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COLLOCATION FOR SINGULAR INTEGRAL EQUATIONS WITH FIXED SINGULARITIES OF PARTICULAR MELLIN TYPE*

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Abstract. This paper is concerned with the stability of collocation methods for Cauchy singular integral equations with fixed singularities on the interval $[-1,1]$. The operator in these equations is supposed to be of the form $a\mathcal{I}+b\mathcal{S}+\mathcal{B}^{\pm}$ with piecewise continuous functions a and b. The operator $\mathcal{S}$ is the Cauchy singular integral operator and $\mathcal{B}^{\pm}$ is a finite sum of integral operators with fixed singularities at the points ± 1 of special kind. The collocation methods search for approximate solutions of the form $\nu(x)p_n(x)$ or $\mu(x)p_n(x)$ with Chebyshev weights $\nu(x)=\sqrt{\frac{1+x}{1-x}}$ or $\mu(x)=\sqrt{\frac{1-x}{1+x}}$, respectively, and collocation with respect to Chebyshev nodes of first and third or fourth kind is considered. For the stability of collocation methods in a weighted $\mathbf{L}^2$-space, we derive necessary and sufficient conditions.

Key words. collocation method, stability, C^*-algebra, notched half plane problem

AMS subject classifications. 65R20, 45E05

1. Introduction. Polynomial collocation methods for singular integral equations with fixed singularities are studied, for example, in [1, 11, 17]. In [11], the stability of a polynomial collocation method is investigated for a class of Cauchy singular integral equations with additional fixed singularities of Mellin convolution type. The papers [1, 17] are more concerned with the computational aspects of these methods. While [17] deals with integral equations of the form

$$u(x)+b(x)\int_{-1}^{1}\mathbf{h}\left(\frac{1+x}{1+y}\right)\frac{u(y)\,dy}{1+y}+\int_{-1}^{1}\mathbf{h}_0(x,y)u(y)\,dy=f(x),\quad -1<x<1,$$

where $\mathbf{h}:(0,\infty)\longrightarrow\mathbb{C}$, $b,f:[-1,1]\longrightarrow\mathbb{C}$, and $\mathbf{h}_0:[-1,1]^2\longrightarrow\mathbb{C}$ are given (continuous) functions, the paper [1] deals with the effective realization of polynomial collocation methods for the equation (see [1, (1.8)])

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}-\frac{1}{2+y+x}+\frac{6(1+x)}{(2+y+x)^2}-\frac{4(1+x)^2}{(2+y+x)^3}\right]u(y)\,dy=f(x),\quad -1<x<1, \tag{1.1}$$

associated with the so-called notched half plane problem; see also [14, Section 37a] and [2, Section 14]; we also refer to [1, Remark 2.6]. In particular, if the right-hand side $f(x)$ of (1.1) is a constant function, then the solution $u(x)$ has a singularity of the form $(1-x)^{-\frac{1}{2}}$ at the endpoint 1 of the integration interval. More detailed, the function $\sqrt{1-x}\,u(x)$ is bounded and satisfies certain smoothness properties; cf. [2, Theorem 14.1]. In [11], singularities of the solutions are considered which can be represented by a Jacobi weight the exponents of which are in the interval $(-\frac{1}{4},\frac{3}{4})$. Hence, the stability results given in [11] are not applicable to the equation (1.1) if we want to represent the asymptotic behaviour $(1-x)^{-\frac{1}{2}}$ of the solution at the right endpoint of the integration interval.

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In the present paper, we investigate the stability of collocation methods applied to a class of Cauchy singular integral equations with additional fixed singularities of Mellin type (of special form) covering equation (1.1) of the notched half plane problem, where the solution $u(x)$ can be represented in the form

$$u(x) = \sqrt{\frac{1+x}{1-x}}\, u_0(x) \qquad \text{or} \qquad u(x) = \sqrt{\frac{1-x}{1+x}}\, u_0(x) \tag{1.2}$$

with sufficiently regular functions $u_0(x)$. Of course, for the problem (1.1), this asymptotic behavior is not the best one, and further investigations are necessary. Let us also mention that other exponents in the weights are of interest depending on the concrete problem; see, for example, [2, Theorem 15.1] or [5, Section 2]. In [11], the stability of the collocation methods is proved by using respective results for Cauchy singular integral equations (cf. [12, 13]) and a representation of the Mellin operators by Bochner integrals. Since the kernels of Mellin operators under consideration in the present paper do not satisfy all assumptions made in [11], we develop here necessary and sufficient conditions for the stability of these methods in a more direct manner taking advantage of the special structure of the Mellin kernels occurring for example in (1.1).

The paper is organized as follows. In Section 2 we introduce the class of integral equations under consideration and describe the polynomial collocation methods we want to apply. In Section 3.1 an algebra of operator sequences is defined for which the stability of these operator sequences is equivalent to its invertibility modulo a suitable ideal and the invertibility of four limit operators associated to the operator sequence. The fact that the operator sequences of our collocation methods belong to this algebra is the topic of Section 3.2, where also the respective four limit operators are presented. Section 3.3 discusses the invertibility of these limit operators and prepares the proof of the main result on the stability of the collocation methods, which is presented in Section 4. Section 5 shows how to deal with the first type of singularities in (1.2) since the previous results are concerned with the second type in (1.2). In Section 6 we discuss some numerical aspects of the investigated collocation methods and present numerical results for their application to the notched half plane problem (1.1) together with a discussion of the numerical results already presented in [1]. The final Sections 7 and 8 give the technical proofs for the results of Section 3.2 and of Lemma 4.8, respectively.

2. The integral equation and a collocation method. Here we consider the Cauchy singular integral equation with fixed singularities of the form

$$\begin{aligned} a(x)u(x) &+ \frac{b(x)}{\pi \mathbf{i}} \int_{-1}^{1} \frac{u(y)}{y-x}\,dy + \sum_{k=1}^{m_-} \frac{\beta_k^-}{\pi \mathbf{i}} \int_{-1}^{1} \frac{(1+x)^{k-1} u(y)\,dy}{(y+x+2)^k} \\ &+ \sum_{k=1}^{m_+} \frac{\beta_k^+}{\pi \mathbf{i}} \int_{-1}^{1} \frac{(1-x)^{k-1} u(y)\,dy}{(y+x-2)^k} = f(x), \quad -1 < x < 1, \end{aligned} \tag{2.1}$$

with given $\beta_k^\pm \in \mathbb{C}$ and nonnegative integers $m_\pm$. In this equation, the coefficient functions a, b belong to the set $\mathbf{PC}$ of piecewise continuous functions[1], the right-hand side function f is assumed to belong to the weighted $\mathbf{L}^2$-space $\mathbf{L}^2_\nu$, and $u \in \mathbf{L}^2_\nu$ stands for the unknown solution. The inner product in the Hilbert space $\mathbf{L}^2_\nu$ is given by

$$\langle u, v \rangle_\nu := \int_{-1}^{1} u(y)\overline{v(y)}\nu(y)\,dy,$$

[1] We call a function $a : [-1,1] \to \mathbb{C}$ piecewise continuous if it is continuous at ± 1, if the one-sided limits $a(x \pm 0)$ exist for all $x \in (-1,1)$, and at least one of them coincides with $a(x)$.

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where $\nu(x)=\sqrt{\frac{1+x}{1-x}}$ is the Chebyshev weight of third kind. Let

$$\mathcal{S}:\mathbf{L}^2_\nu\to\mathbf{L}^2_\nu,\quad u\mapsto\frac{1}{\pi\mathbf{i}}\int_{-1}^1\frac{u(y)}{y-\cdot}\,dy$$

be the Cauchy singular integral operator, $a\mathcal{I}:\mathbf{L}^2_\nu\to\mathbf{L}^2_\nu$, $u\mapsto au$ be the operator of multiplication by a, and

$$\mathcal{B}^\pm_k:\mathbf{L}^2_\nu\longrightarrow\mathbf{L}^2_\nu,\ u\mapsto\frac{1}{\pi\mathbf{i}}\int_{-1}^1\frac{(1\mp\cdot)^{k-1}u(y)\,dy}{(y+\cdot\mp2)^k}$$

be the integral operators with a fixed singularity at ±1. We write equation (2.1) in the form

$$\mathcal{A}u:=\left(a\mathcal{I}+b\mathcal{S}+\sum_{k=1}^{m_-}\beta^-_k\mathcal{B}^-_k+\sum_{k=1}^{m_+}\beta^+_k\mathcal{B}^+_k\right)u=f.$$

It is a well known fact that the single operators $a\mathcal{I}$, $\mathcal{S}$, and $\mathcal{B}^\pm_k$ are bounded in $\mathbf{L}^2_\nu$; see [2, Theorem 1.16 and Remark 8.3]. This means that these operators belong to the Banach algebra $\mathcal{L}(\mathbf{L}^2_\nu)$ of all bounded and linear operators $\mathcal{A}:\mathbf{L}^2_\nu\longrightarrow\mathbf{L}^2_\nu$. In order to get approximate solutions of the integral equation, we use a polynomial collocation method. For this we need some further notions. Let $\sigma(x)=\frac{1}{\sqrt{1-x^2}}$, $\varphi(x)=\sqrt{1-x^2}$, and $\mu(x)=\sqrt{\frac{1-x}{1+x}}$ be the Chebyshev weights of first, second, and fourth kind, respectively. For $n\ge0$ and $\tau\in\{\sigma,\varphi,\nu,\mu\}$, we denote by $p^\tau_n(x)$ the corresponding normalized Chebychev polynomials of degree n with respect to the weight $\tau(x)$ and with positive leading coefficient, which we abbreviate by $T_n(x)=p^\sigma_n(x)$, $U_n(x)=p^\varphi_n(x)$, $R_n(x)=p^\nu_n(x)$, and $P_n(x)=p^\mu_n(x)$. We know that

$$T_0(x)=\frac{1}{\sqrt{\pi}},\quad T_n(\cos s)=\sqrt{\frac{2}{\pi}}\cos ns,\quad n\ge1,\ s\in(0,\pi),$$

and, for $n\ge0$, $s\in(0,\pi)$,

$$U_n(\cos s)=\frac{\sqrt{2}\sin(n+1)s}{\sqrt{\pi}\sin s},\quad R_n(\cos s)=\frac{\cos(n+\frac12)s}{\sqrt{\pi}\cos\frac{s}{2}},\quad P_n(\cos s)=\frac{\sin(n+\frac12)s}{\sqrt{\pi}\sin\frac{s}{2}}.$$

The zeros x^τ_{jn} of $p^\tau_n(x)$ are given by

$$x^\sigma_{jn}=\cos\frac{j-\frac12}{n}\pi,\quad x^\varphi_{jn}=\cos\frac{j\pi}{n+1},\quad x^\nu_{jn}=\cos\frac{j-\frac12}{n+\frac12}\pi,\quad x^\mu_{jn}=\cos\frac{j\pi}{n+\frac12},$$

for $j=1,\cdots,n$. We introduce the Lagrange interpolation operator $\mathcal{L}^\tau_n$ defined for every function $f:(-1,1)\to\mathbb{C}$ by

$$\mathcal{L}^\tau_nf=\sum_{j=1}^n f(x^\tau_{jn})\ell^\tau_{jn},\quad \ell^\tau_{jn}(x)=\frac{p^\tau_n(x)}{(x-x^\tau_{jn})(p^\tau_n)'(x^\tau_{jn})}=\prod_{k=1,k\ne j}^n\frac{x-x^\tau_{kn}}{x^\tau_{jn}-x^\tau_{kn}}\,.$$

We remark that the respective Christoffel numbers $\lambda^\tau_{jn}=\displaystyle\int_{-1}^1\ell^\tau_{jn}(x)\tau(x)\,dx$ are equal to

$$\lambda^\sigma_{jn}=\frac{\pi}{n},\quad \lambda^\varphi_{jn}=\frac{\pi\left[1-(x^\varphi_{jn})^2\right]}{n+1},\quad \lambda^\nu_{jn}=\frac{\pi(1+x^\nu_{jn})}{n+\frac12},\quad \lambda^\mu_{jn}=\frac{\pi(1-x^\mu_{jn})}{n+\frac12}\,.$$

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The collocation method seeks an approximation $u_n \in \mathbf{L}^2_\nu$ of the form

$$u_n(x) = \mu(x) p_n(x), \quad p_n \in \mathbb{P}_n, \tag{2.2}$$

to the exact solution of $\mathcal{A}u = f$ by solving

$$(\mathcal{A}u_n)(x^\tau_{kn}) = f(x^\tau_{kn}), \quad k = 1, 2, \ldots, n, \tag{2.3}$$

where $\mathbb{P}_n$ denotes the set of all algebraic polynomials of degree less than $n \in \mathbb{N}$. We set

$$\widetilde{p}_n(x) := \mu(x) P_n(x), \quad n = 0, 1, 2, \ldots$$

Using the Lagrange basis

$$\widetilde{\ell}^\tau_{kn}(x) = \frac{\mu(x)\ell^\tau_{kn}(x)}{\mu(x^\tau_{kn})}, \quad k = 1, \ldots, n,$$

in $\mu\mathbb{P}_n$, we can write u_n as

$$u_n = \sum_{j=0}^{n-1} \alpha_{jn} \widetilde{p}_j = \sum_{k=1}^{n} \xi_{kn} \widetilde{\ell}^\tau_{kn}.$$

If we introduce the Fourier projections

$$\mathcal{L}_n : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widetilde{p}_j \rangle_\nu \, \widetilde{p}_j$$

and the weighted interpolation operators $\mathcal{M}^\tau_n := \mu \mathcal{L}^\tau_n \mu^{-1} \mathcal{I}$, then the collocation system (2.3) can be written as an operator equation

$$\mathcal{A}^\tau_n := \mathcal{M}^\tau_n \mathcal{A} \mathcal{L}_n u_n = \mathcal{M}^\tau_n f, \quad u_n \in \operatorname{im} \mathcal{L}_n, \tag{2.4}$$

where im denotes the range of an operator. For the relation between the approximate solution and the exact solution, we have to investigate the stability of the collocation method. We call the collocation method stable if the approximation operators $\mathcal{A}^\tau_n$ are invertible for all sufficiently large $n \in \mathbb{N}$ and if the norms $\left\| (\mathcal{A}^\tau_n)^{-1} \mathcal{L}_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}$ are uniformly bounded. If the collocation method is stable, then the strong convergence of $\mathcal{A}^\tau_n \mathcal{L}_n$ to $\mathcal{A} \in \mathcal{L}(\mathbf{L}^2_\nu)$ as well as the convergence $\mathcal{M}^\tau_n f \longrightarrow f$ in $\mathbf{L}^2_\nu$ imply the convergence of the approximations u_n to the exact solution u in $\mathbf{L}^2_\nu$. This can be seen from the estimate

$$\begin{aligned} \|\mathcal{L}_n u - u_n\|_\nu &= \left\| \mathcal{A}^{-1}_n \mathcal{L}_n (\mathcal{A}_n \mathcal{L}_n u - \mathcal{A}_n u_n) \right\|_\nu \\ &\le \left\| \mathcal{A}^{-1}_n \mathcal{L}_n \right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \left(\|\mathcal{A}_n \mathcal{L}_n u - \mathcal{A}u\|_\nu + \|f - \mathcal{M}^\tau_n f\|_\nu \right), \end{aligned}$$

which also shows that, for getting convergence rates, one has to estimate the errors $\mathcal{L}_n u - u$ and $\mathcal{A}_n \mathcal{L}_n u - \mathcal{A}u$ with the solution u and the error $\mathcal{M}^\tau_n f - f$ with the right-hand side f. The technique, which we use to prove stability, includes the proof of strong convergence $\mathcal{A}^\tau_n \mathcal{L}_n \longrightarrow \mathcal{A}$; cf. the definition of the algebra $\mathfrak{F}$ in Section 3.1. For $\mathcal{M}^\tau_n f \longrightarrow f$, see Lemma 7.2. But the focus of the present paper is the stability of the methods under consideration. Proving convergence rates by using certain smoothness properties of the right-hand side f and of the solution u is a further task; cf., for example, [17, Section 5]. The main result of our paper on the stability of the collocation methods (2.4) applied to the integral equation (2.1) is given in Theorem 4.11.

Of course, by making the ansatz (2.2), we are only concerned with the second representation of the solution in (1.2). How to use the corresponding results for the other representation in (1.2) is shown in Section 5.

3. The stability of the collocation methods.

3.1. The Banach algebra framework for the stability of operator sequences. In what follows, the operator sequence, for which we want to prove stability, is considered as an element of a Banach algebra. For the definition of this algebra, we need some spaces as well as some useful operators. By ℓ^2 we denote the Hilbert space of all square summable sequences $\xi = (\xi_j)_{j=0}^\infty$, $\xi_j \in \mathbb{C}$, with the inner product

$$\langle \xi, \eta \rangle = \sum_{j=0}^{\infty} \xi_j \overline{\eta_j}.$$

Additionally, we define the following operators

$$\mathcal{W}_n : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu, \qquad u \mapsto \sum_{j=0}^{n-1} \langle u, \widetilde{p}_{n-1-j} \rangle_\nu \widetilde{p}_j,$$
$$\mathcal{P}_n : \ell^2 \longrightarrow \ell^2, \qquad (\xi_j)_{j=0}^\infty \mapsto (\xi_0, \cdots, \xi_{n-1}, 0, \dots),$$

and, for $\tau \in \{\sigma, \mu\}$,

$$\mathcal{V}_n^\tau : \operatorname{im} \mathcal{L}_n \longrightarrow \operatorname{im} \mathcal{P}_n, \quad u \mapsto \left(\omega_n^\tau \sqrt{1 + x_{1n}^\tau}\, u(x_{1n}^\tau), \dots, \omega_n^\tau \sqrt{1 + x_{nn}^\tau}\, u(x_{nn}^\tau), 0, \dots \right),$$
$$\widetilde{\mathcal{V}}_n^\tau : \operatorname{im} \mathcal{L}_n \longrightarrow \operatorname{im} \mathcal{P}_n, \quad u \mapsto \left(\omega_n^\tau \sqrt{1 + x_{nn}^\tau}\, u(x_{nn}^\tau), \dots, \omega_n^\tau \sqrt{1 + x_{1n}^\tau}\, u(x_{1n}^\tau), 0, \dots \right),$$

where $\omega_n^\sigma = \sqrt{\frac{\pi}{n}}$ and $\omega_n^\mu = \sqrt{\frac{\pi}{n+\frac{1}{2}}}$. Let $T = \{1, 2, 3, 4\}$ and set

$$\mathbf{X}^{(1)} = \mathbf{X}^{(2)} = \mathbf{L}^2_\nu, \quad \mathbf{X}^{(3)} = \mathbf{X}^{(4)} = \ell^2, \quad \mathcal{L}_n^{(1)} = \mathcal{L}_n^{(2)} = \mathcal{L}_n, \quad \mathcal{L}_n^{(3)} = \mathcal{L}_n^{(4)} = \mathcal{P}_n,$$

and define $\mathcal{E}_n^{(t)} : \operatorname{im} \mathcal{L}_n \longrightarrow \mathbf{X}_n^{(t)} := \operatorname{im} \mathcal{L}_n^{(t)}$ for $t \in T$ by

$$\mathcal{E}_n^{(1)} = \mathcal{L}_n, \qquad \mathcal{E}_n^{(2)} = \mathcal{W}_n, \qquad \mathcal{E}_n^{(3)} = \mathcal{V}_n^\tau, \qquad \mathcal{E}_n^{(4)} = \widetilde{\mathcal{V}}_n^\tau.$$

Here and at other places, we use the notion $\mathcal{L}_n, \mathcal{W}_n, \dots$ instead of $\mathcal{L}_n|_{\operatorname{im} \mathcal{L}_n}, \mathcal{W}_n|_{\operatorname{im} \mathcal{L}_n}, \dots,$ respectively. All operators $\mathcal{E}_n^{(t)},\ t \in T$, are invertible with inverses

$$\left(\mathcal{E}_n^{(1)}\right)^{-1} = \mathcal{E}_n^{(1)}, \quad \left(\mathcal{E}_n^{(2)}\right)^{-1} = \mathcal{E}_n^{(2)}, \quad \left(\mathcal{E}_n^{(3)}\right)^{-1} = (\mathcal{V}_n^\tau)^{-1}, \quad \left(\mathcal{E}_n^{(4)}\right)^{-1} = (\widetilde{V}_n^\tau)^{-1},$$

where, for $\xi \in \operatorname{im} \mathcal{P}_n$,

$$(\mathcal{V}_n^\tau)^{-1} \xi = (\omega_n^\tau)^{-1} \sum_{k=1}^{n} \frac{1}{\sqrt{1 + x_{kn}^\tau}} \xi_{k-1} \widetilde{\ell}_{kn}^\tau$$

and

$$(\widetilde{V}_n^\tau)^{-1} \xi = (\omega_n^\tau)^{-1} \sum_{k=1}^{n} \frac{1}{\sqrt{1 + x_{kn}^\tau}} \xi_{n-k} \widetilde{\ell}_{kn}^\tau.$$

Now we can introduce the algebra of operator sequences we are interested in. By $\mathfrak{F}$ we denote the set of all sequences $(\mathcal{A}_n)_{n=1}^\infty =: (\mathcal{A}_n)$ of linear operators $\mathcal{A}_n : \operatorname{im} \mathcal{L}_n \longrightarrow \operatorname{im} \mathcal{L}_n$ for which the strong limits

$$\mathcal{W}^t(\mathcal{A}_n) := \lim_{n\to\infty} \mathcal{E}_n^{(t)} \mathcal{A}_n \left(\mathcal{E}_n^{(t)}\right)^{-1} \mathcal{L}_n^{(t)}, \qquad t \in T,$$
$$\left(\mathcal{W}^t(\mathcal{A}_n)\right)^* = \lim_{n\to\infty} \left(\mathcal{E}_n^{(t)} \mathcal{A}_n \left(\mathcal{E}_n^{(t)}\right)^{-1} \mathcal{L}_n^{(t)} \right)^*, \qquad t \in T,$$

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exist. If we provide $\mathfrak{F}$ with the supremum norm $\|(\mathcal{A}_n)\|_{\mathfrak{F}} := \sup_{n\geq 1} \|\mathcal{A}_n\mathcal{L}_n\|_{\mathfrak{L}(\mathbf{L}^2_\nu)}$ and with operations $(\mathcal{A}_n) + (\mathcal{B}_n) := (\mathcal{A}_n + \mathcal{B}_n)$, $(\mathcal{A}_n)(\mathcal{B}_n) := (\mathcal{A}_n\mathcal{B}_n)$ and $(\mathcal{A}_n)^* := (\mathcal{A}_n^*)$, one can easily check that $\mathfrak{F}$ becomes a C^*-algebra with the identity element $(\mathcal{L}_n)$. Moreover, we introduce the set $\mathfrak{J} \subset \mathfrak{F}$ of all sequences of the form

$$\left(\sum_{t=1}^{4} \left(\mathcal{E}_n^{(t)}\right)^{-1} \mathcal{L}_n^{(t)} \mathcal{T}_t \mathcal{E}_n^{(t)} + \mathcal{C}_n \right),$$

where the linear operators $\mathcal{T}_t : \mathbf{X}^{(t)} \longrightarrow \mathbf{X}^{(t)}$ are compact and $\|\mathcal{C}_n\mathcal{L}_n\|_{\mathfrak{L}(\mathbf{L}^2_\nu)} \longrightarrow 0$ as $n \to \infty$.

PROPOSITION 3.1 (Lemma 2.1 in [10], Theorem 10.33 in [18, 19]). *The set $\mathfrak{J}$ forms a two-sided closed ideal in the C^*-algebra $\mathfrak{F}$. Moreover, a sequence $(\mathcal{A}_n) \in \mathfrak{F}$ is stable if and only if the operators $\mathcal{W}^t(\mathcal{A}_n) : \mathbf{X}^{(t)} \to \mathbf{X}^{(t)}$, $t \in T$, and the coset $(\mathcal{A}_n) + \mathfrak{J} \in \mathfrak{F}/\mathfrak{J}$ are invertible.*

3.2. The collocation sequence as an element of the Banach algebra $\mathfrak{F}$. For the investigation of the stability of the collocation method $(\mathcal{A}_n^\tau) = (\mathcal{M}_n^\tau\mathcal{A}\mathcal{L}_n)$, we have to show that this sequence belongs to the algebra $\mathfrak{F}$, which means to prove the existence of the four limit operators $\mathcal{W}^t(\mathcal{A}_n)$. Regarding the multiplication operator $a\mathcal{I}$ as well as the Cauchy singular integral operator $\mathcal{S}$, Proposition 3.2 below was proved in [10]. To describe the respective limit operators we need some further notation. Define the isometries

$$\begin{aligned}
&\mathcal{J}_1 : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu, \quad u \mapsto \sum_{j=0}^{\infty} \langle u, \widetilde{p}_j\rangle_\nu R_j, \\
&\mathcal{J}_2 : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu, \quad u \mapsto \sum_{j=0}^{\infty} \langle u, \widetilde{p}_j\rangle_\nu \sqrt{1-x}\, U_j, \\
&\mathcal{J}_3 : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu, \quad u \mapsto \sum_{j=0}^{\infty} \langle u, \widetilde{p}_j\rangle_\nu \frac{1}{\sqrt{1+x}}\, T_j,
\end{aligned} \tag{3.1}$$

and the shift operator

$$\mathcal{V} : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu, \qquad u \mapsto \sum_{j=0}^{\infty} \langle u, \widetilde{p}_j\rangle_\nu \widetilde{p}_{j+1}. \tag{3.2}$$

The adjoint operators $\mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^*, \mathcal{V}^* : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu$ are given by

$$\mathcal{J}_1^* u = \mathcal{J}_1^{-1} u = \sum_{j=0}^{\infty} \langle u, R_j\rangle_\nu \,\widetilde{p}_j, \qquad \mathcal{J}_2^* u = \mathcal{J}_2^{-1} u = \sum_{j=0}^{\infty} \left\langle u, \sqrt{1-x}\, U_j\right\rangle_\nu \widetilde{p}_j$$

and

$$\mathcal{J}_3^* u = \mathcal{J}_3^{-1} u = \sum_{j=0}^{\infty} \left\langle u, \frac{1}{\sqrt{1+x}}\, T_j \right\rangle_\nu \widetilde{p}_j, \quad \mathcal{V}^* u = \sum_{j=0}^{\infty} \langle u, \widetilde{p}_{j+1}\rangle_\nu \,\widetilde{p}_j.$$

Finally, we denote by $\mathbf{I} = [\delta_{jk}]_{j,k=0}^{\infty}$ the identity in ℓ^2 and by $\widetilde{\mathbf{S}}, \mathbf{S}^\tau : \ell^2 \longrightarrow \ell^2$ the operators defined by

$$\widetilde{\mathbf{S}} = \left[\frac{1-(-1)^{j-k}}{\pi \mathbf{i}(j-k)} + \frac{1-(-1)^{j+k+1}}{\pi \mathbf{i}(j+k+1)} \right]_{j,k=0}^{\infty}$$

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and

$$\mathbf{S}^\tau = \begin{cases} \left[\dfrac{1-(-1)^{j-k}}{\pi \mathbf{i}(j-k)} - \dfrac{1-(-1)^{j+k+1}}{\pi \mathbf{i}(j+k+1)} \right]_{j,k=0}^{\infty} & : \tau = \sigma, \\ \left[\dfrac{1-(-1)^{j-k}}{\pi \mathbf{i}} \left[\dfrac{1}{j-k} - \dfrac{1}{j+k+2} \right] \right]_{j,k=0}^{\infty} & : \tau = \mu. \end{cases}$$

The following proposition is already known.

PROPOSITION 3.2 (Proposition 3.5 in [10]). *Let* $a, b \in \mathbf{PC}$, $\mathcal{A} = a\mathcal{I} + b\mathcal{S}$, *and* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$. *Then, for* $\tau \in \{\sigma, \mu\}$, *we have* $(\mathcal{A}_n^\tau) \in \mathfrak{F}$ *and*

$$\mathcal{W}^1(\mathcal{A}_n^\tau) = \mathcal{A}, \qquad \mathcal{W}^2(\mathcal{A}_n^\tau) = \begin{cases} \mathcal{J}_1^{-1}(a\mathcal{J}_1 + \mathbf{i}b\mathcal{I}) & : \tau = \sigma, \\ \mathcal{J}_2^{-1}(a\mathcal{J}_2 - \mathbf{i}b\mathcal{J}_3\mathcal{V}) & : \tau = \mu, \end{cases}$$

$$\mathcal{W}^3(\mathcal{A}_n^\tau) = a(1)\mathbf{I} + b(1)\mathbf{S}^\tau, \qquad \mathcal{W}^4(\mathcal{A}_n^\tau) = a(-1)\mathbf{I} - b(-1)\widetilde{\mathbf{S}}.$$

REMARK 3.3. We have to mention that in [10, p. 745, line 13] there is a sign error. One has

$$-\left[\frac{1-(-1)^{j-k}}{2\mathbf{i}n \sin\frac{j-k}{2n}}\pi + \frac{1-(-1)^{j+k+1}}{2\mathbf{i}n \sin\frac{j+k+1}{2n}}\pi \right]_{j,k=0}^{n-1}$$

instead of

$$-\left[\frac{1-(-1)^{j-k}}{2\mathbf{i}n \sin\frac{j-k}{2n}}\pi - \frac{1-(-1)^{j+k+1}}{2\mathbf{i}n \sin\frac{j+k+1}{2n}}\pi \right]_{j,k=0}^{n-1}.$$

This leads to $\mathcal{W}^4(\mathcal{A}_n^\sigma) = a(-1)\mathbf{I} - b(-1)\widetilde{\mathbf{S}}$ and not to $\mathcal{W}^4(\mathcal{A}_n) = a(-1)\mathbf{I} - b(-1)\mathbf{S}^\sigma$ as formulated in [10, Proposition 3.5].

Having in mind Proposition 3.2, our next aim is to show that the sequences $\left(\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n\right)$, $k \in \mathbb{N}$, belong to $\mathfrak{F}$ and to determine their limit operators $\mathcal{W}^j\left(\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n\right)$. As a result, we can state the following proposition, the proof of which is given in Section 7.

PROPOSITION 3.4. *Let* $a, b \in \mathbf{PC}$, $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+$, *and* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$. *Then, for* $\tau \in \{\sigma, \mu\}$, *we have* $(\mathcal{A}_n^\tau) \in \mathfrak{F}$ *and*

$$\begin{aligned} \mathcal{W}^1(\mathcal{A}_n^\tau) &= \mathcal{A}, \\ \mathcal{W}^2(\mathcal{A}_n^\tau) &= \begin{cases} \mathcal{J}_1^{-1}(a\mathcal{J}_1 + \mathbf{i}b\mathcal{I}) & : \tau = \sigma, \\ \mathcal{J}_2^{-1}(a\mathcal{J}_2 - \mathbf{i}b\mathcal{J}_3\mathcal{V}) & : \tau = \mu, \end{cases} \\ \mathcal{W}^3(\mathcal{A}_n^\tau) &= a(1)\mathbf{I} + b(1)\mathbf{S}^\tau + \mathbf{A}^\tau + \mathbf{K}^\tau, \\ \mathcal{W}^4(\mathcal{A}_n^\tau) &= a(-1)\mathbf{I} - b(-1)\widetilde{\mathbf{S}} + \mathbf{A} + \mathbf{K}, \end{aligned}$$

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where the operators $\mathbf{A}, \mathbf{A}^\tau \in \mathcal{L}(\ell^2)$ *are defined as*

$$\mathbf{A} = \sum_{k_0=1}^{m_-} \beta_{k_0}^- \left[2\,\mathbf{h}_{k_0}^- \left(\frac{(j+\frac{1}{2})^2}{(k+\frac{1}{2})^2} \right) \frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2} \right]_{j,k=0}^{\infty}, \tag{3.3}$$

$$\mathbf{A}^\sigma = \sum_{k_0=1}^{m_+} \beta_{k_0}^+ \left[2\,\mathbf{h}_{k_0}^+ \left(\frac{(j+\frac{1}{2})^2}{(k+\frac{1}{2})^2} \right) \frac{1}{k+\frac{1}{2}} \right]_{j,k=0}^{\infty}, \tag{3.4}$$

$$\mathbf{A}^\mu = \sum_{k_0=1}^{m_+} \beta_{k_0}^+ \left[2\,\mathbf{h}_{k_0}^+ \left(\frac{(j+1)^2}{(k+1)^2} \right) \frac{1}{k+1} \right]_{j,k=0}^{\infty}, \tag{3.5}$$

with

$$\mathbf{h}_k^\pm(x) = \frac{(\mp 1)^k}{\pi \mathbf{i}} \frac{x^{k-1}}{(1+x)^k}, \quad x \in (0,\infty),\ k \in \mathbb{N}, \tag{3.6}$$

and where $\mathbf{K}, \mathbf{K}^\tau : \ell^2 \longrightarrow \ell^2$ *are compact operators.*

3.3. The invertibility of the limit operators. In this section we consider the invertibility of the four limit operators. Due to Proposition 3.1, this is necessary for the stability of the collocation method. Thus, our main concern is devoted to necessary and sufficient conditions for the invertibility of these limit operators. At first we consider the operator $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+$. For this, we need the Mellin transform

$$\widehat{y}(z) := \int_0^\infty y(x) x^{z-1}\, dx$$

of a function $y : (0,\infty) \to \mathbb{C}$. With the help of the continuous functions $\mathbf{h}_k^\pm : (0,\infty) \longrightarrow \mathbb{C}$ defined in (3.6), we can write the linear combination of the integral operators $\mathcal{B}_k^\pm$ in (2.1) in the form

$$\begin{aligned} &\sum_{k=1}^{m_-} \beta_k^- (\mathcal{B}_k^- u)(x) + \sum_{k=1}^{m_+} \beta_k^+ (\mathcal{B}_k^+ u)(x) \\ &\quad = \sum_{k=1}^{m_-} \beta_k^- \int_{-1}^1 \mathbf{h}_k^- \left(\frac{1+x}{1+y} \right) \frac{u(y)}{1+y}\, dy + \sum_{k=1}^{m_+} \beta_k^+ \int_{-1}^1 \mathbf{h}_k^+ \left(\frac{1-x}{1-y} \right) \frac{u(y)}{1-y}\, dy. \end{aligned} \tag{3.7}$$

For $\mathbf{h}_k^\pm(x)$, $k \in \mathbb{N}$, the Mellin transform is given by $\widehat{\mathbf{h}}_k^\pm(z) = (\mp 1)^k \widehat{\mathbf{h}}_k(z+k-1)$ with $\mathbf{h}_k(x) = (1+x)^k$, and (see, for example, [4, 6.2.(6)])

$$\widehat{\mathbf{h}}_k(z) = (-1)^{k-1} \binom{z-1}{k-1} \frac{\pi}{\sin(\pi z)}$$

is holomorphic in the strip $0 < \operatorname{Re} z < k$. This implies

$$\widehat{\mathbf{h}}_k^\pm(\beta - \mathbf{i}t) = \binom{\beta - \mathbf{i}t + k - 2}{k-1} \frac{(\mp 1)^k}{\sinh(\pi(\mathbf{i}\beta + t))}, \quad 1-k < \beta < 1,\ t \in \mathbb{R}.$$

We remark that $\widehat{\mathbf{h}}_k^\pm(\beta - \mathbf{i}t)$ is analytic in the strip $0 < \beta < 1$ for all $k \in \mathbb{N}$. Due to (3.7) and by [2, Theorem 9.1] (cf. also [3, 7, 8, 16]), we can state the following proposition.

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PROPOSITION 3.5. *Let* $a, b \in \mathbf{PC}$, $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$.

(a) *The operator* $\mathcal{A}$ *is Fredholm if and only if:*

- *For any* $x \in (-1,1)$, *there holds* $a(x \pm 0) + b(x \pm 0) \neq 0$ *and* $a(x \pm 0) - b(x \pm 0) \neq 0$ *as well as* $a(\pm 1) + b(\pm 1) \neq 0$ *and* $a(\pm 1) - b(\pm 1) \neq 0$.
- *If* a *or* b *has a jump at* $x \in (-1,1)$, *then there holds*

$$\lambda \frac{a(x+0)+b(x+0)}{a(x+0)-b(x+0)} + (1-\lambda)\frac{a(x-0)+b(x-0)}{a(x-0)-b(x-0)} \neq 0, \quad 0 \le \lambda \le 1.$$

- *For* $x = \pm 1$, *there holds*

$$a(1) + b(1)\mathbf{i}\cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+\left(\tfrac{1}{4} - \mathbf{i}\xi\right) \neq 0, \quad -\infty < \xi < \infty,$$

and

$$a(-1) + b(-1)\mathbf{i}\cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^-\left(\tfrac{3}{4} - \mathbf{i}\xi\right) \neq 0, \quad -\infty < \xi < \infty.$$

(b) *If* $\mathcal{A}$ *is Fredholm and if the coefficients* a *and* b *have finitely many jumps, then the Fredholm index of* $\mathcal{A} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *is equal to minus the winding number of the closed continuous curve* $\Gamma_{\mathcal{A}} := \Gamma_- \cup \Gamma_1 \cup \Gamma_1' \cup \ldots \cup \Gamma_N \cup \Gamma_N' \cup \Gamma_{N+1} \cup \Gamma_+$ *with the orientation given by the subsequent parametrization. Here,* N *stands for the number of discontinuity points* $x_i,\ i = 1, \ldots, N$, *of the functions* a *and* b *chosen such that* $x_0 := -1 < x_1 < \cdots < x_N < x_{N+1} := 1$. *Using these* x_i, *the curves* $\Gamma_i,\ i = 1, \ldots, N+1$, *and* $\Gamma_i',\ i = 1, \ldots, N$, *are given by*

$$\Gamma_i := \left\{ \frac{a(y)+b(y)}{a(y)-b(y)} : x_{i-1} < y < x_i \right\},$$

$$\Gamma_i' := \left\{ \lambda \frac{a(x_i+0)+b(x_i+0)}{a(x_i+0)-b(x_i+0)} + (1-\lambda)\frac{a(x_i-0)+b(x_i-0)}{a(x_i-0)-b(x_i-0)} : 0 \le \lambda \le 1 \right\}.$$

The curves $\Gamma_\pm$, *connecting the point* 1 *with one of the end points of* Γ_1 *and* Γ_{N+1}, *are given by the formulas*

$$\Gamma_+ := \left\{ \frac{a(1) + b(1)\mathbf{i}\cot\left(\frac{\pi}{4} - \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+\left(\frac{1}{4} - \mathbf{i}\xi\right)}{a(1) - b(1)} : -\infty \le \xi \le \infty \right\}$$

and

$$\Gamma_- := \left\{ \frac{a(-1) + b(-1)\mathbf{i}\cot\left(\frac{\pi}{4} + \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^-\left(\frac{3}{4} - \mathbf{i}\xi\right)}{a(-1) - b(-1)} : -\infty \le \xi \le \infty \right\}.$$

(c) *If* $\mathcal{A}$ *is Fredholm and if* $m_- = 0$ *or* $m_+ = 0$, *then* $\mathcal{A}$ *is one-sided invertible.*

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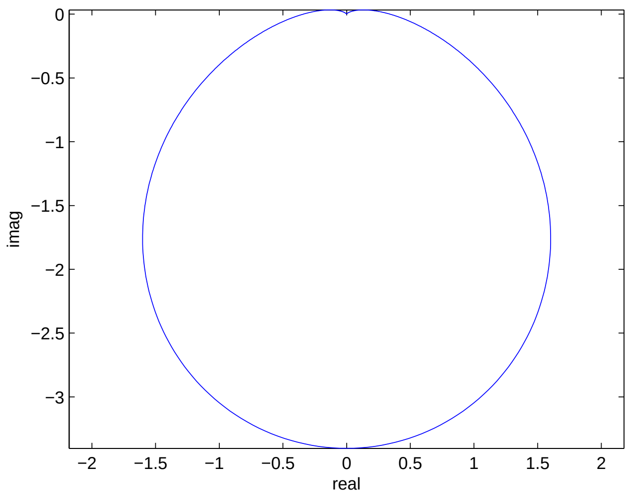

FIG. 3.1. $\left\{\sum_{k=1}^{3} \widehat{\mathbf{h}}_k^- \left(\frac{3}{4} - \mathbf{i}\xi\right) : -\infty \le \xi \le \infty\right\}$.

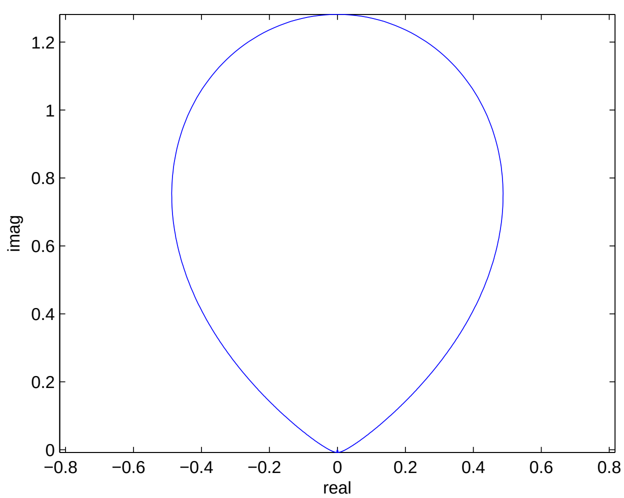

FIG. 3.2. $\left\{\sum_{k=1}^{3} \widehat{\mathbf{h}}_k^+ \left(\frac{1}{4} - \mathbf{i}\xi\right) : -\infty \le \xi \le \infty\right\}$.

Let $z_1, z_2 \in \mathbb{C}$. We denote by $\gamma_{\ell/r}[z_1, z_2]$ the half circle line from z_1 to z_2 lying on the left, respectively, on the right of the segment $[z_1, z_2]$ and by $\gamma[z_1, z_2]$ the circle line with diameter $[z_1, z_2]$ starting in z_1 with clockwise orientation. For given functions $a, b \in \mathbf{PC}$ with $a(x \pm 0) - b(x \pm 0) \neq 0,\ x \in [-1, 1]$, we define

$$c(x) := \frac{a(x) + b(x)}{a(x) - b(x)}. \tag{3.8}$$

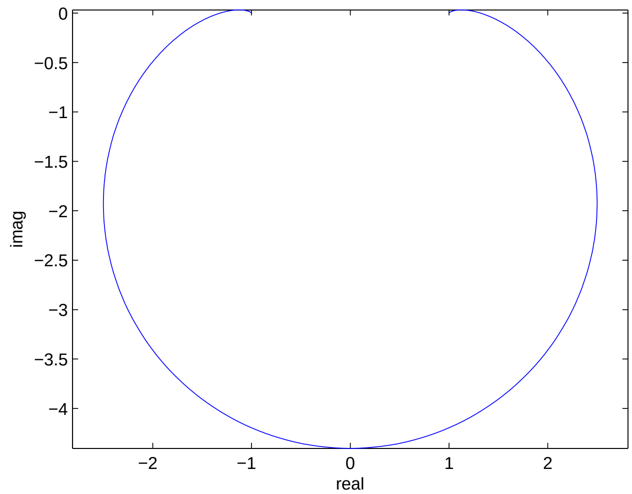

FIG. 3.3. Γ_- *in case of* $a(-1) = 0$, $m_- = 3$, $\beta_k^- = 1$.

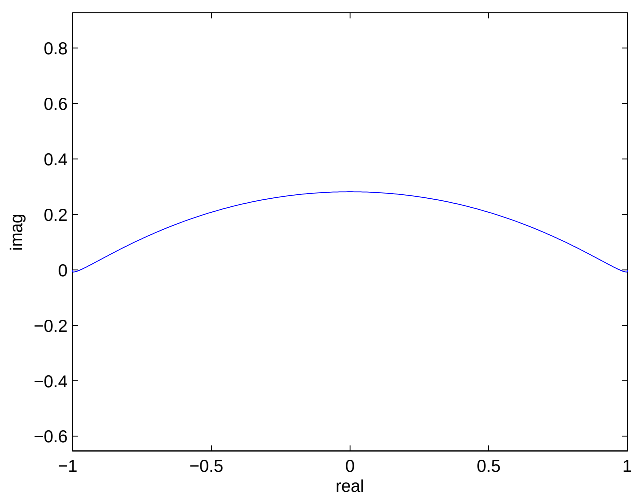

FIG. 3.4. Γ_+ *in case of* $a(1) = 0$, $m_+ = 3$, $\beta_k^+ = 1$.

The equalities

$$\left\{ \frac{a(1) + b(1)\mathbf{i}\cot\left(\frac{\pi}{4} - \mathbf{i}\pi\xi\right)}{a(1) - b(1)} \,:\, -\infty \le \xi \le \infty \right\} = \gamma_r[c(1), 1]. \tag{3.9}$$

and

$$\left\{ \frac{a(-1) + b(-1)\mathbf{i}\cot\left(\frac{\pi}{4} + \mathbf{i}\pi\xi\right)}{a(-1) - b(-1)} \,:\, -\infty \le \xi \le \infty \right\} = \gamma_\ell[1, c(-1)] \tag{3.10}$$

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can easily be shown. The curve Γ_+ is a modified arc from $c(1)$ to 1 and the curve Γ_- is a modified arc from 1 to $c(-1)$. For instance, Figures 3.1 and 3.2 display the images of $\sum_{k=1}^{3} \widehat{\mathbf{h}}_k^{\pm}\left(\frac{3}{4} - \mathbf{i}\xi\right)$ (i.e., $m_\pm = 3$, $\beta_k^\pm = 1$) and Figures 3.3 and 3.4 the respective curves $\Gamma_\pm$ in the case $a(\pm 1) = 0$.

The above proposition enables us to give conditions for the invertibility of the second limit operator $\mathcal{W}^2$. So we derive from [10, Lemma 4.4 and Corollary 4.5].

LEMMA 3.6. *Let* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau(a\mathcal{I} + b\mathcal{S})\mathcal{L}_n$, $\tau \in \{\sigma, \mu\}$.

(a) *The operator* $\mathcal{W}^2(\mathcal{A}_n^\sigma)$ *is invertible in* $\mathbf{L}_\nu^2$ *if and only if* $\mathcal{A} = a\mathcal{I} + b\mathcal{S}$ *has this property.*

(b) *If* $a\mathcal{I} + b\mathcal{S} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *is invertible, then the invertibility of* $\mathcal{W}^2(\mathcal{A}_n^\mu) : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *is equivalent to the condition* $|a(1)| > |b(1)|$, *which is again equivalent to the condition* $\operatorname{Re} c(1) > 0$.

For the index calculation of the second limit operator, we can state the following lemma.

LEMMA 3.7. *Let* $a, b \in \mathbf{PC}$, $\tau \in \{\sigma, \mu\}$, *and* $\mathcal{A} := a\mathcal{I} + b\mathcal{S} : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$, *as well as* $\mathcal{A}_n^\tau := \mathcal{M}_n^\tau \mathcal{A}\mathcal{L}_n$. *If* $\mathcal{A}$ *is Fredholm, then the second limit operator* $\mathcal{W}^2(\mathcal{A}_n^\sigma) : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$ *is also Fredholm, where*

$$\operatorname{ind} \mathcal{W}^2(\mathcal{A}_n^\sigma) = -\operatorname{ind} \mathcal{A}. \tag{3.11}$$

If $\mathcal{A}, \mathcal{W}^2(\mathcal{A}_n^\mu) : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *are Fredholm, then*

$$\operatorname{ind} \mathcal{W}^2(\mathcal{A}_n^\mu) = \begin{cases} -\operatorname{ind} \mathcal{A} & : \operatorname{Re}\, c(1) > 0\,, \\ -\operatorname{ind} \mathcal{A} - 1 & : \operatorname{Re}\, c(1) < 0\,. \end{cases} \tag{3.12}$$

Proof. Let $\operatorname{ind} \mathcal{A} = \kappa$. For $\lambda \in [0, 1]$, define

$$\mathbf{c}(x, \lambda) = \begin{cases} c(x-0)(1-\lambda) + c(x+0)\lambda & : x \in (-1, 1), \\ c(1) + [1 - c(1)]\mathbf{f}_{-\frac{1}{2}}(\lambda) & : x = +1, \\ 1 + [c(-1) - 1]\mathbf{f}_{\frac{1}{2}}(\lambda) & : x = -1, \end{cases} \tag{3.13}$$

where $\mathbf{f}_\alpha(\lambda) = \dfrac{\sin \pi\alpha\lambda}{\sin \pi\alpha}\, e^{-\mathbf{i}\pi\alpha(\lambda-1)}$ and $c(x)$ is defined in (3.8). Note that, for $z_1, z_2 \in \mathbb{C}$, the image of the function $z_1 + (z_2 - z_1)\mathbf{f}_\alpha(\lambda)$, $\lambda \in [0, 1]$, describes the circular arc from z_1 to z_2 such that the straight line segment $[z_1, z_2]$ is seen from the points of the arc under the angle $\pi(1 + \alpha)$, i.e., in case $\alpha \in (-1, 0)$, the arc lies on the right of the segment $[z_1, z_2]$ and, in case $\alpha \in (0, 1)$, on the left. By (3.9), (3.10), and Proposition 3.5, it follows that $\Gamma_{\mathcal{A}} = \{\mathbf{c}(x, \lambda) : (x, \lambda) \in [-1, 1] \times [0, 1]\}$. Moreover, we denote the winding number of this curve with respect to the origin of the complex plane by $\operatorname{wind} \mathbf{c}(x, \lambda)$. Due to the fact that every piecewise continuous function can be approximated by a function with finitely many jumps, we can assume that $-1 < x_1 < \cdots < x_N < 1$ are the only discontinuities of $c(x)$. Define the piecewise continuous argument function $\alpha(x) = \frac{1}{2\pi} \arg c(x)$ in such a way that

$$|\alpha(x_k + 0) - \alpha(x_k - 0)| < \frac{1}{2}\,,\ k = 1, \ldots, N, \qquad \text{and} \qquad \alpha(-1) \in \left(-\frac{3}{4}, \frac{1}{4}\right). \tag{3.14}$$

For the winding number, we derive

$$\operatorname{wind} \mathbf{c}(x, \lambda) \in \mathbb{Z} \cap \left(\alpha(1) - \frac{1}{4}, \alpha(1) + \frac{3}{4}\right). \tag{3.15}$$

Due to Proposition 3.5, we have $\kappa = -\operatorname{wind} \mathbf{c}(x, \lambda)$.

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In case $\tau=\sigma$, set $d(x)=\dfrac{b(x)-a(x)}{b(x)+a(x)}$ and define $\mathbf{d}(x,\lambda)$ analogously to (3.13). Then (cf. the proof of [10, Lemma 4.4]), $\mathbf{d}(x,\lambda)\neq 0,\ \forall\,(x,\lambda)\in[-1,1]\times[0,1]$, if and only if $\mathbf{c}(x,\lambda)\neq 0,\ \forall\,(x,\lambda)\in[-1,1]\times[0,1]$. Define the piecewise continuous argument function $\beta(x)=\frac{1}{2\pi}\arg d(x)$ satisfying the respective conditions (3.14). Since (cf. again the proof of [10, Lemma 4.4])

$$\operatorname{ind}\mathcal{W}^2(\mathcal{A}_n^\sigma)=\operatorname{ind}(b\mathcal{I}-a\mathcal{S})\quad\text{and}\quad\beta(x)=-\frac{1}{2}-\alpha(x),$$

we have

$$-\operatorname{ind}\mathcal{W}^2(\mathcal{A}_n^\sigma)=\operatorname{wind}\mathbf{d}(x,\lambda)\in\mathbb{Z}\cap\left(-\frac{3}{4}-\alpha(1),\frac{1}{4}-\alpha(1)\right),$$

proving, together with (3.15), the relation (3.11).

Let us turn to the case $\tau=\mu$, and assume that $\mathcal{W}^2(A_n^\mu):\mathbf{L}^2_\nu\longrightarrow\mathbf{L}^2_\nu$ is Fredholm. From the proof of [10, Lemma 4.5], we have

$$-\operatorname{ind}\mathcal{W}^2(\mathcal{A}_n^\mu)\in\mathbb{Z}\cap\left(-\alpha(1)-\frac{1}{4},-\alpha(1)+\frac{3}{4}\right).$$

In view of (3.15), we get $\kappa\in\left(-\alpha(1)-\frac{1}{4},-\alpha(1)+\frac{3}{4}\right)$ if and only if

$$\alpha(1)-\frac{1}{4}<\operatorname{wind}\mathbf{c}(x,\mu)<\alpha(1)+\frac{1}{4},$$

which is equivalent to $\operatorname{Re}c(1)>0$. Analogously, $\kappa+1\in\left(-\alpha(1)-\frac{1}{4},-\alpha(1)+\frac{3}{4}\right)$ if and only if $\alpha(1)+\frac{1}{4}<\operatorname{wind}\mathbf{c}(x,\mu)<\alpha(1)+\frac{3}{4}$, i.e., $\operatorname{Re}c(1)<0$.

Observe that the Fredholmness of the operator $\mathcal{A}:=a\mathcal{I}+b\mathcal{S}:\mathbf{L}^2_\nu\to\mathbf{L}^2_\nu$ implies that the half circle line $\gamma_r[c(1),1]$ does not contain 0, which implies $c(1)\notin\{iy\,:\,y\geq 0\}$. Moreover, by the Fredholmness of (cf. [10, (4.4)])

$$\mathcal{W}^2(\mathcal{A}_n^\mu)=\mathcal{J}_2^{-1}\frac{1}{\sqrt{2}}\left(\left[a(\sqrt{1+x}+\mathbf{i}b\sqrt{1-x}\right]-\left[\mathbf{i}a\sqrt{1-x}+b\sqrt{1+x}\right]\mathcal{S}\right),$$

we get $0\notin\gamma_r\left[\frac{1}{c(1)},1\right]$, i.e., $c(1)\notin\{\mathbf{i}y\,:\,y\leq 0\}$. Hence, (3.12) is proved. □

We also need conditions for the Fredholmness of the operators $\mathcal{W}^{3/4}(\mathcal{A}_n^\tau)$. For that, we consider the C^*-algebra $\mathcal{L}(\ell^2)$ of all linear and continuous operators in ℓ^2. By $\operatorname{alg}\mathcal{T}(\mathbf{PC})$ we denote the smallest C^*-subalgebra of $\mathcal{L}(\ell^2)$ generated by the Toeplitz matrices $\left[\ \widehat{g}_{j-k}\ \right]_{j,k=0}^\infty$ with piecewise continuous generating functions $g(t):=\sum_{\ell\in\mathbb{Z}}\widehat{g}_\ell t^\ell$ defined on the unit circle $\mathbb{T}:=\{t\in\mathbb{C}\,:\,|t|=1\}$ and being continuous on $\mathbb{T}\setminus\{\pm1\}$.

PROPOSITION 3.8 (Theorem 16.2 in [15]). *There exists a (continuous) map* smb *from* $\operatorname{alg}\mathcal{T}(\mathbf{PC})$ *into a set of complex valued functions defined on* $\mathbb{T}\times[0,1]$, *which sends each* $\mathbf{R}\in\operatorname{alg}\mathcal{T}(\mathbf{PC})$ *to the function* $\operatorname{smb}_{\mathbf{R}}(t,\lambda)$, *which is called symbol of* $\mathbf{R}$ *and which satisfies the following properties:*

(a) *For each fixed* $(t,\lambda)\in\mathbb{T}\times[0,1]$, *the map* $\operatorname{alg}\mathcal{T}(\mathbf{PC})\longrightarrow\mathbb{C}$, $\mathbf{R}\mapsto\operatorname{smb}_{\mathbf{R}}(t,\lambda)$ *is a multiplicative linear functional on* $\operatorname{alg}\mathcal{T}(\mathbf{PC})$.

(b) *For any* $t\neq\pm1$, *the value* $\operatorname{smb}_{\mathbf{R}}(t,\lambda)$ *is independent of* λ, *and the function* $t\mapsto\operatorname{smb}_{\mathbf{R}}(t,0)$ *is continuous on* $\{t\in\mathbb{T}\,:\,\operatorname{Im}t>0\}$ *and on* $\{t\in\mathbb{T}\,:\,\operatorname{Im}t<0\}$

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with the limits

$$\operatorname{smb}_{\mathbf{R}}(1+0,0) := \lim_{t\to+1, \operatorname{Im} t>0} \operatorname{smb}_{\mathbf{R}}(t,0) = \operatorname{smb}_{\mathbf{R}}(1,1),$$
$$\operatorname{smb}_{\mathbf{R}}(1-0,0) := \lim_{t\to+1, \operatorname{Im} t<0} \operatorname{smb}_{\mathbf{R}}(t,0) = \operatorname{smb}_{\mathbf{R}}(1,0),$$
$$\operatorname{smb}_{\mathbf{R}}(-1+0,0) := \lim_{t\to-1, \operatorname{Im} t<0} \operatorname{smb}_{\mathbf{R}}(t,0) = \operatorname{smb}_{\mathbf{R}}(-1,1),$$
$$\operatorname{smb}_{\mathbf{R}}(-1-0,0) := \lim_{t\to-1, \operatorname{Im} t>0} \operatorname{smb}_{\mathbf{R}}(t,0) = \operatorname{smb}_{\mathbf{R}}(-1,0).$$

(c) *An operator* $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ *is Fredholm if and only if* $\operatorname{smb}_{\mathbf{R}}(t,\lambda) \neq 0$ *for all* $(t,\lambda) \in \mathbb{T}\times[0,1]$.

(d) *For any Fredholm operator* $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$*, the index of* $\mathbf{R}$ *is the negative winding number of the closed curve*

$$\begin{aligned}\Gamma_{\mathbf{R}} : = &\{\operatorname{smb}_{\mathbf{R}}(e^{\mathbf{i}s},0) : 0<s<\pi\} \cup \{\operatorname{smb}_{\mathbf{R}}(-1,s) : 0\le s\le 1\} \\ &\cup \{\operatorname{smb}_{\mathbf{R}}(-e^{\mathbf{i}s},0) : 0<s<\pi\} \cup \{\operatorname{smb}_{\mathbf{R}}(1,s) : 0\le s\le 1\},\end{aligned} \tag{3.16}$$

where the orientation of $\Gamma_{\mathbf{R}}$ *is given in a natural way by the parametrization of* $\mathbb{T}$ *and* $[0,1]$.

(e) *An operator* $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ *is compact if and only if the symbol* $\operatorname{smb}_{\mathbf{R}}(t,\lambda)$ *vanishes for all* $(t,\lambda) \in \mathbb{T}\times[0,1]$.

In what follows, we show that the limit operators $\mathcal{W}^{3/4}(\mathcal{A}_n^\tau)$ belong to $\operatorname{alg}\mathcal{T}(\mathbf{PC})$ and consider their symbols as well as the respective curves (3.16). Using the results of [10, Section 4] and the relations

$$\mathbf{i}\cot\left(\frac{\pi}{4} \pm \frac{\mathbf{i}}{4}\log\frac{\lambda}{1-\lambda}\right) = \pm(2\lambda-1) + 2\mathbf{i}\sqrt{\lambda(1-\lambda)}\,, \quad 0\le\lambda\le 1\,,$$

as well as

$$\left\{\mathbf{i}\cot\left(\frac{\pi}{4} - \mathbf{i}\xi\right) : -\infty \le \xi \le \infty\right\} = \gamma_r[1,-1]$$

and

$$\left\{\mathbf{i}\cot\left(\frac{\pi}{4} + \mathbf{i}\xi\right) : -\infty \le \xi \le \infty\right\} = \gamma_\ell[-1,1]$$

(cf. also (3.9), (3.10)), we get the following lemma.

LEMMA 3.9. *Let* $\tau \in \{\sigma,\mu\}$ *and* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau(a\mathcal{I}+b\mathcal{S})\mathcal{L}_n$*. The limit operators* $\mathcal{W}^t(\mathcal{A}_n^\tau)$*,* $t\in\{3,4\}$*, belong to the algebra* $\operatorname{alg}\mathcal{T}(\mathbf{PC})$*, and their symbols are given by*

$$\operatorname{smb}_{\mathcal{W}^3(\mathcal{A}_n^\tau)}(t,\lambda) = a(1)+b(1)\cdot\begin{cases} 1 & : & \operatorname{Im} t>0,\\ -1 & : & \operatorname{Im} t<0,\\ \mathbf{i}\cot\left(\frac{\pi}{4}+\frac{\mathbf{i}}{4}\log\frac{\lambda}{1-\lambda}\right) & : \tau\in\{\sigma,\mu\}, & t=1,\\ \mathbf{i}\cot\left(\frac{\pi}{4}-\frac{\mathbf{i}}{4}\log\frac{\lambda}{1-\lambda}\right) & : \tau=\sigma, & t=-1,\\ -\mathbf{i}\cot\left(\frac{\pi}{4}+\frac{\mathbf{i}}{4}\log\frac{\lambda}{1-\lambda}\right) & : \tau=\mu, & t=-1,\end{cases}$$

and

$$\operatorname{smb}_{\mathcal{W}^4(\mathcal{A}_n^\tau)}(t,\lambda) = a(-1)-b(-1)\cdot\begin{cases} 1 & : \operatorname{Im} t>0,\\ -1 & : \operatorname{Im} t<0,\\ -\mathbf{i}\cot\left(\frac{\pi}{4}-\frac{\mathbf{i}}{4}\log\frac{\lambda}{1-\lambda}\right) & : t=1,\\ -\mathbf{i}\cot\left(\frac{\pi}{4}+\frac{\mathbf{i}}{4}\log\frac{\lambda}{1-\lambda}\right) & : t=-1.\end{cases}$$

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The respective closed curves (3.16) *are*

$$\begin{aligned}
\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\sigma)} &= \gamma_r[a(1)+b(1),a(1)-b(1)] \cup \gamma_\ell[a(1)-b(1),a(1)+b(1)],\\
\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\mu)} &= \gamma[a(1)+b(1),a(1)-b(1)],\\
\Gamma_{\mathcal{W}^4(\mathcal{A}_n^\sigma)} &= \Gamma_{\mathcal{W}^4(\mathcal{A}_n^\mu)}\\
&= \gamma_\ell[a(-1)-b(-1),a(-1)+b(-1)] \cup \gamma_r[a(-1)+b(-1),a(-1)-b(-1)].
\end{aligned}$$

We remark that the limit operators $\mathcal{W}^t(\mathcal{M}_n^\tau(a\mathcal{I}+b\mathcal{S})\mathcal{L}_n)$, $t=3,4$, are invertible if they are Fredholm with index 0 [10, Corollary 4.9].

LEMMA 3.10 (Lemma 4.2 and Lemma 4.6 in [10]). *Let* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau(a\mathcal{I}+b\mathcal{S})\mathcal{L}_n$, *where* $\tau \in \{\sigma,\mu\}$.

(a) *If* $a\mathcal{I}+b\mathcal{S} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *is Fredholm, then* $\mathcal{W}^3(\mathcal{A}_n^\sigma)$ *and* $\mathcal{W}^4(\mathcal{A}_n^\tau)$ *are invertible.*
(b) *The operator* $\mathcal{W}^3(\mathcal{A}_n^\mu)$ *is invertible if and only if* $|a(1)| > |b(1)|$.

We turn to the limit operators of $\mathcal{B}_k^\pm$ and verify that $\mathbf{A}, \mathbf{A}^\sigma, \mathbf{A}^\mu \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$. For this we recall the following lemma.

LEMMA 3.11 (Lemma 7.1 in [12] and Lemma 4.5 in [13]). *Suppose that the Mellin transform* $\widehat{y}(z)$ *of the function* $y:(0,\infty)\longrightarrow\mathbb{C}$ *is analytic in the strip*

$$\frac{1}{2}-\varepsilon < \operatorname{Re} z < \frac{1}{2}+\varepsilon$$

for some $\varepsilon>0$ *and that*

$$\sup_{\frac{1}{2}-\varepsilon<\operatorname{Re} z<\frac{1}{2}+\varepsilon} \left| \frac{d^k}{dz^k}\widehat{y}(z)(1+|z|)^k \right| < \infty, \quad k=0,1,\ldots$$

Then, $y:(0,\infty)\longrightarrow\mathbb{C}$ *is infinitely differentiable, the operators* $\mathbf{M}_{\pm1}, \widetilde{\mathbf{M}}_{\pm1} \in \mathcal{L}(\ell^2)$ *defined by*

$$\mathbf{M}_{+1} := \left[y\left(\frac{j+\frac12}{k+\frac12}\right)\frac{1}{k+\frac12}\right]_{j,k=0}^\infty, \qquad \widetilde{\mathbf{M}}_{+1} := \left[y\left(\frac{j+1}{k+1}\right)\frac{1}{k+1}\right]_{j,k=0}^\infty,$$

and

$$\mathbf{M}_{-1} := \left[(-1)^{j-k} y\left(\frac{j+\frac12}{k+\frac12}\right)\frac{1}{k+\frac12}\right]_{j,k=0}^\infty, \quad \widetilde{\mathbf{M}}_{-1} := \left[(-1)^{j-k} y\left(\frac{j+1}{k+1}\right)\frac{1}{k+1}\right]_{j,k=0}^\infty$$

belong to the algebra $\operatorname{alg}\mathcal{T}(\mathbf{PC})$, *and their symbols are given by*

$$\operatorname{smb}_{\mathbf{M}_{+1}}(t,\lambda) = \operatorname{smb}_{\widetilde{\mathbf{M}}_{+1}}(t,\lambda) = \begin{cases} \widehat{y}\left(\frac12 + \frac{\mathbf{i}}{2\pi}\log\frac{\lambda}{1-\lambda}\right) & : t=1,\\ 0 & : t\in\mathbb{T}\setminus\{1\},\end{cases}$$

and

$$\operatorname{smb}_{\mathbf{M}_{-1}}(t,\lambda) = \operatorname{smb}_{\widetilde{\mathbf{M}}_{-1}}(t,\lambda) = \begin{cases} \widehat{y}\left(\frac12 + \frac{\mathbf{i}}{2\pi}\log\frac{\lambda}{1-\lambda}\right) & : t=-1,\\ 0 & : t\in\mathbb{T}\setminus\{-1\}.\end{cases}$$

For $k\in\mathbb{N}$, set $\mathbf{g}_k^-(x) := 2\mathbf{h}_k^-(x^2)x$ and $\mathbf{g}_k^+(x) := 2\mathbf{h}_k^+(x^2)$ such that

$$\widehat{\mathbf{g}}_k^-(z) = \widehat{\mathbf{h}}_k^-\left(\tfrac{z+1}{2}\right) \quad\text{and}\quad \widehat{\mathbf{g}}_k^+(z) = \widehat{\mathbf{h}}_k^+\left(\tfrac{z}{2}\right).$$

Since the Mellin transforms $\widehat{\mathbf{h}}_k^{\pm}(z)$ are analytic in the strip $0 < \operatorname{Re} z < 1$, it follows that $\widehat{\mathbf{g}}_k^{-}(z)$ is analytic in the strip $-1 < \operatorname{Re} z < 1$ and $\widehat{\mathbf{g}}_k^{+}(z)$ is analytic in the strip $0 < \operatorname{Re} z < 2$. Hence, we can apply Lemma 3.11 and obtain that $\mathbf{A}, \mathbf{A}^\sigma, \mathbf{A}^\mu \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ with symbols

$$\operatorname{smb}_{\mathbf{A}}(t,\lambda) = \begin{cases} \displaystyle\sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^- \left(\frac{3}{4} + \frac{\mathbf{i}}{4\pi} \log \frac{\lambda}{1-\lambda}\right) & : t = 1, \\ 0 & : t \in \mathbb{T}\backslash\{1\}, \end{cases}$$

$$\operatorname{smb}_{\mathbf{A}^\tau}(t,\lambda) = \begin{cases} \displaystyle\sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+ \left(\frac{1}{4} + \frac{\mathbf{i}}{4\pi} \log \frac{\lambda}{1-\lambda}\right) & : t = 1, \\ 0 & : t \in \mathbb{T}\backslash\{1\}. \end{cases}$$

From Proposition 3.4, Lemma 3.9, Lemma 3.11, and $\left\{\frac{1}{4\pi} \log \frac{\lambda}{1-\lambda} : \lambda \in (0,1)\right\} = \mathbb{R}$, we conclude the following assertion.

LEMMA 3.12. *Let $\tau \in \{\sigma, \mu\}$ and*

$$\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ \right) \mathcal{L}_n.$$

Then, the limit operators $\mathcal{W}^3(\mathcal{A}_n^\tau)$ and $\mathcal{W}^4(\mathcal{A}_n^\tau)$ belong to the algebra $\operatorname{alg}\mathcal{T}(\mathbf{PC})$ with

$$\begin{aligned}\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\sigma)} &= \left\{ a(1) + b(1)\mathbf{i}\cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \\ &\quad \cup \left\{ a(1) + b(1)\mathbf{i}\cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+ \left(\tfrac{1}{4} + \mathbf{i}\xi\right) : -\infty \le \xi \le \infty \right\},\end{aligned}$$

$$\begin{aligned}\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\mu)} &= \left\{ a(1) - b(1)\mathbf{i}\cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \\ &\quad \cup \left\{ a(1) + b(1)\mathbf{i}\cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+ \left(\tfrac{1}{4} + \mathbf{i}\xi\right) : -\infty \le \xi \le \infty \right\},\end{aligned}$$

and

$$\begin{aligned}\Gamma_{\mathcal{W}^4(\mathcal{A}_n^\tau)} &= \left\{ a(-1) + b(-1)\mathbf{i}\cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \\ &\quad \cup \left\{ a(-1) + b(-1)\mathbf{i}\cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^- \left(\tfrac{3}{4} + \mathbf{i}\xi\right) : -\infty \le \xi \le \infty \right\}.\end{aligned}$$

COROLLARY 3.13. *Let $\tau \in \{\sigma, \mu\}$ and*

$$\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ \right) \mathcal{L}_n.$$

For $t \in \{3,4\}$, the limit operator $\mathcal{W}^t(\mathcal{A}_n^\tau)$ is invertible if and only if the closed curve $\Gamma_{\mathcal{W}^t(\mathcal{A}_n^\tau)}$ does not contain the zero point, its winding number vanishes, and the null space of the operator $\mathcal{W}^t(\mathcal{A}_n^\tau) \in \mathcal{L}(\ell^2)$ is trivial.

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In the following three propositions, we give our final results concerning the invertibility of the limit operators.

PROPOSITION 3.14. *Let* $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *be invertible,* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$, *and let* $\mathcal{W}^4(\mathcal{A}_n^\tau)$ *be Fredholm with index zero. Then,*
(a) *in case* $\tau = \sigma$, $\mathcal{W}^2(\mathcal{A}_n^\sigma)$ *and* $\mathcal{W}^3(\mathcal{A}_n^\sigma)$ *are invertible,*
(b) *in case* $\tau = \mu$, $\mathcal{W}^2(\mathcal{A}_n^\mu)$ *and* $\mathcal{W}^3(\mathcal{A}_n^\mu)$ *are invertible if and only if* $|a(1)| > |b(1)|$.

Proof. Write $\Gamma_\mathcal{A} = \Gamma_- \cup \Gamma_c \cup \Gamma_+$, with $\Gamma_c = \Gamma_1 \cup \Gamma_1' \cup \ldots \cup \Gamma_{N+1}$; cf. Proposition 3.5(b). In the present situation we have $\Gamma_+ = \gamma_r[c(1), 1]$. Then (see Lemma 3.12),

$$\Gamma_{\mathcal{W}^4(\mathcal{A}_n^\tau)} = [a(-1) - b(-1)] \left(\gamma_\ell[1, c(-1)] \cup \widetilde{\Gamma}_- \right)$$

and

$$\Gamma_{a\mathcal{I}+b\mathcal{S}} = \gamma_\ell[1, c(-1)] \cup \Gamma_c \cup \gamma_r[c(1), 1],$$

where $\widetilde{\Gamma}_-$ is Γ_- with reverse orientation. In view of Proposition 3.5 and Corollary 3.13, the invertibility of $\mathcal{W}^1(\mathcal{A}_n^\tau)$ and the vanishing index of $\mathcal{W}^4(\mathcal{A}_n^\tau)$ imply the invertibility of $a\mathcal{I} + b\mathcal{S} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$. Since the second and third limit operators are independent of $\mathcal{B}_k^-$ (see Proposition 3.4), it remains to apply Lemma 3.6 and Lemma 3.10. □

The following two propositions can be proved analogously.

PROPOSITION 3.15. *Let* $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *be invertible,* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$, *and let* $\mathcal{W}^3(\mathcal{A}_n^\tau)$ *be Fredholm with index zero. Then,*
(a) *in case* $\tau = \sigma$, $\mathcal{W}^2(\mathcal{A}_n^\sigma)$ *and* $\mathcal{W}^4(\mathcal{A}_n^\tau)$ *are invertible,*
(b) *in case* $\tau = \mu$, $\mathcal{W}^2(\mathcal{A}_n^\mu)$ *is invertible if and only if* $|a(1)| > |b(1)|$.

PROPOSITION 3.16. *Let* $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ *be invertible,* $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$, *and let* $\mathcal{W}^3(\mathcal{A}_n^\tau)$ *as well as* $\mathcal{W}^4(\mathcal{A}_n^\tau)$ *be Fredholm with index zero. Then,*
(a) *in case* $\tau = \sigma$, $\mathcal{W}^2(\mathcal{A}_n^\sigma)$ *is invertible,*
(b) *in case* $\tau = \mu$, $\mathcal{W}^2(\mathcal{A}_n^\mu)$ *is invertible if and only if* $|a(1)| > |b(1)|$.

EXAMPLE 3.17. Consider $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{3} \mathcal{B}_k^- : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$ and $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$, $\tau \in \{\sigma, \mu\}$, where $a(x) = \mathbf{i}\sqrt{1-x} + 1$ and $b(x) = -\sqrt{1+x} - 1$. In Figure 3.5, the curve $\Gamma_\mathcal{A} = \Gamma_- \cup \Gamma_c \cup \gamma_r[c(1), 1]$ (blue, dashed, and red lines) is given.

The winding number of $\Gamma_\mathcal{A}$ vanishes. Thus, the operator $\mathcal{A}$ is invertible (see Proposition 3.5(c)). If we replace the bloated arc Γ_- (blue) by the circular arc $\gamma_\ell[1, c(-1)]$ (green), we get the curve concerning the operator $a\mathcal{I} + b\mathcal{S} : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$. Consequently, $a\mathcal{I} + b\mathcal{S}$ is Fredholm with index -1 and, in particular, not invertible. As a consequence of Lemma 3.7, we derive

$$\operatorname{ind} \mathcal{W}^2(\mathcal{A}_n^\sigma) = 1 \quad \text{and} \quad \operatorname{ind} \mathcal{W}^2(\mathcal{A}_n^\mu) = 0.$$

$\mathcal{W}^3(\mathcal{A}_n^\tau)$ does not depend on the Mellin part of the operator and, due to Lemma 3.12,

$$\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\sigma)} = [a(1) - b(1)] \left(\gamma_r[c(1), 1] \cup \gamma_\ell[1, c(1)] \right)$$

and

$$\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\mu)} = [a(1) - b(1)] \left(\gamma_\ell[c(1), 1] \cup \gamma_\ell[1, c(1)] \right).$$

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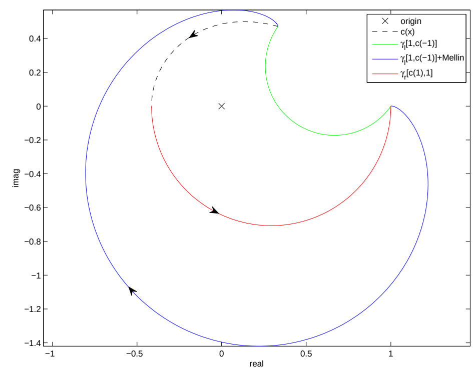

FIG. 3.5. $\Gamma_{\mathcal{A}}$ *for Example 3.17.*

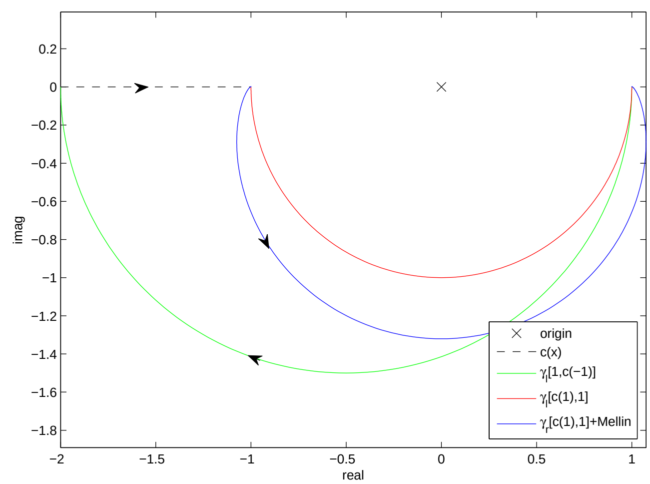

FIG. 3.6. $\Gamma_{\mathcal{A}}$ *for Example 3.18.*

Thus, by Proposition 3.8 and Lemma 3.9, $\operatorname{ind}\mathcal{W}^3(\mathcal{A}_n^\sigma) = 0$ and $\operatorname{ind}\mathcal{W}^3(\mathcal{A}_n^\mu) = 1$. The winding number of the curve $\Gamma_{\mathcal{W}^4(\mathcal{A}_n^\tau)} = [a(-1) - b(-1)]\left(\gamma_\ell[1, c(-1)] \cup \widetilde{\Gamma}_-\right)$, where $\widetilde{\Gamma}_-$ equals Γ_- but with reverse orientation, is equal to 1. Thus, in view of Proposition 3.8 and Lemma 3.12, the limit operators $\mathcal{W}^4(\mathcal{A}_n^\tau)$, $\tau \in \{\sigma, \mu\}$, are Fredholm with index -1.

EXAMPLE 3.18. Consider the functions $a(x) = 1 - x$ and $b(x) = 5 - x$. Let

$$\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{3} \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$$

and $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$, $\tau \in \{\sigma, \mu\}$. The image Γ_c of the function $c(x)$, $x \in [-1, 1]$, is the straight segment from -2 to -1. The curve $\Gamma_{\mathcal{A}} = \gamma_\ell[1, c(-1)] \cup \Gamma_c \cup \Gamma_+$ (green, dashed, and blue lines) and the curve $\Gamma_{a\mathcal{I}+b\mathcal{S}} = \gamma_\ell[1, c(-1)] \cup \Gamma_c \cup \gamma_r[c(1), 1]$ (green, dashed, and red lines) are given in Figure 3.6.

In view of Proposition 3.5, the operators $\mathcal{A} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ and $a\mathcal{I} + b\mathcal{S} : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$ are invertible. By Lemma 3.7, $\mathcal{W}^2(\mathcal{A}_n^\sigma)$ is invertible and $\mathcal{W}^2(\mathcal{A}_n^\mu)$ is Fredholm with index -1.

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For the fourth limit operators, we have (see Lemma 3.12)

$$\Gamma_{\mathcal{W}^4(\mathcal{A}_n^\tau)} = [a(-1) - b(-1)]\,(\gamma_\ell[1, c(-1)] \cup \gamma_r[c(-1), 1])$$

implying their invertibility. Furthermore, $\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\sigma)} = [a(1) - b(1)] \left(\gamma_r[c(1), 1] \cup \widetilde{\Gamma}_+\right)$ and $\Gamma_{\mathcal{W}^3(\mathcal{A}_n^\mu)} = [a(1) - b(1)] \left(\gamma_\ell[c(1), 1] \cup \widetilde{\Gamma}_+\right)$, where $\tilde{\Gamma}_+$ equals Γ_+ but with reverse orientation. Thus, in view of Proposition 3.8 and Lemma 3.12, the limit operator $\mathcal{W}^3(\mathcal{A}_n^\sigma)$ is Fredholm with index 0, and the limit operator $\mathcal{W}^3(\mathcal{A}_n^\mu)$ is Fredholm with index 1.

4. The main theorem for the stability of the collocation methods. In this section we investigate the invertibility of the coset $(\mathcal{A}_n^\tau) + \mathfrak{J}$ in the algebra $\mathfrak{F}/\mathfrak{J}$, where $\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ \right) \mathcal{L}_n$ is one of the considered collocation methods. For this, we need some other operator sequences. Let $\mathbf{R} \in \operatorname{alg} \mathcal{T}(\mathbf{PC})$. We define the finite sections $\mathbf{R}_n := \mathcal{P}_n \mathbf{R} \mathcal{P}_n \in \mathcal{L}(\operatorname{im} \mathcal{P}_n)$ and set $\mathbf{R}_n^t := (\mathcal{E}_n^{(t)})^{-1} \mathbf{R}_n \mathcal{E}_n^{(t)}$, $t \in \{3, 4\}$.

LEMMA 4.1 (Lemma 5.4 in [10]). *For $\mathbf{R} \in \operatorname{alg} \mathcal{T}(\mathbf{PC})$ and $t \in \{3, 4\}$, the sequences $(\mathbf{R}_n^t)$ belong to the algebra $\mathfrak{F}$.*

Let $m_\pm \in \mathbb{N}$ be fixed. Now we denote by $\mathfrak{A}$ the smallest $\mathcal{C}^*$-subalgebra of $\mathfrak{F}$ generated by all sequences of the ideal $\mathfrak{J}$, all sequences $(\mathbf{R}_n^t)$ with $t \in \{3, 4\}$ and $\mathbf{R} \in \operatorname{alg} \mathcal{T}(\mathbf{PC})$ as well as by all sequences $(\mathcal{A}_n)$ with $\mathcal{A}_n = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ \right) \mathcal{L}_n$, $a, b \in \mathbf{PC}$. Moreover, let $\mathfrak{A}_0$ be the smallest C^*-subalgebra of $\mathfrak{F}$ containing all sequences from $\mathfrak{J}$ and all sequences $(\mathcal{A}_n)$ with $\mathcal{A}_n = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ \right) \mathcal{L}_n$, $a, b \in \mathbf{PC}$. For the coset $(\mathcal{A}_n) + \mathfrak{J}$, we use the abbreviation $(\mathcal{A}_n)^o$. As a main tool for proving invertibility in the quotient algebra $\mathfrak{A}/\mathfrak{J}$, we use the local principle of Allan and Douglas. For this, we have to find a C^*-subalgebra of the center of $\mathfrak{A}/\mathfrak{J}$ as well as its maximal ideal space.

Let $0 < \varepsilon < \frac{1}{2}$ and define $\mathbf{C}_\varepsilon^-$ ($\mathbf{C}_\varepsilon^+$) as the Banach space of all continuous functions $f : (-1, 1] \longrightarrow \mathbb{C}$ ($f : [-1, 1) \longrightarrow \mathbb{C}$) satisfying

$$\lim_{x \to -1+0} (1 + x)^\varepsilon f(x) = 0 \qquad \left(\lim_{x \to 1-0} (1 - x)^\varepsilon f(x) = 0 \right)$$

with the norm

$$\|f\|_{\infty,\varepsilon,\pm} := \sup \left\{ (1 \mp x)^\varepsilon |f(x)| : -1 < x < 1 \right\}.$$

Remark that $\mathbf{C}_\varepsilon^\pm$ is continuously embedded into $\mathbf{L}_\nu^2$.

LEMMA 4.2. *For a polynomial p, the operators $\mathcal{B}_k^\pm p\mathcal{I} - p\mathcal{B}_k^\pm : \mathbf{L}_\nu^2 \longrightarrow \mathbf{C}_\varepsilon^\pm$ are compact.*

Proof. For example, the operator $\mathcal{B}_k^- p\mathcal{I} - p\mathcal{B}_k^-$ is an integral operator with kernel function

$$h_{-,k}(x, y) = \frac{[p(y) - p(x)](1 + x)^{k-1}}{(2 + y + x)^k}.$$

Since $(1+x)^\varepsilon h_{-,k}(x, y)$ is continuous on $[-1, 1] \times [-1, 1]$, we obtain the assertion by Arzela-Ascoli's theorem. □

LEMMA 4.3. *For $f \in \mathbf{C}[-1, 1]$, the cosets $(\mathcal{M}_n^\tau f \mathcal{L}_n)^o$ belong to the center of $\mathfrak{A}/\mathfrak{J}$.*

Proof. We have to show that

$$(\mathcal{M}_n^\tau f \mathcal{L}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{M}_n^\tau f \mathcal{L}_n) \in \mathfrak{J} \tag{4.1}$$

for all generating sequences $(\mathcal{A}_n)$ of $\mathfrak{A}$. In the cases $\mathcal{A}_n = \mathcal{M}_n^\tau(a\mathcal{I} + b\mathcal{S})\mathcal{L}_n$, $a, b \in \mathbf{PC}$, and $\mathcal{A}_n = \mathcal{R}_n^{(t)}$, $t = 3, 4$, this was proved in [10, Lemma 5.7]; cf. also [12, 13]. It remains to consider $\mathcal{A}_n = \mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n$. In view of the estimate (see, [10, (3.11)])

$$\|\mathcal{M}_n^\tau f_1 \mathcal{L}_n - \mathcal{M}_n^\tau f_2 \mathcal{L}_n\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \le \mathrm{const}\, \|f_1 - f_2\|_\infty ,$$

and the closedness of $\mathfrak{J}$, it is sufficient to verify (4.1) for polynomials f. Thus, let p be a polynomial with $\deg p \le m$. By $\mathcal{M}_n^\tau p \mathcal{L}_{n-m} = p\mathcal{L}_{n-m}$, $n > m$, and $\mathcal{L}_n - \mathcal{L}_{n-m} = \mathcal{W}_n \mathcal{L}_m \mathcal{W}_n$, we derive

$$\begin{aligned}
&\mathcal{M}_n^\tau p \mathcal{L}_n \mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n - \mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n \mathcal{M}_n^\tau p \mathcal{L}_n \\
&\quad = -\mathcal{M}_n^\tau (\mathcal{B}_k^\pm p - p\mathcal{B}_k^\pm)\mathcal{L}_n + \mathcal{M}_n^\tau \mathcal{B}_k^\pm (\mathcal{I} - \mathcal{M}_n^\tau) p (\mathcal{L}_n - \mathcal{L}_{n-m}) \\
&\quad = -\mathcal{M}_n^\tau (\mathcal{B}_k^\pm p - p\mathcal{B}_k^\pm)\mathcal{L}_n \\
&\qquad + \left[\mathcal{M}_n^\tau (\mathcal{B}_k^\pm p - p\mathcal{B}_k^\pm)\mathcal{L}_n + \mathcal{M}_n^\tau p \mathcal{B}_k^\pm \mathcal{L}_n - \mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n \mathcal{M}_n^\tau p \mathcal{L}_n\right] \mathcal{W}_n \mathcal{L}_m \mathcal{W}_n .
\end{aligned}$$

The application of the ideal property together with Lemma 4.2 completes the proof. □

Lemma 4.3 shows that the set $\mathfrak{C} := \{(\mathcal{M}_n^\tau f \mathcal{L}_n)^o : f \in \mathbf{C}[-1,1]\}$ forms a $\mathcal{C}^*$-subalgebra of the center of $\mathfrak{A}/\mathfrak{J}$. This subalgebra is via the mapping $(\mathcal{M}_n^\tau f \mathcal{L}_n)^o \to f$ *-isomorphic to $\mathbf{C}[-1,1]$. Consequently, the maximal ideal space of $\mathfrak{C}$ is equal to $\{\mathfrak{T}_\omega : \omega \in [-1,1]\}$ with

$$\mathfrak{T}_\omega := \{(\mathcal{M}_n^\tau f \mathcal{L}_n)^o \,:\, f \in \mathbf{C}[-1,1], f(\omega) = 0\} .$$

By $\mathfrak{J}_\omega$ we denote the smallest closed ideal of $\mathfrak{A}/\mathfrak{J}$ which contains $\mathfrak{T}_\omega$, i.e., $\mathfrak{J}_\omega$ is equal to

$$\mathrm{clos}_{\mathfrak{A}/\mathfrak{J}} \left\{ \sum_{j=1}^m \left(\mathcal{A}_n^j \mathcal{M}_n^\tau f_j \mathcal{L}_n\right)^o \,:\, \left(\mathcal{A}_n^j\right) \in \mathfrak{A},\ f_j \in \mathbf{C}[-1,1],\ f_j(\omega) = 0,\ m = 1, 2, \dots \right\} .$$

The local principle of Allan and Douglas claims the following.

PROPOSITION 4.4 (cf. Sections 1.4.4, 1.4.6 in [9]). *For all $\omega \in [-1,1]$, the ideal $\mathfrak{J}_\omega$ is a proper ideal in $\mathfrak{A}/\mathfrak{J}$. An element $(\mathcal{A}_n)^o$ of $\mathfrak{A}/\mathfrak{J}$ is invertible if and only if $(\mathcal{A}_n)^o + \mathfrak{J}_\omega$ is invertible in $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_\omega$ for all $\omega \in [-1,1]$.*

LEMMA 4.5. *The cosets $(\mathcal{M}_n^\tau \mathcal{B}_k^- \mathcal{L}_n)^o$, $1 \le k \le m_-$, are contained in $\mathfrak{J}_\omega$, $-1 < \omega \le 1$, and the cosets $(\mathcal{M}_n^\tau \mathcal{B}_k^+ \mathcal{L}_n)^o$, $1 \le k \le m_+$, are contained in $\mathfrak{J}_\omega$, $-1 \le \omega < 1$.*

Proof. Consider the case $\mathcal{B} = \mathcal{B}_k^-$. (The case $\mathcal{B} = \mathcal{B}_k^+$ has to be handled in the same way.) Let $-1 < \omega \le 1$ and let χ be a smooth function which vanishes in some neighborhood of -1 and satisfies $\chi(\omega) = 1$. Since $\chi\mathcal{B} : \mathbf{L}^2_\nu \to \mathbf{C}[-1,1]$ is compact, the operator norm $\|(\mathcal{L}_n - \mathcal{M}_n^\tau)\chi\mathcal{B}\mathcal{L}_n\|_{\mathcal{L}(\mathbf{L}^2_\nu)}$ tends to zero. Due to the definition of the ideal $\mathfrak{J}$, we get $(\mathcal{M}_n^\tau \chi \mathcal{B} \mathcal{L}_n) \in \mathfrak{J}$. Thus,

$$(\mathcal{M}_n^\tau \mathcal{B} \mathcal{L}_n)^o = (\mathcal{L}_n \mathcal{M}_n^\tau \mathcal{B} \mathcal{L}_n - \mathcal{M}_n^\tau \chi \mathcal{B} \mathcal{L}_n)^o = \left(\mathcal{M}_n^\tau (1-\chi) \mathcal{L}_n \mathcal{M}_n^\tau \mathcal{B} \mathcal{L}_n\right)^o \in \mathfrak{J}_\omega .$$

The lemma is proved. □

As a consequence of Lemma 4.5, for $-1 < \omega < 1$, the invertibility of the coset

$$\left(\mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+\right) \mathcal{L}_n\right)^o + \mathfrak{J}_\omega$$

is equivalent to the invertibility of $(\mathcal{M}_n^\tau (a\mathcal{I} + b\mathcal{S})\mathcal{L}_n)^o + \mathfrak{J}_\omega$. In the same manner as in [10, Corollary 5.13], we can state the following.

LEMMA 4.6. *Let $(\mathcal{A}_n) \in \mathfrak{A}_0$. If the limit operator $\mathcal{W}^1(\mathcal{A}_n) : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is Fredholm, then for all $\omega \in (-1,1)$, the coset $(\mathcal{A}_n)^o + \mathfrak{J}_\omega$ is invertible in $(\mathfrak{A}_0/\mathfrak{J})/\mathfrak{J}_\omega$.*

Now, we consider the invertibility of $(\mathcal{A}_n^\tau)^o + \mathfrak{J}_{\pm 1}$ in $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{\pm 1}$. To this end, we show that the invertibility of the limit operators $\mathcal{W}^3(\mathcal{A}_n^\tau)$ and $\mathcal{W}^4(\mathcal{A}_n^\tau)$ implies the invertibility of $(\mathcal{A}_n^\tau)^o + \mathfrak{J}_{+1}$ and $(\mathcal{A}_n^\tau)^o + \mathfrak{J}_{-1}$, respectively.

LEMMA 4.7 (Lemma 5.9 in [10]). *Let $a \in \mathbf{PC}[-1,1]$ be continuous at the point $\omega \in [-1,1]$ with $a(\omega) = 0$. Then, $(\mathcal{M}_n^\tau a \mathcal{L}_n)^o \in \mathfrak{J}_\omega$.*

By $\mathbf{C}_{\pm 1}$ we refer to the set all of continuous functions $f \in \mathbf{C}[-1,1]$ with $f(\pm 1) = 1$ and $0 \le f(x) \le 1$ for all $x \in [-1,1]$. For an arbitrary $(\mathcal{A}_n)^o + \mathfrak{J}_{\pm 1} \in (\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{\pm 1}$, we have, due to Lemma 4.7,

$$\|(\mathcal{A}_n)^o + \mathfrak{J}_{\pm 1}\|_{(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{\pm 1}} \le \inf_{f \in \mathbf{C}_{\pm 1}} \|(\mathcal{M}_n^\tau f \mathcal{L}_n)^o (\mathcal{A}_n)^o\|_{\mathfrak{A}/\mathfrak{J}} . \tag{4.2}$$

The proof of the following lemma is given in Section 8.

LEMMA 4.8. *Let $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ and let*

$$\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k_0=1}^{m_-} \beta_{k_0}^- \mathcal{B}_{k_0}^- + \sum_{k_0=1}^{m_+} \beta_{k_0}^+ \mathcal{B}_{k_0}^+ \right) \mathcal{L}_n,$$

$\mathbf{S} := \mathcal{W}^3(\mathcal{A}_n^\tau)$, and $\mathbf{T} := \mathcal{W}^4(\mathcal{A}_n^\tau)$.

(a) *If $\mathbf{R}$ is invertible, then the coset $\left([\mathbf{R}^{-1}]_n^{3/4}\right)^o + \mathfrak{J}_{\pm 1}$ is the inverse of $\left(\mathbf{R}_n^{3/4}\right)^o + \mathfrak{J}_{\pm 1}$ in $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{\pm 1}$.*

(b) *We have $\left(\mathbf{S}_n^3\right)^o + \mathfrak{J}_1 = (\mathcal{A}_n^\tau)^o + \mathfrak{J}_1$ and $\left(\mathbf{T}_n^4\right)^o + \mathfrak{J}_{-1} = (\mathcal{A}_n^\tau)^o + \mathfrak{J}_{-1}$.*

For the generating sequences of $\mathfrak{A}_0$, we know that the limit operators with $t \in \{3,4\}$ belong to $\operatorname{alg}\mathcal{T}(\mathbf{PC})$; cf. Lemma 3.12. Since the mappings $\mathcal{W}^{3/4} : \mathfrak{F} \longrightarrow \mathfrak{L}(\ell^2)$ are continuous *-homomorphisms (see [10, Corollary 2.4]), we have $W^{3/4}(\mathcal{A}_n) \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ if $(\mathcal{A}_n) \in \mathfrak{A}_0$. Thus, by Lemma 4.8 and the closedness of $\mathfrak{J}_{\pm 1}$, we get the following corollary.

COROLLARY 4.9. *Let $(\mathcal{A}_n) \in \mathfrak{A}_0$. Then, the invertibility of $\mathcal{W}^3(\mathcal{A}_n)$ and $\mathcal{W}^4(\mathcal{A}_n)$ implies, respectively, the invertibility of $(\mathcal{A}_n)^o + \mathfrak{J}_{+1}$ and $(\mathcal{A}_n)^o + \mathfrak{J}_{-1}$ in $(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{\pm 1}$.*

Now, we are able to prove the stability theorem for sequences of the algebra $\mathfrak{A}_0$, in particular, for the collocation method $(\mathcal{A}_n^\tau) = \left(\mathcal{M}_n^\tau (a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+) \mathcal{L}_n \right)$. Indeed, with the help of Proposition 3.1, Lemma 4.6, Corollary 4.9, and the local principle of Allan and Douglas, we can state the following theorem.

THEOREM 4.10. *A sequence $(\mathcal{A}_n) \in \mathfrak{A}_0$ is stable if and only if all operators $\mathcal{W}^t(\mathcal{A}_n) : \mathbf{X}^{(t)} \longrightarrow \mathbf{X}^{(t)}$, $t = 1,2,3,4$, are invertible.*

Having in mind Proposition 3.4, we set

$$\mathbf{A}_- := a(-1)\mathbf{I} - b(-1)\tilde{\mathbf{S}} + \mathbf{A} + \mathbf{K} \quad \text{and} \quad \mathbf{A}_+^\tau := a(1)\mathbf{I} + b(1)\mathbf{S}^\tau + \mathbf{A}^\tau + \mathbf{K}^\tau .$$

Moreover, we define the curves

$$\begin{aligned}
\Gamma_- &= \left\{ a(-1) + b(-1)\mathbf{i}\cot\left(\pi[\tfrac14 + \mathbf{i}\xi]\right) : -\infty < \xi < \infty \right\} \\
&\qquad \cup \left\{ a(-1) + b(-1)\mathbf{i}\cot\left(\pi[\tfrac14 - \mathbf{i}\xi]\right) + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^-(\tfrac34 + \mathbf{i}\xi) : -\infty \le \xi \le \infty \right\}, \\
\Gamma_+^\sigma &= \left\{ a(1) + b(1)\mathbf{i}\cot\left(\pi[\tfrac14 - \mathbf{i}\xi]\right) : -\infty < \xi < \infty \right\} \\
&\qquad \cup \left\{ a(1) + b(1)\mathbf{i}\cot\left(\pi[\tfrac14 + \mathbf{i}\xi]\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+(\tfrac14 + \mathbf{i}\xi) : -\infty \le \xi \le \infty \right\},
\end{aligned}$$

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$$\Gamma_+^\mu = \left\{a(1) - b(1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4}+\mathbf{i}\xi]\right) : -\infty < \xi < \infty\right\}$$
$$\cup \left\{a(1) + b(1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4}+\mathbf{i}\xi]\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+(\tfrac{1}{4}+\mathbf{i}\xi) : -\infty \le \xi \le \infty\right\}.$$

With the help of Theorem 4.10, Corollary 3.13, Proposition 3.16, and Proposition 3.4, we derive the following.

THEOREM 4.11. *Let* $a, b \in \mathbf{PC}$ *and* $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$. *Then, the collocation method* $\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n$, $\tau \in \{\sigma, \mu\}$, *is stable if and only if*

(a) *the operator* $\mathcal{A} \in \mathcal{L}(\mathbf{L}_\nu^2)$ *is invertible (cf. Proposition 3.5),*
(b) *the closed curves* Γ_- *and* Γ_+^τ *do not contain the zero point and their winding numbers vanish,*
(c) *the null spaces of the operators* $\mathbf{A}_-, \mathbf{A}_+^\tau \in \mathcal{L}(\ell^2)$ *are trivial,*
(d) *in case* $\tau = \mu$, *the relation* $|a(1)| > |b(1)|$ *is fulfilled.*

5. Approximate solutions of the form $\nu(x)p_n(x)$**.** In this section we consider the integral equation (2.1) in the space $\mathbf{L}_\mu^2$, which stands for the weighted $\mathbf{L}^2$-space referring to the fourth Chebyshev weight $\mu(x)$. Again we apply a collocation method to the integral equation. However, this time the collocation method seeks an approximation $u_n \in \mathbf{L}_\mu^2$ of the form

$$u_n(x) = \nu(x)p_n(x)$$

with a polynomial $p_n(x)$ of degree less than n. Setting

$$\widehat{p}_n(x) := \nu(x)R_n(x),\ n = 0, 1, 2, \dots \quad \text{and} \quad \widehat{\ell}_{kn}^\tau = \frac{\nu(x)\ell_{kn}^\tau(x)}{\nu(x_{kn}^\tau)},\ k = 1, \dots, n,$$

we can write u_n in the form

$$u_n = \sum_{j=0}^{n-1} \alpha_{jn}\widehat{p}_j = \sum_{k=1}^{n} \xi_{kn}\widehat{\ell}_{kn}^\tau.$$

Introducing the Fourier projections

$$\widehat{\mathcal{L}}_n : \mathbf{L}_\mu^2 \longrightarrow \mathbf{L}_\mu^2, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widehat{p}_j \rangle_\mu \widehat{p}_j$$

and the weighted interpolation operators $\widehat{\mathcal{M}}_n^\tau := \nu\mathcal{L}_n^\tau\nu^{-1}\mathcal{I}$ instead of the collocation method (2.3), we consider the collocation method

$$\widehat{\mathcal{A}}_n^\tau := \widehat{\mathcal{M}}_n^\tau \widehat{\mathcal{A}} \widehat{\mathcal{L}}_n u_n = \widehat{\mathcal{M}}_n^\tau \widehat{f}, \quad u_n \in \operatorname{im} \widehat{\mathcal{L}}_n \tag{5.1}$$

for solving the operator equation

$$\widehat{\mathcal{A}}u = \widehat{f} \quad \text{in} \quad \mathbf{L}_\nu^2$$

approximately, where

$$\widehat{\mathcal{A}} := a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\mu^2 \longrightarrow \mathbf{L}_\mu^2.$$

Here and in what follows, for $\mathcal{I}, \mathcal{S}, \mathcal{B}_k^\pm : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ and for $\mathcal{I}, \mathcal{S}, \mathcal{B}_k^\pm : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\mu$, we use the same notation, which does not lead to any confusion. For the investigation of the stability of the collocation method (5.1), we introduce the isometric isomorphism $\mathcal{J} : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\nu$, $u(x) \mapsto u(-x)$, and we get

$$\mathcal{J}\mathcal{S}\mathcal{J}^{-1} = -\mathcal{S}, \quad \mathcal{J}\mathcal{B}_k^\pm\mathcal{J}^{-1} = (-1)^k\mathcal{B}_k^\mp, \quad \text{and} \quad \mathcal{J}\hat{\mathcal{L}}_n\mathcal{J}^{-1} = \mathcal{L}_n,$$

where, for the last equality, we took into account the relation $\widehat{p}_n(-x) = (-1)^n\widetilde{p}_n(x)$. Since $x^\sigma_{n-j+1,n} = -x^\sigma_{jn}$ and $x^\nu_{n-j+1,n} = -x^\mu_{jn}$, $j = 1, \ldots, n$, it follows that

$$\ell^\sigma_{n-j+1,n}(-x) = \ell^\sigma_{j,n}(x) \quad \text{and} \quad \ell^\nu_{n-j+1,n}(-x) = \ell^\mu_{j,n}(x),$$

which implies, for every function $f : (-1, 1) \to \mathbb{C}$, the relations $\mathcal{J}\mathcal{L}^\sigma_n\mathcal{J}^{-1}f = \mathcal{L}^\sigma_n f$ and $\mathcal{J}\mathcal{L}^\nu_n\mathcal{J}^{-1}f = \mathcal{L}^\mu_n f$. Consequently, using $\mathcal{J}\nu\mathcal{J}^{-1} = \mu\mathcal{I}$,

$$\mathcal{J}\widehat{\mathcal{M}}^\sigma_n\mathcal{J}^{-1}f = \mathcal{M}^\sigma_n f \quad \text{and} \quad \mathcal{J}\widehat{\mathcal{M}}^\nu_n\mathcal{J}^{-1}f = \mathcal{M}^\mu_n f.$$

Thus, we arrive at the following result: the collocation method (5.1) is equivalent to the method

$$\mathcal{A}_n v_n = \begin{cases} \mathcal{M}^\sigma_n f & : \tau = \sigma, \\ \mathcal{M}^\mu_n f & : \tau = \nu, \end{cases}$$

for the approximate solution of

$$\mathcal{A}v = f \quad \text{in} \quad \mathbf{L}^2_\mu,$$

where $v_n = \mathcal{J}u_n \in \operatorname{im}\mathcal{L}_n$, $f = \mathcal{J}\widehat{f}$, and

$$\mathcal{A}_n = \begin{cases} \mathcal{M}^\sigma_n\mathcal{A}\mathcal{L}_n & : \tau = \sigma, \\ \mathcal{M}^\mu_n\mathcal{A}\mathcal{L}_n & : \tau = \nu, \end{cases} \tag{5.2}$$

as well as

$$\mathcal{A} := \widetilde{a}\mathcal{I} - \widetilde{b}\mathcal{S} + \sum_{k=1}^{m_-}(-1)^k\beta_k^-\mathcal{B}_k^+ + \sum_{k=1}^{m_+}(-1)^k\beta_k^+\mathcal{B}_k^-, \quad \widetilde{a}(x) := a(-x), \ \widetilde{b}(x) := b(-x).$$

LEMMA 5.1. *The collocation method* $\big(\widehat{\mathcal{A}}^\tau_n\big)$ *given by* (5.1) *is stable in* $\mathbf{L}^2_\mu$ *if and only if the respective method* $(\mathcal{A}_n)$ *defined by* (5.2) *is stable in* $\mathbf{L}^2_\nu$.

This lemma enables us to check the stability of the method (5.1) in $\mathbf{L}^2_\mu$ by applying Theorem 4.11 to the sequence (5.2).

6. Computational aspects and numerical results. In this section we want to discuss computational aspects of the collocation methods (2.4). In particular, we are interested in a fast computation of the approximate solutions u_n of the collocation methods. Moreover, we will present numerical results for specifically chosen $a, b, \beta_k^\pm$ as well as for a specifically chosen right-hand side f. First of all, if we want to compute the solutions u_n, we can solve the corresponding system of linear equations

$$\widetilde{\mathbb{A}}_n\widetilde{\xi}_n = \widetilde{\eta}_n, \tag{6.1}$$

where the involved matrices and vectors are given by

$$\widetilde{\mathbb{A}}_n = \left[\ (\mathcal{A}\widetilde{\ell}^\tau_{kn})(x^\tau_{jn})\ \right]_{j,k=1}^n, \quad \widetilde{\xi}_n = \left[\ \xi_{kn}\ \right]_{k=1}^n, \quad \widetilde{\eta}_n = \left[\ f(x^\tau_{kn})\ \right]_{k=1}^n,$$

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and where ξ_{kn} are the coefficients of u_n in the basis $\left\{\widetilde{\ell}_{kn}^\tau ; k=1,\dots,n\right\}$ of the space $\operatorname{im}\mathcal{L}_n$. Since this basis is not orthonormal, the stability of the collocation method does not imply the uniform boundedness of the condition numbers of the matrices $\widetilde{\mathbb{A}}_n$. With the help of the Gaussian quadrature rule, one can show that the set $\left\{(\omega_n^\tau)^{-1}(1+x_{kn}^\tau)^{-\frac{1}{2}}\,\widetilde{\ell}_{kn}^\tau : k=1,\dots,n\right\}$ forms an orthonormal basis of $\operatorname{im}\mathcal{L}_n$. The matrix representation of the operators $\mathcal{M}_n^\tau\mathcal{A}\mathcal{L}_n$ in this basis is given by the matrix

$$\mathbb{A}_n := \left[\ \sqrt{\tfrac{1+x_{jn}^\tau}{1+x_{kn}^\tau}}(\mathcal{A}\widetilde{\ell}_{kn}^\tau)(x_{jn}^\tau)\ \right]_{j,k=1}^n ,$$

which is equal to the operator $(\mathcal{V}_n^\tau)^{-1}\mathcal{M}_n^\tau\mathcal{A}\mathcal{L}_n\mathcal{V}_n^\tau : \operatorname{im}\mathcal{L}_n \longrightarrow \operatorname{im}\mathcal{L}_n$. Consequently, in case of stability of the collocation method $(\mathcal{A}_n^\tau)$, the system (6.1) can be preconditioned with the help of the diagonal matrices $\mathbb{D}_n = \operatorname{diag}\left[\ \sqrt{1+x_{jn}^\tau}\ \right]_{j=1}^n$, and we have to solve the system

$$\mathbb{A}_n\xi_n = \eta_n,$$

where

$$\mathbb{A}_n = \mathbb{D}_n\widetilde{\mathbb{A}}_n\mathbb{D}_n^{-1}, \quad \eta_n = \mathbb{D}_n\widetilde{\eta}_n, \quad \text{and} \quad \xi_n = \mathbb{D}_n\widetilde{\xi}_n.$$

Thus, we are mainly interested in the fast computation of the entries of the matrices

$$\mathbb{S}_n^\tau = \left[\ \sqrt{\tfrac{1+x_{jn}^\tau}{1+x_{kn}^\tau}}\left(\mathcal{S}\widetilde{\ell}_{kn}^\tau\right)(x_{jn}^\tau)\ \right]_{j,k=1}^n$$

and

$$\mathbb{B}_{k_0,n}^{\tau,\pm} = \left[\ \sqrt{\tfrac{1+x_{jn}^\tau}{1+x_{kn}^\tau}}\left(\mathcal{B}_{k_0}^\pm\widetilde{\ell}_{kn}^\tau\right)(x_{jn}^\tau)\ \right]_{j,k=1}^n$$

or/and in the fast application of these matrices to a vector. Let us represent the weighted fundamental Lagrange interpolation polynomials in the form

$$\widetilde{\ell}_{kn}^\tau(x) = \sum_{m=1}^n \varepsilon_{mk}^\tau\widetilde{p}_{m-1}(x), \quad \varepsilon_{mk}^\tau = \left\langle \widetilde{\ell}_{kn}^\tau, \widetilde{p}_{m-1}\right\rangle_\nu .$$

Then,

$$\mathbb{S}_n^\tau = \mathbb{D}_n\mathbb{H}_n^\tau\mathbb{J}_n^\tau\mathbb{D}_n^{-1} \quad \text{and} \quad \mathbb{B}_{k_0,n}^{\tau,\pm} = \mathbb{D}_n\mathbb{H}_{k_0,n}^{\tau,\pm}\mathbb{J}_n^\tau\mathbb{D}_n^{-1}, \tag{6.2}$$

where

$$\mathbb{J}_n^\tau = \left[\ \varepsilon_{mk}^\tau\ \right]_{m,k=1}^n , \quad \mathbb{H}_n^\tau = \left[\ (\mathcal{S}\widetilde{p}_{m-1})(x_{jn}^\tau)\ \right]_{j,m=1}^n ,$$

and

$$\mathbb{H}_{k_0,n}^{\tau,\pm} = \left[\ (-1)^{k_0-1}\,h_{m-1}^{k_0}(\pm 2 - x_{jn}^\tau)\ \right]_{j,m=1}^n .$$

In view of (7.9) we have

$$(\mathcal{S}\widetilde{p}_{m-1})(x_{jn}^\tau) = \mathbf{i}R_{m-1}(x_{jn}^\tau) = \frac{\mathbf{i}}{\sqrt{1+x_{jn}^\tau}}\sqrt{\frac{2}{\pi}}\begin{cases}\cos\frac{(2j-1)(2m-1)\pi}{4n} & : \tau=\sigma,\\ \cos\frac{j(2m-1)\pi}{2n+1} & : \tau=\mu.\end{cases}$$

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Defining the matrices

$$\mathbb{C}_n^{\sigma,8} = \left[\cos\frac{(2j-1)(2k-1)\pi}{4n}\right]_{j,k=1}^n \quad \text{and} \quad \mathbb{C}_n^{\mu,7} = \left[\cos\frac{j(2k-1)\pi}{2n+1}\right]_{j,k=1}^n,$$

we can write

$$\mathbb{H}_n^\tau = \begin{cases} \mathbf{i}\sqrt{\frac{2}{\pi}}\,\mathbb{D}_n^{-1}\mathbb{C}_n^{\sigma,8} & : \tau=\sigma, \\ \mathbf{i}\sqrt{\frac{2}{\pi}}\,\mathbb{D}_n^{-1}\mathbb{C}_n^{\mu,7} & : \tau=\mu. \end{cases} \tag{6.3}$$

The entries ε_{mk}^τ of $\mathbb{J}_n^\tau$ can be computed with the help of the respective Gaussian rules, namely

$$\varepsilon_{mk}^\sigma = \frac{\lambda_{kn}^\sigma}{\mu(x_{kn}^\sigma)}(1-x_{kn}^\sigma)P_{m-1}(x_{kn}^\sigma) = \frac{\sqrt{2\pi}}{n}\sqrt{1+x_{kn}^\sigma}\,\sin\frac{(2m-1)(2k-1)\pi}{4n}$$

and

$$\varepsilon_{mk}^\mu = \frac{\lambda_{kn}^\mu}{\mu(x_{kn}^\mu)}P_{m-1}(x_{kn}^\mu) = \frac{\sqrt{2\pi}}{n+\frac{1}{2}}\sqrt{1+x_{kn}^\mu}\,\sin\frac{(2m-1)k\pi}{2n+1}.$$

Thus,

$$\mathbb{J}_n^\sigma = \frac{\sqrt{2\pi}}{n}\,\mathbb{S}_n^{\sigma,8}\mathbb{D}_n \quad \text{and} \quad \mathbb{J}_n^\mu = \frac{\sqrt{2\pi}}{n+\frac{1}{2}}\,\mathbb{S}_n^{\mu,7}\mathbb{D}_n, \tag{6.4}$$

where the matrices $\mathbb{S}_n^{\sigma,8}$ and $\mathbb{S}_n^{\mu,7}$ are defined by

$$\mathbb{S}_n^{\sigma,8} = \left[\sin\frac{(2j-1)(2k-1)\pi}{4n}\right]_{j,k=1}^n \quad \text{and} \quad \mathbb{S}_n^{\mu,7} = \left[\sin\frac{(2j-1)k\pi}{2n+1}\right]_{j,k=1}^n.$$

From (6.2), together with (6.3) and (6.4), we conclude

$$\mathbb{S}_n^\sigma = \frac{4\mathbf{i}}{2n}\,\mathbb{C}_n^{\sigma,8}\mathbb{S}_n^{\sigma,8} \quad \text{and} \quad \mathbb{S}_n^\mu = \frac{4\mathbf{i}}{2n+1}\,\mathbb{C}_n^{\mu,7}\mathbb{S}_n^{\mu,7}$$

as well as

$$\mathbb{B}_{k_0,n}^{\sigma,\pm} = \frac{\sqrt{2\pi}}{n}\,\mathbb{D}_n\mathbb{H}_{k_0,n}^{\sigma,\pm}\mathbb{S}_n^{\sigma,8} \quad \text{and} \quad \mathbb{B}_{k_0,n}^{\mu,\pm} = \frac{\sqrt{2\pi}}{n+\frac{1}{2}}\,\mathbb{D}_n\mathbb{H}_{k_0,n}^{\mu,\pm}\mathbb{S}_n^{\mu,7}.$$

The matrices $\mathbb{C}_n^{\sigma,8}$, $\mathbb{C}_n^{\mu,7}$, $\mathbb{S}_n^{\sigma,4}$, and $\mathbb{S}_n^{\mu,8}$ represent well-known discrete cosine and sine transforms. This enables us to apply them to a vector of length n with $\mathcal{O}(n\log n)$ complexity. So, it remains to consider the matrices $\mathbb{H}_{k_0,n}^{\tau,\pm}$. For this, we use the recurrence relations (cf. (7.1) and (7.2))

$$h_1^1(x) - (2x+1)h_0^1(x) = \frac{2}{\pi\mathbf{i}}\int_{-1}^1 \mu(y)P_0(y)\,dy = \frac{2}{\sqrt{\pi}\mathbf{i}}, \tag{6.5}$$

$$h_{n+1}^1(x) - 2xh_n^1(x) + h_{n-1}^1(x) = \frac{2}{\pi\mathbf{i}}\int_{-1}^1 \mu(y)P_n(y)\,dy = 0, \quad n\ge 1, \tag{6.6}$$

and, for $k>1$,

$$h_1^k(x) - (2x+1)h_0^k(x) = 2(1-|x|)h_0^{k-1}(x), \tag{6.7}$$

$$h_{n+1}^k(x) - 2xh_n^k(x) + h_{n-1}^k(x) = 2(1-|x|)h_n^{k-1}(x), \quad n\ge 1. \tag{6.8}$$

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To compute the entries of $\mathbb{H}_{k_0,n}^{\tau,\pm}$, we solve the linear systems

$$\begin{bmatrix} -(2z+1) & 1 & & & \\ 1 & -2z & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2z & 1 \\ & & & 1 & -2z \end{bmatrix} \begin{bmatrix} h_0^1(z) \\ h_1^1(z) \\ \vdots \\ h_{n-3}^1(z) \\ h_{n-2}^1(z) \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{\pi}\mathbf{i}} \\ 0 \\ \vdots \\ 0 \\ -h_{n-1}^1(z) \end{bmatrix}$$

and, for $k = 2, 3, \dots$,

$$\begin{bmatrix} -(2z+1) & 1 & & & \\ 1 & -2z & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2z & 1 \\ & & & 1 & -2z \end{bmatrix} \begin{bmatrix} h_0^k(z) \\ h_1^k(z) \\ \vdots \\ h_{n-3}^k(z) \\ h_{n-2}^k(z) \end{bmatrix} = \begin{bmatrix} 2(1-|z|)h_0^{k-1}(z) \\ 2(1-|z|)h_1^{k-1}(z) \\ \vdots \\ 2(1-|z|)h_{n-3}^{k-1}(z) \\ 2(1-|z|)h_{n-2}^{k-1}(z) - h_{n-1}^k(z) \end{bmatrix}$$

for $z = \pm 2 - x_{jn}^\tau$, $j = 1, \dots, n$. To use these systems and not the forward recurrences suggested by (6.5), (6.6) and (6.7), (6.8) is motivated by the following fact. In case $|x| > 1$, the roots of the characteristic polynomial $\lambda^2 - 2x\lambda + 1$ of the second order difference equation (6.6) or (6.8) are equal to $\lambda_{1/2} = x \pm \sqrt{x^2-1}$. Consequently, one of these roots has an absolute value greater than 1, and it is well known that this leads to instabilities in the forward computation; cf. also the discussion in [17, pp. 362, 363].

Of course, the values $h_{n-1}^k(z)$ have to be precomputed. For this we can use appropriate Gaussian rules of sufficiently high order N. For example,

$$\begin{aligned} h_{n-1}^{k_0}(2-x) &= \frac{(x-1)^{k_0-1}}{\pi\mathbf{i}} \int_{-1}^1 \frac{(1-y)P_{n-1}(y)}{(y+x-2)^{k_0}} \frac{dy}{\sqrt{1-y^2}} \\ &\approx \frac{(x-1)^{k_0-1}}{\mathbf{i}N} \sum_{k=1}^N \frac{(1-x_{kn}^\sigma)P_{n-1}(x_{kn}^\sigma)}{(x_{kn}^\sigma + x - 2)^{k_0}} \end{aligned}$$

with N sufficiently large.

Now we turn back to the integral equation

$$\frac{1}{\pi}\int_{-1}^1 \left[\frac{1}{y-x} - \frac{1}{2+y+x} + \frac{6(1+x)}{(2+y+x)^2} - \frac{4(1+x)^2}{(2+y+x)^3}\right] u(y)\,dy = f(x), \qquad -1 < x < 1, \tag{6.9}$$

of the notched half plane problem already mentioned in the introduction of this paper. As stated there, we should take into account that the solution has a singularity of the form $(1-x)^{-\frac{1}{2}}$. Thus, we try to apply a collocation method in which the approximate solution has the form $u_n(x) = \nu(x)p_n(x)$ with a polynomial $p_n(x)$ of degree less than n. With the notations of Section 5, this means that we consider the collocation method

$$\widehat{\mathcal{A}}_n^\tau u_n := \widehat{\mathcal{M}}_n^\tau \mathcal{A}\widehat{\mathcal{L}}_n u_n = \widehat{\mathcal{M}}_n^\tau f, \quad u_n \in \operatorname{im}\widehat{\mathcal{L}}_n, \quad \tau \in \{\sigma, \nu\}, \tag{6.10}$$

where

$$\widehat{\mathcal{A}} := \mathbf{i}\mathcal{S} - \mathbf{i}\mathcal{B}_1^- + 6\mathbf{i}\mathcal{B}_2^- - 4\mathbf{i}\mathcal{B}_3^-.$$

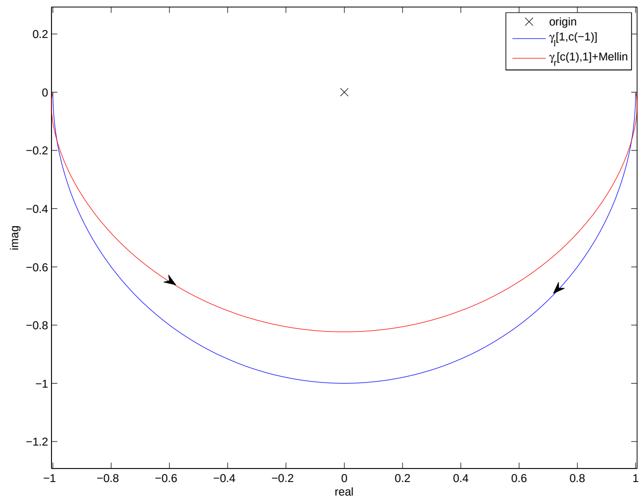

FIG. 6.1. $\Gamma_{\mathcal{A}}$ *for the operator* (6.11).

Due to Lemma 5.1, the stability of the collocation method (6.10) in $\mathbf{L}^2_\mu$ is equivalent to the stability of the method $(\mathcal{A}_n)$ in $\mathbf{L}^2_\nu$, where

$$\mathcal{A}_n = \begin{cases} \mathcal{M}_n^\sigma \mathcal{A} \mathcal{L}_n & : \tau = \sigma, \\ \mathcal{M}_n^\mu \mathcal{A} \mathcal{L}_n & : \tau = \nu, \end{cases}$$

and

$$\mathcal{A} = \mathcal{S} - \mathcal{B}_1^+ - 6\mathcal{B}_2^+ - 4\mathcal{B}_3^+ . \tag{6.11}$$

Let us verify the conditions of Theorem 4.11. Due to Proposition 3.5, for the invertibility of $\mathcal{A} : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$, we have to show that the closed curve

$$\Gamma_{\mathcal{A}} = \left\{ -\mathbf{i} \cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\}$$
$$\cup \left\{ -\mathbf{i} \cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) + \widehat{\mathbf{h}}_1^+\left(\tfrac{1}{4} - \mathbf{i}\xi\right) + 6\widehat{\mathbf{h}}_2^+\left(\tfrac{1}{4} - \mathbf{i}\xi\right) + 4\widehat{\mathbf{h}}_3^+\left(\tfrac{1}{4} - \mathbf{i}\xi\right) : -\infty \le \xi \le \infty \right\}$$

does not contain 0 and that its winding number is equal to zero, but this can be seen from Figure 6.1. (Of course, the invertibility of $\mathcal{A} : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is equivalent to the invertibility of $\widehat{\mathcal{A}} : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\mu$, which was already verified in [1, p. 101].) Obviously, condition (d) of Theorem 4.11 is not fulfilled, which implies that the sequence $\left(\widehat{\mathcal{A}}_n^\nu\right)$ for the collocation with respect to the Chebyshev nodes of third kind is not stable. Therefore, we concentrate on the case $\tau = \sigma$. Due to condition (b) of Theorem 4.11, we have to consider the curves

$$\begin{aligned} \Gamma_- &= \left\{ \mathbf{i} \cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \cup \left\{ \mathbf{i} \cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \\ &= \gamma_\ell[-1,1] \cup \gamma_r[1,-1] = \{ z \in \mathbb{C} : |z| = 1, \operatorname{Im} z \ge 0 \} \end{aligned}$$

and

$$\Gamma_+^\sigma = \left\{ \cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\}$$
$$\cup \left\{ \cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) - \widehat{\mathbf{h}}_1^+\left(\tfrac{1}{4} + \mathbf{i}\xi\right) - 6\widehat{\mathbf{h}}_2^+\left(\tfrac{1}{4} + \mathbf{i}\xi\right) - 4\widehat{\mathbf{h}}_3^+\left(\tfrac{1}{4} + \mathbf{i}\xi\right) : -\infty \le \xi \le \infty \right\} .$$

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TABLE 6.1
Collocation for (6.9) *with* $\tau = \sigma$.

n	$\mathrm{cond}(\widetilde{\mathbb{A}}_n)$	$\mathrm{cond}(\mathbb{A}_n)$
8	13.5	1.32
16	27.1	1.35
32	54.1	1.37
64	108.1	1.38
128	216.1	1.40
256	432.2	1.41
512	864.3	1.41
1024	1728.5	1.42
2048	3456.9	1.42

TABLE 6.2
Collocation for (6.9) *with* $\tau = \nu$.

n	$\mathrm{cond}(\widetilde{\mathbb{A}}_n)$	$\mathrm{cond}(\mathbb{A}_n)$
8	16.1	5.00
16	30.3	7.37
32	58.7	10.66
64	115.5	15.24
128	229.1	21.63
256	456.3	30.66
512	910.9	43.51
1024	1820.0	61.79
2048	3638.4	87.63

Of course, the winding number of Γ_- is equal to zero and since $\Gamma_+^\sigma = -\Gamma_{\mathcal{A}}$ (with reverse direction), this is also true for Γ_+^σ. Due to the complicated structure of the compact operators $\mathbf{K}_{k_0}^\sigma$ and $\mathbf{K}_{k_0}$ (cf. (7.34) and (7.36)) in the definition of $\mathbf{A}_+^\sigma$ and $\mathbf{A}_-$ (cf. (3.3), (3.4), and (7.37)), we are not able to check if condition (c) of Theorem 4.11 holds true. But, the results shown in Table 6.1 (already presented in [1, table on page 112], where one can also find computed values for the stress intensity factor and the crack opening displacement which are of practical interest) suggest that the collocation method (6.10) with $\tau = \sigma$ applied to (6.9) is stable.

On the other hand, Table 6.2 confirms our theoretical result that the collocation method (6.10) for $\tau = \nu$ applied to (6.9) is not stable.

7. Proof of Proposition 3.4. At a first step, we compute the values $\mathcal{B}_k^\pm \widetilde{p}_n$ for the complete orthonormal system $(\widetilde{p}_n)_{n=0}^\infty$ (in $\mathbf{L}_\nu^2$). Define

$$h_n^k(x) := \frac{(1-|x|)^{k-1}}{\pi \mathbf{i}} \int_{-1}^1 \frac{\mu(y) P_n(y)}{(y-x)^k}\, dy, \quad n \ge 0,\ |x|>1,\ k=1,2,\dots,$$

and set $h_n(x) := h_n^1(x)$. It is well known that the following recursion formula holds

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x),\ n\ge 1, \quad P_0(x) = \frac{1}{\sqrt{\pi}},\ P_1(x) = \frac{1}{\sqrt{\pi}}(2x+1).$$

Using this formula, we get

$$h_{n+1}(x) - 2xh_n(x) + h_{n-1}(x) = \frac{2}{\pi \mathbf{i}} \int_{-1}^1 \mu(y) P_n(y)\, dy = 0, \quad n \ge 1 \tag{7.1}$$

and

$$h_1(x) - (2x+1)h_0(x) = \frac{2}{\pi \mathbf{i}} \int_{-1}^1 \mu(y) P_0(y)\, dy = \frac{2}{\sqrt{\pi}\mathbf{i}}\,. \tag{7.2}$$

In order to solve these recursion formulas, we have to determine the values

$$\mathbf{i}\sqrt{\pi}\, h_0(x) := \frac{1}{\pi} \int_{-1}^1 \frac{\mu(y)\, dy}{y-x}.$$

Setting

$$\gamma(x) := \frac{1}{\pi} \int_{-1}^1 \frac{\sigma(y)\, dy}{y-x},$$

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we easily obtain

$$\mathbf{i}\sqrt{\pi}\, h_0(x) = (1-x)\gamma(x) - 1.$$

Let us compute $\gamma(x)$. Using the substitutions $y = \cos s$ as well as $z = \tan \frac{s}{2}$, we get

$$\gamma(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sigma(y)\,dy}{y-x} = \frac{-2}{\pi(x+1)}\int_0^\infty \frac{dz}{\frac{x-1}{x+1}+z^2} = -\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}}, \quad |x|>1.$$

Consequently,

$$\mathbf{i}\sqrt{\pi}\, h_0(x) = \sqrt{\frac{x-1}{x+1}} - 1, \quad |x|>1.$$

Using the recursion formulas (7.1), (7.2), we are now able to compute $h_n(x)$. The zeros of the characteristic polynomial $p(t) = t^2 - 2xt + 1$ for the recursion (7.1) are given by $x \pm \sqrt{x^2-1}$. Thus, the solution has the form

$$\mathbf{i}\sqrt{\pi}\, h_n(x) = \delta_0\big(x+\sqrt{x^2-1}\big)^n + \delta_1\big(x-\sqrt{x^2-1}\big)^n,$$

where the δ_i are determined by the initial values $h_0(x)$ and $h_1(x)$. Formula (7.2) gives

$$\mathbf{i}\sqrt{\pi} h_1(x) = (2x+1)\left(\sqrt{\frac{x-1}{x+1}} - 1\right) + 2,$$

and so, for $|x|>1$, we get

$$\mathbf{i}\sqrt{\pi}\, h_n(x) = \left(\sqrt{\frac{x-1}{x+1}} - 1\right)\left(x - \operatorname{sgn}(x)\sqrt{x^2-1}\right)^n. \tag{7.3}$$

If $k \geq 2$, we can use the relations

$$\int_{-1}^{1} \frac{\mu(y)P_n(y)}{(y-x)^k}\,dy = \frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\int_{-1}^{1}\frac{\mu(y)P_n(y)}{y-x}\,dy, \quad k\geq 1,\ |x|>1. \tag{7.4}$$

For the determination of the derivatives, we state the following lemma.

LEMMA 7.1. *Let $k \in \mathbb{N}$ be arbitrary. Then the following equation is true*

$$\frac{d^k}{dx^k}\left(x \pm \sqrt{x^2-1}\right)^n = \left(x\pm\sqrt{x^2-1}\right)^n \sum_{s=0}^{k-1} \frac{n^{k-s}\,p_s^k(x)}{(x^2-1)^{\frac{k+s}{2}}}, \quad |x|>1,$$

where $p_s^k(x)$ are polynomials with $p_0^k(x) = (\pm 1)^k$ and $\deg p_s^k \leq s$.

Proof. Let us proof this fact by induction with respect to $k \in \mathbb{N}$. For $k=1$, we have

$$\begin{aligned}\frac{d}{dx}(x\pm\sqrt{x^2-1})^n &= \left(x\pm\sqrt{x^2-1}\right)^{n-1} n\left(1 \pm \frac{x}{\sqrt{x^2-1}}\right)\\ &= \left(x\pm\sqrt{x^2-1}\right)^n \frac{\pm n}{\sqrt{x^2-1}}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\left(x\pm\sqrt{x^2-1}\right)^n &= \frac{d}{dx}\left[\left(x\pm\sqrt{x^2-1}\right)^n \sum_{s=0}^{k-1}\frac{n^{k-s}\,p_s^k(x)}{(x^2-1)^{\frac{k+s}{2}}}\right]\\ &= \left(x\pm\sqrt{x^2-1}\right)^n\left[\sum_{s=0}^{k-1}\frac{\pm n^{k+1-s}\,p_s^k(x)}{(x^2-1)^{\frac{k+1+s}{2}}} + \sum_{s=0}^{k-1}\frac{n^{k-s}\,q_{s+1}^k(x)}{(x^2-1)^{\frac{k+s}{2}+1}}\right],\end{aligned}$$

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where

$$q_{s+1}^k(x) := (x^2-1)\,\frac{dp_s^k(x)}{dx} - (k+s)x\,p_s^k(x) = (x^2-1)^{\frac{k+s}{2}+1}\,\frac{d}{dx}\,\frac{p_s^k(x)}{(x^2-1)^{\frac{k+s}{2}}}\,.$$

It follows that

$$\begin{aligned}
&\frac{d^{k+1}}{dx^{k+1}}\left(x\pm\sqrt{x^2-1}\right)^n\\
&= \left(x\pm\sqrt{x^2-1}\right)^n\left[\frac{\pm n^{k+1}\,p_0^k(x)}{(x^2-1)^{\frac{k+1}{2}}} \pm \sum_{s=1}^{k-1}\frac{n^{k+1-s}\,p_s^k(x)}{(x^2-1)^{\frac{k+1+s}{2}}} + \sum_{s=1}^{k}\frac{n^{k+1-s}\,q_s^k(x)}{(x^2-1)^{\frac{k+1+s}{2}}}\right]\\
&= \left(x\pm\sqrt{x^2-1}\right)^n\left[\frac{\pm n^{k+1}\,p_0^k(x)}{(x^2-1)^{\frac{k+1}{2}}} + \sum_{s=1}^{k-1}\frac{n^{k+1-s}\left[q_s^k(x)\pm p_s^k(x)\right]}{(x^2-1)^{\frac{k+1+s}{2}}} + \frac{n\,q_k^k(x)}{(x^2-1)^{k+\frac{1}{2}}}\right].
\end{aligned}$$

If we set

$$\begin{aligned}
&p_s^{k+1}(x) := q_s^k(x)\pm p_s^k(x),\ 1\le s\le k-1,\\
&p_k^{k+1}(x) := q_k^k(x), \qquad p_0^{k+1}(x) := \pm p_0^k(x) = (\pm 1)^{k+1},
\end{aligned}$$

then we arrive at

$$\frac{d^{k+1}}{dx^{k+1}}\left(x\pm\sqrt{x^2-1}\right)^n = \left(x\pm\sqrt{x^2-1}\right)^n\sum_{s=0}^{k}\frac{n^{k+1-s}\,p_s^{k+1}(x)}{(x^2-1)^{\frac{k+1+s}{2}}}.$$

The lemma is proved. □

We mention that there exist polynomials $p_{k-1}(x)$ with $p_{-1}\equiv 1$ and $\deg p_{k-1}\le k-1$, $k\ge 1$, such that

$$\frac{d^k}{dx^k}\left(\sqrt{\frac{x-1}{x+1}}\right) = \sqrt{\frac{x-1}{x+1}}\,\frac{p_{k-1}(x)}{(x^2-1)^k},\quad |x|>1, \tag{7.5}$$

which can also be proved by induction. With the help of (7.3), (7.4), (7.5), and Lemma 7.1 we can write, for $|x|>1$ and $k\ge 1$,

(7.6)

$$\begin{aligned}
h_n^k(x) &= \frac{(1-|x|)^{k-1}}{\sqrt{\pi}\mathbf{i}(k-1)!}\,\frac{d^{k-1}}{dx^{k-1}}\left[\left(\sqrt{\frac{x-1}{x+1}}-1\right)\left(x-\operatorname{sgn}(x)\sqrt{x^2-1}\right)^n\right]\\
&= \frac{(1-|x|)^{k-1}}{\sqrt{\pi}\mathbf{i}(k-1)!}\left(x-\operatorname{sgn}(x)\sqrt{x^2-1}\right)^n\\
&\quad\cdot\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\left[\sqrt{\frac{x-1}{x+1}}\,\frac{p_{k-2-\ell}(x)}{(x^2-1)^{k-1-\ell}}-\delta_{k-1,\ell}\right]\sum_{s=0}^{\ell-1}\frac{n^{\ell-s}\,p_s^\ell(x)}{(x^2-1)^{\frac{\ell+s}{2}}}\\
&= \frac{(-1)^{k-1}}{\sqrt{\pi}\mathbf{i}(k-1)!}\left[\sqrt{\frac{x-1}{x+1}}\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\sum_{s=0}^{\ell-1}\frac{p_{k-2-\ell}(x)\,p_s^\ell(x)\,n^{\ell-s}(|x|-1)^{\frac{\ell-s}{2}}}{(|x|+1)^{k-1-\frac{\ell-s}{2}}}\right.\\
&\quad\left.-\sum_{s=0}^{k-2}\frac{p_s^{k-1}(x)\,n^{k-1-s}(|x|-1)^{\frac{k-1-s}{2}}}{(|x|+1)^{\frac{k-1+s}{2}}}\right]\left(x-\operatorname{sgn}(x)\sqrt{x^2-1}\right)^n,\quad n\ge 1,
\end{aligned}$$

and

$$h_0^k(x) = \frac{(-1)^{k-1}}{\sqrt{\pi}\mathbf{i}(k-1)!}\sqrt{\frac{x-1}{x+1}}\,\frac{p_{k-2}(x)}{(|x|+1)^{k-1}} - \frac{1}{\sqrt{\pi}\mathbf{i}}\delta_{1,k}, \tag{7.7}$$

where the sums in (7.6) with negative upper limit are equal to 1. For example, the values $h_n^k(x)$, $k = 2, 3$, $x < -1$, are given by

$$\mathbf{i}\sqrt{\pi}\, h_n^2(x) = \left[-\frac{1}{\sqrt{x^2-1}} - n\left(1 - \sqrt{\frac{x+1}{x-1}}\right)\right]\left(x + \sqrt{x^2-1}\right)^n,$$

and

$$\begin{aligned}\mathbf{i}\sqrt{\pi}\, h_n^3(x) = \Bigg\{&\sqrt{\frac{x+1}{x-1}}\left[\frac{3}{2}\frac{1}{x^2-1} - \frac{1}{x-1}\right] + \frac{n}{2}\left(1 - \frac{1}{x-1} - \frac{x}{(x-1)}\sqrt{\frac{x+1}{x-1}}\right)\\ &- \frac{n^2}{2}\left(\frac{x+1}{x-1} - \sqrt{\frac{x+1}{x-1}}\right)\Bigg\}\left(x + \sqrt{x^2-1}\right)^n.\end{aligned}$$

For the determination of the limits $\mathcal{W}^{3/4}(\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n)$ in case $\tau = \sigma$, we have to compute the values

$$\widetilde{h}_n^k(x) := \frac{(1-|x|)^{k-1}}{\pi\mathbf{i}}\int_{-1}^1 \frac{\mu(y)T_n(y)}{(y-x)^k}\,dy, \quad n \ge 0,\ |x| > 1,\ k \ge 1.$$

Again set $\widetilde{h}_n(x) := \widetilde{h}_n^1(x)$. The following relation

$$c_n\widetilde{h}_n(x) = (1-x)\widetilde{\gamma}_n(x) - \delta_{0,n}$$

holds, where

$$\widetilde{\gamma}_n(x) = \frac{c_n}{\pi\mathbf{i}}\int_{-1}^1 \frac{\sigma(y)T_n(y)}{y-x}\,dy$$

and $c_0 = \sqrt{\pi}\,\mathbf{i}$, $c_n = \sqrt{\frac{\pi}{2}}\,\mathbf{i}$, $n \ge 1$. We can state the recursion formula

$$\widetilde{\gamma}_{n-1}(x) - 2x\widetilde{\gamma}_n(x) + \widetilde{\gamma}_{n+1} = 0, \quad n \ge 1,$$

as well as

$$-x\widetilde{\gamma}_0(x) + \widetilde{\gamma}_1(x) = 1.$$

Now we can solve the formula as in the previous case. For $|x| > 1$, we obtain

$$\widetilde{\gamma}_n^1(x) = -\operatorname{sgn}(x)\,\frac{\left(x - \operatorname{sgn}(x)\sqrt{x^2-1}\right)^n}{\sqrt{x^2-1}}.$$

Thus,

$$c_n\widetilde{h}_n(x) = \sqrt{\frac{x-1}{x+1}}\left(x - \operatorname{sgn}(x)\sqrt{x^2-1}\right)^n - \delta_{0,n}.$$

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We obtain, for $n \geq 1$,

(7.8)
$$
\begin{aligned}
& c_n \widetilde{h}_n^k(x) \\
&\quad = \frac{(1-|x|)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \left[\sqrt{\frac{x-1}{x+1}} \left(x - \operatorname{sgn}(x) \sqrt{x^2-1} \right)^n \right] \\
&\quad = \frac{(-1)^{k-1}}{(k-1)!} \left(x - \operatorname{sgn}(x) \sqrt{x^2-1} \right)^n \sqrt{\frac{x-1}{x+1}} \\
&\qquad\qquad \cdot \sum_{l=0}^{k-1} \binom{k-1}{l} \sum_{s=0}^{l-1} \frac{p_{k-l-2}(x)\, p_s^l(x)}{(|x|+1)^{k-1-\frac{l-s}{2}}} \, n^{l-s} (|x|-1)^{\frac{l-s}{2}}
\end{aligned}
$$

and

$$
c_0 \widetilde{h}_0^k(x) = \frac{(-1)^{k-1}}{(k-1)!} \sqrt{\frac{x-1}{x+1}} \frac{p_{k-2}(x)}{(|x|+1)^{k-1}} - \delta_{1,k}.
$$

For instance, if $x < -1$, then

$$
c_n \widetilde{h}_n^2(x) = -\left(x + \sqrt{x^2-1} \right)^n \left\{ n + \frac{1}{\sqrt{x^2-1}} \right\}
$$

and

$$
\begin{aligned}
c_n \widetilde{h}_n^3(x) = \Bigg\{ & \sqrt{\frac{x-1}{x+1}} \left[\frac{3}{2(x-1)^2} + \frac{x+1}{(x-1)^2} \right] \\
& + \frac{n}{2} \left(\frac{x+1}{x-1} - \frac{3}{x-1} \right) + \frac{n^2}{2} \sqrt{\frac{x+1}{x-1}} \Bigg\} \left(x + \sqrt{x^2-1} \right)^n.
\end{aligned}
$$

Let $\mathbf{R} = \mathbf{R}(-1,1)$ denote the set of all functions $f : (-1,1) \longrightarrow \mathbb{C}$, which are bounded and Riemann integrable on each closed subinterval $[a,b] \subset (-1,1)$.

LEMMA 7.2 (Corollary 3.3 in [10]). *Let $f \in \mathbf{R}$ and*

$$
|f(x)| \leq \text{const}\, (1-x)^{\varepsilon - \frac{1}{4}} (1+x)^{\varepsilon - \frac{3}{4}}, \quad -1 < x < 1,
$$

for some $\varepsilon > 0$. Then,

$$
\mathcal{M}_n^\tau f \longrightarrow f \quad \text{in} \quad \mathbf{L}_\nu^2 \quad \text{for } \tau \in \{\sigma, \mu\}.
$$

Let $k : [-1,1] \times [-1,1] \longrightarrow \mathbb{C}$ be a continuous kernel function and

$$
(\mathcal{K}u)(x) := \int_{-1}^{1} k(x,y) u(y)\, dy, \quad u \in \mathbf{L}_\nu^2,
$$

the associated integral operator.

LEMMA 7.3. *If $k : [-1,1] \times [-1,1] \longrightarrow \mathbb{C}$ is continuous, then $\mathcal{K} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{C}[-1,1]$ is a compact operator. In particular, $(\mathcal{M}_n^\tau \mathcal{K} \mathcal{L}_n) \in \mathfrak{J}$ for $\tau \in \{\sigma, \mu\}$.*

Proof. Let $u \in \mathbf{L}_\nu^2$ and $\varepsilon > 0$. Since $k(x,y)$ is uniformly continuous on $[-1,1] \times [-1,1]$, there exists a positive number $\delta = \delta(\varepsilon)$ such that $|x - x'| < \delta$ and $x, x' \in [-1,1]$ implies $|k(x,y) - k(x',y)| < \varepsilon$. Thus, for $|x - x'| < \delta$, we have

$$
\begin{aligned}
|(\mathcal{K}u)(x) - (\mathcal{K}u)(x')| &\leq \int_{-1}^{1} \frac{|k(x,y) - k(x',y)|}{\sqrt{\nu(y)}} \sqrt{\nu(y)} |u(y)|\, dy \\
&\leq \varepsilon \|1\|_\mu \|u\|_\nu = \text{const}\, \varepsilon \|u\|_\nu.
\end{aligned}
$$

Consequently, the set $\{\mathcal{K}u \,:\, u \in \mathbf{L}^2_\nu,\ \|u\|_\nu \le 1\}$ is bounded and equicontinuous. The Arzela-Ascoli theorem yields the compactness of $\mathcal{K} : \mathbf{L}^2_\nu \longrightarrow \mathbf{C}[-1,1]$. With the help of Lemma 7.2, we get $\|\mathcal{M}^\tau_n \mathcal{K}\mathcal{L}_n - \mathcal{L}_n\mathcal{K}\mathcal{L}_n\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \longrightarrow 0$. Hence, $(\mathcal{M}^\tau_n \mathcal{K}\mathcal{L}_n - \mathcal{L}_n\mathcal{K}\mathcal{L}_n) \in \mathfrak{J}$, which implies $(\mathcal{M}^\tau_n \mathcal{K}\mathcal{L}_n) \in \mathfrak{J}$. □

If $A(x,n,\dots)$ and $B(x,n,\dots)$ are two positive functions depending on certain variables $x, n, \dots$, then we write $A \sim_{x,n,\dots} B$ if there is a constant $\mathcal{C} \ne \mathcal{C}(x,n,\dots) > 0$ such that

$$\mathcal{C}^{-1}B(x,n,\dots) \le A(x,n,\dots) \le \mathcal{C}B(x,n,\dots)$$

holds.

LEMMA 7.4. *Let $n \in \mathbb{N}$, $k = 0,1,\dots,n$, and $\tau \in \{\sigma,\varphi,\nu,\mu\}$. For the zeros x^τ_{kn} of the orthogonal polynomials p^τ_n, we have, for all sufficiently large n,*

$$\int_{x^\tau_{k+1,n}}^{x^\tau_{kn}} \sqrt{\frac{1+x}{1-x}}\,dx \ \sim_{k,n}\ \frac{1}{n}(1+x^\tau_{kn}),$$

where $x^\tau_{0n} := 1,\ x^\tau_{n+1,n} := -1$.

Proof. Of course,

$$\int_{x^\tau_{k+1,n}}^{x^\tau_{kn}} \sqrt{\frac{1+x}{1-x}}\,dx = \int_{x^\tau_{k+1,n}}^{x^\tau_{kn}} \frac{1+x}{\sqrt{1-x^2}}\,dx \le (1+x^\tau_{kn})\frac{\pi}{n}.$$

Moreover, in case $\tau = \sigma$ and for sufficiently large n,

$$\begin{aligned}
\int_{x^\sigma_{k+1,n}}^{x^\sigma_{kn}} \sqrt{\frac{1+x}{1-x}}\,dx &= \int_{x^\sigma_{k+1,n}}^{x^\sigma_{kn}} \frac{1}{\sqrt{1-x^2}}\,dx + \int_{x^\sigma_{k+1,n}}^{x^\sigma_{kn}} \frac{x}{\sqrt{1-x^2}}\,dx \\
&\ge \frac{1}{2}\left(\frac{\pi}{n} + 2\cos\tfrac{k\pi}{n}\sin\tfrac{\pi}{2n}\right) \ge \mathcal{C}\frac{\pi}{n}\left(1+\cos\frac{k\pi}{n}\right) \\
&= \mathcal{C}\frac{\pi}{n}\left(1+\cos\frac{k-\frac{1}{2}}{n}\pi\cos\frac{\pi}{2n} - \sin\frac{k-\frac{1}{2}}{n}\pi\sin\frac{\pi}{2n}\right) \\
&\ge \mathcal{C}\frac{\pi}{n}\left(1+\cos\frac{k-\frac{1}{2}}{n}\pi\cos\frac{\pi}{2n} - \sin\frac{\pi}{2n}\right) \\
&\sim_{k,n}\ \frac{1}{n}\left(1+\cos\frac{k-\frac{1}{2}}{n}\pi\right) = \frac{1}{n}(1+x^\sigma_{kn}).
\end{aligned}$$

The proof for the other nodes is analogous. □

Define $\widetilde{\mathcal{B}}^\pm_k$, $k \in \mathbb{N}$, by

$$(\widetilde{\mathcal{B}}^\pm_k u)(x) = \frac{1}{\pi\mathbf{i}}\int_{-1}^{1} \frac{(1\mp y)^{k-1}u(y)\,dy}{(y+x\mp 2)^k}, \quad -1 < x < 1.$$

By induction one can show that there exist constants $c^\pm_{kj}$ and $d^\pm_{kj}$ such that

$$\mathcal{B}^\pm_k = \sum_{j=1}^{k} c^\pm_{kj}\widetilde{\mathcal{B}}_j \quad \text{and} \quad \widetilde{\mathcal{B}}^\pm_k = \sum_{j=1}^{k} d^\pm_{kj}\mathcal{B}^\pm_j.$$

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Since, for $u \in \mathbf{L}^2_\nu$ and $x, x_0 \in (-1,1]$,

$$\begin{aligned}
&\left|(\widetilde{\mathcal{B}}_k^- u)(x) - (\widetilde{\mathcal{B}}_k^- u)(x_0)\right| \\
&\quad \le \frac{1}{\pi}\int_{-1}^1 \left| \frac{(1+y)^{k-1}\sum_{j=0}^k \binom{k}{j}(1+y)^{k-j}\left[(1+x_0)^j-(1+x)^j\right]u(y)}{(1+y+1+x)^k(1+y+1+x_0)^k}\right| dy \\
&\quad \le \frac{1}{\pi(1+x)^k(1+x_0)^k} \\
&\qquad \cdot \sqrt{\int_{-1}^1 \left|(1+y)^{k-1}\sum_{j=0}^k \binom{k}{j}(1+y)^{k-j}\left[(1+x_0)^j-(1+x)^j\right]\right|^2 \mu(y)\,dy}\; \|u\|_\nu\,,
\end{aligned}$$

the functions $(\widetilde{\mathcal{B}}_k^- u)(x)$ and $(\mathcal{B}_k^- u)(x)$ are continuous for $-1 < x \le 1$ if $u \in \mathbf{L}^2_\nu$. The same is true for $(\widetilde{\mathcal{B}}_k^+ u)(x)$ and $(\mathcal{B}_k^+ u)(x)$, $-1 \le x < 1$, and $u \in \mathbf{L}^2_\nu$. Hence, the values $(\mathcal{B}_k^\pm u)(x_{jn}^\tau)$ are well defined.

LEMMA 7.5. *Let $\tau \in \{\sigma,\mu\}$. Then, for $k \in \mathbb{N}$,*

$$\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n \longrightarrow \mathcal{B}_k^\pm, \qquad (\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n)^* \longrightarrow (\mathcal{B}_k^\pm)^*$$

and

$$\mathcal{M}_n^\tau \widetilde{\mathcal{B}}_k^\pm \mathcal{L}_n \longrightarrow \widetilde{\mathcal{B}}_k^\pm, \qquad (\mathcal{M}_n^\tau \widetilde{\mathcal{B}}_k^\pm \mathcal{L}_n)^* \longrightarrow (\widetilde{\mathcal{B}}_k^\pm)^*$$

in the sense of strong convergence in $\mathbf{L}^2_\nu$.

Proof. Let $u \in \mathbf{L}^2_\nu$. With the help of the exactness of the Gaussian rule for polynomials of degree less than $2n$, we have

$$\|\mathcal{M}_n^\sigma \widetilde{\mathcal{B}}_k^\pm u\|_\nu^2 = \|\sqrt{1-x}\mathcal{L}_n^\sigma \nu \widetilde{\mathcal{B}}_k^\pm u\|_\sigma^2 = \frac{\pi}{n}\sum_{k=1}^n (1+x_{kn}^\sigma)|(\widetilde{\mathcal{B}}_k^\pm u)(x_{kn}^\sigma)|^2$$

and

$$\|\mathcal{M}_n^\mu \widetilde{\mathcal{B}}_k^\pm u\|_\nu^2 = \|\mathcal{L}_n^\mu \nu \widetilde{\mathcal{B}}_k^\pm u\|_\mu^2 = \frac{\pi}{n+\frac{1}{2}}\sum_{k=1}^n (1+x_{kn}^\mu)|(\widetilde{\mathcal{B}}_k^\pm u)(x_{kn}^\mu)|^2.$$

Taking into account Lemma 7.4, we can estimate for $\tau=\sigma$, $\tau=\mu$, and for all sufficiently large n

$$\begin{aligned}
\|\mathcal{M}_n^\tau \widetilde{\mathcal{B}}_k^- u\|_\nu^2 &\le \text{const}\sum_{k=1}^n \int_{x_{k+1,n}^\tau}^{x_{kn}^\tau} \sqrt{\frac{1+x}{1-x}}\,dx \left(\frac{1}{\pi}\int_{-1}^1 \frac{(1+y)^{k-1}|u(y)|\,dy}{(y+x_{kn}^\tau+2)^k}\right)^2 \\
&\le \text{const}\sum_{k=1}^n \int_{x_{k+1,n}^\tau}^{x_{kn}^\tau} \sqrt{\frac{1+x}{1-x}} \left(\frac{1}{\pi}\int_{-1}^1 \frac{(1+y)^{k-1}|u(y)|\,dy}{(y+x+2)^k}\right)^2 dx \\
&\le \text{const}\int_{-1}^1 \nu(x)\left(\frac{1}{\pi}\int_{-1}^1 \frac{(1+y)^{k-1}|u(y)|\,dy}{(y+x+2)^k}\right)^2 dx \\
&= \text{const}\left\|\widetilde{\mathcal{B}}_k^-|u|\right\|_\nu^2 \le \text{const}\left\|\widetilde{\mathcal{B}}_k^-\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}^2 \|u\|_\nu^2\,.
\end{aligned}$$

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To handle $\|\mathcal{M}_n^\tau \widetilde{\mathcal{B}}_k^+ u\|_\nu$, we use the estimate

$$\int_{x_{kn}^\tau}^{x_{k-1,n}^\tau} \sqrt{\frac{1+x}{1-x}}\, dx \ge \frac{\pi}{n}\, (1 + x_{kn}^\tau)\,, \quad k = 1, \ldots, n,$$

and get

$$\begin{aligned}
\|\mathcal{M}_n^\tau \widetilde{\mathcal{B}}_k^+ u\|_\nu^2 &\le \sum_{k=1}^n \int_{x_{kn}^\tau}^{x_{k-1,n}^\tau} \sqrt{\frac{1+x}{1-x}}\, dx \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1-y)^{k-1}|u(y)|\, dy}{(2-y-x_{kn}^\tau)^k} \right)^2 \\
&\le \sum_{k=1}^n \int_{x_{kn}^\tau}^{x_{k-1,n}^\tau} \sqrt{\frac{1+x}{1-x}} \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1-y)^{k-1}|u(y)|\, dy}{(2-y-x)^k} \right)^2 dx \\
&\le \int_{-1}^1 \nu(x) \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1-y)^{k-1}|u(y)|\, dy}{(2-y-x)^k} \right)^2 dx \\
&= \left\| \widetilde{\mathcal{B}}_k^+ |u| \right\|_\nu^2 \le \left\| \widetilde{\mathcal{B}}_k^+ \right\|_{\mathcal{L}(\mathbf{L}_\nu^2)}^2 \|u\|_\nu^2 \,.
\end{aligned}$$

Since the operators $\widetilde{\mathcal{B}}_k^\pm : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ are bounded, we obtain uniform boundedness of the sequences $\left(\mathcal{M}_n^\tau \widetilde{\mathcal{B}}_k^\pm\right)$ and $\left(\mathcal{M}_n^\tau \mathcal{B}_k^\pm\right)$. If $u \in \mathbf{L}^\infty$, then the continuous function $\mathcal{B}_k^\pm \mu u : (-1,1) \longrightarrow \mathbb{C}$ satisfies the estimate

$$|(\mathcal{B}_k^\pm \mu u)(x)| \le \text{const} \left| h_0^k(-x \pm 2) \right| \le \text{const} \left[(1 \mp x)^{\pm\frac{1}{2}} + 1 \right],$$

where we took into account (7.7). Thus, applying Lemma 7.2,

$$\lim_{n\to\infty} \mathcal{M}_n^\tau \mathcal{B}_k^\pm \mu u = \mathcal{B}_k^\pm \mu u \qquad \text{in } \mathbf{L}_\nu^2, \text{ for } u \in \mathbf{L}^\infty.$$

With the help of the Banach-Steinhaus theorem, we get strong convergence of $\mathcal{M}_n^\tau \mathcal{B}_k^\pm \longrightarrow \mathcal{B}_k^\pm$ as well as $\mathcal{M}_n^\tau \widetilde{\mathcal{B}}_k^\pm \longrightarrow \widetilde{\mathcal{B}}_k^\pm$ in $\mathbf{L}_\nu^2$.

For convergence of the adjoint operators, we remark that, for a continuous function $\chi : [-1,1] \longrightarrow \mathbb{C}$ vanishing in a neighborhood of ± 1, the operator $\chi \mathcal{B}_k^\pm$ is an integral operator with continuous kernel so that $\chi \mathcal{B}_k^\pm : \mathbf{L}_\nu^2 \longrightarrow \mathbf{C}[-1,1]$ is compact. Hence, by Lemma 7.3, $\left(\mathcal{M}_n^\tau \chi \mathcal{B}_k^\pm \mathcal{L}_n\right) \in \mathfrak{J}$. In particular, $(\mathcal{M}_n^\tau \chi \mathcal{B}_k \mathcal{L}_n)^*$ converges strongly in $\mathbf{L}_\nu^2$. Now, let $f \in \mathbf{C}[-1,1]$ be vanishing in a neighborhood of ± 1. The set of these functions is dense in $\mathbf{L}_\nu^2$. Moreover, there exists a continuous and real valued function χ vanishing in a neighborhood of ± 1 such that $\chi f = f$. Due to $(\mathcal{M}_n^\tau \chi \mathcal{L}_n)^* = \mathcal{M}_n^\tau \chi \mathcal{L}_n$ (cf. [10, (3.13)]) and the uniform boundedness of the sequence $\left(\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n\right)$, we get

$$(\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n)^* (\mathcal{L}_n f - \mathcal{M}_n^\tau f) \longrightarrow 0$$

and

$$\begin{aligned}
(\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n)^* \mathcal{M}_n^\tau f &= (\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n)^* \mathcal{M}_n^\tau \chi \mathcal{L}_n \mathcal{M}_n^\tau f \\
&= (\mathcal{M}_n^\tau \chi \mathcal{B}_k^\pm \mathcal{L}_n)^* \mathcal{M}_n^\tau f \longrightarrow (\chi \mathcal{B}_k^\pm)^* f = (\mathcal{B}_k^\pm)^* f
\end{aligned}$$

in $\mathbf{L}_\nu^2$. The lemma is proved. □

Recall the following version of Lebesgue's dominant convergence theorem.

LEMMA 7.6. *If* $\xi, \eta \in \ell^2$, $\xi^n = (\xi_k^n)_{k=1}^\infty$, $|\xi_k^n| \le |\eta_k|\ \forall\, k = 0,1,2,\ldots, \forall\, n \ge n_0$ *and if* $\lim_{n\to\infty} \xi_k^n = \xi_k\ \forall\, k = 0,1,2,\ldots$, *then* $\lim_{n\to\infty} \|\xi^n - \xi\|_{\ell^2} = 0$.

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For the determination of the limit operators $\mathcal{W}^{3/4}(\mathcal{M}_n^\tau \mathcal{B}_k^\pm \mathcal{L}_n)$, we need the well known relations

(7.9) $$\mathcal{S}\sigma T_j = -\mathbf{i} U_{j-1}, \quad \mathcal{S}\widetilde{p}_j = \mathbf{i} R_j, \quad j = 0, 1, 2, \ldots, \quad U_{-1} := 0\,.$$

LEMMA 7.7. *Let $j \geq 0$ be fixed and $\tau \in \{\sigma, \mu\}$. Then,*

(7.10) $$\lim_{n\to\infty} \left(2 + x_{n-j,n}^\tau - \sqrt{(2 + x_{n-j,n}^\tau)^2 - 1}\right)^n = e^{-\left(j+\frac{1}{2}\right)\pi}.$$

Moreover,

(7.11) $$\left(2 + x_{jn}^\tau - \sqrt{(2 + x_{jn}^\tau)^2 - 1}\right)^\ell \leq e^{-\ell + \left(j - \frac{1}{2}\right)\frac{\ell}{n}}, \quad 1 \leq j \leq n,\ n, \ell \in \mathbb{N}.$$

Proof. Set

$$\gamma_n = \gamma_n^\tau := 1 + x_{n-j,n}^\tau = \begin{cases} 2\sin^2 \frac{j+\frac{1}{2}}{2n}\pi & : \tau = \sigma, \\ 2\sin^2 \frac{j+\frac{1}{2}}{2\left(n+\frac{1}{2}\right)}\pi & : \tau = \mu. \end{cases}$$

Then,

$$\left(2 + x_{n-j,n}^\tau - \sqrt{(2 + x_{n-j,n}^\tau)^2 - 1}\right)^n = \left(1 + \gamma_n - \sqrt{(1+\gamma_n)^2 - 1}\right)^n = \frac{1}{(1+\delta_n)^n},$$

where $\delta_n = \gamma_n + \sqrt{(1+\gamma_n)^2 - 1}$, $\lim_{n\to\infty} n\delta_n = \left(j + \frac{1}{2}\right)\pi$, and

$$(1+\delta_n)^n = \left[(1+\delta_n)^{\frac{1}{\delta_n}}\right]^{n\delta_n} \longrightarrow e^{\left(j+\frac{1}{2}\right)\pi}.$$

This proves (7.10). For the second relation we define the mapping $h : [0, \pi] \longrightarrow \mathbb{R}$ by

$$h(s) = \frac{2 + \cos s - \sqrt{(2+\cos s)^2 - 1}}{e^{\frac{s}{\pi}}} = \frac{1}{e^{\frac{s}{\pi}}\left(2 + \cos s + \sqrt{(2+\cos s)^2 - 1}\right)} =: \frac{1}{f(s)}.$$

We show that $f(s) \geq e$ for $s \in [0, \pi]$. Indeed, since

$$\begin{aligned} f'(s) &= \frac{e^{\frac{s}{\pi}}}{\pi}\left(2 + \cos s + \sqrt{(2+\cos s)^2 - 1}\right) + e^{\frac{s}{\pi}}\left(-\sin s - \frac{(2+\cos s)\sin s}{\sqrt{(2+\cos s)^2 - 1}}\right) \\ &= e^{\frac{s}{\pi}}\left((2 + \cos s + \sqrt{(2+\cos s)^2 - 1}\right)\left(\frac{1}{\pi} - \frac{\sin s}{\sqrt{(2+\cos s)^2 - 1}}\right), \end{aligned}$$

we have $f'(s) = 0$ if and only if $s = s^* \in (0, \pi)$ such that $\sqrt{(2 + \cos s^*)^2 - 1} = \pi \sin s^*$. It follows that

$$f(s^*) = g(s^*) \quad \text{with} \quad g(s) = e^{\frac{s}{\pi}}\left(2 + \cos s + \pi \sin s\right).$$

Now,

$$g'(s) = e^{\frac{s}{\pi}}\left[\frac{2}{\pi} + \left(\frac{1}{\pi} + \pi\right)\cos s\right] = 0$$

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if and only if $s = s_0 \in (0, \pi)$ such that $\cos s_0 = -\frac{2}{1+\pi^2}$. Hence,

$$\begin{aligned} g(s_0) &= e^{\frac{s_0}{\pi}} \left(2 - \frac{2}{1+\pi^2} + \pi \sqrt{1 - \frac{4}{(1+\pi^2)^2}} \right) \\ &= e^{\frac{s_0}{\pi}} \left(\frac{2\pi^2}{1+\pi^2} + \frac{\pi}{1+\pi^2} \sqrt{(1+\pi^2)^2 - 4} \right) \\ &\geq e^{\frac{s_0}{\pi}} \left[\frac{2\pi^2}{1+\pi^2} + \frac{\pi\left(\pi + \frac{3}{\pi}\right)}{1+\pi^2} \right] > e. \end{aligned}$$

Together with $g(0) = 3 > e$ and $g(\pi) = e$, we get $g(s) \geq e$ for $s \in [0, \pi]$. This implies $f(s^*) \geq e$, which, together with $f(0) = 3 + \sqrt{8} > e$ and $f(\pi) = e$, yields $f(s) \geq e$ for all $s \in [0, \pi]$. Using this result we conclude that

$$\frac{\left(2 + x_{jn}^\sigma - \sqrt{(2 + x_{jn}^\sigma)^2 - 1}\right)^\ell}{e^{\left(j - \frac{1}{2}\right)\frac{\ell}{n}}} \leq e^{-\ell},$$

which proves the second relation. The case $\tau = \mu$ can be treated in the same way. □

Since

$$1 - x_{jn}^\tau = \begin{cases} 2\sin^2 \frac{j - \frac{1}{2}}{2n}\pi & : \tau = \sigma, \\ 2\sin^2 \frac{j\pi}{2\left(n + \frac{1}{2}\right)} & : \tau = \mu, \end{cases}$$

one can analogously prove the following lemma.

LEMMA 7.8. *Let $j \geq 1$ be fixed and $\tau \in \{\sigma, \mu\}$. Then,*

$$\lim_{n\to\infty} \left(2 - x_{jn}^\tau - \sqrt{(2 - x_{jn}^\tau)^2 - 1}\right)^n = \begin{cases} e^{-\left(j - \frac{1}{2}\right)\pi} & : \tau = \sigma, \\ e^{-j\pi} & : \tau = \mu. \end{cases} \tag{7.12}$$

Moreover,

$$\left(2 - x_{jn}^\tau - \sqrt{(2 - x_{jn}^\tau)^2 - 1}\right)^\ell \leq e^{-\left(j - \frac{1}{2}\right)\frac{\ell}{n}}, \quad 1 \leq j \leq n,\ n, \ell \in \mathbb{N}. \tag{7.13}$$

For $k \in \mathbb{N}$, $a, b \in \mathbb{C}$, and $b \neq a$, it follows from the formula

$$b^k - a^k = (b - a) \sum_{s=1}^{k} a^{k-s} b^{s-1}$$

by dividing by $a^k b^k (a - b)$ that

$$\frac{1}{(a - b)a^k} = \frac{1}{b^k} \left(\frac{1}{a - b} - \sum_{s=1}^{k} \frac{b^{s-1}}{a^s} \right).$$

In case $a = y + x \pm 2$ and $b = x_0 + x \pm 2$, we conclude

$$\frac{1}{(y - x_0)(y + x \pm 2)^k} = \frac{1}{(x_0 + x \pm 2)^k} \left[\frac{1}{y - x_0} - \sum_{t=1}^{k} \frac{(x_0 + x \pm 2)^{t-1}}{(y + x \pm 2)^t} \right]. \tag{7.14}$$

LEMMA 7.9. *For $\tau \in \{\sigma, \mu\}$ and $k_0 \in \mathbb{N}$, the strong limits*

$$\mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n) = \Theta \quad \textit{and} \quad \mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{B}_{k_0}^+ \mathcal{L}_n) = \left[\ a_{jk}^\tau\ \right]_{j,k=0}^\infty$$

exist, where Θ is the zero operator in ℓ^2 and where

$$\begin{aligned} a_{jk}^\sigma = {} & \frac{2}{\pi \mathbf{i}} \frac{(-1)^{k_0} \left(k+\frac{1}{2}\right)\left(j+\frac{1}{2}\right)^{2k_0-2}}{\left[\left(k+\frac{1}{2}\right)^2 + \left(j+\frac{1}{2}\right)^2\right]^{k_0}} \\ & - (-1)^k e^{-(j+\frac{1}{2})\pi} \sum_{t=1}^{k_0} \frac{2}{\pi \mathbf{i}} \frac{(-1)^{k_0-t}\left(j+\frac{1}{2}\right)^{2k_0-2t+1}}{\left[\left(k+\frac{1}{2}\right)^2 + \left(j+\frac{1}{2}\right)^2\right]^{k_0-t+1}} \\ & \cdot \frac{1}{(t-1)!} \sum_{l=0}^{t-1} \binom{t-1}{l} \sum_{s=0}^{l-1} \frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}} t \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{l-s}, \end{aligned}$$

$$\begin{aligned} a_{jk}^\mu = {} & \frac{2}{\pi \mathbf{i}} \frac{(-1)^{k_0}(k+1)(j+1)^{2k_0-2}}{\left[(k+1)^2 + (j+1)^2\right]^{k_0}} \\ & + (-1)^k e^{-(j+1)\pi} \sum_{t=1}^{k_0} \frac{2}{\pi \mathbf{i}} \frac{(-1)^{k_0-t}(k+1)(j+1)^{2k_0-2t}}{\left[(k+1)^2 + (j+1)^2\right]^{k_0-t}} \\ & \cdot \frac{1}{(t-1)!} \sum_{s=0}^{t-2} \frac{\chi_s^{t-1}}{2^{\frac{t-1+s}{2}}} \left[\frac{(j+1)\pi}{\sqrt{2}}\right]^{t-1-s}, \end{aligned}$$

with certain real numbers χ_t and χ_s^t. Moreover, the respective sequences of the adjoint operators converge strongly.

Proof. We have to show that the sequence $\left(\mathcal{E}_n^{(3)} \mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n (\mathcal{E}_n^{(3)})^{-1} \mathcal{L}_n^{(3)}\right)$ converges strongly to Θ in ℓ^2. For that, we investigate the convergence at the elements $e_m = (\delta_{j,m})_{j=0}^\infty$, $m = 0, 1, 2, \ldots$, of the standard basis of ℓ^2. For $n > m \geq 0$, we can write

$$\begin{aligned} \mathcal{E}_n^{(3)} \mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n (\mathcal{E}_n^{(3)})^{-1} \mathcal{L}_n^{(3)} e_m &= \mathcal{V}_n^\tau \mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n (\mathcal{V}_n^\tau)^{-1} \mathcal{P}_n e_m \\ &= \left[\ \sqrt{\frac{1+x_{jn}^\tau}{1+x_{m+1,n}^\tau}}\, \mathcal{B}_{k_0}^- \widetilde{\ell}_{m+1,n}^\tau(x_{jn}^\tau)\ \right]_{j=1}^n, \end{aligned}$$

where $\left[\ \xi_{j-1}\ \right]_{j=1}^n \in \mathbb{C}^n$ is identified with $(\xi_0, \ldots, \xi_{n-1}, 0, \ldots) \in \ell^2$. Let us consider the case $\tau = \sigma$. For $k_0 \in \mathbb{N}$ fixed, we compute $\mathcal{B}_{k_0}^- \widetilde{\ell}_{kn}^\sigma(x_{jn}^\sigma)$, $j, k = 1, \ldots, n$. Using, for $n \geq 1$,

$$T_n'(x_{kn}^\sigma) = \sqrt{\frac{2}{\pi}} \frac{n(-1)^{k+1}}{\varphi(x_{kn}^\sigma)}$$

and

$$\frac{1}{\pi} \int_{-1}^1 \frac{1-y}{y-x} \sigma(y) T_n(y)\, dy = \frac{1}{\pi} \int_{-1}^1 \frac{1-x-(y-x)}{y-x} \sigma(y) T_n(y)\, dy = (1-x) U_{n-1}(x), \quad -1 < x < 1,$$

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we find by applying equation (7.14)

$$\begin{aligned}
\mathcal{B}_{k_0}^{\pm}\widetilde{\ell}_{kn}^{\sigma}(x) &= \frac{\nu(x_{kn}^{\sigma})}{T_n'(x_{kn}^{\sigma})}\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)T_n(y)}{(y-x_{kn}^{\sigma})(y+x\mp 2)^{k_0}}\,dy \\
&= \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n}\varphi(x_{kn}^{\sigma})\nu(x_{kn}^{\sigma})\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)T_n(y)}{(y-x_{kn}^{\sigma})(y+x\mp 2)^{k_0}}\,dy \\
&= \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n}\frac{1+x_{kn}^{\sigma}}{(x_{kn}^{\sigma}+x\mp 2)^{k_0}}\left[\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)T_n(y)}{y-x_{kn}^{\sigma}}\,dy\right. \\
&\qquad\left.-\sum_{t=1}^{k_0}(x_{kn}^{\sigma}+x\mp 2)^{t-1}\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)T_n(y)}{(y+x\mp 2)^{t}}\,dy\right] \\
&= \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n\mathbf{i}}\frac{1+x_{kn}^{\sigma}}{(x_{kn}^{\sigma}+x\mp 2)^{k_0}}\left[(1\mp x)^{k_0-1}(1-x_{kn}^{\sigma})U_{n-1}(x_{kn}^{\sigma})\right. \\
&\qquad\left.-\sum_{t=1}^{k_0}(x_{kn}^{\sigma}+x\mp 2)^{t-1}(1\mp x)^{k_0-t}\frac{1}{\pi}\int_{-1}^{1}\frac{(1\mp x)^{t-1}\mu(y)T_n(y)}{(y+x\mp 2)^{t}}\,dy\right].
\end{aligned}$$

Thus, since $U_{n-1}(x_{kn}^{\sigma}) = (-1)^{k+1}\sqrt{\frac{2}{\pi}}\,\sigma(x_{kn}^{\sigma})$,

$$\begin{aligned}
\mathcal{B}_{k_0}^{\pm}\widetilde{\ell}_{kn}^{\sigma}(x) &= \frac{1}{n\mathbf{i}}\frac{\varphi(x_{kn}^{\sigma})(1\mp x)^{k_0-1}}{(x_{kn}^{\sigma}+x\mp 2)^{k_0}} - \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n}\frac{1+x_{kn}^{\sigma}}{(x_{kn}^{\sigma}+x\mp 2)^{k_0}} \\
&\qquad\cdot\sum_{t=1}^{k_0}(x_{kn}^{\sigma}+x\mp 2)^{t-1}(1\mp x)^{k_0-t}(-1)^{t-1}\widetilde{h}_n^t(-x\pm 2).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sqrt{\frac{1+x_{jn}^{\sigma}}{1+x_{kn}^{\sigma}}}\mathcal{B}_{k_0}^{\pm}\widetilde{\ell}_{kn}^{\sigma}(x_{jn}^{\sigma}) \\
&= \frac{1}{n\mathbf{i}}\frac{\sqrt{1-x_{kn}^{\sigma}}\sqrt{1+x_{jn}^{\sigma}}(1\mp x_{jn}^{\sigma})^{k_0-1}}{(x_{kn}^{\sigma}+x_{jn}^{\sigma}\mp 2)^{k_0}} - \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n}\frac{\sqrt{1+x_{jn}^{\sigma}}\sqrt{1+x_{kn}^{\sigma}}}{(x_{kn}^{\sigma}+x_{jn}^{\sigma}\mp 2)^{k_0}} \\
&\qquad\cdot\sum_{t=1}^{k_0}(x_{kn}^{\sigma}+x_{jn}^{\sigma}\mp 2)^{t-1}(1\mp x_{jn}^{\sigma})^{k_0-t}(-1)^{t-1}\widetilde{h}_n^t(-x_{jn}^{\sigma}\pm 2) \\
&= \frac{1}{n\mathbf{i}}\frac{\sqrt{1-x_{kn}^{\sigma}}\sqrt{1+x_{jn}^{\sigma}}}{x_{kn}^{\sigma}+x_{jn}^{\sigma}\mp 2}\frac{(1\mp x_{jn}^{\sigma})^{k_0-1}}{(x_{kn}^{\sigma}+x_{jn}^{\sigma}\mp 2)^{k_0-1}} - \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n}\frac{\sqrt{1+x_{kn}^{\sigma}}}{x_{kn}^{\sigma}+x_{jn}^{\sigma}\mp 2} \\
&\qquad\cdot\sum_{t=1}^{k_0}\frac{(1\mp x_{jn}^{\sigma})^{k_0-t}}{(x_{kn}^{\sigma}+x_{jn}^{\sigma}\mp 2)^{k_0-t}}\sqrt{1+x_{jn}^{\sigma}}(-1)^{t-1}\widetilde{h}_n^t(-x_{jn}^{\sigma}\pm 2)\,.
\end{aligned} \tag{7.15}$$

Now we consider $\tau = \mu$. For the determination of $\mathcal{B}_{k_0}^{\pm}\widetilde{\ell}_{kn}^{\mu}(x_{jn}^{\mu}),\ j,k = 1,\ldots,n$, we need

$$P_n'(x_{kn}^{\mu}) = \sqrt{\frac{2}{\pi}}\frac{n+\frac{1}{2}}{\varphi(x_{kn}^{\mu})}\frac{(-1)^{k+1}}{\sqrt{1-x_{kn}^{\mu}}} \quad\text{and}\quad R_n(x_{kn}^{\mu}) = \sqrt{\frac{2}{\pi}}\frac{(-1)^k}{\sqrt{1+x_{kn}^{\mu}}}.$$

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We have

$$\begin{aligned}
\mathcal{B}^{\pm}_{k_0}\widetilde{\ell}^{\mu}_{kn}(x) &= \frac{\nu(x^{\mu}_{kn})}{P'_n(x^{\mu}_{kn})}\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)P_n(y)}{(y-x^{\mu}_{kn})(y+x\mp 2)^{k_0}}\,dy \\
&= \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}\nu(x^{\mu}_{kn})\varphi(x^{\mu}_{kn})\sqrt{1-x^{\mu}_{kn}}}{n+\frac{1}{2}}\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)P_n(y)}{(y-x^{\mu}_{kn})(y+x\mp 2)^{k_0}}\,dy \\
&= \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n+\frac{1}{2}}\frac{(1+x^{\mu}_{kn})\sqrt{1-x^{\mu}_{kn}}}{(x^{\mu}_{kn}+x\mp 2)^{k_0}}\left[\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)P_n(y)}{y-x^{\mu}_{kn}}\,dy\right. \\
&\qquad\left.-\sum_{t=1}^{k_0}(x^{\mu}_{kn}+x\mp 2)^{t-1}\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{(1\mp x)^{k_0-1}\mu(y)P_n(y)}{(y+x\mp 2)^{t}}\,dy\right] \\
&\overset{(7.9)}{=} \sqrt{\frac{\pi}{2}}\frac{(-1)^{k}}{(n+\frac{1}{2})\,\mathbf{i}}\frac{(1+x^{\mu}_{kn})\sqrt{1-x^{\mu}_{kn}}}{(x^{\mu}_{kn}+x\mp 2)^{k_0}}\left[(1\mp x)^{k_0-1}R_n(x^{\mu}_{kn})\right. \\
&\qquad\left.+\sum_{t=1}^{k_0}(x^{\mu}_{kn}+x\mp 2)^{t-1}(1\mp x)^{k_0-t}\frac{1}{\pi}\int_{-1}^{1}\frac{(1\mp x)^{t-1}\mu(y)P_n(y)}{(y+x\mp 2)^{t}}\,dy\right] \\
&= \frac{1}{(n+\frac{1}{2})\,\mathbf{i}}\frac{\varphi(x^{\mu}_{kn})(1\mp x)^{k_0-1}}{(x^{\mu}_{kn}+x\mp 2)^{k_0}} + \sqrt{\frac{\pi}{2}}\frac{(-1)^{k}}{n+\frac{1}{2}}\frac{(1+x^{\mu}_{kn})\sqrt{1-x^{\mu}_{kn}}}{x^{\mu}_{kn}+x\mp 2} \\
&\qquad\cdot\sum_{t=1}^{k_0}(x^{\mu}_{kn}+x\mp 2)^{t-1}(1\mp x)^{k_0-t}\frac{1}{\mathbf{i}\pi}\int_{-1}^{1}\frac{(1\mp x)^{t-1}\mu(y)P_n(y)}{(y+x\mp 2)^{t}}\,dy\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sqrt{\frac{1+x^{\mu}_{jn}}{1+x^{\mu}_{kn}}}\mathcal{B}^{\pm}_{k_0}\widetilde{\ell}^{\mu}_{kn}(x^{\mu}_{jn}) \\
&\quad= \frac{1}{(n+\frac{1}{2})\,\mathbf{i}}\frac{\sqrt{1-x^{\mu}_{kn}}\sqrt{1+x^{\mu}_{jn}}(1\mp x^{\mu}_{jn})^{k_0-1}}{(x^{\mu}_{kn}+x^{\mu}_{jn}\mp 2)^{k_0}} \\
&\qquad+\sqrt{\frac{\pi}{2}}\frac{(-1)^{k}}{n+\frac{1}{2}}\frac{\sqrt{1+x^{\mu}_{jn}}\varphi(x^{\mu}_{kn})}{(x^{\mu}_{kn}+x^{\mu}_{jn}\mp 2)^{k_0}} \\
&\qquad\quad\cdot\sum_{t=1}^{k_0}(x^{\mu}_{kn}+x^{\mu}_{jn}\mp 2)^{t-1}(1\mp x^{\mu}_{jn})^{k_0-t}(-1)^{t-1}h^{t}_{n}(-x^{\mu}_{jn}\pm 2)
\end{aligned}\tag{7.16}$$

$$\begin{aligned}
&\quad= \frac{1}{(n+\frac{1}{2})\,\mathbf{i}}\frac{\sqrt{1-x^{\mu}_{kn}}\sqrt{1+x^{\mu}_{jn}}}{x^{\mu}_{kn}+x^{\mu}_{jn}\mp 2}\frac{(1\mp x^{\mu}_{jn})^{k_0-1}}{(x^{\mu}_{kn}+x^{\mu}_{jn}\mp 2)^{k_0-1}} \\
&\qquad+\sqrt{\frac{\pi}{2}}\frac{(-1)^{k}}{n+\frac{1}{2}}\frac{\varphi(x^{\mu}_{kn})}{x^{\mu}_{kn}+x^{\mu}_{jn}\mp 2} \\
&\qquad\quad\cdot\sum_{t=1}^{k_0}\frac{(1\mp x^{\mu}_{jn})^{k_0-t}}{(x^{\mu}_{kn}+x^{\mu}_{jn}\mp 2)^{k_0-t}}\sqrt{1+x^{\mu}_{jn}}\,(-1)^{t-1}h^{t}_{n}(-x^{\mu}_{jn}\pm 2)\,.
\end{aligned}$$

Let $-1 < x < 1$. By (7.6) we have

$$
\begin{aligned}
&\left|h_n^t(-x-2)\right| \\
&\le \text{const}\left|\sqrt{\frac{x+3}{x+1}}\sum_{l=0}^{t-1}\binom{t-1}{l}\sum_{s=0}^{l-1}\frac{p_{t-l-2}(-x-2)\,p_s^l(-x-2)}{(x+3)^{t-1-\frac{l-s}{2}}}\,n^{l-s}(x+1)^{\frac{l-s}{2}}\right. \\
&\qquad\left.-\sum_{s=0}^{t-2}\frac{p_s^{t-1}(-x-2)}{(x+3)^{\frac{t-1+s}{2}}}\,n^{t-1-s}(x+1)^{\frac{t-1-s}{2}}\right|\left(x+2-\sqrt{(x+2)^2-1}\right)^n \\
&\le \text{const}\left[\frac{1}{\sqrt{x+1}}\sum_{l=0}^{t-1}\sum_{s=0}^{l-1}n^{l-s}\,(x+1)^{\frac{l-s}{2}}+\sum_{s=0}^{t-2}n^{t-1-s}(x+1)^{\frac{t-1-s}{2}}\right] \\
&\qquad\cdot\left(x+2-\sqrt{(x+2)^2-1}\right)^n.
\end{aligned}
$$

Estimate (7.11) leads to

$$
\begin{aligned}
&\left|\sqrt{1+x_{jn}^\tau}\,h_m^t(-x_{jn}^\tau-2)\right| \\
&\le \text{const}\left[\sum_{l=0}^{t-1}\sum_{s=0}^{l-1}m^{l-s}(x_{jn}^\tau+1)^{\frac{l-s}{2}}+\sqrt{1+x_{jn}^\tau}\sum_{s=0}^{t-2}m^{t-1-s}(x_{jn}^\tau+1)^{\frac{t-1-s}{2}}\right] \\
&\qquad\cdot\left(2+x_{jn}^\tau-\sqrt{(2+x_{jn}^\tau)^2-1}\right)^m \\
&\le \text{const}\left[\sum_{l=0}^{t-1}\sum_{s=0}^{l-1}m^{l-s}\left(\frac{n+1-j}{n}\right)^{l-s}\right. \\
&\qquad\left.+\frac{n+1-j}{n}\sum_{s=0}^{t-2}m^{t-1-s}\left(\frac{n+1-j}{n}\right)^{t-s}\right]e^{-m+\left(j+\frac{1}{2}\right)\frac{m}{n}},
\end{aligned}
\tag{7.17}
$$

where we used $\sqrt{1+x_{jn}^\tau} \le \text{const}\frac{n+1-j}{n}$. Consequently,

$$
\begin{aligned}
&\left|\sqrt{1+x_{jn}^\tau}\,h_n^t(-x_{jn}^\tau-2)\right| \\
&\le \text{const}\left[\sum_{l=0}^{t-1}\sum_{s=0}^{l-1}(n+1-j)^{l-s}+\frac{1}{n}\sum_{s=0}^{t-2}(n+1-j)^{t-s}\right]e^{-n+j} \\
&\le \text{const}(n+1-j)^{t-1}e^{-n+j} \le \text{const}, \quad j=1,\dots,n.
\end{aligned}
\tag{7.18}
$$

Analogously, one can show that

$$
\left|\sqrt{1+x_{jn}^\tau}\,\widetilde{h}_n^t(-x_{jn}^\tau-2)\right| \le \text{const}(n+1-j)^{t-1}e^{-n+j} \le \text{const}, \tag{7.19}
$$

$j=1,\dots,n$, as well as

$$
\left|\frac{\widetilde{h}_n^t(-x_{jn}^\tau+2)}{\sqrt{1-x_{jn}^\tau}}\right| \le \text{const}\; j^{t-1}e^{-j} \le \text{const}, \quad j=1,\dots,n, \tag{7.20}
$$

$$
\left|h_n^t(-x_{jn}^\tau+2)\right| \le \text{const}\; j^{t-1}e^{-j} \le \text{const}, \quad j=1,\dots,n. \tag{7.21}
$$

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Thus, for fixed $m \in \mathbb{N}_0$, the terms in (7.16) without the factor $\frac{1}{n+\frac{1}{2}}$ are bounded and

$$\left| \sqrt{\frac{1+x^\tau_{jn}}{1+x^\tau_{m+1,n}}} \mathcal{B}^-_{k_0} \widetilde{\ell}^\tau_{m+1,n}(x^\tau_{jn}) \right| \leq \text{const}\, \frac{1}{n} \leq \text{const}\, \frac{1}{j}, \quad j = 1, \ldots, n.$$

With the help of Lemma 7.6, we obtain $\mathcal{W}^3(\mathcal{M}^\tau_n \mathcal{B}^-_{k_0} \mathcal{L}_n) = \Theta$.

Now, we turn to the sequences $(\mathcal{M}^\tau_n \mathcal{B}^+_{k_0} \mathcal{L}_n)$. Defining

$$\widetilde{s}^{(n)}_{jk} := \frac{1}{n\mathbf{i}} \frac{\sqrt{1-x^\tau_{jn}}}{2-x^\tau_{kn}-x^\tau_{jn}} \quad \text{and} \quad \widetilde{b}^{(n)}_{jk} := \frac{1-x^\tau_{jn}}{2-x^\tau_{kn}-x^\tau_{jn}},$$

we have, for fixed $j, k \geq 1$,

$$\lim_{n\to\infty} \widetilde{s}^{(n)}_{jk} = \frac{\sqrt{2}}{\pi\mathbf{i}} \begin{cases} \dfrac{j-\frac{1}{2}}{\left(k-\frac{1}{2}\right)^2 + \left(j-\frac{1}{2}\right)^2} & : \tau = \sigma, \\ \dfrac{j}{k^2+j^2} & : \tau = \mu, \end{cases} \tag{7.22}$$

and

$$\lim_{n\to\infty} \widetilde{b}^{(n)}_{jk} = \begin{cases} \dfrac{\left(j-\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)^2 + \left(j-\frac{1}{2}\right)^2} & : \tau = \sigma, \\ \dfrac{j^2}{k^2+j^2} & : \tau = \mu. \end{cases} \tag{7.23}$$

Furthermore,

$$\lim_{n\to\infty} n\sqrt{1-x^\tau_{jn}} = \frac{1}{\sqrt{2}} \begin{cases} \left(j-\frac{1}{2}\right)\pi & : \tau = \sigma, \\ j\pi & : \tau = \mu. \end{cases}$$

With the help of (7.15), we obtain

$$\begin{aligned} &\sqrt{\frac{1+x^\sigma_{jn}}{1+x^\sigma_{kn}}}\, \mathcal{B}^+_{k_0} \widetilde{\ell}^\sigma_{kn}(x^\sigma_{jn}) \\ &\quad = -\widetilde{s}^{(n)}_{kj} \sqrt{1+x^\sigma_{jn}} (-1)^{k_0-1} \left[\widetilde{b}^{(n)}_{jk}\right]^{k_0-1} + \frac{(-1)^{k+1}}{n\mathbf{i}} \frac{\sqrt{1+x^\sigma_{kn}}}{2-x^\sigma_{kn}-x^\sigma_{jn}} \\ &\qquad \cdot \sum_{t=1}^{k_0} (-1)^{k_0-t} \left[\widetilde{b}^{(n)}_{jk}\right]^{k_0-t} \sqrt{1+x^\sigma_{jn}}\, (-1)^{t-1} c_n \widetilde{h}^t_n(2-x^\sigma_{jn}), \end{aligned}$$

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where, in view of (7.8) and (7.12),

$$\begin{aligned}
&\frac{c_n \widetilde{h}_n^t(2-x_{jn}^\sigma)}{n(2-x_{kn}^\sigma-x_{jn}^\sigma)} \\
&\quad = \frac{(-1)^{t-1}}{(t-1)!}\left(2-x_{jn}^\sigma-\sqrt{(2-x_{jn}^\sigma)^2-1}\right)^n \frac{1}{\sqrt{3-x_{jn}^\sigma}}\,\frac{1}{n\sqrt{1-x_{jn}^\sigma}} \\
&\qquad \cdot \widetilde{b}_{jk}^{(n)} \sum_{l=0}^{t-1}\binom{t-1}{l}\sum_{s=0}^{l-1}\frac{p_{t-l-2}(2-x_{jn}^\sigma)p_s^l(2-x_{jn}^\sigma)}{(3-x_{jn}^\sigma)^{t-1-\frac{l-s}{2}}}\left(n\sqrt{1-x_{jn}^\sigma}\right)^{l-s} \\
&\quad \longrightarrow \frac{(-1)^{t-1}}{(t-1)!}\,e^{-(j-\frac{1}{2})\pi}\frac{1}{\left(j-\frac{1}{2}\right)\pi}\,\frac{\left(j-\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)^2+\left(j-\frac{1}{2}\right)^2} \\
&\qquad\qquad \cdot \sum_{l=0}^{t-1}\binom{t-1}{l}\sum_{s=0}^{l-1}\frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}}\left[\frac{\left(j-\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{l-s}.
\end{aligned}$$

Together with (7.22) and (7.23), this leads to

$$\lim_{n\to\infty}\sqrt{\frac{1+x_{jn}^\sigma}{1+x_{kn}^\sigma}}\,\mathcal{B}_{k_0}^+\widetilde{\ell}_{kn}^\sigma(x_{jn}^\sigma)=a_{j-1,k-1}^\sigma, \quad j,k\ge 1.$$

In case $\tau=\mu$ we have, due to (7.16),

$$\begin{aligned}
&\sqrt{\frac{1+x_{jn}^\mu}{1+x_{kn}^\mu}}\,\mathcal{B}_{k_0}^+\widetilde{\ell}_{kn}^\mu(x_{jn}^\mu) \\
&\quad = \frac{n}{n+\frac{1}{2}}(-1)^{k_0}\widetilde{s}_{kj}^{(n)}\left[\widetilde{b}_{jk}^{(n)}\right]^{k_0-1}\sqrt{1+x_{jn}^\mu}-\sqrt{\frac{\pi}{2}}\frac{(-1)^k n}{n+\frac{1}{2}}\,\frac{\varphi(x_{kn}^\mu)}{n(2-x_{kn}^\mu-x_{jn}^\mu)} \\
&\qquad\qquad \cdot \sum_{t=1}^{k_0}(-1)^{k_0-t}\left[\widetilde{b}_{jk}^{(n)}\right]^{k_0-t}\sqrt{1+x_{jn}^\mu}(-1)^{t-1}h_n^t(2-x_{jn}^\mu),
\end{aligned}$$

where, taking into account (7.6) and (7.12),

$$\begin{aligned}
&h_n^t(2-x_{jn}^\mu) \\
&\quad = \frac{(-1)^{t-1}}{\sqrt{\pi}\mathbf{i}(t-1)!}\left[\sqrt{\frac{1-x_{jn}^\mu}{3-x_{jn}^\mu}}\sum_{l=0}^{t-1}\binom{t-1}{l}\cdot\right. \\
&\qquad\qquad \cdot\sum_{s=0}^{l-1}\frac{p_{t-l-2}(2-x_{jn}^\mu)p_s^l(2-x_{jn}^\mu)}{(3-x_{jn}^\mu)^{t-1-\frac{l-s}{2}}}\left(n\sqrt{1-x_{jn}^\mu}\right)^{l-s} \\
&\qquad \left. -\sum_{s=0}^{t-2}\frac{p_s^{t-1}(2-x_{jn}^\mu)}{(3-x_{jn}^\mu)^{\frac{t-1+s}{2}}}\left(n\sqrt{1-x_{jn}^\mu}\right)^{t-1-s}\right]\left(2-x_{jn}^\mu-\sqrt{(2-x_{jn}^\mu)^2-1}\right)^n \\
&\quad \longrightarrow \frac{(-1)^t}{(t-1)!\sqrt{\pi}\mathbf{i}}\sum_{s=0}^{t-2}\frac{p_s^{t-1}(1)}{2^{\frac{t-1+s}{2}}}\left(\frac{j\pi}{\sqrt{2}}\right)^{t-1-s} e^{-j\pi}.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty}\sqrt{\frac{1+x_{jn}^\mu}{1+x_{kn}^\mu}}\,\mathcal{B}_{k_0}^+\widetilde{\ell}_{kn}^\mu(x_{jn}^\mu)=a_{j-1,k-1}^\mu, \quad j,k\ge 1.$$

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With the help of Lemma 7.12, equation (7.13), and equations (7.20), (7.21), one can show that (cf. the end of the proof of Lemma 7.10)

$$\left|\sqrt{\frac{1+x_{jn}^\tau}{1+x_{kn}^\tau}}\,\mathcal{B}_{k_0}^+\widetilde{\ell}_{kn}^\tau(x_{jn}^\tau)\right| \le \frac{\text{const}}{j}, \quad 1 \le j \le n.$$

Now, Lemma 7.6 yields $\mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{B}_{k_0}^+ \mathcal{L}_n) = \left[\ a_{jk}^\tau\ \right]_{j,k=0}^\infty$. The convergence of the sequences of the adjoint operators can be seen analogously. □

LEMMA 7.10. *Let $\tau \in \{\sigma,\mu\}$ and $k_0 \in \mathbb{N}$. Then,*

$$\mathcal{W}^4(\mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n) = \left[\ a_{jk}\ \right]_{j,k=0}^\infty \quad \textit{and} \quad \mathcal{W}^4(\mathcal{M}_n^\tau \mathcal{B}_{k_0}^+ \mathcal{L}_n) = \Theta,$$

where

$$\begin{aligned} a_{jk} = {}& \frac{2}{\pi\mathbf{i}} \frac{\left(j+\frac12\right)^{2k_0-1}}{\left[\left(k+\frac12\right)^2+\left(j+\frac12\right)^2\right]^{k_0}} \\ &+ (-1)^k e^{-(j+\frac12)\pi} \sum_{t=1}^{k_0} \frac{2}{\pi\mathbf{i}} \frac{\left(k+\frac12\right)\left(j+\frac12\right)^{2k_0-2t}}{\left[\left(k+\frac12\right)^2+\left(j+\frac12\right)^2\right]^{k_0-t+1}} \\ &\qquad \cdot \frac{1}{(t-1)!} \sum_{l=0}^{t-1} \binom{t-1}{l} \sum_{s=0}^{l-1} \frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}} t \left[\frac{\left(j+\frac12\right)\pi}{\sqrt2}\right]^{l-s}, \end{aligned}$$

with certain real numbers χ_t and χ_s^t and where also the respective sequences of the respective adjoint operators converge strongly.

Proof. Again, we have to check the convergence of $\left(\mathcal{E}_n^{(4)}\mathcal{M}_n^\tau\mathcal{B}_{k_0}^-\mathcal{L}_n(\mathcal{E}_n^{(4)})^{-1}\mathcal{L}_n^{(4)}\right)$ at the elements e_m, $m = 0,1,2,\ldots$, of the standard basis of ℓ^2. For $n > m \ge 0$, we can write

$$\begin{aligned} \mathcal{E}_n^{(4)}\mathcal{M}_n^\tau\mathcal{B}_{k_0}^-\mathcal{L}_n(\mathcal{E}_n^{(4)})^{-1}\mathcal{L}_n^{(4)}e_m &= \widetilde{\mathcal{V}}_n^\tau \mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n (\widetilde{\mathcal{V}}_n^\tau)^{-1}\mathcal{P}_n e_m \\ &= \left[\sqrt{\frac{1+x_{n-j,n}^\tau}{1+x_{n-m,n}^\tau}}\,\mathcal{B}_{k_0}^- \widetilde{\ell}_{n-m,n}^\tau(x_{n-j,n}^\tau)\right]_{j=0}^{n-1}. \end{aligned}$$

Let j,k be fixed. Define

$$\begin{aligned} s_{jk}^{(n)} &:= \frac{1}{n\mathbf{i}} \frac{\sqrt{1+x_{jn}^\tau}}{x_{kn}^\tau + x_{jn}^\tau + 2} \\ &= \frac{1}{n\mathbf{i}}\frac{1}{\sqrt2} \begin{cases} \dfrac{\cos\frac{j-\frac12}{2n}\pi}{\cos^2\frac{k-\frac12}{2n}\pi + \cos^2\frac{j-\frac12}{2n}\pi} & : \tau = \sigma, \\[2ex] \dfrac{\cos\frac{j}{2(n+\frac12)}\pi}{\cos^2\frac{k}{2(n+\frac12)}\pi + \cos^2\frac{j}{2(n+\frac12)}\pi} & : \tau = \mu. \end{cases} \end{aligned} \tag{7.24}$$

Thus, we have

$$s_{n-j,n-k}^{(n)} = \frac{1}{n\mathbf{i}}\frac{1}{\sqrt{2}}\begin{cases} \dfrac{\sin\frac{j+\frac12}{2n}\pi}{\sin^2\frac{k+\frac12}{2n}\pi+\sin^2\frac{j+\frac12}{2n}\pi} & : \tau=\sigma, \\ \dfrac{\sin\frac{j+\frac12}{2(n+\frac12)}\pi}{\sin^2\frac{k+\frac12}{2(n+\frac12)}\pi+\sin^2\frac{j+\frac12}{2(n+\frac12)}\pi} & : \tau=\mu. \end{cases}$$

We obtain convergence of $s_{n-j,n-k}^{(n)}$,

$$s_{n-j,n-k}^{(n)} \longrightarrow \frac{1}{\sqrt{2}}\frac{2}{\pi\mathbf{i}}\frac{j+\frac12}{(k+\frac12)^2+(j+\frac12)^2}, \quad n\longrightarrow\infty, \quad \tau\in\{\sigma,\mu\}. \tag{7.25}$$

For $n \geq 1$, we set

$$b_{jk}^{(n)} := \frac{1+x_{jn}^\tau}{x_{kn}^\tau+x_{jn}^\tau+2} = \begin{cases} \dfrac{\cos^2\frac{j-\frac12}{2n}\pi}{\cos^2\frac{k-\frac12}{2n}\pi+\cos^2\frac{j-\frac12}{2n}\pi} & : \tau=\sigma, \\ \dfrac{\cos^2\frac{j}{2(n+\frac12)}\pi}{\cos^2\frac{k}{2(n+\frac12)}\pi+\cos^2\frac{j}{2(n+\frac12)}\pi} & : \tau=\mu, \end{cases} \tag{7.26}$$

and get

$$b_{n-j,n-k}^{(n)} \longrightarrow \frac{(j+\frac12)^2}{(k+\frac12)^2+(j+\frac12)^2}, \quad n\longrightarrow\infty.$$

Moreover,

$$n\sqrt{1+x_{n-j,n}^\tau} \longrightarrow \frac{(j+\frac12)\pi}{\sqrt{2}}, \quad n\longrightarrow\infty. \tag{7.27}$$

For $\tau=\sigma$, equation (7.8) leads to

$$\begin{aligned} &(-1)^n\sqrt{1+x_{n-j,n}^\sigma}\,c_n\widetilde{h}_n^t(-x_{n-j,n}^\sigma-2) \\ &= \frac{(-1)^{t-1}}{(t-1)!}\left(2+x_{n-j,n}^\sigma-\sqrt{(2+x_{n-j,n}^\sigma)^2-1}\right)^n\sqrt{x_{n-j,n}^\sigma+3} \\ &\quad\cdot\sum_{l=0}^{t-1}\binom{t-1}{l}\sum_{s=0}^{l-1}\frac{p_{t-l-2}(-x_{n-j,n}^\sigma-2)p_s^l(-x_{n-j,n}^\sigma-2)}{(x_{n-j,n}^\sigma+3)^{t-1-\frac{l-s}{2}}}n^{l-s}\left(x_{n-j,n}^\sigma+1\right)^{\frac{l-s}{2}}. \end{aligned}$$

Hence, by (7.10) we have

$$\begin{aligned} &(-1)^{n+t-1}\sqrt{1+x_{n-j,n}^\sigma}\,c_n\widetilde{h}_n^t(-x_{n-j,n}^\sigma-2) \\ &\quad\longrightarrow \frac{\sqrt{2}}{(t-1)!}e^{-(j+\frac12)\pi}\sum_{l=0}^{t-1}\binom{t-1}{l}\sum_{s=0}^{l-1}\frac{p_{t-l-2}(-1)\,p_s^l(-1)}{2^{t-1-\frac{l-s}{2}}}\left[\frac{(j+\frac12)\,\pi}{\sqrt{2}}\right]^{l-s}. \end{aligned}$$

Since, due to (7.15),

$$\begin{aligned} &\sqrt{\frac{1+x^\sigma_{n-j,n}}{1+x^\sigma_{n-k,n}}}\mathcal{B}^-_{k_0}\widetilde{\ell}^\sigma_{n-k,n}(x^\sigma_{n-j,n}) \\ &= \sqrt{1-x^\sigma_{n-k,n}}\, s^{(n)}_{n-j,n-k}\left[b^{(n)}_{n-j,n-k}\right]^{k_0-1} \\ &\quad + (-1)^k s^{(n)}_{n-k,n-j}\sum_{t=1}^{k_0}\left[b^{(n)}_{n-j,n-k}\right]^{k_0-t} \\ &\qquad \cdot (-1)^{n+t-1}\sqrt{1+x^\sigma_{n-j,n}}\, c_n \widetilde{h}^t_n(-x^\sigma_{n-j,n}-2), \end{aligned} \tag{7.28}$$

we obtain, together with (7.25) and (7.27), the convergence result

$$\begin{aligned} &\sqrt{\frac{1+x^\sigma_{n-j,n}}{1+x^\sigma_{n-k,n}}}\,\mathcal{B}^-_{k_0}\widetilde{\ell}^\sigma_{n-k,n}(x^\sigma_{n-j,n}) \\ &\longrightarrow \frac{2}{\pi\mathbf{i}}\frac{(j+\frac12)^{2k_0-1}}{[(k+\frac12)^2+(j+\frac12)^2]^{k_0}} + (-1)^k e^{-(j+\frac12)\pi}\sum_{t=1}^{k_0}\frac{2}{\pi\mathbf{i}}\frac{(k+\frac12)(j+\frac12)^{2k_0-2t}}{\left[(k+\frac12)^2+(j+\frac12)^2\right]^{k_0-t+1}} \\ &\qquad \cdot \frac{1}{(t-1)!}\sum_{l=0}^{t-1}\binom{t-1}{l}\sum_{s=0}^{l-1}\frac{p_{t-l-2}(-1)\,p^l_s(-1)}{2^{t-1-\frac{l-s}{2}}}\left[\frac{(j+\frac12)\pi}{\sqrt2}\right]^{l-s}, \end{aligned}$$

for j,k fixed and n tending to infinity. For the case $\tau=\mu$, we have due to (7.16)

$$\begin{aligned} &\sqrt{\frac{1+x^\mu_{n-j,n}}{1+x^\mu_{n-k,n}}}\mathcal{B}^-_{k_0}\widetilde{\ell}^\mu_{n-k,n}(x^\mu_{n-j,n}) \\ &= \sqrt{1-x^\mu_{n-k,n}}\frac{n}{n+\frac12}\, s^{(n)}_{n-j,n-k}\left[b^{(n)}_{n-j,n-k}\right]^{k_0-1} \\ &\quad + \frac{(-1)^k n}{n+\frac12}s^{(n)}_{n-k,n-j}\sqrt{1-x^\mu_{n-k,n}}\sqrt{\frac{\pi}{2}}\sum_{t=1}^{k_0}\left[b^{(n)}_{n-j,n-k}\right]^{k_0-t} \\ &\qquad \cdot (-1)^{n+t-1}\sqrt{1+x^\mu_{n-j,n}}\, h^t_n(-x^\mu_{n-j,n}-2). \end{aligned} \tag{7.29}$$

So we easily derive that

$$\sqrt{\frac{1+x^\mu_{n-j,n}}{1+x^\mu_{n-k,n}}}\mathcal{B}^-_{k_0}\widetilde{\ell}^\mu_{n-k,n}(x^\mu_{n-j,n}) \longrightarrow a_{jk}, \quad n\longrightarrow\infty.$$

Let $k\ge 0$ be fixed and $n>k$. In view of Lemma 7.11, we have

$$|s^{(n)}_{n-j,n-k}| \le \frac{\sqrt2}{\pi}\frac{j+\frac12}{(k+\frac12)^2+(j+\frac12)^2} + \text{const}\left(\frac{j+1}{n^2}\right) \le \frac{\text{const}}{j+1}, \quad 0\le j<n$$

as well as

$$|s^{(n)}_{n-k,n-j}| \le \frac{1}{\sqrt2\pi}\frac{1}{j+\frac12} + \text{const}\left(\frac{k+1}{n^2}\right) \le \frac{\text{const}}{j+1}, \quad 0\le j<n.$$

Moreover, due to the estimates (7.18) and (7.19),

$$\left|\sqrt{1+x^\sigma_{n-j,n}}\,\widetilde{h}^t_n(-x^\sigma_{n-j,n}-2)\right| \le \text{const}, \quad \left|\sqrt{1+x^\mu_{n-j,n}}\, h^t_n(-x^\mu_{n-j,n}-2)\right| \le \text{const},$$

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for $0 \le j < n$. With the help of (7.28) and (7.29) as well as

$$\sqrt{1 - x_{n-k,n}} \le \text{const}\, \frac{n+1-k}{n}\,, \quad \sqrt{1 + x_{n-k,n}} \le \text{const}\, \frac{k}{n}\,, \quad |b_{n-j,n-k}| \le \frac{\text{const}}{j},$$

we arrive at

$$\left| \sqrt{\frac{1 + x^\tau_{n-j,n}}{1 + x^\tau_{n-k,n}}}\, \mathcal{B}^-_{k_0} \widetilde{\ell}^\tau_{n-k,n}(x^\tau_{n-j,n}) \right| \le \frac{\text{const}}{j+1}, \quad j = 0, \dots, n-1.$$

Applying Lemma 7.6, we obtain $\mathcal{W}^4(\mathcal{M}^\tau_n \mathcal{B}^-_{k_0} \mathcal{L}_n) = \left[\ a_{jk}\ \right]_{j,k=0}^\infty$.

The proof of $\mathcal{W}^4(\mathcal{M}^\tau_n \mathcal{B}^+_{k_0} \mathcal{L}_n) = \Theta$ can be performed in the same way as the proof of $\mathcal{W}^3(\mathcal{M}^\tau_n \mathcal{B}^-_{k_0} \mathcal{L}_n) = \Theta$; cf. the proof of Lemma 7.9. The convergence of the sequences of the adjoint operators can be seen analogously. □

Let $n \in \mathbb{N}$. In what follows, we study the asymptotic behavior of the values $s^{(n)}_{jk}$ defined in (7.24).

LEMMA 7.11. *Let d be fixed with $0 < d < 1$. For $n \longrightarrow \infty$, we have*

$$s^{(n)}_{jk} = \begin{cases} \mathcal{O}\left(\frac{n+1-j}{n^2}\right) & : k \le d\,n,\ j \ge 1 \\ \mathcal{O}\left(\frac{1}{n}\right) & : k \ge 1,\ j \le d\,n \end{cases} \tag{7.30}$$

and

$$s^{(n)}_{jk} = \frac{1}{\sqrt{2}}\,\frac{2}{\pi \mathbf{i}}\, \frac{n + \frac12 - j}{(n+\frac12 - j)^2 + (n + \frac12 - k)^2} + \mathcal{O}\left(\frac{n+1-j}{n^2}\right), \quad 1 \le j,k \le n, \tag{7.31}$$

where the constants regarding the $\mathcal{O}$-terms are bounded by a constant which is independent of j, k, and n.

Proof. We prove the assertion only for $\tau = \sigma$. The proof for the other case is analogous. Define

$$g_1 : [0,1) \longrightarrow \mathbb{R}, \quad s \mapsto \frac{\cos \frac{\pi s}{2}}{1-s}.$$

Since this mapping is bounded and positive, we get

$$\sqrt{1 + x^\sigma_{jn}} = \sqrt{2} \cos \tfrac{j - \frac12}{2n}\pi = \sqrt{2}\, g_1\left(\tfrac{j-\frac12}{n}\right)\left(1 - \tfrac{j - \frac12}{n}\right) \le \text{const}\left(\tfrac{n-j+1}{n}\right). \tag{7.32}$$

Thus, we easily derive (7.30). Define the mapping

$$g_2 : \left[0, \frac{\pi}{2}\right]^2 \longrightarrow \mathbb{R}, \quad (s,t) \mapsto \frac{\sin s}{s}\, \frac{1}{\sin^2 s + \sin^2 t} - \frac{1}{s^2 + t^2}.$$

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One can show that this mapping is bounded. Thus,

$$\begin{aligned}
s^{(n)}_{n-j,n-k} &= \frac{1}{n\mathbf{i}}\frac{1}{\sqrt{2}}\frac{\sin\frac{j+\frac{1}{2}}{2n}\pi}{\sin^2\frac{k+\frac{1}{2}}{2n}\pi+\sin^2\frac{j+\frac{1}{2}}{2n}\pi} \\
&= \frac{\left(\frac{j+\frac{1}{2}}{2n}\pi\right)}{n\mathbf{i}}\frac{1}{\sqrt{2}}\left[\frac{1}{\left(\frac{k+\frac{1}{2}}{2n}\pi\right)^2+\left(\frac{j+\frac{1}{2}}{2n}\pi\right)^2}\right. \\
&\qquad\left.+\frac{\sin\frac{j+\frac{1}{2}}{2n}\pi/\frac{j+\frac{1}{2}}{2n}\pi}{\sin^2\frac{k+\frac{1}{2}}{2n}\pi+\sin^2\frac{j+\frac{1}{2}}{2n}\pi}-\frac{1}{\left(\frac{k+\frac{1}{2}}{2n}\pi\right)^2+\left(\frac{j+\frac{1}{2}}{2n}\pi\right)^2}\right] \\
&= \frac{1}{\sqrt{2}}\frac{2}{\pi\mathbf{i}}\frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2+(j+\frac{1}{2})^2}+g_2\left(\frac{k+\frac{1}{2}}{2n}\pi,\frac{j+\frac{1}{2}}{2n}\pi\right)\frac{\left(\frac{j+\frac{1}{2}}{2n}\pi\right)}{n\mathbf{i}}\frac{1}{\sqrt{2}}.
\end{aligned}$$

Replacing j and k by $n-j$ and $n-k$, respectively, we get (7.31). □

Analogously, we can prove the following lemma.

LEMMA 7.12. *Let d be fixed with $0 < d \le 1$. For $n \longrightarrow \infty$, we have*

$$\widetilde{s}^{(n)}_{jk} = \begin{cases} \mathcal{O}\left(\frac{1}{n^2}\right) & : k \ge d\,n,\ j \ge 1, \\ \mathcal{O}\left(\frac{1}{n}\right) & : k \ge 1,\ j \ge d\,n \end{cases}$$

and

$$\widetilde{s}^{(n)}_{jk} = \begin{cases} \frac{1}{\sqrt{2}}\frac{2}{\pi\mathbf{i}}\frac{j-\frac{1}{2}}{(j-\frac{1}{2})^2+(k-\frac{1}{2})^2}+\mathcal{O}\left(\frac{j}{n^2}\right) & : \tau=\sigma, \\ \frac{1}{\sqrt{2}}\frac{2}{\pi\mathbf{i}}\frac{j}{j^2+k^2}+\mathcal{O}\left(\frac{j}{n^2}\right) & : \tau=\mu, \end{cases} \qquad 1 \le j,k \le n,$$

where the constants regarding the $\mathcal{O}$-terms are bounded by a constant which is independent of j, k, and n.

LEMMA 7.13 (Satz III.5.1 in [6]). *Let $\vartheta(x) = (1-x)^\gamma(1+x)^\delta$ with $\gamma, \delta > -1$. If $f \in \mathbf{R}$ satisfies*

$$|f(x)| \le \mathrm{const}(1-x)^{\varepsilon-\frac{1+\gamma}{2}}(1+x)^{\varepsilon-\frac{1+\delta}{2}}, \quad -1 < x < 1,$$

for some $\varepsilon > 0$, then $\lim_{n\to\infty}\left\|\mathcal{L}^\vartheta_n f - f\right\|_\vartheta = 0$.

LEMMA 7.14. *Let $\tau \in \{\sigma,\mu\}$ and $k_0 \in \mathbb{N}$. Then, $\mathcal{W}^2(\mathcal{M}^\tau_n\mathcal{B}^\pm_{k_0}\mathcal{L}_n) = \Theta$, where Θ is the zero operator in $\mathbf{L}^2_\nu$ and where also the sequences of the adjoint operators converge strongly.*

Proof. Let $k_0 \in \mathbb{N}$. It suffices to show convergence on a dense subset. With the help of equation (7.17) and the Gaussian quadrature rule, we derive, for fixed $m < n$,

$$\begin{aligned}
&\left\|\mathcal{W}_n\mathcal{M}_n^\sigma\mathcal{B}_{k_0}^-\mathcal{L}_n\mathcal{W}_n\widetilde{p}_m\right\|_\nu^2 \\
&\quad= \left\|\mathcal{M}_n^\sigma\mathcal{B}_{k_0}^-\widetilde{p}_{n-1-m}\right\|_\nu^2 = \left\|\mathcal{L}_n^\sigma\nu\mathcal{B}_{k_0}^-\widetilde{p}_{n-1-m}\right\|_\mu^2 = \left\|\sqrt{1-x}\mathcal{L}_n^\sigma\nu\mathcal{B}_{k_0}^-\widetilde{p}_{n-1-m}\right\|_\sigma^2 \\
&\quad= \frac{\pi}{n}\sum_{j=1}^n(1+x_{jn}^\sigma)\left|\mathcal{B}_{k_0}^-\widetilde{p}_{n-1-m}(x_{jn}^\sigma)\right|^2 = \frac{\pi}{n}\sum_{j=1}^n(1+x_{jn}^\sigma)\left|h_{n-1-m}^{k_0}(-2-x_{jn}^\sigma)\right|^2 \\
&\quad\le \text{const}\,\frac{\pi}{n}\sum_{j=1}^n\left[\sum_{l=0}^{k_0-1}\sum_{s=0}^{l-1}(n-1-m)^{l-s}\left(\frac{n+1-j}{n}\right)^{l-s}\right. \\
&\qquad\left.+\sum_{s=0}^{k_0-2}(n-1-m)^{k_0-1-s}\left(\frac{n+1-j}{n}\right)^{k_0-s}\right]^2 e^{-2(n-1-m)+(2j-1)\frac{n-1-m}{n}} \\
&\quad\le \text{const}\,\frac{\pi}{n}\sum_{j=1}^n(n+1-j)^{2k_0-2}e^{-2(n-j)} \longrightarrow 0 \quad \text{for} \quad n\longrightarrow\infty.
\end{aligned}$$

In the last step we used the fact that the sequence $\sum_{j=1}^n j^p x^j$, is convergent for any $p\in\mathbb{N}_0$ and $|x|<1$. Using (7.21) and (7.13) for $\mathcal{B}_{k_0}^+$, we easily derive

$$\begin{aligned}
\left\|\mathcal{W}_n\mathcal{M}_n^\sigma\mathcal{B}_{k_0}^+\mathcal{L}_n\mathcal{W}_n\widetilde{p}_m\right\|_\nu^2 &= \frac{\pi}{n}\sum_{j=1}^n(1+x_{jn}^\sigma)\left|\mathcal{B}_{k_0}^+\widetilde{p}_{n-1-m}(x_{jn}^\sigma)\right|^2 \\
&\le \frac{2\pi}{n}\sum_{j=1}^n\left|h_{n-1-m}^{k_0}(2-x_{jn}^\sigma)\right|^2 \\
&\le \text{const}\,\frac{\pi}{n}\sum_{j=1}^n j^{2k_0-2}e^{-2j} \longrightarrow 0 \quad \text{for} \quad n\longrightarrow\infty.
\end{aligned}$$

The case $\tau=\mu$ can be treated in the same way.

Let us turn to the convergence of the adjoint operators. Let $f\in\mathbf{C}[-1,1]$ be vanishing in a neighborhood of ±1. Set $\widetilde{f}=\mathcal{J}_1^{-1}f$, where $\mathcal{J}_1$ is the isometry from (3.1). The set of such functions $\widetilde{f}$ is dense in $\mathbf{L}_\nu^2$. Now we choose a smooth function $\chi:[-1,1]\longrightarrow\mathbb{C}$ which vanishes in a neighborhood of ±1 such that $\chi f=f$. Since

$$\left\|\mathcal{M}_n^\tau\chi\mathcal{B}_k^\pm\mathcal{L}_n-\mathcal{L}_n\chi\mathcal{B}_k^\pm\mathcal{L}_n\right\|_\nu\longrightarrow 0$$

holds (cf. the proof of Lemma 7.3), convergence of

$$\begin{aligned}
\mathcal{W}_n(\mathcal{M}_n^\tau\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{M}_n^\tau\chi\mathcal{L}_n\mathcal{W}_n\widetilde{f} &= \mathcal{W}_n(\mathcal{M}_n^\tau\mathcal{B}_k^\pm\mathcal{L}_n)^*(\mathcal{M}_n^\tau\chi\mathcal{L}_n)^*\mathcal{W}_n\widetilde{f} \\
&= \mathcal{W}_n(\mathcal{M}_n^\tau\chi\mathcal{L}_n\mathcal{M}_n^\tau\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n\widetilde{f} \\
&= \mathcal{W}_n(\mathcal{M}_n^\tau\chi\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n\widetilde{f}\longrightarrow 0 \quad \text{in} \quad \mathbf{L}_\nu^2
\end{aligned} \tag{7.33}$$

is equivalent to

$$\mathcal{W}_n(\mathcal{L}_n\chi\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n\widetilde{f}=\mathcal{W}_n\left(\chi\mathcal{B}_k^\pm\right)^*\mathcal{W}_n\widetilde{f}\longrightarrow 0 \quad \text{in} \quad \mathbf{L}_\nu^2.$$

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But this convergence takes place since $\mathcal{W}_n$ tends weakly to zero and $\chi\mathcal{B}_k^\pm : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ is compact. Taking into account Lemma 7.13 , the relation

$$\mathcal{W}_n\mathcal{M}_n^\sigma\chi\mathcal{L}_n\mathcal{W}_n = J_1^{-1}\mathcal{L}_n^\sigma\chi\mathcal{J}_1$$

(cf. [10, (3.19)]), and the above convergence result, we derive

$$\begin{aligned}
&\left\|\mathcal{W}_n(\mathcal{M}_n^\sigma\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{M}_n^\sigma\chi\mathcal{L}_n\mathcal{W}_n\widetilde{f} - \mathcal{W}_n(\mathcal{M}_n^\sigma\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n\widetilde{f}\right\|_\nu \\
&\quad = \left\|\mathcal{W}_n(\mathcal{M}_n^\sigma\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n\mathcal{W}_n\mathcal{M}_n^\sigma\chi\mathcal{L}_n\mathcal{W}_n\widetilde{f} - \mathcal{W}_n(\mathcal{M}_n^\sigma\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n\widetilde{f}\right\|_\nu \\
&\quad \le \text{const}\left\|\mathcal{W}_n\mathcal{M}_n^\sigma\chi\mathcal{L}_n\mathcal{W}_n\widetilde{f} - \widetilde{f}\right\|_\nu = \text{const}\left\|J_1^{-1}\mathcal{L}_n^\sigma\chi f - J_1^{-1}f\right\|_\nu \\
&\quad = \text{const}\left\|\mathcal{L}_n^\sigma f - f\right\|_\nu \le \text{const}\left\|\mathcal{L}_n^\sigma f - f\right\|_\sigma \longrightarrow 0.
\end{aligned}$$

This implies together with convergence in (7.33) that $\mathcal{W}_n(\mathcal{M}_n^\sigma\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n \longrightarrow 0$ in $\mathbf{L}^2_\nu$. In case of nodes of the fourth kind, we use the formula

$$\mathcal{W}_n\mathcal{M}_n^\mu\chi\mathcal{L}_n\mathcal{W}_n = J_2^{-1}\sqrt{1-x}\,\mathcal{L}_n^\sigma\frac{1}{\sqrt{1-x}}\chi\mathcal{J}_2$$

(cf. [10, (3.20)]) and Lemma 7.2. We get

$$\begin{aligned}
&\left\|\mathcal{W}_n(\mathcal{M}_n^\mu\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{M}_n^\mu\chi\mathcal{L}_n\mathcal{W}_n\widetilde{f} - \mathcal{W}_n(\mathcal{M}_n^\mu\mathcal{B}_k^\pm\mathcal{L}_n)^*\mathcal{W}_n\widetilde{f}\right\|_\nu \\
&\quad \le \text{const}\left\|\sqrt{1-x}\,\mathcal{L}_n^\mu\frac{1}{\sqrt{1-x}}f - f\right\|_\nu = \text{const}\left\|\sqrt{1+x}\,\mathcal{M}_n^\mu\frac{1}{\sqrt{1+x}}f - f\right\|_\nu \\
&\quad \le \text{const}\left\|\mathcal{M}_n^\mu\frac{f}{\sqrt{1+x}} - \frac{f}{\sqrt{1+x}}\right\|_\nu \longrightarrow 0.
\end{aligned}$$

This completes the proof. □

For $k_0 \in \mathbb{N}$ and $\tau \in \{\sigma, \mu\}$, we define, in accordance with Lemma 7.9 and Lemma 7.10, the matrices $\mathbf{K}^\tau_{k_0} := \left[\ d^\tau_{jk}\ \right]_{j,k=0}^\infty$ and $\mathbf{K}_{k_0} := \left[\ d_{jk}\ \right]_{j,k=0}^\infty$ by

$$\begin{aligned}
d^\sigma_{jk} &= -(-1)^k e^{-(j+\frac12)\pi}\sum_{t=1}^{k_0}\frac{2}{\pi\mathbf{i}}\frac{(-1)^{k_0-t}\left(j+\frac12\right)^{2k_0-2t+1}}{\left[\left(k+\frac12\right)^2+\left(j+\frac12\right)^2\right]^{k_0-t+1}} \\
&\qquad\cdot\frac{1}{(t-1)!}\sum_{l=0}^{t-1}\binom{t-1}{l}\sum_{s=0}^{l-1}\frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}}\left[\frac{\left(j+\frac12\right)\pi}{\sqrt2}\right]^{l-s},
\end{aligned} \tag{7.34}$$

$$\begin{aligned}
d^\mu_{jk} &= (-1)^k e^{-(j+1)\pi}\sum_{t=1}^{k_0}\frac{2}{\pi\mathbf{i}}\frac{(-1)^{k_0-t}(k+1)(j+1)^{2k_0-2t}}{[(k+1)^2+(j+1)^2]^{k_0-t}} \\
&\qquad\cdot\frac{1}{(t-1)!}\sum_{s=0}^{t-2}\frac{\chi_s^{t-1}}{2^{\frac{t-1+s}{2}}}\left[\frac{(j+1)\pi}{\sqrt2}\right]^{t-1-s},
\end{aligned} \tag{7.35}$$

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and

$$d_{jk} = (-1)^k e^{-(j+\frac{1}{2})\pi} \sum_{t=1}^{k_0} \frac{2}{\pi \mathbf{i}} \frac{\left(k+\frac{1}{2}\right)\left(j+\frac{1}{2}\right)^{2k_0-2t}}{\left[\left(k+\frac{1}{2}\right)^2+\left(j+\frac{1}{2}\right)^2\right]^{k_0-t+1}} \cdot \frac{1}{(t-1)!} \sum_{l=0}^{t-1} \binom{t-1}{l} \sum_{s=0}^{l-1} \frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}} \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{l-s}. \tag{7.36}$$

LEMMA 7.15. *The operators* $\mathbf{K}_{k_0}^\tau, \mathbf{K}_{k_0} : \ell^2 \longrightarrow \ell^2$, $k_0 \in \mathbb{N}$, $\tau \in \{\sigma, \mu\}$, *are compact.*

Proof. We consider only $\mathbf{K}_{k_0}$. The proof for $\mathbf{K}_{k_0}^\tau$ is completely analogous. The estimates

$$|d_{jk}| \le \text{const}\, e^{-j\pi} \sum_{t=1}^{k_0} \frac{k+\frac{1}{2}}{(k+\frac{1}{2})^2+(j+\frac{1}{2})^2} \frac{(j+\frac{1}{2})^{2k_0-2t}(j+\frac{1}{2})^{k_0}}{\left[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\right]^{k_0-t}} \le \text{const} \frac{e^{-j\pi}(j+\frac{1}{2})^{k_0}}{k+\frac{1}{2}}$$

and

$$\sum_{j,k=0}^{n} |d_{jk}|^2 \le \text{const} \sum_{j=0}^{n} e^{-2j\pi}(j+\tfrac{1}{2})^{2k_0} \sum_{k=0}^{n} \frac{1}{(k+\frac{1}{2})^2}$$

show that

$$\sum_{j,k=0}^{\infty} |d_{jk}|^2 < \infty.$$

Consequently, for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\|\mathbf{K}_{k_0} - \mathcal{P}_n \mathbf{K}_{k_0} \mathcal{P}_n\|_{\mathcal{L}(\ell^2)} \le \|\mathbf{K}_{k_0} - \mathcal{P}_n \mathbf{K}_{k_0} \mathcal{P}_n\|_F < \varepsilon \quad \forall n > n_0,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Since the operators $\mathcal{P}_n \mathbf{K}_{k_0} \mathcal{P}_n$ are compact, the operator $\mathbf{K}_{k_0}$ is compact, too. □

Let us define the operators $\mathbf{A}, \mathbf{A}^\tau : \ell^2 \longrightarrow \ell^2$ as in (3.3), (3.4), (3.5) as well as the operators

$$\mathbf{K} = \sum_{k_0=1}^{m_-} \beta_{k_0}^- \mathbf{K}_{k_0} \quad \text{and} \quad \mathbf{K}^\tau = \sum_{k_0=1}^{m_+} \beta_{k_0}^+ \mathbf{K}_{k_0}^\tau, \quad \tau \in \{\sigma, \mu\}. \tag{7.37}$$

Then, using Proposition 3.2 and the Lemmas 7.5, 7.9, 7.10, 7.14, 7.15, we get Proposition 3.4.

8. Proof of Lemma 4.8. Assertion (a) was established in [10, Corollary 5.15]. In the case that the operators $\mathcal{B}_k^\pm$ do not occur, assertion (b) was also proved in [10, Lemma 5.16]. According to Propositions 3.2 and 3.4, it suffices to show that

$$\left([\mathbf{A}_{k_0}]_n^4 + [\mathbf{K}_{k_0}]_n^4 - \mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n\right)^o \in \mathfrak{J}_{-1}, \tag{8.1}$$

and

$$\begin{aligned} \left([\mathbf{A}_{k_0}^\sigma]_n^3 + [\mathbf{K}_{k_0}^\sigma]_n^3 - \mathcal{M}_n^\sigma \mathcal{B}_{k_0}^+ \mathcal{L}_n\right)^o &\in \mathfrak{J}_1, \\ \left([\mathbf{A}_{k_0}^\mu]_n^3 + [\mathbf{K}_{k_0}^\mu]_n^3 - \mathcal{M}_n^\mu \mathcal{B}_{k_0}^+ \mathcal{L}_n\right)^o &\in \mathfrak{J}_1, \end{aligned} \tag{8.2}$$

where

$$\mathbf{h}_{k_0}^{\pm}(x) = \frac{(\mp 1)^{k_0}}{\pi \mathbf{i}} \frac{x^{k_0-1}}{(1+x)^{k_0}},$$

$$\mathbf{A}_{k_0} = \left[2 \cdot \mathbf{h}_{k_0}^{-} \left(\frac{(j+\frac{1}{2})^2}{(k+\frac{1}{2})^2} \right) \frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2} \right]_{j,k=0}^{\infty},$$

and

$$\mathbf{A}_{k_0}^{\sigma} = \left[2 \cdot \mathbf{h}_{k_0}^{+} \left(\frac{(j+\frac{1}{2})^2}{(k+\frac{1}{2})^2} \right) \frac{1}{k+\frac{1}{2}} \right]_{j,k=0}^{\infty},$$

$$\mathbf{A}_{k_0}^{\mu} = \left[2 \cdot \mathbf{h}_{k_0}^{+} \left(\frac{(j+1)^2}{(k+1)^2} \right) \frac{1}{k+1} \right]_{j,k=0}^{\infty},$$

and where $\mathbf{K}_{k_0}^{\tau}$, $\mathbf{K}_{k_0}$ are defined by (7.34), (7.35), (7.36). At first we consider relation (8.1). Define the matrices (cf. (7.24) and (7.26))

$$\mathbf{B}_n := \left[\frac{2}{\pi \mathbf{i}} \frac{(j+\frac{1}{2})^{2k_0-1}}{\left[(k+\frac{1}{2})^2 + (j+\frac{1}{2})^2\right]^{k_0}} - \sqrt{1 - x_{n-k,n}^{\sigma}}\, s_{n-j,n-k}^{(n)} \left[b_{n-j,n-k}^{(n)} \right]^{k_0-1} \right]_{j,k=0}^{n-1},$$

and

$$\mathbf{C}_n := \Bigg[(-1)^k e^{-(j+\frac{1}{2})\pi} \sum_{t=1}^{k_0} \frac{2}{\pi \mathbf{i}} \frac{(k+\frac{1}{2})(j+\frac{1}{2})^{2k_0-2t}}{\left[(k+\frac{1}{2})^2 + (j+\frac{1}{2})^2\right]^{k_0-t+1}} \cdot \frac{1}{(t-1)!} \sum_{l=0}^{t-1} \binom{t-1}{l} \sum_{s=0}^{l-1} \frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}} \left[\frac{(j+\frac{1}{2})\pi}{\sqrt{2}} \right]^{l-s}$$
$$- (-1)^k s_{n-k,n-j}^{(n)} \sum_{t=1}^{k_0} \left[b_{n-j,n-k}^{(n)} \right]^{k_0-t} (-1)^{n+t-1} \sqrt{1 + x_{n-j,n}^{\sigma}} \sqrt{\frac{\pi}{2}} \cdot \widetilde{h}_n^t(-2 - x_{n-j,n}^{\sigma}) \Bigg]_{j,k=0}^{n-1}.$$

Regarding Lemma 7.10 and equation (7.28), we derive

$$\tilde{\mathcal{V}}_n^{\sigma} \left([\mathbf{A}_{k_0}]_n^4 + [\mathbf{K}_{k_0}]_n^4 - \mathcal{M}_n^{\sigma} \mathcal{B}_{k_0}^{-} \mathcal{L}_n \right) (\tilde{\mathcal{V}}_n^{\sigma})^{-1} = \mathbf{B}_n + \mathbf{C}_n. \tag{8.3}$$

The matrix $\mathbf{B}_n$ can be written in the form

$$\mathbf{B}_n = \Bigg[\frac{2}{\pi \mathbf{i}} \frac{(j+\frac{1}{2})^{2k_0-1}}{\left[(k+\frac{1}{2})^2 + (j+\frac{1}{2})^2\right]^{k_0}} - \frac{\sqrt{1 - x_{n-k,n}^{\sigma}}}{\sqrt{2}} \left\{ \frac{2}{\pi \mathbf{i}} \frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2 + (j+\frac{1}{2})^2} + \mathcal{O}\left(\frac{1}{n}\right) \right\} \cdot \left[n\sqrt{1 + x_{n-j,n}^{\sigma}} \left\{ \frac{\sqrt{2}}{\pi} \frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2 + (j+\frac{1}{2})^2} + \mathcal{O}\left(\frac{1}{n}\right) \right\} \right]^{k_0-1} \Bigg]_{j,k=0}^{n-1},$$

where we used

$$b^{(n)}_{n-j,n-k} = n\mathbf{i}\,\sqrt{1+x^\sigma_{n-j,n}}\,s^{(n)}_{n-j,n-k}$$

and Lemma 7.11. Since (cf. (7.32))

$$n\,\sqrt{1+x^\sigma_{n-j,n}}\left(j+\frac{1}{2}\right)^{-1} \le \text{const}, \quad 0 \le j \le n-1, \quad n \in \mathbb{N}, \tag{8.4}$$

we conclude that

$$\begin{aligned}
\mathbf{B}_n = &\left[\frac{2}{\pi\mathbf{i}}\frac{(j+\frac{1}{2})^{2k_0-1}}{\left[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\right]^{k_0}} - \frac{\sqrt{1-x^\sigma_{n-k,n}}}{\sqrt{2}}\left\{\frac{2}{\pi\mathbf{i}}\frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2+(j+\frac{1}{2})^2}\right\}\right.\\
&\left.\cdot\left[n\,\sqrt{1+x^\sigma_{n-j,n}}\left\{\frac{\sqrt{2}}{\pi}\frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2+(j+\frac{1}{2})^2}\right\}\right]^{k_0-1}\right]_{j,k=0}^{n-1} + \left[\mathcal{O}\left(\frac{1}{n}\right)\right]_{j,k=0}^{n-1}\\
= &\left[\frac{2}{\pi\mathbf{i}}\frac{(j+\frac{1}{2})^{2k_0-1}}{\left[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\right]^{k_0}} - \frac{\sqrt{1-x^\sigma_{n-k,n}}}{\sqrt{2}}\left\{\frac{n\,\sqrt{2(1+x^\sigma_{n-j,n})}}{\pi\left(j+\frac{1}{2}\right)}\right\}^{k_0-1}\right.\\
&\left.\cdot\frac{2}{\pi\mathbf{i}}\frac{(j+\frac{1}{2})^{2k_0-1}}{\left[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\right]^{k_0}}\right]_{j,k=0}^{n-1} + \left[\mathcal{O}\left(\frac{1}{n}\right)\right]_{j,k=0}^{n-1}.
\end{aligned}$$

We remark that the constants in $\mathcal{O}\left(\frac{1}{n}\right)$ are independent of j, k, and n. Define the matrices

$$\mathbf{D}_n = \left[\left\{\frac{n\,\sqrt{2(1+x^\sigma_{n-j,n})}}{\pi\left(j+\frac{1}{2}\right)}\right\}^{k_0-1}\delta_{j,k}\right]_{j,k=0}^{n-1},$$

$$\mathbf{E}_n = \left[\left\{\frac{\sqrt{1-x^\sigma_{n-k,n}}}{\sqrt{2}} - 1\right\}\delta_{j,k}\right]_{j,k=0}^{n-1},$$

$$\mathbf{F}_n = \left[\left\{1 - \left(\frac{n\,\sqrt{2(1+x^\sigma_{n-j,n})}}{\pi\left(j+\frac{1}{2}\right)}\right)^{k_0-1}\right\}\delta_{j,k}\right]_{j,k=0}^{n-1},$$

$$\mathbf{G}_n = \left[\frac{2}{\pi\mathbf{i}}\frac{(j+\frac{1}{2})^{2k_0-1}}{\left[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\right]^{k_0}}\right]_{j,k=0}^{n-1},$$

and $\mathbf{H}_n = \left[\mathcal{O}\left(\frac{1}{n}\right)\right]_{j,k=0}^{n-1}$ such that

$$\mathbf{B}_n = -\mathbf{D}_n\mathbf{G}_n\mathbf{E}_n + \mathbf{F}_n\mathbf{G}_n + \mathbf{H}_n. \tag{8.5}$$

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Now we turn to the matrix $\mathbf{C}_n$. Due to Lemma 7.11, this matrix can be written as

$$\mathbf{C}_n := \Bigg[(-1)^k e^{-(j+\frac{1}{2})\pi} \sum_{t=1}^{k_0} \frac{2}{\pi \mathbf{i}} \frac{(k+\frac{1}{2})(j+\frac{1}{2})^{2k_0-2t}}{\big[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\big]^{k_0-t+1}} \frac{1}{(t-1)!} \sum_{l=0}^{t-1} \binom{t-1}{l} \cdot \sum_{s=0}^{l-1} \frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}} \left[\frac{(j+\frac{1}{2})\pi}{\sqrt{2}}\right]^{l-s} - \frac{(-1)^k}{\sqrt{2}} \left\{\frac{2}{\pi \mathbf{i}} \frac{k+\frac{1}{2}}{(k+\frac{1}{2})^2+(j+\frac{1}{2})^2} + \mathcal{O}\left(\frac{1}{n}\right)\right\} \cdot \sum_{t=1}^{k_0} \left[n\sqrt{1+x_{n-j}^\sigma}\left\{\frac{\sqrt{2}}{\pi} \frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2+(j+\frac{1}{2})^2} + \mathcal{O}\left(\frac{1}{n}\right)\right\}\right]^{k_0-t} \cdot (-1)^{n+t-1}\sqrt{1+x_{n-j,n}^\sigma}\sqrt{\frac{\pi}{2}}\,\widetilde{h}_n^t(-2-x_{n-j,n}^\sigma)\Bigg]_{j,k=0}^{n-1}.$$

Taking into account the relations (7.19) and (8.4), we get

$$\mathbf{C}_n := \Bigg[(-1)^k e^{-(j+\frac{1}{2})\pi} \sum_{t=1}^{k_0} \frac{2}{\pi \mathbf{i}} \frac{(k+\frac{1}{2})(j+\frac{1}{2})^{2k_0-2t}}{\big[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\big]^{k_0-t+1}} \frac{1}{(t-1)!} \sum_{l=0}^{t-1} \binom{t-1}{l} \cdot \sum_{s=0}^{l-1} \frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}} \left[\frac{(j+\frac{1}{2})\pi}{\sqrt{2}}\right]^{l-s} - (-1)^k \sum_{t=1}^{k_0} \left\{\frac{n\cdot\sqrt{2(1+x_{n-j}^\sigma)}}{\pi\,(j+\frac{1}{2})}\right\}^{k_0-t} \cdot \frac{2}{\pi \mathbf{i}} \frac{(k+\frac{1}{2})(j+\frac{1}{2})^{2k_0-2t}}{\big[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\big]^{k_0-t+1}} \frac{(-1)^{n+t-1}}{\sqrt{2}}\sqrt{1+x_{n-j,n}^\sigma} \cdot \sqrt{\frac{\pi}{2}}\,\widetilde{h}_n^t(-2-x_{n-j,n}^\sigma)\Bigg]_{j,k=0}^{n-1} + \left[\mathcal{O}\left(\frac{1}{n}\right)\right]_{j,k=0}^{n-1}.$$

Set

$$\mathbf{K}_n^t = \left[\left(\left\{\frac{n\sqrt{2(1+x_{n-j}^\sigma)}}{\pi\,(j+\frac{1}{2})}\right\}^{k_0-t} - 1\right)\delta_{j,k}\right]_{j,k=0}^{n-1},$$

$$\mathbf{L}_n^t = \left[(-1)^k \frac{2}{\pi \mathbf{i}} \frac{(k+\frac{1}{2})(j+\frac{1}{2})^{2k_0-2t}}{\big[(k+\frac{1}{2})^2+(j+\frac{1}{2})^2\big]^{k_0-t+1}}\right]_{j,k=0}^{n-1},$$

$$\mathbf{M}_n^t = \left[\frac{(-1)^{n+t-1}}{\sqrt{2}}\sqrt{1+x_{n-j,n}^\sigma}\sqrt{\frac{\pi}{2}}\,\widetilde{h}_n^t(-2-x_{n-j,n}^\sigma)\delta_{j,k}\right]_{j,k=0}^{n-1},$$

$$\mathbf{N}_n^t = \Bigg[\Bigg(\frac{e^{-(j+\frac{1}{2})\pi}}{(t-1)!} \sum_{l=0}^{t-1} \binom{t-1}{l} \sum_{s=0}^{l-1} \frac{\chi_{t-l-2}\chi_s^l}{2^{t-1-\frac{l-s}{2}}} \left[\frac{(j+\frac{1}{2})\pi}{\sqrt{2}}\right]^{l-s} - \frac{(-1)^{n+t-1}}{\sqrt{2}}\sqrt{1+x_{n-j,n}^\sigma}\sqrt{\frac{\pi}{2}}\,\widetilde{h}_n^t(-2-x_{n-j,n}^\sigma)\Bigg)\delta_{j,k}\Bigg]_{j,k=0}^{n-1},$$

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and $\mathbf{P}_n = \left[\mathcal{O}\left(\frac{1}{n}\right)\right]_{j,k=0}^{n-1}$. We get

$$\mathbf{C}_n = \sum_{t=1}^{k_0} \mathbf{N}_n^t \cdot \mathbf{L}_n^t - \sum_{t=1}^{k_0} \mathbf{K}_n^t \cdot \mathbf{M}_n^t \cdot \mathbf{L}_n^t + \mathbf{P}_n. \tag{8.6}$$

The diagonal operators $\mathbf{M}_n^t, \mathbf{D}_n : \ell^2 \longrightarrow \ell^2$ are uniformly bounded due to (7.19) and (8.4). The same holds true for $\mathbf{L}_n^t, \mathbf{G}_n : \ell^2 \longrightarrow \ell^2$ in view of Lemma 3.11. Hence, using (8.3), (8.5), (8.6), (4.2), and taking into account the unitarity of $\widetilde{\mathcal{V}}_n^\sigma : \mathrm{im}\ \mathcal{L}_n \longrightarrow \mathrm{im}\ \mathcal{P}_n$, we can write

$$\begin{aligned}
&\left\| \left([\mathbf{A}_{k_0}]_n^4 + [\mathbf{K}_{k_0}]_n^4 - \mathcal{M}_n^\tau \mathcal{B}_{k_0}^- \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\le \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{B}_n \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} + \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{C}_n \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\le \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{D}_n \mathbf{G}_n \mathbf{E}_n \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\quad + \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{F}_n \mathbf{G}_n \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\quad + \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{H}_n \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\quad + \sum_{t=1}^{k_0} \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{N}_n^t \mathbf{L}_n^t \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\quad + \sum_{t=1}^{k_0} \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{K}_n^t \mathbf{M}_n^t \mathbf{L}_n^t \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\quad + \left\| \left((\tilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{P}_n \tilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} \\
&\le \inf_{f \in \mathbf{C}_{-1}} \sup_{n=1,2,\ldots} \left\| \mathbf{D}_n \mathbf{G}_n \mathbf{E}_n \left[\ f(x_{n-j,n}^\sigma) \delta_{j,k}\ \right]_{j,k=0}^{n-1} \right\|_{\mathfrak{L}(\ell^2)} \\
&\quad + \inf_{f \in \mathbf{C}_{-1}} \sup_{n=1,2,\ldots} \left\| \left[\ f(x_{n-j,n}^\sigma) \delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{F}_n \mathbf{G}_n \right\|_{\mathfrak{L}(\ell^2)} \\
&\quad + \inf_{f \in \mathbf{C}_{-1}} \sup_{n=1,2,\ldots} \left\| \left[\ f(x_{n-j,n}^\sigma) \delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{H}_n \right\|_{\mathfrak{L}(\ell^2)} \\
&\quad + \sum_{t=1}^{k_0} \inf_{f \in \mathbf{C}_{-1}} \sup_{n=1,2,\ldots} \left\| \left[\ f(x_{n-j,n}^\sigma) \delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{N}_n^t \mathbf{L}_n^t \right\|_{\mathfrak{L}(\ell^2)} \\
&\quad + \sum_{t=1}^{k_0} \inf_{f \in \mathbf{C}_{-1}} \sup_{n=1,2,\ldots} \left\| \left[\ f(x_{n-j,n}^\sigma) \delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{K}_n^t \mathbf{M}_n^t \mathbf{L}_n^t \right\|_{\mathfrak{L}(\ell^2)} \\
&\quad + \inf_{f \in \mathbf{C}_{-1}} \sup_{n=1,2,\ldots} \left\| \left[\ f(x_{n-j,n}^\sigma) \delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{P}_n \right\|_{\mathfrak{L}(\ell^2)}
\end{aligned}$$

$$\leq \text{const}\Bigg(\inf_{f\in\mathbf{C}_{-1}} \sup_{n=1,2,\dots} \left\| \mathbf{E}_n \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \right\|_{\mathfrak{L}(\ell^2)}$$
$$+ \inf_{f\in\mathbf{C}_{-1}} \sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{F}_n \right\|_{\mathfrak{L}(\ell^2)}$$
$$+ \inf_{f\in\mathbf{C}_{-1}} \sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{H}_n \right\|_{\mathfrak{L}(\ell^2)}$$
$$+ \sum_{t=1}^{k_0} \inf_{f\in\mathbf{C}_{-1}} \sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{N}^t_n \right\|_{\mathfrak{L}(\ell^2)}$$
$$+ \sum_{t=1}^{k_0} \inf_{f\in\mathbf{C}_{-1}} \sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{K}^t_n \right\|_{\mathfrak{L}(\ell^2)}$$
$$+ \inf_{f\in\mathbf{C}_{-1}} \sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{P}_n \right\|_{\mathfrak{L}(\ell^2)} \Bigg).$$

Choosing $f \in \mathbf{C}_{-1}$ with $\operatorname{supp}(f \circ \cos) \subset [\pi - \varepsilon, \pi]$ and with $\varepsilon \in (0,1)$, a simple Frobenius norm estimate shows that

$$\sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{H}_n \right\|_{\mathfrak{L}(\ell^2)} \leq \text{const}\cdot\varepsilon$$

and

$$\sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{P}_n \right\|_{\mathfrak{L}(\ell^2)} \leq \text{const}\cdot\varepsilon.$$

Moreover, there exist a $\delta \in (0,1)$ and a function $f \in \mathbf{C}_{-1}$ with $\operatorname{supp}(f\circ\cos) \subset [\pi-\delta,\pi]$, such that

$$\sup_{n=1,2,\dots} \left\| \mathbf{E}_n \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \right\|_{\mathfrak{L}(\ell^2)} \leq \varepsilon,$$

$$\sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{F}_n \right\|_{\mathfrak{L}(\ell^2)} \leq \varepsilon,$$

$$\sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{N}^t_n \right\|_{\mathfrak{L}(\ell^2)} \leq \varepsilon,$$

and

$$\sup_{n=1,2,\dots} \left\| \left[\ f(x^\sigma_{n-j,n})\delta_{j,k}\ \right]_{j,k=0}^{n-1} \mathbf{K}^t_n \right\|_{\mathfrak{L}(\ell^2)} \leq \varepsilon.$$

For example, let us consider the matrices $\mathbf{N}^t_n = \left[\ (\alpha_j - \beta_j)\delta_{j,k}\ \right]_{j,k=0}^{n-1}$, where

$$\alpha_j = \frac{e^{-(j+\frac12)\pi}}{(t-1)!} \sum_{\ell=0}^{t-1} \binom{t-1}{\ell} \sum_{s=0}^{\ell-1} \frac{\chi_{t-\ell-2}\chi_s^\ell}{2^{t-1-\frac{\ell-s}{2}}} \left[\frac{\left(j+\frac12\right)\pi}{\sqrt2} \right]^{\ell-s}$$

and (cf. (7.8))

$$\beta_j = \frac{(-1)^{n+t-1}}{\sqrt2} \sqrt{1+x^\sigma_{n-j,n}} \sqrt{\frac{\pi}{2}}\, \tilde h^t_n(-2-x^\sigma_{n-j,n})$$
$$= \left(2 + x^\sigma_{n-j,n} - \sqrt{(2+x^\sigma_{n-j,n})^2-1}\right)^n \sqrt{\frac{3+x^\sigma_{n-j,n}}{2}}\, \frac{1}{(t-1)!}$$
$$\cdot \sum_{\ell=0}^{t-1} \binom{t-1}{\ell} \sum_{s=0}^{\ell-1} \frac{p_{t-\ell-2}(-2-x^\sigma_{n-j,n})p^\ell_s(-2-x^\sigma_{n-j,n})}{(3+x^\sigma_{n-j,n})^{t-1-\frac{\ell-s}{2}}} \left(n\sqrt{1+x^\sigma_{n-j,n}}\right)^{\ell-s}.$$

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Remember that $\chi_s = p_s(-1)$ and $\chi_s^\ell = p_s^\ell(-1)$, where $p_s(x)$ and $p_s^\ell(x)$ are polynomials. This yields

$$\begin{aligned}
&\alpha_j - \beta_j \\
&= \frac{1}{(t-1)!}\sum_{\ell=0}^{t-1}\binom{t-1}{\ell} \\
&\quad \cdot \sum_{s=0}^{\ell-1}\left[\frac{p_{t-\ell-2}(-1)p_s^\ell(-1)}{2^{t-1-\frac{\ell-s}{2}}} - \sqrt{\frac{3+x_{n-j,n}^\sigma}{2}}\,\frac{p_{t-\ell-2}(-2-x_{n-j,n}^\sigma)p_s^\ell(-2-x_{n-j,n}^\sigma)}{(3+x_{n-j,n}^\sigma)^{t-1-\frac{\ell-s}{2}}}\right] \\
&\qquad\qquad \cdot e^{-(j+\frac{1}{2})\pi}\left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} \\
&\quad + \frac{1}{(t-1)!}\sum_{\ell=0}^{t-1}\binom{t-1}{\ell}\sum_{s=0}^{\ell-1}\left[\sqrt{\frac{3+x_{n-j,n}^\sigma}{2}}\,\frac{p_{t-\ell-2}(-2-x_{n-j,n}^\sigma)p_s^\ell(-2-x_{n-j,n}^\sigma)}{(3+x_{n-j,n}^\sigma)^{t-1-\frac{\ell-s}{2}}}\right] \\
&\quad \cdot \left\{e^{-(j+\frac{1}{2})\pi}\left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s}\right. \\
&\qquad\qquad \left. - \left(2+x_{n-j,n}^\sigma - \sqrt{(2+x_{n-j,n}^\sigma)^2-1}\right)^n\left(n\sqrt{1+x_{n-j,n}^\sigma}\right)^{\ell-s}\right\}.
\end{aligned}$$

Since, for $\operatorname{supp}(f\circ\cos) \subset [\pi-\delta,\pi]$, $f(x_{n-j,n}^\sigma) \neq 0$ is equivalent to $\frac{(j+\frac{1}{2})\pi}{n} \le \delta$, it is sufficient to show that $\varepsilon_{jn} \le \mathrm{const}\,\varepsilon$ for

$$\begin{aligned}
\varepsilon_{jn} := &\left| e^{-(j+\frac{1}{2})\pi}\left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s}\right. \\
&\qquad \left. - \left(2+x_{n-j,n}^\sigma - \sqrt{(2+x_{n-j,n}^\sigma)^2-1}\right)^n\left(n\sqrt{1+x_{n-j,n}^\sigma}\right)^{\ell-s}\right|
\end{aligned}$$

and for $n\in\mathbb{N}$, $\frac{(j+\frac{1}{2})\pi}{n} \le \delta$, and $\delta > 0$ sufficiently small. Using the same notations as in the proof of (7.10), we can write

$$\begin{aligned}
\varepsilon_{jn} &= e^{-(j+\frac{1}{2})\pi}\left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s}\left\{1-\left[\frac{\sin\frac{(j+\frac{1}{2})\pi}{2n}}{\frac{(j+\frac{1}{2})\pi}{2n}}\right]^{\ell-s}\right\} \\
&\quad + \left(n\sqrt{1+x_{n-j,n}^\sigma}\right)^{\ell-s}\left|e^{-(j+\frac{1}{2})\pi} - \left(2+x_{n-j,n}^\sigma - \sqrt{(2+x_{n-j,n}^\sigma)^2-1}\right)^n\right| \\
&= \varepsilon_{jn}^{(1)} + \varepsilon_{jn}^{(2)},
\end{aligned}$$

where

$$\varepsilon_{jn}^{(1)} = e^{-(j+\frac{1}{2})\pi}\left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s}\left\{1-\left[\frac{\sin\frac{(j+\frac{1}{2})\pi}{2n}}{\frac{(j+\frac{1}{2})\pi}{2n}}\right]^{\ell-s}\right\}$$

and

$$\varepsilon_{jn}^{(2)} = \left(n\sqrt{1+x_{n-j,n}^\sigma}\right)^{\ell-s} e^{-(j+\frac{1}{2})\pi}\left|1-\frac{e^{(j+\frac{1}{2})\pi}}{\left[(1+\delta_n)^{\frac{1}{\delta_n}}\right]^{n\delta_n}}\right|.$$

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Of course, $\varepsilon_{jn}^{(1)} \leq \varepsilon$ if $\delta > 0$ is small enough. Furthermore,

$$\begin{aligned}\varepsilon_{jn}^{(2)} \leq & \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} e^{-\left(j+\frac{1}{2}\right)\pi} \\ & \cdot \left| \left(\left[\frac{e}{(1+\delta_n)^{\frac{1}{\delta_n}}}\right]^{n\delta_n} - 1 \right) e^{\left(j+\frac{1}{2}\right)\pi - n\delta_n} + e^{\left(j+\frac{1}{2}\right)\pi - n\delta_n} - 1 \right| \\ = & \, \varepsilon_{jn}^{(3)} + \varepsilon_{jn}^{(4)},\end{aligned}$$

where

$$\varepsilon_{jn}^{(3)} = \left(\left[\frac{e}{(1+\delta_n)^{\frac{1}{\delta_n}}}\right]^{n\delta_n} - 1 \right) \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} e^{-n\delta_n},$$

$$\varepsilon_{jn}^{(4)} = \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} e^{-\left(j+\frac{1}{2}\right)\pi} \left| e^{\left(j+\frac{1}{2}\right)\pi - n\delta_n} - 1 \right| .$$

For sufficiently small $\delta > 0$, we have $\dfrac{e}{(1+\delta_n)^{\frac{1}{\delta_n}}} \leq 1+\varepsilon$. Consequently, since

$$\left| n\delta_n - \left(j+\frac{1}{2}\right)\pi \right| \leq c_1\delta \left(j+\frac{1}{2}\right)$$

and $(1+\varepsilon)^{n\delta_n} - 1 \leq n\delta_n 2^{n\delta_n - 1}\varepsilon \leq c_2 \left(j+\frac{1}{2}\right) 2^{n\delta_n}\varepsilon$ with constants c_1 and c_2, we get

$$\begin{aligned}\varepsilon_{jn}^{(3)} & \leq c_2 \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} \left(j+\frac{1}{2}\right) \left(\frac{2}{e}\right)^{n\delta_n} \varepsilon \\ & \leq c_2 \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} \left(j+\frac{1}{2}\right) \left(\frac{2}{e}\right)^{(\pi - c_1\delta)\left(j+\frac{1}{2}\right)} \varepsilon \leq c_3\varepsilon \quad \text{if} \quad c_1\delta < \frac{\pi}{2}.\end{aligned}$$

Finally, to estimate $\varepsilon_{jn}^{(4)}$, we remark that there are constants c_4 and $A > 0$ such that

$$\left[\frac{x\pi}{\sqrt{2}}\right]^{\ell-s} e^{-x\pi} \leq c_4 \quad \text{for all } x \geq 0 \qquad \text{and} \qquad \left[\frac{x\pi}{\sqrt{2}}\right]^{\ell-s} e^{-\frac{x\pi}{2}} \leq \varepsilon \quad \text{for } x \geq A.$$

Choose $\delta > 0$ such that $c_1\delta \leq \frac{\pi}{2}$ and $\left|e^{\pm c_1\delta A} - 1\right| \leq \varepsilon$. Then, $\varepsilon_{jn}^{(4)} \leq c_4\varepsilon$ if $j+\frac{1}{2} \leq A$ and

$$\varepsilon_{jn}^{(4)} \leq \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} e^{-\left(j+\frac{1}{2}\right)\pi} e^{c_1\delta\left(j+\frac{1}{2}\right)} \leq \left[\frac{\left(j+\frac{1}{2}\right)\pi}{\sqrt{2}}\right]^{\ell-s} e^{-\left(j+\frac{1}{2}\right)\frac{\pi}{2}} \leq \varepsilon$$

if $j+\frac{1}{2} \geq A$. Consequently,

$$\left\| \left([\mathbf{A}_{k_0}]_n^4 + [\tilde{\mathbf{A}}_{k_0}]_n^4 - \mathcal{M}_n^\tau \mathcal{B}_{k_0} \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{F}/\mathfrak{J})/\mathfrak{J}_{-1}} = 0.$$

Hence, (8.1) is proved in case $\tau = \sigma$. The proofs of (8.1) in case $\tau = \mu$ and of (8.2) can be done in the same way.

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