Omega)} \leq C \|\nabla_h^D u\|_D \tag{3.33}$$

and

$$\|u^P - m_\Omega^P(u)\|_{L^2(\Omega)} \leq C \|\nabla_h^D u\|_D, \tag{3.34}$$

where $m_\Omega^T(u)$ (resp. $m_\Omega^P(u)$) is the mean-value of u^T (resp. u^P) on Ω.

Proof. Since Ω is polygonal, there exists a finite number of disjoint convex polygonal sets, denoted by $\{\Omega_1, ..., \Omega_n\}$, such that $\overline{\Omega} = \cup_{i=1}^n \overline{\Omega}_i$. Let $\mathcal{I}_{i,j} = \overline{\Omega}_i \cap \overline{\Omega}_j$ and B be the set of couples $(i,j) \in \{1, ..., n\}^2$ such that $i \neq j$ and the one-dimensional Lebesgue measure of $\mathcal{I}_{i,j}$, denoted by $|\mathcal{I}_{i,j}|$, is strictly positive.

Let m_i denote the mean value of u^T on Ω_i, $i \in \{1, ..., n\}$, and $m_{i,j}$ denote the mean value of u^T on $\mathcal{I}_{i,j}$, $(i,j) \in B$. Note that $m_{i,j} = m_{j,i}$ for all $(i,j) \in B$. Theorem 3.3 gives the existence of C_i, $i \in \{1, ..., n\}$, only depending on Ω (since the Ω_i only depend on Ω) and θ^*, such that

$$\|u^T - m_i\|_{L^2(\Omega_i)} \leq C_i \, \|\nabla_h^D u\|_D, \quad \forall i \in \{1, ..., n\}. \tag{3.35}$$

Applying the Cauchy-Schwarz inequality, we have

$$\|u^T - m_i\|_{L^1(\Omega_i)} \leq |\Omega_i|^{1/2} C_i \, \|\nabla_h^D u\|_D, \quad \forall i \in \{1, ..., n\}.$$

Moreover, Theorem 3.5 gives the existence of $C_{i,j}$, $(i,j) \in B$, only depending on Ω and θ^*, such that

$$\|u^T - m_{i,j}\|_{L^1(\Omega_i)} \leq C_{i,j} \, \|\nabla_h^D u\|_D, \quad \forall (i,j) \in B.$$

Then, one has, by the triangle inequality

$$|\Omega_i| \, |m_i - m_{i,j}| = \|m_i - m_{i,j}\|_{L^1(\Omega_i)} \leq \Big(|\Omega_i|^{1/2} C_i + C_{i,j} \Big) \, \|\nabla_h^D u\|_D, \tag{3.36}$$

for all $(i,j) \in B$. Applying again the triangular inequality and using the fact that $m_{i,j} = m_{j,i}$, we get from (3.36) that there exists a constant $C'_{i,j}$, only depending on Ω and θ^*, such that

$$|m_i - m_j| \leq C'_{i,j} \, \|\nabla_h^D u\|_D, \tag{3.37}$$

for all $(i,j) \in B$.

Since Ω is connected, we can always connect any $(i,j) \in \{1, ..., n\}^2$ by a finite set of couples belonging to B. Applying the triangular inequality and related inequalities (3.37), we obtain the existence of $K_{i,j}$, only depending on Ω and θ^*, such that $|m_i - m_j| \leq K_{i,j} \|\nabla_h^D u\|_D$, for all $(i,j) \in \{1, ..., n\}^2$, and therefore the existence of a constant M_i, only depending on Ω and θ^*, such that

$$\big| m_\Omega^T(u) - m_i \big| = \left| \frac{1}{|\Omega|} \sum_{j \in [1,n]} |\Omega_j| (m_j - m_i) \right| \leq M_i \|\nabla_h^D u\|_D. \tag{3.38}$$

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Then, (3.35), (3.38) and the triangle inequality yield

$$
\begin{aligned}
\|u^T - m^T_\Omega(u)\|_{L^2(\Omega_i)} &\le \|u^T - m_i\|_{L^2(\Omega_i)} + |\Omega_i|^{1/2} \left| m^T_\Omega(u) - m_i \right| \\
&\le \left(C_i + M_i |\Omega_i|^{1/2} \right) \|\nabla^D_h u\|_D.
\end{aligned}
\tag{3.39}
$$

Summing up the squares of inequalities (3.39) over $i \in \{1, ..., n\}$ yields (3.33). We obtain (3.34) in a similar way. This completes the proof. ∎

COROLLARY 3.7. *Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^2$; let $u = (u^T, u^P)$ be in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ and such that*

$$
\sum_{i=1}^{I} |T_i| u^T_i = \sum_{k=1}^{K} |P_k| u^P_k = 0.
$$

Let θ^* be defined through Definition 2.1. Then there exists a constant C only depending on Ω and θ^* such that

$$
\|u\|_{T,P} \le C \|\nabla^D_h u\|_D.
$$

4. Applications. The so-called "div–curl" problem, which consists of finding a velocity field from the knowledge of its divergence and curl, together with appropriate boundary conditions, has important applications in electrostatics and magnetostatics as well as in fluid dynamics; the discrete duality discretization allows us to solve this problem numerically on arbitrary 2D meshes; see [9]. The next theorem shows the stability of such a numerical procedure.

THEOREM 4.1 (Discrete Div-Curl stability). *Let Ω be a two-dimensional polygonal domain with exterior boundary denoted by Γ_0 and internal connected components denoted by Γ_q, with $q \in [1, Q]$. There exists a constant C depending only on Ω and θ^* defined by Definition 2.1, such that for any discrete vector field $(\mathbf{v}_j)_{j\in[1,J]}$ with $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ and $(\mathbf{v}\cdot\boldsymbol{\tau}, 1)_{\Gamma_q,h} = 0$, for all $q \in [1, Q]$, there holds*

$$
||\mathbf{v}||_D \le C \left(||\nabla^{T,P} \cdot \mathbf{v}||_{T,P} + ||\nabla^{T,P} \times \mathbf{v}||_{T,P} \right).
\tag{4.1}
$$

Proof. Let $(\mathbf{v}_j)_{j\in[1,J]}$ be given with $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ and $(\mathbf{v}\cdot\boldsymbol{\tau}, 1)_{\Gamma_q,h} = 0$, for all $q \in [1, Q]$. According to Theorem 2.6, there exists $\phi = (\phi^T_i, \phi^P_k)_{i\in[1,I+J^\Gamma],k\in[1,K]}$, $\psi = (\psi^T_i, \psi^P_k)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ and $(c^T_q, c^P_q)_{q\in[1,Q]}$ such that (2.2) holds, the decomposition being orthogonal. Then, there holds

$$
||\mathbf{v}||^2_D = (\mathbf{v}, \nabla^D_h \phi)_D + (\mathbf{v}, \nabla^D_h \times \psi)_D,
\tag{4.2}
$$

$$
||\nabla^D_h \phi||_D \le ||\mathbf{v}||_D, \quad \text{and} \quad ||\nabla^D_h \times \psi||_D = ||\nabla^D_h \psi||_D \le ||\mathbf{v}||_D.
\tag{4.3}
$$

Using the discrete integration by part properties (2.5) and (2.6) in (4.2), we obtain

$$
||\mathbf{v}||^2_D = -(\nabla^{T,P}_h \cdot \mathbf{v}, \phi^{T,P})_{T,P} + (\mathbf{v}\cdot\mathbf{n}, \tilde{\phi})_{\Gamma,h} + (\nabla^{T,P}_h \times \mathbf{v}, \psi^{T,P})_{T,P} - (\mathbf{v}\cdot\boldsymbol{\tau}, \tilde{\psi})_{\Gamma,h}.
\tag{4.4}
$$

In (4.4), both boundary terms vanish. The first because $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ. As far as the second is concerned, from (2.4) and the definition of the boundary scalar product (2.1) we have

$$
(\mathbf{v}\cdot\boldsymbol{\tau}, \tilde{\psi})_{\Gamma,h} = (\mathbf{v}\cdot\boldsymbol{\tau}, \tilde{\psi})_{\Gamma_0,h} + \sum_{q\in[1,Q]} \left(\frac{c^T_q + c^P_q}{2} \right) (\mathbf{v}\cdot\boldsymbol{\tau}, 1)_{\Gamma_q,h},
$$

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so that (2.3) and the fact that $(\mathbf{v}\cdot\boldsymbol{\tau},1)_{\Gamma_q,h}=0$, for all $q\in[1,Q]$, allow us to conclude that $(\mathbf{v}\cdot\boldsymbol{\tau},\tilde{\psi})_{\Gamma,h}=0$. Thus, we have

$$||\mathbf{v}||_D^2=-(\nabla_h^{T,P}\cdot\mathbf{v},\phi^{T,P})_{T,P}+(\nabla_h^{T,P}\times\mathbf{v},\psi^{T,P})_{T,P}. \tag{4.5}$$

Using the Cauchy-Schwarz inequality in (4.5), and applying Theorem 3.2 for ψ and Corollary 3.7 for ϕ, we get (4.1) from (4.3). ◻

Appendix A. Proof of Lemma 3.4. We shall only give the proof of the first inequality in Lemma 3.4, since the proof of the other inequality follows exactly the same lines. If the four points (A,B,C,D) are distinct, then we may denote by I the intersection of AC and BD, and the angle α between the diagonals AC and BD is different from 0. This is also the case for the angles β_i and γ_i displayed in Figure A.1. If ω is a triangle, up to a rotation we have that $A=B$ and we set $I=A=B$. Then, the angles α, β_1, and γ_1 are all different from 0 and evaluating the term G in (A.1) reduces to the evaluation of H_1, which simplifies the proof. Let us go back to the general case. We set

$$G=\int_{y_B}^{y_D}\int_{x_A}^{x_C}|u^T(x_{AC})-u^T(y_{BD})|\,dx_1dy_2=H_1+H_2+H_3+H_4, \tag{A.1}$$

where

$$\begin{aligned}
H_1&=\int_{y_I}^{y_D}\int_{x_I}^{x_C}|u^T(x_{AC})-u^T(y_{BD})|\,dx_1dy_2,\\
H_2&=\int_{y_I}^{y_D}\int_{x_A}^{x_I}|u^T(x_{AC})-u^T(y_{BD})|\,dx_1dy_2,\\
H_3&=\int_{y_B}^{y_I}\int_{x_A}^{x_I}|u^T(x_{AC})-u^T(y_{BD})|\,dx_1dy_2,\\
H_4&=\int_{y_B}^{y_I}\int_{x_I}^{x_C}|u^T(x_{AC})-u^T(y_{BD})|\,dx_1dy_2.
\end{aligned}$$

We only estimate the first term in the right-hand side of equation (A.1), since the others may be treated similarly. For any $x_{AC}\in IC$ and $y_{BD}\in ID$, let x_M (resp. y_P) be the intersection of DC with the straight line going though x_{AC} (resp. y_{BD}) and parallel to the segment $[ID]$ (resp. $[IC]$), and let x_{M_1} (resp. y_{P_1}) be the intersection of ID (resp. IC) with the straight line going through x_M (resp. x_P) and parallel to the segment IC (resp. ID). Then, we shall examine two cases, according to where the broken line $x_{AC}x_Mx_{M_1}$ intersects the broken line $y_{BD}y_Py_{P_1}$ at point N.

Case 1: The broken line $x_{AC}x_Mx_{M_1}$ intersects DC at x_M before it intersects the broken line $y_{BD}y_Py_{P_1}$; see Figure A.1. Then, the triangle inequality leads to

$$\begin{aligned}
|u^T(x_{AC})-u^T(y_{BD})|\le{}&|u^T(x_{AC})-u^T(x_M)|+|u^T(x_M)-u^T(N)|\\
&+|u^T(N)-u^T(y_P)|+|u^T(y_P)-u(y_{BD})|.
\end{aligned}$$

Let the function χ_j from $\mathbb{R}^2\times\mathbb{R}^2$ to $\{0,1\}$ be defined by

$$\chi_j(x,y)=\begin{cases}1, & \text{if } [x,y]\cap A_j\neq\emptyset,\\ 0, & \text{if } [x,y]\cap A_j=\emptyset.\end{cases}$$

Recalling once again the notation $v_j=|(\boldsymbol{\nabla}_h^D u^T)_j\cdot\overrightarrow{G_{i_1(j)}G_{i_2(j)}}|$, we have that

$$|u^T(x_M)-u^T(N)|\le\sum_{j=1}^{J}\chi_j(x_M,N)\,v_j\le\sum_{j=1}^{J}\chi_j(x_M,x_{M_1})\,v_j, \tag{A.2}$$

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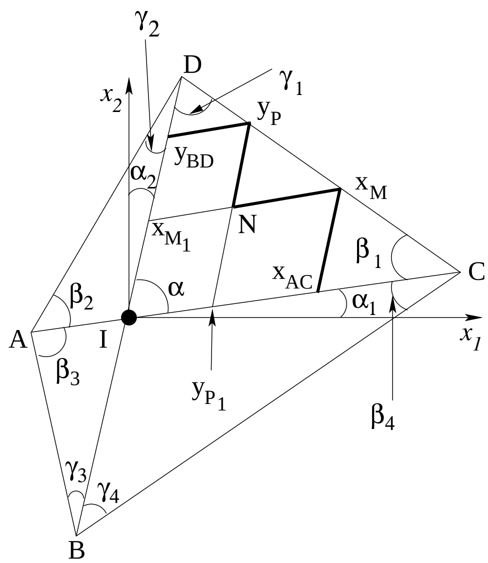

FIGURE A.1. $x_{AC}x_Mx_{M_1}$ *intersects* DC *before it intersects* $y_{BD}y_Py_{P_1}$.

due to the fact that since $N \in [x_M x_{M_1}]$ then $\chi_j(x_M, N) \le \chi_j(x_M, x_{M_1})$. Similarly, we obtain that

$$|u^T(N) - u^T(y_P)| \le \sum_{j=1}^{J} \chi_j(y_P, y_{P_1})\, v_j. \tag{A.3}$$

We also have

$$|u^T(x_{AC}) - u^T(x_M)| \le \sum_{j=1}^{J} \chi_j(x_{AC}, x_M)\, v_j \tag{A.4}$$

and

$$|u^T(y_P) - u^T(y_{BD})| \le \sum_{j=1}^{J} \chi_j(y_{BD}, y_P)\, v_j. \tag{A.5}$$

From (A.2)–(A.5), we have

$$\begin{aligned}|u^T(x_{AC}) - u^T(y_{BD})| \le \sum_{j=1}^{J} \chi_j(x_{AC}, x_M)\, v_j + \sum_{j=1}^{J} \chi_j(x_M, x_{M_1})\, v_j \\ + \sum_{j=1}^{J} \chi_j(y_{BD}, y_P)\, v_j + \sum_{j=1}^{J} \chi_j(y_P, y_{P_1})\, v_j.\end{aligned}$$

Case 2: The broken line $x_{AC}x_Mx_{M_1}$ intersects the broken line $y_{BD}y_Py_{P_1}$ at N before it intersects DC; see Figure A.2. We use the triangle inequality to obtain

$$|u^T(x_{AC}) - u^T(y_{BD})| \le |u^T(x_{AC}) - u^T(N)| + |u^T(N) - u^T(y_{BD})|. \tag{A.6}$$

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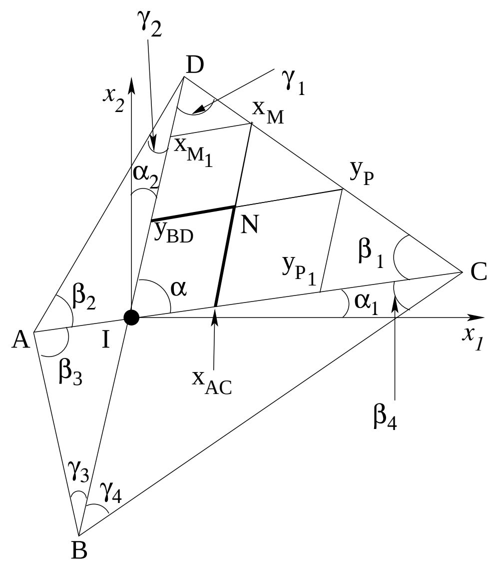

FIGURE A.2. $x_{AC}x_Mx_{M_1}$ *intersects* $y_{BD}y_Py_{P_1}$ *before it intersects* DC.

Similarly to Case 1, since $N \in [x_{AC}x_M]$ and $N \in [y_{BD}y_P]$, there holds

$$|u^T(x_{AC}) - u^T(N)| \leq \sum_{j=1}^{J} \chi_j(x_{AC}, x_M)\, v_j, \tag{A.7}$$

$$|u^T(N) - u^T(y_{BD})| \leq \sum_{j=1}^{J} \chi_j(y_{BD}, y_P)\, v_j. \tag{A.8}$$

Adding (A.7) to (A.8), and combining with (A.6), we have

$$|u^T(x_{AC}) - u^T(y_{BD})| \leq \sum_{j=1}^{J} \chi_j(x_{AC}, x_M)\, v_j + \sum_{j=1}^{J} \chi_j(y_{BD}, y_P)\, v_j.$$

Thus, in both cases, we always obtain

$$\begin{aligned} |u^T(x_{AC}) - u^T(y_{BD})| \leq \sum_{j=1}^{J} \chi_j(x_{AC}, x_M)\, v_j + \sum_{j=1}^{J} \chi_j(x_M, x_{M_1})\, v_j \\ + \sum_{j=1}^{J} \chi_j(y_{BD}, y_P)\, v_j + \sum_{j=1}^{J} \chi_j(y_P, y_{P_1})\, v_j. \end{aligned}$$

We thus always have

$$H_1 = \int_{y_I}^{y_D} \int_{x_I}^{x_C} |u^T(x_{AC}) - u^T(y_{BD})|\, dx_1 dy_2 \leq L_1 + L_2 + L_3 + L_4, \tag{A.9}$$

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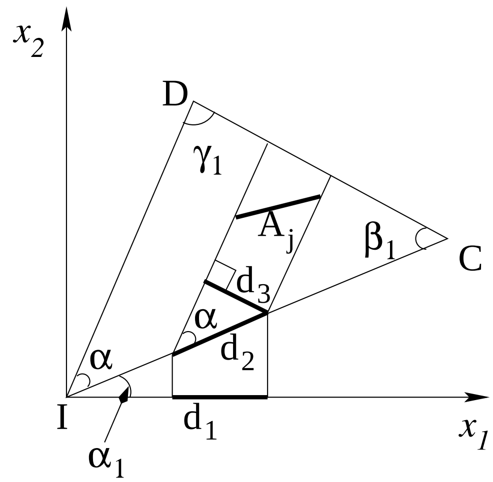

FIGURE A.3. *How to estimate the term* $\int_{x_I}^{x_C} \chi_j(x_{AC}, x_M) dx_1$.

where L_1, L_2, L_3, and L_4 are defined as follows:

$$L_1 = \int_{y_I}^{y_D} \int_{x_I}^{x_C} \sum_{j=1}^{J} \chi_j(x_{AC}, x_M)\, v_j dx_1 dy_2,$$

$$L_2 = \int_{y_I}^{y_D} \int_{x_I}^{x_C} \sum_{j=1}^{J} \chi_j(x_M, x_{M_1})\, v_j dx_1 dy_2,$$

$$L_3 = \int_{y_I}^{y_D} \int_{x_I}^{x_C} \sum_{j=1}^{J} \chi_j(y_{BD}, y_P)\, v_j dx_1 dy_2,$$

$$L_4 = \int_{y_I}^{y_D} \int_{x_I}^{x_C} \sum_{j=1}^{J} \chi_j(y_P, y_{P_1})\, v_j dx_1 dy_2.$$

Observing that $\chi_j(x_{AC}, x_M)$ only depends on the variable x_1, we find

$$\begin{aligned} L_1 &\leq (y_D - y_I) \int_{x_I}^{x_C} \sum_{j=1}^{J} \chi_j(x_{AC}, x_M)\, v_j dx_1 \\ &= (y_D - y_I) \sum_{j=1}^{J} \int_{x_I}^{x_C} \chi_j(x_{AC}, x_M) dx_1 v_j. \end{aligned}$$

Let us take a look at Figure A.3 and its associated notations. Simple geometrical arguments show that

$$\int_{x_I}^{x_C} \chi_j(x_{AC}, x_M) dx_1 =: d_1 = d_2 \cos\alpha_1 = d_3 \frac{\cos\alpha_1}{\sin\alpha} \leq \frac{\cos\alpha_1 |A_j|}{\sin\alpha}.$$

This results in

$$L_1 \leq (y_D - y_I) \frac{\cos\alpha_1}{\sin\alpha} \left(\sum_{j=1}^{J} |A_j| v_j \right). \tag{A.10}$$

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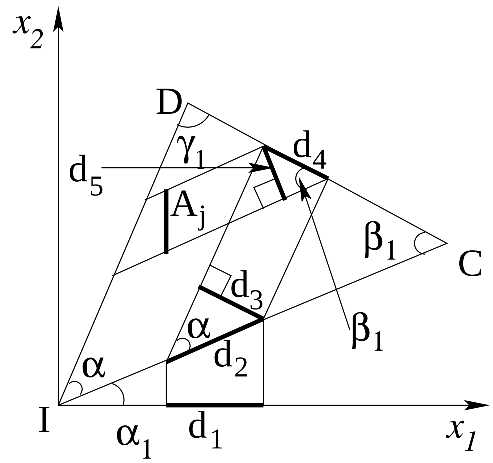

FIGURE A.4. *How to estimate the term* $\int_{x_I}^{x_C} \chi_j(x_M, x_{M_1})dx_1$.

Moreover, there holds

$$L_2 \le (y_D - y_I) \int_{x_I}^{x_C} \sum_{j=1}^{J} \chi_j(x_M, x_{M_1})\, v_j dx_1$$
$$= (y_D - y_I) \sum_{j=1}^{J} \int_{x_I}^{x_C} \chi_j(x_M, x_{M_1}) dx_1 v_j.$$

Let us take a look at Figure A.4 and its associated notations. Simple geometrical arguments show that

$$\int_{x_I}^{x_C} \chi_j(x_M, x_{M_1})dx_1 =: d_1 = d_2 \cos\alpha_1 = d_3 \frac{\cos\alpha_1}{\sin\alpha}$$
$$= d_4 \frac{\cos\alpha_1 \sin\gamma_1}{\sin\alpha} = d_5 \frac{\cos\alpha_1 \sin\gamma_1}{\sin\alpha\, \sin\beta_1} \le \frac{\cos\alpha_1 \sin\gamma_1 |A_j|}{\sin\alpha\, \sin\beta_1},$$

so that there holds

$$L_2 \le \frac{\cos\alpha_1 \sin\gamma_1}{\sin\alpha\, \sin\beta_1} (y_D - y_I) \left(\sum_{j=1}^{J} |A_j| v_j \right). \tag{A.11}$$

Similarly,

$$L_3 \le \frac{\cos\alpha_2}{\sin\alpha} (x_C - x_I) \left(\sum_{j=1}^{J} |A_j| v_j \right), \tag{A.12}$$

$$L_4 \le \frac{\cos\alpha_2 \sin\beta_1}{\sin\alpha\, \sin\gamma_1} (x_C - x_I) \left(\sum_{j=1}^{J} |A_j| v_j \right). \tag{A.13}$$

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From (A.9)–(A.13), we conclude that there exists a constant C depending only on the geometry of ω (since the angles depend only on the geometry of ω) such that

$$H_1 \leq C\,\mathrm{diam}(\omega)\left(\sum_{j=1}^{J}|A_j|v_j\right). \tag{A.14}$$

Using similar techniques, we also obtain that

$$H_2 \leq C\,\mathrm{diam}(\Omega)\left(\sum_{j=1}^{J}|A_j|v_j\right), \tag{A.15}$$

$$H_3 \leq C\,\mathrm{diam}(\Omega)\left(\sum_{j=1}^{J}|A_j|v_j\right), \tag{A.16}$$

$$H_4 \leq C\,\mathrm{diam}(\Omega)\left(\sum_{j=1}^{J}|A_j|v_j\right). \tag{A.17}$$

Combining (A.14)–(A.17) with (A.1), we have

$$\int_{y_B}^{y_D}\int_{x_A}^{x_C}|u^T(x_{AC})-u^T(y_{BD})|\,dx_1dy_2 \leq C_1\,\mathrm{diam}(\Omega)\left(\sum_{j=1}^{J}|A_j|v_j\right),$$

where $C_1 = 4C$, which concludes the proof of Lemma 3.4.

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