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PARAMETER ESTIMATION OF MONOMIAL-EXPONENTIAL SUMS*

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Abstract. In this paper we propose a matrix-pencil method for the identification of parameters and coefficients of a monomial-exponential sum which can be considered as an extension of existing matrix-pencil methods for the parameter estimation of exponential sums. The technique adopted is based on properties of the finite difference equations and it overcomes the difficulty of their extension via the invertibility of the generalized Vandermonde matrix. As a result, a matrix-pencil method based on the GSVD or the SVD is proposed which allows us to identify both simple and multiple parameters. Applications of this method to various examples show its effectiveness.

Key words. nonlinear approximation, parameter estimation, matrix pencils

AMS subject classifications. 41A46, 15A22, 65F15

1. Introduction.

Because of its relevance in science and engineering, several papers have been published on parameter estimation in exponential sums of the form

$$h(x) = \sum_{j=1}^{n} c_j e^{f_j x}, \tag{1.1}$$

where $\{c_j\}_{j=1}^n$ are complex or real numbers and $\{f_j\}_{j=1}^n$ are distinct complex parameters. The problem is to identify the $2n$ parameters $\{c_j,\ f_j\}$ from a given set of $2N$ ($N \geq n$) values of $h(x)$ in equidistant points of $\mathbb{R}$. This problem arises, for instance, from the propagation of signals [13, 15], electromagnetics [2, 21] and high-resolution imaging of moving targets [11].

The two mostly used methods are the Prony-like (or polynomial) methods and the matrix-pencil methods. The first ones are based on the paper by G. de Prony [6] who was the first to investigate this problem. The method is principally based on the solution of two linear systems characterized by a Hankel and a Vandermonde matrix, respectively. The first system furnishes the coefficients of a polynomial (the so-called Prony polynomial) whose roots z_j allow one to determine the parameters f_j (being $z_j = e^{f_j}$), while the second system provides the coefficients c_j. Several extensions have been proposed (see, for instance, [10, pp. 458-462], [5, 22, 23, 25], and more recently [16, 17, 18]) to apply this polynomial method also to the case where n is only approximately known or the data are affected by noise. The matrix-pencil technique has been developed more recently [12]. As the Prony-like methods, one recovers the coefficients c_j by solving a Vandermonde system. But the computation of the parameters f_j is reduced to only one step; see, for instance, [21]. In fact, it allows one to estimate the zeros of the Prony polynomial and then f_j without passing through the computation of its coefficients. This is the main difference with the Prony-like methods and it makes this kind of method more computationally efficient.

A close connection between the two methods mentioned above has been observed in [19], which allows one to obtain a unified approach in the case where an approximate upper bound of n is given. In this context two algorithms have been proposed [19], based on a QR factorization and on a SVD decomposition of a rectangular Hankel matrix, respectively. The second algorithm makes it equivalent to the ESPRIT (Estimation of Signal Parameters via Rotational Invariance Techniques) method (see, for instance, [20]) which is often used by engineers. But this method fails whenever two or more eigenvalues coincide, since this implies

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that the corresponding parameters f_j are equal, in contrast with the assumption in (1.1). This difficulty can be overcome by generalizing (1.1) to a monomial-exponential sums of the type (1.2) below. This problem is of primary interest, for instance in the direct scattering problem concerning the solution of nonlinear partial differential equations (NPDEs) of integrable type. In this field, which is of major interest to us, we need to estimate the parameters of monomial-exponential sums of the form (see more details in Section 4.1)

$$h(x) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} x^s e^{f_j x}, \tag{1.2}$$

where $\{c_{js}\}_{j=1,s=0}^{n,m_j-1}$ and $\{f_j\}_{j=1}^{n}$ are complex or real parameters and $\{m_j\}_{j=1}^{n}$ are positive integers. In the case $m_1 = m_2 = \cdots = m_n = 1$, the monomial exponential sum $h(x)$ reduces to the exponential sum (1.1). More precisely, setting

$$M = m_1 + m_2 + \cdots + m_n,$$

the problem we are addressing is to recover the $M + n$ parameters of h given $2N$ ($N \geq M$) observed data.

Some attempts to treat problem (1.2) with limitations on the choice of the integers m_j have been made in particular in [3, 4]. However, these papers do not contain any proof of unique reconstruction of the parameters from the data matrix, nor a comparison of numerical results obtained by other methods.

In this paper we propose a matrix-pencil method which allows one to solve this problem, overcoming the difficulty of the matrix-pencil methods mentioned above in the case of repeated values. For noiseless data, we prove the uniqueness of the recovery of the parameters from the data matrix. In the presence of noisy data, our numerical experiments show that our method is also effective under the hypothesis that a reasonable upper bound of M is known.

This paper is organized as follows. In Section 2 we illustrate our numerical method, assuming that $M = m_1 + m_2 + \cdots + m_n$ is exactly known. In Section 3 we describe the changes needed if only an upper bound of M is known. Section 4 is devoted to the results of our numerical experiments, with conclusions following in Section 5.

2. The numerical method. The numerical method we propose to recover all parameters appearing in the monomial-exponential sum (1.2) reduces the non-linear approximation problem to two problems of linear algebra. The first one is a generalized eigenvalue problem, which allows us to recover n, f_j and m_j. The second one is the solution of a linear system with a Casorati matrix to compute the parameters c_{js}.

First we note that, setting $z_j = e^{f_j} \neq 0$, we can rewrite the monomial exponential sum (1.2) as a monomial-power sum

$$h(x) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} x^s z_j^x.$$

Moreover, we assume that $2N$ sampled data with $N \geq M$, $M = m_1 + \ldots + m_n$,

$$h(k) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} k^s z_j^k, \quad 0^0 \equiv 1$$

are given for the $2N$ values $k = k_0, k_0+1, \ldots, k_0+2N-1$ with $k_0 \in \mathbb{N}^+ = \{0, 1, 2, ..., k_0, ...\}$. Preliminarily, we arrange the $2N$ given data in the following square Hankel matrices of order

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$$\mathbf{H}_{NN}^{k_0} = \begin{bmatrix} h(k_0) & h(k_0+1) & \dots & h(k_0+N-1) \\ h(k_0+1) & h(k_0+2) & \dots & h(k_0+N) \\ \vdots & \vdots & \vdots & \vdots \\ h(k_0+N-1) & h(k_0+N) & \dots & h(k_0+2N-2) \end{bmatrix}$$

$$= [\mathbf{h}_{k_0}, \mathbf{h}_{k_0+1}, \dots, , \mathbf{h}_{k_0+N-1}], \tag{2.1}$$

$$\mathbf{H}_{NN}^{k_0+1} = \begin{bmatrix} h(k_0+1) & h(k_0+2) & \dots & h(k_0+N) \\ h(k_0+2) & h(k_0+3) & \dots & h(k_0+N+1) \\ \vdots & \vdots & \vdots & \vdots \\ h(k_0+N) & h(k_0+N+1) & \dots & h(k_0+2N-1) \end{bmatrix}$$

$$= [\mathbf{h}_{k_0+1}, \mathbf{h}_{k_0+2}, \dots, \mathbf{h}_{k_0+N}]. \tag{2.2}$$

Notice that $\mathbf{H}_{NN}^{k_0+1}$ is essentially a shift of $\mathbf{H}_{NN}^{k_0}$, as the first $N-1$ columns of $\mathbf{H}_{NN}^{k_0+1}$ coincide with the last $N-1$ columns of $\mathbf{H}_{NN}^{k_0}$ apart from the last entry.

In the following we will often write $\mathbf{H}_{NM}^{k_0}$ and $\mathbf{H}_{NM}^{k_0+1}$, each of order $N \times M$ with $N \geq M$, for the submatrices of the Hankel matrices $\mathbf{H}_{NN}^{k_0}$ and $\mathbf{H}_{NN}^{k_0+1}$ formed by their first M columns, respectively.

The next lemma contains two properties of these Hankel matrices that are relevant to our method. Such properties have already been proved in [18, 19] for the special case of distinct z_j values. Here we prove the result in the general case, by giving a proof based on the theory of linear difference equations with constant coefficients.

LEMMA 2.1. *Assume M is known and the sampled data are noiseless. Then*

(a) The matrices (2.1) and (2.2) have rank M, that is

$$\operatorname{rank} \mathbf{H}_{NN}^{k_0} = \operatorname{rank} \mathbf{H}_{NN}^{k_0+1} = M.$$

(b) The following relation holds

$$\mathbf{H}_{NM}^{k_0+1} = \mathbf{H}_{NM}^{k_0} \mathbf{C}_M(P), \tag{2.3}$$

where $\mathbf{C}_M(P)$ is the companion matrix of the Prony polynomial, i.e.,

$$\mathbf{C}_M(P) = \begin{bmatrix} 0 & 0 & \dots & 0 & -p_0 \\ 1 & 0 & \dots & 0 & -p_1 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & -p_{M-1} \end{bmatrix}.$$

Proof. To prove (a), we interpret $h(k)$ as the general solution of the following homogeneous linear difference equation of order M

$$\sum_{k=0}^{M} p_k h_{k+m} = 0, \quad p_M = 1, \tag{2.4}$$

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whose characteristic polynomial is the Prony polynomial, i.e., the monic polynomial of degree M having z_j as the jth zero with multiplicity m_j

$$P(z) = \prod_{j=1}^{n} (z - z_j)^{m_j} = \sum_{k=0}^{M} p_k z^k, \quad p_M \equiv 1. \tag{2.5}$$

It is well known (see, for instance, [14]) that, regardless of the values $\{p_k\}_{k=0}^{M-1}$, (2.4) has a unique solution h_k, for each given set of M initial conditions $h_{k_0}, h_{k_0+1}, \ldots, h_{k_0+M-1}$. Since (2.5) is the characteristic polynomial of (2.4), each function $h_{j,s}(k) = k^s z_j^k, j = 1, \ldots, n, s = 0, 1, \ldots, m_j - 1$, is a solution of (2.4). Moreover, they are linearly independent [14, Theorem 2.2.3] and represent a basis for the vector space of solutions of (2.4). Hence the function $h(k)$ is the general solution of (2.4) and its coefficients $\{c_{js}\}_{j=1,s=0}^{n,m_j-1}$ can be uniquely determined by fixing M initial values $h(k_0), h(k_0+1), \cdots, h(k_0+M-1)$. Then, if we consider the first M columns $\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{M-1}$ of $\mathbf{H}_{NN}^{k_0}$ as initial data, we can see that each column $\mathbf{h}_M, \mathbf{h}_{M+1}, \ldots, \mathbf{h}_N$ is a linear combination of the first M ones. As a result, $\operatorname{rank} \mathbf{H}_{NN}^{k_0} = M$. The same conclusion holds if k_0 is replaced by $k_0 + 1$, and $\operatorname{rank} \mathbf{H}_{NN}^{k_0+1} = M$.

Relation (2.3) is immediate by observing that the product between $\mathbf{H}_{NM}^{k_0}$ and the jth column of $\mathbf{C}_M(P)$ gives the $(j+1)$th column of $\mathbf{H}_{NM}^{k_0+1}$ and further, by (2.4), we have

$$-\sum_{k=0}^{M-1} p_k h_{k+k_0} = h_{k_0+M}. \qquad \square$$

The next theorem contains two results essential to our method.

THEOREM 2.2. *The zeros of the Prony polynomial, counting their multiplicities, are exactly the eigenvalues, with the same multiplicity, of the matrix-pencil*

$$\mathbf{H}_{MM}(z) = (\mathbf{H}_{NM}^{k_0})^* (\mathbf{H}_{NM}^{k_0+1} - z\mathbf{H}_{NM}^{k_0}), \tag{2.6}$$

where the asterisk denotes the conjugate transpose.

Moreover, the coefficients c_{js} in (1.2) are the solutions of the linear system

$$\mathbf{K}_M^{k_0} \mathbf{c} = \mathbf{h}_{k_0}, \tag{2.7}$$

where $\mathbf{c} = [c_{1,0}, \ldots, c_{1,n_1-1}, \ldots, c_{M,0}, \ldots, c_{M,n_n-1}]^T$, $\mathbf{h}_{k_0} = [h(k_0), h(k_0+1), \ldots, h(k_0+M-1)]^T$ *and* $\mathbf{K}_M^{k_0}$ *is the Casorati matrix*

(2.8)
$$\mathbf{K}_M^{k_0} = \begin{bmatrix} z_1^{k_0} & k_0 z_1^{k_0} & \cdots & k_0^{n_1-1} z_1^{k_0} & \cdots & z_n^{k_0} & k_0 z_n^{k_0} & \cdots & k_0^{n_n-1} z_n^{k_0} \\ z_1^{k_1} & k_1 z_1^{k_1} & \cdots & k_1^{n_1-1} z_1^{k_1} & \cdots & z_n^{k_1} & k_1 z_n^{k_1} & \cdots & k_1^{n_n-1} z_n^{k_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^{k_{M-1}} & k_{M-1} z_1^{k_{M-1}} & \cdots & k_{M-1}^{n_1-1} z_1^{k_{M-1}} & \cdots & z_n^{k_{M-1}} & k_{M-1} z_n^{k_{M-1}} & \cdots & k_{M-1}^{n_n-1} z_n^{k_{M-1}} \end{bmatrix}.$$

Proof. By using (2.3), we can write

$$\mathbf{H}_{MM}(z) = (\mathbf{H}_{NM}^{k_0})^* \mathbf{H}_{NM}^{k_0} (\mathbf{C}_M(P) - z\mathbf{I}_{MM}),$$

where $\mathbf{I}_{MM}$ is the identity matrix of order M. Hence, the first statement follows by noting that

$$\det \mathbf{H}_{MM}(z) = \det((\mathbf{H}_{NM}^{k_0})^* \mathbf{H}_{NM}^{k_0}) \det(\mathbf{C}_M(P) - z\mathbf{I}_{MM}) = \det((\mathbf{H}_{NM}^{k_0})^* \mathbf{H}_{NM}^{k_0}) P(z),$$

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and by taking into account that $\det((\mathbf{H}_{NM}^{k_0})^*\mathbf{H}_{NM}^{k_0}) \neq 0$ as $\mathbf{H}_{NM}^{k_0}$ has full rank. Concerning system (2.7), we note that $\mathbf{K}_M^{k_0}$ is nonsingular regardless of the k_0 value since it is the Casorati matrix, which plays in the theory of difference equations the same role as the Wronskian matrix in the theory of differential equations. Notice that the Casorati matrix coincides with the Vandermonde matrix $\mathbf{V}_M = [z_j^{k_i}]_{i=0,j=1}^{M-1,n}$ when all zeros z_j are simple ($m_j \equiv 1$). ∎

2.1. Computation of $\{z_j, m_j\}$ via the Generalized Singular Value Decomposition (GSVD). Knowing M, finding those parameters can be carried out by solving the following generalized eigenvalue problem:

$$(\mathbf{H}_{NM}^{k_0})^*\mathbf{H}_{NM}^{k_0+1}\mathbf{x} = z(\mathbf{H}_{NM}^{k_0})^*\mathbf{H}_{NM}^{k_0}\mathbf{x}, \quad \mathbf{x} \neq \mathbf{0}.$$

To this end, considering that both matrices $\mathbf{H}_{NM}^{k_0+1}$ and $\mathbf{H}_{NM}^{k_0}$ have rank M, we implement a simultaneous factorization of the two matrices by means of the Generalized Singular Value Decomposition (GSVD) [8]

$$\mathbf{H}_{NM}^{k_0+1} = \mathbf{U}_{NN}\begin{bmatrix}\mathbf{\Sigma}_{MM}^{k_0+1}\\ \mathbf{0}_{N-M,M}\end{bmatrix}\mathbf{X}_{MM}, \tag{2.9}$$

$$\mathbf{H}_{NM}^{k_0} = \mathbf{V}_{NN}\begin{bmatrix}\mathbf{\Sigma}_{MM}^{k_0}\\ \mathbf{0}_{N-M,M}\end{bmatrix}\mathbf{X}_{MM}, \tag{2.10}$$

where $\mathbf{\Sigma}_{MM}^{k_0+1}$ and $\mathbf{\Sigma}_{MM}^{k_0}$ are two non-negative diagonal matrices of order M, $\mathbf{U}_{NN}$ and $\mathbf{V}_{NN}$ are two square unitary matrices of size N, $\mathbf{X}_{MM}$ is a nonsingular matrix of order M and $\mathbf{0}_{N-M,M}$ is the null matrix of order $(N-M)\times M$.

Thus, by using (2.9) and (2.10), we can rewrite the matrix-pencil as

$$\begin{aligned}\mathbf{H}_{MM}(z) &= (\mathbf{X}_{MM})^*\begin{bmatrix}(\mathbf{\Sigma}_{MM}^{k_0})^* & \mathbf{0}_{M,N-M}\end{bmatrix}(\mathbf{V}_{NN})^*\mathbf{U}_{NN}\begin{bmatrix}\mathbf{\Sigma}_{MM}^{k_0+1}\\ \mathbf{0}_{N-M,M}\end{bmatrix}\mathbf{X}_{MM}\\ &\quad - z(\mathbf{X}_{MM})^*\begin{bmatrix}(\mathbf{\Sigma}_{MM}^{k_0})^* & \mathbf{0}_{M,N-M}\end{bmatrix}\begin{bmatrix}\mathbf{\Sigma}_{MM}^{k_0}\\ \mathbf{0}_{N-M,M}\end{bmatrix}\mathbf{X}_{MM}\\ &= (\mathbf{X}_{MM})^*(\mathbf{\Sigma}_{MM}^{k_0})^*\left[(\mathbf{V}_{NM})^*\mathbf{U}_{NM}\mathbf{\Sigma}_{MM}^{k_0+1} - z\mathbf{\Sigma}_{MM}^{k_0}\right]\mathbf{X}_{MM}\\ &= (\mathbf{\Sigma}_{MM}^{k_0}\mathbf{X}_{MM}))^*\left[(\mathbf{V}_{NM})^*\mathbf{U}_{NM}\mathbf{\Sigma}_{MM}^{k_0+1}(\mathbf{\Sigma}_{MM}^{k_0})^{-1} - z\mathbf{I}_{MM}\right]\mathbf{\Sigma}_{MM}^{k_0}\mathbf{X}_{MM}.\end{aligned}$$

As a result, the generalized eigenvalues of the matrix-pencil, and thus the zeros of the Prony polynomial, are exactly the eigenvalues of the matrix

$$\mathbf{A}_{MM} = (\mathbf{V}_{NM})^*\mathbf{U}_{NM}\mathbf{\Sigma}_{MM}^{k_0+1}(\mathbf{\Sigma}_{MM}^{k_0})^{-1},$$

which can be effectively computed by using the *eig* algorithm of MATLAB.

We note that $\mathbf{A}_{MM}$ is well conditioned, as the generalized singular values of $\mathbf{H}_{NM}^{k_0}$ and $\mathbf{H}_{NM}^{k_0+1}$ are close to each other and the matrices $\mathbf{V}_{NM}$ and $\mathbf{U}_{NM}$ are sub-matrices of unitary matrices.

In this way we compute the zeros z_j with their multiplicities m_j and of course n. The computation of f_j is immediate as $z_j = e^{f_j}$, $j = 1, \ldots, n$.

It is worthwhile to note that if $N = M$, the zeros z_j of the Prony polynomial can be computed by considering the simple matrix-pencil

$$\widehat{\mathbf{H}}_{MM}(z) = \mathbf{H}_{MM}^{k_0+1} - z\mathbf{H}_{MM}^{k_0}.$$

In this case, as $\mathbf{H}_{MM}^{k_0+1}$ and $\mathbf{H}_{MM}^{k_0}$ are symmetric, the QZ technique [8] is effective as well and numerically stable as explained in [7]. In this paper, we do not consider this case because in the applications that interest us N is larger than M and furthermore our numerical experiments show that using all available data $h(k)$ is more effective.

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2.2. Computation of $\{z_j,\, m_j\}$ via the Singular Value Decomposition (SVD). Using the SVD in place of the GSVD generates a second effective technique for computing the parameters $\{z_j,\, m_j\}$.

The starting point, as in [19], is to consider the augmented Hankel matrix

$$\mathbf{H}_{N,M+1} = (\mathbf{H}_{NM}^{k_0},\, \mathbf{h}_{k_0+M}) = (\mathbf{h}_{k_0},\, \mathbf{H}_{NM}^{k_0+1}) \tag{2.11}$$

and factorize it by means of the SVD.

Considering that $\mathbf{h}_{k_0+M}$ is a linear combination of the columns of $\mathbf{H}_{NM}^{k_0}$, and $\mathbf{h}_{k_0}$ is a linear combination of the columns of $\mathbf{H}_{NM}^{k_0+1}$, and recalling that $\operatorname{rank}(\mathbf{H}_{NM}^{k_0}) = \operatorname{rank}(\mathbf{H}_{NM}^{k_0+1}) = M$, we obtain

$$\mathbf{H}_{N,M+1} = \mathbf{U}_{NM}\mathbf{\Sigma}_{MM}\mathbf{V}_{M,M+1}^{*}, \tag{2.12}$$

where $\mathbf{\Sigma}_{MM} = diag(\sigma_1, \sigma_2, \ldots, \sigma_M)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_M > 0$, $\mathbf{U}_{NM}^{*}\mathbf{U}_{NM} = \mathbf{I}_{MM}$ and $\mathbf{V}_{M,M+1}^{*}\mathbf{V}_{M,M+1} = \mathbf{I}_{M+1,M+1}$.

Hence (2.11) and (2.12) imply that

$$\mathbf{H}_{NM}^{k_0} = \mathbf{U}_{NM}\Sigma_{MM}\mathbf{V}_{MM}^{*,k_0},$$

and

$$\mathbf{H}_{NM}^{k_0+1} = \mathbf{U}_{NM}\Sigma_{MM}\mathbf{V}_{MM}^{*,k_0+1},$$

where $\mathbf{V}_{MM}^{*,k_0}$ and $\mathbf{V}_{MM}^{*,k_0+1}$ are obtained by deleting the last and the first column of $\mathbf{V}_{M,M+1}^{*}$, respectively.

As a result, the matrix pencil (2.6) can be rewritten as

$$\mathbf{H}_{MM}(z) = (\mathbf{V}_{MM}^{*,k_0})^{*}(\mathbf{\Sigma_M})^2(\mathbf{V}_{MM}^{*,k_0+1} - z\mathbf{V}_{MM}^{*,k_0})$$

and thus the parameters $\{z_j, m_j\}$ are the eigenvalues with the respective multiplicities of the square matrix

$$\mathbf{B}_{MM} = (\mathbf{V}_{MM}^{*,k_0})^{-1}\mathbf{V}_{MM}^{*,k_0+1},$$

whose condition number is close to that of $\mathbf{A}_{MM}$. Let us note that $\mathbf{B}_{MM}$ is formally analogous to the matrix $\mathbf{F}_M^{SVD}$ introduced in the formulation of the ESPRIT method for (1.1) proposed in [19, p. 1033]. It is also worthwhile to note that when applying GSVD or SVD, the multiplicity of the singular values is irrelevant, unlike what happens in [19], as this contrasts the hypothesis that the parameters $\{f_j\}$ are distinct.

2.3. Computation of $\{c_{js}\}$. Once $\{n,\, z_j,\, m_j\}$ have been computed, we are able to evaluate the coefficients c_{js}, given $h(k)$ at M distinct points $\{k_0,\, k_0+1,\, \ldots,\, k_0+M-1\}$. Indeed, we can write down the Casorati matrix and then solve the linear system (2.7).

Although theoretically not necessary, our numerical tests suggest to use more than $2M$ data. For this reason, whenever it is possible we prefer to use $2N$ $(N > M)$ sampled data and compute the eigenvalues by solving, in the least squares sense, the overdetermined linear system

$$\mathbf{K}_{2N,M}^{k_0}\mathbf{c} = \mathbf{h}_{k_0},$$

where $\mathbf{h}_{k_0} = [h(k_0),\, h(k_0+1),\, \ldots,\, h(k_0+2N-1)]$ and $\mathbf{K}_{NM}^{k_0}$ is the Casorati matrix of order $2N \times M$ $(N > M)$, obtained as a natural extension of (2.8). As expected, this extension becomes more important when the noise/signal ratio increases.

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3. Without knowing the value of M. Now we assume that M, the exact number of terms in (1.2), is unknown and that, as in most applications, only a reasonable upper bound $\widehat{M}$ of M is given.

Under this hypothesis, we want to recover all of the parameters and coefficients $\{n, m_j, f_j, c_{js}\}$ of (1.2), given an estimate of $h(k)$ in a set of $2N$ data $\{k_0,\ k_0+1,\ \ldots,\ k_0+2N-1\} \in \mathbb{N}^+$ with $N \geq \widehat{M}$. In this case we first have to estimate M, which can be done by using the following theorem.

THEOREM 3.1. *In the absence of noise in the data, the rank of the $N \times \widehat{M}$ Hankel matrix*

$$\mathbf{H}_{N\widehat{M}}^{k_0} = \begin{bmatrix} h(k_0) & h(k_0+1) & \ldots & h(k_0+\widehat{M}-1) \\ h(k_0+1) & h(k_0+2) & \ldots & h(k_0+\widehat{M}) \\ \vdots & \vdots & \vdots & \vdots \\ h(k_0+N-1) & h(k_0+N) & \ldots & h(k_0+N+\widehat{M}-2) \end{bmatrix} = [\mathbf{h}_{k_0}, \mathbf{h}_{h_0+1}, \ldots, \mathbf{h}_{k_0+\widehat{M}-1}],$$

which is a natural extension of $\mathbf{H}_{NM}^{k_0}$ ($\widehat{M} \geq M$), is exactly M.

Proof. By virtue of (2.4), taking the entries of the first M arrays $[\mathbf{h}_{k_0}, \ldots, \mathbf{h}_{k_0+M-2}]$ of $\mathbf{H}_{N\widehat{M}}^{k_0}$ as initial data, we get $\mathbf{h}_{k_0+M-1}$ as a linear combination of these vectors. By changing M into $M+1$ and using $[\mathbf{h}_{k_0+1}, \ldots, \mathbf{h}_{k_0+M-1}]$ as initial data for (2.4), we get $\mathbf{h}_{k_0+M}$ as a linear combination of such vectors and then of $[\mathbf{h}_{k_0}, \ldots, \mathbf{h}_{k_0+M-2}]$. Iterating the procedure we obtain that each column vector $[\mathbf{h}_{k_0+M-1}, \ldots, \mathbf{h}_{k_0+\widehat{M}-1}]$ is a linear combination of $[\mathbf{h}_{k_0}, \ldots, \mathbf{h}_{k_0+M-2}]$, which means that $\operatorname{rank}(\mathbf{H}_{N\widehat{M}}^{k_0}) = M = \operatorname{rank}(\mathbf{H}_{NM}^{k_0})$. ∎

In order to evaluate the rank of our matrices we use the *rank* algorithm of MATLAB which only gives the exact value of M in the noiseless data case. In the presence of noise we do not obtain the exact value of M. Nevertheless, the numerical results are satisfactory whenever a reliable overestimate of M is known. This assumption, as in many applications [19], is always satisfied in the application considered in Section 4 which mainly motivates our interest in the identification of parameters in the sum (1.2).

4. Numerical results. In this section we present the results of extensive numerical experiments of various examples, with some examples already presented in the literature and others, to our knowledge, never considered before.

To ascertain the effectiveness of our method, in the application considered below we estimate the relative error in the exponents f_j and the coefficients c_{js} for $j = 1, \ldots, n$, $s = 0, \ldots, m_j - 1$, by using the following error estimates

$$e(\mathbf{f}) = \max_{j=1,\ldots,n} \left| 1 - \frac{f_j}{f_j^*} \right|, \quad e(\mathbf{c}) = \max_{\substack{j=1,\ldots,n \\ s=0,\ldots,m_j-1}} \left| 1 - \frac{c_{js}}{c_{js}^*} \right|,$$

where f_j^* and c_{js}^* denote the exact values of the parameters. Moreover, denoting by $[0, b]$ the domain of $h(x)$ of main interest, we adopt the following relative error estimate of the monomial-exponential sum:

$$e(\mathbf{h}) = \max_{x \in X} \left| 1 - \frac{h(x)}{h^*(x)} \right|,$$

where $X = \{x_i = i\frac{b}{50},\ i = 1, \ldots, 50\}$.

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For Examples 2 and 3 to be discussed in Section 4.2, to compare with the results reported in [18, 19], we instead adopted the following error estimators:

$$e(\mathbf{f}) = \frac{\max\limits_{j=1,\dots,n} \left| f_j - f_j^* \right|}{\max\limits_{j=1,\dots,n} |f_j^*|}, \quad e(\mathbf{c}) = \frac{\max\limits_{\substack{j=1,\dots,n \\ s=0,\dots,m_j-1}} \left| c_{js} - c_{js}^* \right|}{\max\limits_{\substack{j=1,\dots,n \\ s=0,\dots,m_j-1}} |c_{js}^*|}, \tag{4.1}$$

and

$$e(\mathbf{h}) = \frac{\max\limits_{x\in[0,2N]} |h(x) - h^*(x)|}{\max\limits_{x\in[0,2N]} |h^*(x)|}, \tag{4.2}$$

which are the natural extension of those used there, and f_j^*, c_{js}^* and h^* being as mentioned before.

For each test function we assume that we only know a reliable estimate $\widehat{M}$ of M and consider both exact data and noisy data. In the latter case we consider white noise, i.e., we assume

$$h(k) = \tilde{h}(k) + \delta e_k, \quad k = k_0, \dots, k_0 + 2N - 1,$$

where $\tilde{h}(k)$ denotes the exact values of the monomial exponential sum, $e_k \in [0, 1]$ is a normally distributed random array and δ is the standard deviation of the sampled data.

Moreover, since the numerical results do not change significantly with respect to $k_0 \in \mathbb{N}^+$, we take $k_0 = 0$ in all the examples for the sake of simplicity.

Our numerical experiments highlight that, as expected, using GSVD or SVD in the computation of $\{z_j, m_j\}$ is essentially equivalent, with a minor difference in favor of GSVD. Hence, for the sake of simplicity, unless otherwise specified, the results quoted in the following tables refer to using GSVD.

We also note that the values of $\widehat{M}$ reported in each table, which represents an overestimate of M, are assumed known in advance and that all computations have been carried out in MATLAB with $\epsilon_{machine} = 2.22 \cdot 10^{-16}$.

4.1. An application to NPDEs of integrable type. An important area where effective methods for parameter identification in sums of monomial-exponential functions can be very useful is the class of non-linear partial differential equations (NPDEs) of integrable type. In this context the non-linear Schrödinger equation (NLS), which governs the signal transmission in optical fibers [9], plays a special role.

The main characteristic of this class is the fact that any initial value problem associated to an NPDE of integrable type can theoretically be solved by using the inverse scattering transform technique (IST). This technique is primarily based on the solution of a direct scattering problem and then on the solution of an inverse scattering problem, starting from the spectral data previously obtained by time evolution. From the numerical point of view, the first one is actually the most challenging, at least for the NLS, since the second one can be solved by using the numerical method proposed in [1].

The numerical solution of the direct scattering problem for the NLS is primarily based on the computation of the initial Marchenko kernels from the left and from the right, respectively, see [24]. These kernels, whenever the solution of the NLS is represented by one soliton or by

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a multisoliton (the so-called reflectionless case), can be represented by

$$\Omega_\ell(x) = \sum_{j=1}^{n} e^{-a_j x} \sum_{s=0}^{m_j-1} (\Gamma_\ell)_{js} \frac{x^s}{s!}, \quad x \in \mathbb{R}^+,$$

$$\Omega_r(x) = \sum_{j=1}^{n} e^{a_j x} \sum_{s=0}^{m_j-1} (\Gamma_r)_{js} \frac{x^s}{s!}, \quad x \in \mathbb{R}^-,$$

where $0^0 \equiv 1$ and a_j are complex or real parameters with $Re(a_j) > 0$.

The application of our method to Ω_ℓ allows us to estimate $\{n, m_j, (\Gamma_\ell)_{js}\}$, given Ω_ℓ in $2N$ ($N > M$) positive integer points, and then to recover $(\Gamma_r)_{js}$ by solving a linear system of order $N \times M$, given Ω_r in $2N$ ($N > M$) negative integer nodes, in the least squares sense. The same results can certainly also be obtained by applying first the method to $\Omega_r(x)$ to identify $\{n, m_j, (\Gamma_r)_{js}\}$ and then to $\Omega_\ell(x)$ to identify $(\Gamma_\ell)_{js}$.

Tables 4.1 and 4.2 show the error estimates obtained in the identification of Ω_ℓ parameters with coefficients in the following two cases (representative of four-solitons with 4 simple bound states, and with a double and two simple bound states, respectively):

(a) $n = 4, \quad m_1 = \ldots = m_4 = 1,$
$\mathbf{a} = \frac{1}{10}[1+7i, 1.2+3i, 1.4+6i, 3+1.6i]$ and $\mathbf{\Gamma}_\ell = [1+i, 2+i, 3+i, 4+i],$

(b) $n = 3, \quad m_1 = 2, \quad m_2 = m_3 = 1,$
$\mathbf{a} = \frac{1}{10}[1+7i, 1.4+6i, 3+1.6i]$ and $\mathbf{\Gamma}_\ell = [1+i, 2+i, 3+i, 4+i].$

In both cases we considered $[0,\ 5]$ as the interval of effective interest and assumed $b = 5$. Let us note that we assume $a \in \mathbb{C}^+$, where $\mathbb{C}^+$ is the complex upper half plane, as this hypothesis is always satisfied for the NPDEs of integrable type.

TABLE 4.1
Error estimates in the multisoliton case (a).

N	δ	$\widehat{M}$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
4	0	4	1.02e-10	1.28e-09	4.76e-15
8	0	7	1.33e-11	1.58e-10	1.08e-14
16	0	7	9.90e-14	1.11e-12	3.24e-15
32	0	7	5.86e-13	7.15e-12	3.44e-15
4	10^{-9}	4	7.13e-05	9.83e-04	2.48e-09
8	10^{-9}	7	2.70e-07	3.44e-06	2.85e-10
16	10^{-9}	7	8.14e-08	1.01e-06	2.02e-09
32	10^{-9}	7	5.79e-09	9.85e-08	3.69e-10
4	10^{-7}	4	4.56e-03	6.41e-02	9.25e-08
8	10^{-7}	7	3.32e-05	4.63e-04	4.32e-08
16	10^{-7}	7	7.33e-06	1.17e-04	1.21e-07
32	10^{-7}	7	1.13e-06	1.89e-05	6.06e-08

The results reported in Table 4.1 highlight that the identification of parameters and coefficients is satisfactory in case (a). Table 4.2 shows that the situation is more difficult in the presence of multiple bound states (case (b)), as known by people working in the NPDEs area of integrable type. Nevertheless, the results that we obtained are very good in the absence of noise and still reliable in the presence of noise, even when M is not known in advance.

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TABLE 4.2
Error estimates in the multisoliton case (b).

N	δ	$\widehat{M}$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
4	0	4	5.13e-06	5.43e-04	4.90e-08
8	0	7	1.49e-06	1.76e-04	1.66e-07
16	0	7	4.85e-07	7.14e-05	2.63e-07
32	0	7	3.18e-07	5.34e-05	3.06e-07
4	10^{-9}	4	3.17e-04	5.38e-02	3.09e-04
8	10^{-9}	7	2.45e-04	2.91e-02	2.73e-05
16	10^{-9}	7	4.04e-05	5.96e-03	2.20e-05
32	10^{-9}	7	2.49e-05	4.19e-03	2.40e-05
4	10^{-7}	4	2.44e-02	2.25e+00	2.17e-04
8	10^{-7}	7	3.44e-03	2.95e-01	3.82e-04
16	10^{-7}	7	8.83e-04	1.29e-01	4.81e-04
32	10^{-7}	7	3.41e-04	5.76e-02	3.28e-04

4.2. Other examples.

Example 1. We consider now the identification of $\{z_j\}$ and $\{c_j\}$ in the exponential-sum

$$h(x) = \sum_{j=1}^{30} c_j z_j^x,$$

already considered in [18, pp. 624-625], where the coefficients c_j are random values on $[0, 1]$ and the values z_j are equidistant nodes on three circles having radius $r = 0.7, 0.8$ and 0.9. Taking $N = 200$ and $\widehat{M} = 40$ we obtain the results reported in Figure 4.1, where the exact nodes are depicted as circles and their recoveries by solid dots on the left for the exact data and on the right for inexact data. The figure shows that the evaluation of z_j is very accurate in the absence of noise and reliable in the presence of noise, and increasingly more accurate as r decreases compared with those reported in [18, Figure 1]. The reason is that the distance between the nodes reduces with the decrease of r and the method in [18] fails if two of them coincide.

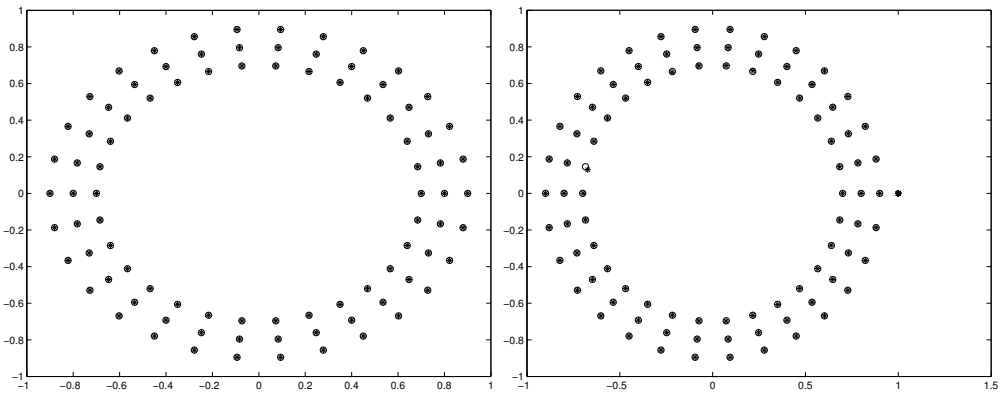

FIG. 4.1. *Graphic representation of the nodes of Example 1 for exact data (to the left) and for noisy data with* $\delta = 10^{-10}$ *(to the right).*

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TABLE 4.3
Error estimates with exact data for Example 2.

	N	$\widehat{M}$	R	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
via GSVD	6	6	6	2.02e-09	1.07e-09	8.63e-15
	12	10	6	2.31e-12	2.18e-12	1.84e-13
	24	10	6	8.33e-14	3.16e-13	1.82e-13
via SVD	6	6	6	5.37e-09	3.16e-09	4.06e-15
	12	10	6	2.44e-11	2.19e-11	1.71e-12
	24	10	6	5.03e-13	3.90e-12	2.28e-12

TABLE 4.4
Error estimates with noisy data for Example 2.

	N	δ	$\widehat{M}$	R	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
via GSVD	6	10^{-9}	6	6	6.47e-05	6.94e-05	9.56e-10
	12	10^{-9}	10	11	4.19e-08	1.27e-07	2.14e-10
	24	10^{-9}	10	11	2.64e-10	1.70e-09	4.24e-10
via SVD	6	10^{-9}	6	6	7.34e-04	4.48e-04	8.31e-11
	12	10^{-9}	10	11	5.67e-07	2.84e-07	6.27e-10
	24	10^{-9}	10	11	1.94e-09	7.06e-09	4.30e-09

Example 2. Let us now consider the exponential sum described in [19, p. 1034] having the following coefficients c_j and zeros z_j:

$$\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 0.9856 - 0.1628i \\ 0.9856 + 0.1628i \\ 0.8976 - 0.4305i \\ 0.8976 + 0.4305i \\ 0.8127 - 0.5690i \\ 0.8127 + 0.5690i \end{bmatrix}.$$

In this case, for an immediate comparison of our results with those reported in [19], we adopted the error estimator proposed there. Considering data without and with noise we obtain the results reported in Table 4.3 and in Table 4.4, respectively, where $\widehat{M}$ is the upper estimate of $M = 6$ and R is the rank value of $\mathbf{H}_{N,\widehat{M}+1}$ obtained by using the *rank* algorithm of MATLAB.

It is worthwhile noting that, in the absence of noise, our method identifies the exact values of M regardless of the number of data. Table 4.4 shows that, if the data are noisy, as it is to be expected, the estimate of M is exact in the case $N = M$ and overestimated whenever $N > M$. Nevertheless, as Table 4.4 shows, the identification of both the parameters and the coefficients is very accurate provided that we can find a reliable estimate of M.

Tables 4.3 and Table 4.4 also show that the results obtained by using GSVD are slighthly better than using SVD.

Let us finally note that, as expected, when the zeros z_j are well separated the results in [19] are equivalent to ours.

Example 3. In order to show that our method is effective in case of multiple zeros, we now modify Example 2 by setting $z_1 = z_2 = 0.9856 - 0.1628i$, $z_3 = z_4 = 0.8976 - 0.4305i$, and leaving z_5 and z_6 unchanged. In this example, as in the previous one, the errors are evaluated by using (4.1) and (4.2). Our numerical results in Tables 4.5 and 4.6, show that

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TABLE 4.5
Error estimates with exact data for Example 3.

	N	$\widehat{M}$	R	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
via GSVD	6	6	6	1.30e-04	3.00e-03	3.09e-07
	12	10	6	1.26e-05	4.29e-04	7.52e-07
	24	10	6	2.51e-06	1.58e-04	5.89e-06
via SVD	6	6	6	1.40e-04	3.35e-03	3.43e-07
	12	10	6	4.76e-06	8.06e-05	3.67e-07
	24	10	6	2.52e-06	2.23e-04	7.66e-06

TABLE 4.6
Error estimates with noisy data for Example 3.

	N	δ	$\widehat{M}$	R	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
via GSVD	6	10^{-9}	6	6	3.51e-02	1.98e+00	1.93e-03
	12	10^{-9}	10	11	5.52e-04	1.30e-02	2.65e-05
	24	10^{-9}	10	11	3.17e-05	2.72e-03	5.85e-05
via SVD	6	10^{-9}	6	6	6.98e-02	2.93e+00	1.90e-03
	12	10^{-9}	10	11	7.10e-04	1.10e-02	5.39e-05
	24	10^{-9}	10	11	1.39e-04	1.31e-02	3.45e-04

the recovery of the parameters is still accurate even if two zeros are double and $\widehat{M}$ is a large overestimate of M.

5. Conclusions. The main contribution of this paper is the extension of matrix-pencil methods, introduced originally for the identification of the parameters of the exponential sums, to monomial-exponential sums. An effective matrix-pencil method based on GSVD or SVD is proposed to identify simple and multiple zeros of the associated Prony polynomials. The effectiveness of GSVD is in general superior to that of SVD, even though only slightly. Nevertheless the availability of the two algorithmic variants of the same method is useful in the sense that they can be used for a mutual validation of the results.

Our extensive numerical experiments show that our method allows us to estimate the parameters and coefficients of a monomial-exponential sum with good precision, even when the number of terms is not known, provided with only a reasonable overestimation of it. This method furnishes very accurate results in the absence of noise and acceptable results in the presence of a moderately high level of noise, whenever a relatively large number of data, compared with the number of parameters and coefficients to identify, is available. Finally, we point out that our method can be considered equivalent to the matrix-pencil SVD version of the ESPRIT method, recently proposed in [18] if the nodes are well separated, and is increasingly more effective when the distance between the nodes diminishes.

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