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CONVERGENCE ANALYSIS OF THE OPERATIONAL TAU METHOD FOR ABEL-TYPE VOLTERRA INTEGRAL EQUATIONS*

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Abstract. In this paper, a spectral Tau method based on Jacobi basis functions is proposed and its stability and convergence properties are considered for obtaining an approximate solution of Abel-type integral equations. This work is organized in two parts. First, we present a stable operational Tau method based on Jacobi basis functions that provides an efficient approximate solution for the Abel-type integral equations by using a reduced set of matrix operations. We also provide a rigorous error analysis for the proposed method in the weighted L^2- and uniform norms under more general regularity assumptions on the exact solution. It is shown that the proposed method converges, but since the solutions of these equations have a singularity near the origin, a loss in the convergence order of the Tau method is expected. To overcome this drawback we then propose a regularization process, in which the original equation is changed into a new equation which possesses a smooth solution, by applying a suitable variable transformation such that the spectral Tau method can be applied conveniently. We also prove that after this regularization technique, the numerical solution of the new equation based on the operational Tau method has exponential rate of convergence. Some standard examples are provided to confirm the reliability of the proposed method.

Key words. Operational Tau method, Abel-type Volterra integral equations

AMS subject classifications. 45E10, 41A25

1. Introduction. This paper deals with the numerical treatment of the following class of Abel-type Volterra integral equations

$$u(x) = f(x) + \lambda \int_0^x \frac{K(x,t)}{\sqrt[\mu]{x-t}}\, u(t)dt, \quad x \in I = [0,1], \tag{1.1}$$

where $f(x)$ and $K(x,t)$ are a given continuous function and a sufficiently smooth kernel function, respectively, $u(x)$ is the unknown function, and $\mu \in \mathbb{N}$ with $\mu \geq 2$ ($\mathbb{N}$ is the collections of all natural numbers). This equation is a particular case of linear Volterra weakly singular integral equations of the second kind. Several numerical methods have been proposed for (1.1); see [2, 3, 5, 6, 8, 9, 11].

From the well-known existence and uniqueness theorems, it can be concluded that in Abel-type integral equations we must expect some derivatives of the solution to have a discontinuity at the origin; see [3, 8]. As discussed in [3, 8], the first derivative of the solution $u(x)$ behaves like $u'(x) \sim x^{-\frac{1}{\mu}}$ and $u'(x) \notin C(I)$.

In order to approximate the solution of (1.1) we propose a strategy mainly consisting in two steps. In the first step, we introduce the operational Tau method based upon Jacobi basis functions to (1.1). This strategy is an application of the matrix-vector product approach in spectral approximation. The main characteristic behind this approach is that it reduces such problems to those of solving a system of linear algebraic equations. In this step, we also investigate convergence and stability behavior of the numerical solution of (1.1). We can deduce convergence of the Tau method at this point, but due to the fact that the solutions of these equations usually have a weak singularity at the origin, this method leads to very poor numerical results. Thus, it is necessary to introduce a regularization procedure that allows us

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to improve the smoothness of the given functions and then to approximate the solution with a satisfactory order of convergence. In [8], Tao Tang introduced and discussed a simple variable transformation for the regularization of solutions of Abel-type Volterra integral equations of the second kind, such that the resulting equation possesses a smooth solution. Then, in the second step we transform (1.1) by using this coordinate transformation and we prove that after this regularization technique the numerical solution of the new equation by the operational Tau method has exponential rate of convergence.

The organization of this paper is as follows: in Section 2, we introduce the operational Tau method and its application to the Abel-type integral equation (1.1). In addition, we analyze convergence and stability behavior of the numerical solution of (1.1). In Section 3, by applying a suitable regularization process, an operational scheme for obtaining the numerical solution of the transformed equation is considered and an exponential rate of convergence for the proposed regularized scheme is proved. In Section 4, some numerical examples are considered which confirm our theoretical predictions.

2. Operational Tau method for Abel integral equations. In this section, we present a numerical solution of (1.1) by using the operational Tau method based on Jacobi basis functions.

2.1. Numerical treatment. Let $\underline{V}^{\alpha,\beta} := [v_0^{\alpha,\beta}(x), v_1^{\alpha,\beta}(x), ..., v_N^{\alpha,\beta}(x), ...]^T$ with parameters $\alpha, \beta \in (-1, 1)$ be an arbitrary Jacobi polynomial bases with respect to the inner product

$$(v_i^{\alpha,\beta}, v_j^{\alpha,\beta})_{\alpha,\beta} = \int_I v_i^{\alpha,\beta}(x) v_j^{\alpha,\beta}(x) w^{\alpha,\beta}(x) dx,$$

where $w^{\alpha,\beta}(x) = (2-2x)^\alpha (2x)^\beta$ is the Jacobi weight function; see [4, 10]. Clearly, we can write $\underline{V}^{\alpha,\beta} := V^{\alpha,\beta}\underline{X}$, where $V^{\alpha,\beta}$ is an infinitely non-singular lower triangular coefficient matrix with degree $(v_i^{\alpha,\beta}(x)) \le i$, for $i = 0, 1, 2, \ldots$, and $\underline{X} = [1, x, x^2, \ldots, x^N, \ldots]^T$.

Suppose that $f(x)$ is a given polynomial and consider

$$f(x) = \sum_{i=0}^{N_f} f_i v_i^{\alpha,\beta}(x) = \underline{f}\,\underline{V}^{\alpha,\beta} = \underline{f} V^{\alpha,\beta} \underline{X}, \quad \underline{f} = (f_0, f_1, f_2, \ldots, f_{N_f}, 0, \ldots). \tag{2.1}$$

If $f(x)$ is not polynomial, then it can be approximated by polynomials to any degree of accuracy by interpolation or any other suitable method. Let $u(x) = \sum_{i=0}^{\infty} a_i v_i^{\alpha,\beta}(x) = \underline{a} V^{\alpha,\beta} \underline{X}$ be the Jacobi series expansion of the exact solution of (1.1), where $\underline{a} = (a_0, a_1, ...)$ and $u_N^{\alpha,\beta}(x)$ is a Tau approximation of degree N for $u(x)$ as

$$u_N^{\alpha,\beta}(x) = \sum_{i=0}^{\infty} a_i v_i^{\alpha,\beta}(x) = \underline{a_N} V^{\alpha,\beta} \underline{X}, \quad \underline{a_N} = (a_0, a_1, ..., a_N, 0, ...). \tag{2.2}$$

We define

$$L(u(x)) = u(x) - \lambda \int_0^x \frac{K(x,t)}{\sqrt[\mu]{x-t}} u(t) dt \tag{2.3}$$

and assume that

$$K(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{ij} v_i^{\alpha,\beta}(x) v_j^{\alpha,\beta}(t),$$

which can be rearranged as

$$K(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tilde{k}_{ij} x^i t^j.$$

Now, the following theorem shows how to replace (1.1) by a matrix formulation of the operational Tau method.

THEOREM 2.1. *Let $v_j^{\alpha,\beta}(x)$ be the orthogonal Jacobi polynomials with respect to the weight function $w^{\alpha,\beta}(x)$ in I and let the approximate solution $u_N^{\alpha,\beta}(x)$ be given by the relation (2.2). Then we have*

$$L(u_N^{\alpha,\beta}(x)) = \underline{a_N}\left(\mathrm{id} - \lambda V^{\alpha,\beta}\Pi^{\alpha,\beta}\right)\underline{V}^{\alpha,\beta},$$

where id *is the infinite identity matrix,*

$$\Pi^{\alpha,\beta} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tilde{k}_{ij} \mathcal{B}^j \mathcal{C}^{i,j},$$

$\mathcal{B}^j$ is the infinite diagonal matrix with diagonal entries $\mathcal{B}^j_{ss} = B(j+s+1, \frac{\mu-1}{\mu})$, for all $j, s = 0,1,2,\ldots$ (here $B(a,b)$ denotes the Beta function), and

$$\mathcal{C}^{i,j} = \left(c_{lk}\right)_{l,k=0}^{\infty} = \frac{\left(x^{i+j+l+1-\frac{1}{\mu}}, v_k^{\alpha,\beta}\right)_{\alpha,\beta}}{\left(v_k^{\alpha,\beta}, v_k^{\alpha,\beta}\right)_{\alpha,\beta}}.$$

Proof. From relations (2.2) and (2.3) we can write

$$L(u_N^{\alpha,\beta}(x)) = \underline{a_N}\left(\underline{V}^{\alpha,\beta} - \lambda V^{\alpha,\beta}\int_0^x \frac{K(x,t)}{\sqrt[\mu]{x-t}}\,\underline{X_t}\,dt\right), \tag{2.4}$$

where $\underline{X_t} = [1, t, t^2, ..., t^N, ...]^T$. Thus, the integral term of (2.4) can be written as

$$\int_0^x \frac{K(x,t)}{\sqrt[\mu]{x-t}}\,\underline{X_t}\,dt = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tilde{k}_{ij} x^i \left[\int_0^x \frac{t^{j+l}}{\sqrt[\mu]{x-t}}dt\right]_{l=0}^{\infty}. \tag{2.5}$$

By using the relation

$$\int_0^x \frac{t^{j+l}}{\sqrt[\mu]{x-t}}dt = x^{j+l+1-\frac{1}{\mu}} B(j+l+1, \frac{\mu-1}{\mu}),$$

we can rewrite (2.5) as

$$\begin{aligned}\int_0^x \frac{K(x,t)}{\sqrt[\mu]{x-t}}\,\underline{X_t}\,dt &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tilde{k}_{ij}\left[x^{i+j+l+1-\frac{1}{\mu}} B(j+l+1, \frac{\mu-1}{\mu})\right]_{l=0}^{\infty} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tilde{k}_{ij}\mathcal{B}^j\left[x^{i+j+l+1-\frac{1}{\mu}}\right]_{l=0}^{\infty},\end{aligned} \tag{2.6}$$

Substituting $x^{i+j+l+1-\frac{1}{\mu}}$, $i, j, l = 0, 1, 2, \ldots$, by its orthogonal Jacobi projection polynomial of degree N we get (see [4, 10])

$$\left[x^{i+j+l+1-\frac{1}{\mu}}\right]_{l=0}^{\infty} = \left[\sum_{k=0}^{\infty} c_{lk} v_k^{\alpha,\beta}(x)\right]_{l=0}^{\infty} = \mathcal{C}^{i,j}\underline{V}^{\alpha,\beta}, \tag{2.7}$$

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where

$$\mathcal{C}^{ij} = (c_{lk})_{l,k=0}^{\infty} = \frac{\left(x^{i+j+l+1-\frac{1}{\mu}}, v_k^{\alpha,\beta}\right)_{\alpha,\beta}}{\left(v_k^{\alpha,\beta}, v_k^{\alpha,\beta}\right)_{\alpha,\beta}}, \qquad i,j = 0,1,2,\dots\,.$$

By substituting (2.7) into (2.6) we obtain

$$\int_0^x \frac{K(x,t)}{\sqrt[\mu]{x-t}} \underline{X_t} dt = \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{k}_{ij} \mathcal{B}^j \mathcal{C}^{i,j} \right) \underline{V}^{\alpha,\beta} = \Pi^{\alpha,\beta} \underline{V}^{\alpha,\beta}, \tag{2.8}$$

where $\Pi^{\alpha,\beta} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{k}_{ij} \mathcal{B}^j \mathcal{C}^{i,j}$, and inserting (2.8) into (2.4) we can conclude the result. $\square$

We are now ready to obtain the following algebraic form of the operational Tau discretization of (1.1) based on Jacobi polynomials. According to the proposed method, following Theorem 2.1 and relation (2.1), we obtain

$$\underline{a_N} \left(\mathrm{id} - \lambda V^{\alpha,\beta} \Pi^{\alpha,\beta} \right) \underline{V}^{\alpha,\beta} = \underline{f} \underline{V}^{\alpha,\beta}. \tag{2.9}$$

If we let $\mathcal{A}^{\alpha,\beta} = \mathrm{id} - \lambda V^{\alpha,\beta} \Pi^{\alpha,\beta}$, then, because of the orthogonality of $\{v_k^{\alpha,\beta}(x)\}_{k=0}^{\infty}$ (see [4, 10]), projecting (2.9) onto $\{v_k^{\alpha,\beta}(x)\}_{k=0}^{N}$ yields

$$\underline{a_N} \mathcal{A}_i^{\alpha,\beta} = f_i,$$

where $\mathcal{A}_i^{\alpha,\beta}$ is the i-th column of $\mathcal{A}^{\alpha,\beta}$. By setting

$$\tilde{\mathcal{A}}_N^{\alpha,\beta} = [\mathcal{A}_0^{\alpha,\beta}, \mathcal{A}_1^{\alpha,\beta}, \dots, \mathcal{A}_N^{\alpha,\beta}], \qquad \tilde{f}_N = [f_0, f_1, \dots, f_N],$$

we obtain $\underline{a_N} \tilde{\mathcal{A}}_N^{\alpha,\beta} = \tilde{f}_N$, which gives us the unknown vector $(a_0, a_1, \dots, a_N)$.

2.2. Stability and convergence analysis. In this section, we provide a suitable stability and error analysis which theoretically justifies stability and convergence of the proposed method for the numerical solution in the special case of (1.1) when $K(x,t) = 1$, i.e.,

$$u(x) = f(x) + \lambda \int_0^x \frac{u(t)}{\sqrt[\mu]{x-t}} dt, \quad x \in I. \tag{2.10}$$

In our subsequent analysis, some preliminary results are needed. Throughout the paper, C will denote a generic positive constant that is independent of N. Let $L^2_{\alpha,\beta}(I)$ be the space of functions whose square is Lebesque integrable in I relative to the weight function $w^{\alpha,\beta}(x)$. The latter is a Banach space with the norm (see [4])

$$\|v\|^2_{\alpha,\beta} = (v,v)_{\alpha,\beta} = \int_I v^2(x) w^{\alpha,\beta}(x) dx, \quad \forall v \in L^2_{\alpha,\beta}(I).$$

$B^k_{\alpha,\beta}(I)$ denotes the non-uniform Jacobi-Sobolev space of all functions $v(x)$ on I such that $v(x)$ and its weak derivatives of order s are in $L^2_{\alpha+s,\beta+s}(I)$ for $0 \le s \le k$. We define the norm (see [10])

$$\|v\|^2_{k,\alpha,\beta} = \sum_{s=1}^{k} \left\| v^{(s)} \right\|^2_{\alpha+s,\beta+s}.$$

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It is trivial that $\|v\|_{0,\alpha,\beta} = \|v\|_{\alpha,\beta}$ and $\|v^{(k)}\|_{\alpha+k,\beta+k} \leq \|v\|_{k,\alpha,\beta}$. The Hölder space $C^{k,\gamma}(I)$ where $k \geq 0$ is an integer, consists of those functions on I having continuous derivatives up to order k and such that the kth derivative is Hölder continuous with exponent γ where $0 < \gamma \leq 1$. If the Hölder constant

$$|v|_{0,\gamma} = \sup_{x \neq t \in I} \frac{|v(x) - v(t)|}{|x-t|^{\gamma}},$$

is finite, then the function v is said to be Hölder continuous with exponent γ in I. If the function v and its derivatives up to order k are bounded on I, then the Hölder space $C^{k,\gamma}(I)$ can be assigned the norm

$$\|v\|_{k,\gamma} = \|v\|_k + \max_{|\eta|=k} |D^{\eta} v|_{0,\gamma},$$

where η ranges over multi-indices and $\|v\|_k = \max_{|\eta| \leq k} \sup_{x \in I} |D^{\eta} v|$. When $\gamma = 0$, $C^{k,0}(I)$ denotes the space of functions with k continuous derivatives on I, also denoted by $C^k(I)$ and with norm $\|.\|_k$; see [1]. It can be easily seen that the function $v(x) = x^s$ with $0 < s \leq 1$ defined on I belongs to the space $C^{0,\gamma}(I)$ for $0 < \gamma \leq s$. We further define a linear weakly singular integral operator

$$\mathcal{K}(v) = \int_0^x \frac{\tilde{K}(x,t)}{\sqrt[\mu]{x-t}} v(t) dt, \quad x, t \in I,$$

where $\tilde{K}(x,t)$ is sufficiently smooth kernel function and μ is a natural number with $\mu \geq 2$. It can be proven that for any continuous function v, there exist a positive constant C such that

$$\mathcal{K}(v) \in C^{0,\gamma}(I), \quad \|\mathcal{K}(v)\|_{0,\gamma} \leq C\|v\|_{\infty}, \quad \gamma \in \left(0, 1 - \frac{1}{\mu}\right), \tag{2.11}$$

where $\|.\|_{\infty}$ is the usual uniform norm. The proof of (2.11) can be found in [5].

Let $\mathcal{P}_N$ be the space of all algebraic polynomials of degree up to N. Now, we introduce the orthogonal projection $\mathcal{P}_N^{\alpha,\beta} : L^2_{\alpha,\beta}(I) \to \mathcal{P}_N$ which is a mapping such that for any $v \in L^2_{\alpha,\beta}(I)$,

$$(v - \mathcal{P}_N^{\alpha,\beta} v, \phi)_{\alpha,\beta} = 0, \quad \forall \phi \in \mathcal{P}_N.$$

Concerning the truncation error of a Jacobi series, the following estimates hold (see [1, 10])

$$\|v - \mathcal{P}_N^{\alpha,\beta} v\|_{k,\alpha,\beta} \leq C N^{k-s} \left\|v^{(s)}\right\|_{\alpha+s,\beta+s}, \; v \in B^s_{\alpha,\beta}(I), \quad s \geq k, \quad k \geq 0, \tag{2.12}$$

$$\begin{aligned} \left\|v - \mathcal{P}_N^{\alpha,\beta} v\right\|_{\infty} \leq C\big(1 + \sigma_p(N)\big) N^{-(k+\gamma)} \|v\|_{k,\gamma}, \\ v \in C^{k,\gamma}(I), \quad k \geq 0, \quad \gamma \in [0,1], \end{aligned} \tag{2.13}$$

where

$$\sigma_p(N) = \begin{cases} O(\log(N)) & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ O(N^{s+\frac{1}{2}}) & \text{otherwise,} \end{cases}$$

with $s = \max\{\alpha, \beta\}$, is the well known Lebesgue constant.

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In our analysis we shall apply the Hardy and Gronwall inequalities (see [5, 7]):

LEMMA 2.2 (Generalized Hardy's inequality [7]). *For measurable functions $g \geq 0$, the following generalized Hardy's inequality*

$$\left(\int_a^b |(\lambda g)(t)|^q w_1(t)dt\right)^{1/q} \leq C\left(\int_a^b |g(t)|^p w_2(t)dt\right)^{1/p},$$

holds if and only if

$$\sup_{a<t<b}\left(\int_t^b w_1(t)dt\right)^{1/q}\left(\int_a^t w_2^{1-p'}(t)\right)^{1/p'} < \infty, \qquad p' = \frac{p}{p-1},$$

for $1 < p \leq q < \infty$. Here, λ is an operator of the form

$$(\lambda g)(t) = \int_a^t k(t,s)g(s)ds,$$

with given kernel $k(t,s)$ and weight functions $w_1(t)$, $w_2(t)$ for $-\infty \leq a < b \leq \infty$.

LEMMA 2.3 (Gronwall inequality [5]). *Assume that $v(x)$ is a non-negative, locally integrable function defined on I which satisfies*

$$v(x) \leq b(x) + B\int_0^x (x-t)^m t^n v(t)dt, \quad t \in I,$$

where $b(x) \geq 0$ and $B \geq 0$. Then there exist a constant C such that

$$v(x) \leq b(x) + C\int_0^x (x-t)^m t^n b(t)dt, \quad t \in I.$$

Now, we state and prove the main results of this section regarding the stability and error analysis of the proposed method for the numerical solution of (2.10).

THEOREM 2.4 (Stability). *Let $u_N^{\alpha,\beta}(x)$ be the Tau approximation (2.2) to the exact solution $u(x)$ of the Abel integral equation (2.10). Assume that the function $f(x)$ is continuous and $|\lambda| < 1$. Also suppose $\tilde{u} \in \mathcal{P}_N$ and $\tilde{f} \in C(I)$ are the errors of $u_N^{\alpha,\beta}$ and f, respectively. Then, we have*

$$\|\tilde{u}\|_{\alpha,\beta} \leq C\big\|\tilde{f}\big\|_{\alpha,\beta}.$$

Proof. We know that $u_N^{\alpha,\beta}$ and $u_N^{\alpha,\beta} + \tilde{u}$ satisfy the following equations

$$u_N^{\alpha,\beta}(x) = \mathcal{P}_N^{\alpha,\beta} f(x) + \lambda\mathcal{P}_N^{\alpha,\beta}\int_0^x \frac{u_N^{\alpha,\beta}(t)}{\sqrt[\mu]{x-t}}dt, \tag{2.14}$$

$$u_N^{\alpha,\beta}(x) + \tilde{u}(x) = \mathcal{P}_N^{\alpha,\beta}\big(f(x) + \tilde{f}(x)\big) + \lambda\mathcal{P}_N^{\alpha,\beta}\int_0^x \frac{\big(u_N^{\alpha,\beta}(t) + \tilde{u}(t)\big)}{\sqrt[\mu]{x-t}}dt. \tag{2.15}$$

Subtracting (2.15) form (2.14) we get

$$\tilde{u}(x) = \mathcal{P}_N^{\alpha,\beta}\tilde{f}(x) + \lambda\mathcal{P}_N^{\alpha,\beta}\int_0^x \frac{\tilde{u}(t)}{\sqrt[\mu]{x-t}}dt$$

and then

$$\|\tilde{u}\|_{\alpha,\beta} \leq \big\|\mathcal{P}_N^{\alpha,\beta}\tilde{f}\big\|_{\alpha,\beta} + |\lambda|\left\|\mathcal{P}_N^{\alpha,\beta}\int_0^x \frac{\tilde{u}(t)}{\sqrt[\mu]{x-t}}dt\right\|_{\alpha,\beta}. \tag{2.16}$$

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Since $\mathcal{P}_N^{\alpha,\beta}$ is an orthogonal projection, $\|\mathcal{P}_N^{\alpha,\beta}\|_{\alpha,\beta} = 1$ and from (2.16) we obtain

$$\|\tilde{u}\|_{\alpha,\beta} \leq \|\tilde{f}\|_{\alpha,\beta} + |\lambda| \left\| \int_0^x \frac{\tilde{u}(t)}{\sqrt[\mu]{x-t}} dt \right\|_{\alpha,\beta}.$$

Using Hardy's inequality (i.e., Lemma 2.2) in the integral term of the (2.13), we obtain the desired result. $\square$

In the following theorem we provide an error analysis which theoretically justifies convergence of the proposed scheme for the numerical solution of (2.10) in the L^2 and uniform norms.

THEOREM 2.5 (Convergence). *Let $u_N^{\alpha,\beta}(x)$ be the Tau approximation (2.2) to the exact solution $u(x)$ of the Abel integral equation (2.10). If $u(x) \in C^{k,\gamma}(I) \cap B^m_{\alpha,\beta}(I)$ with $k \geq 0$, $\gamma \in (0, 1]$, and $m \geq 0$, then we have*

$$\left\|e_N^{\alpha,\beta}(u)\right\|_\infty \leq \begin{cases} C \log(N) N^{-(\gamma+k)} \|u\|_{k,\gamma} & \text{for } -1 < \alpha, \beta \leq -\frac{1}{2}, \\ C N^{\frac{1}{2}+s-\gamma-k} \|u\|_{k,\gamma} & \text{for } \gamma + k > \frac{1}{2},\ s = \max\{\alpha, \beta\} < 0, \end{cases}$$

as well as,

$$\left\|e_N^{\alpha,\beta}(u)\right\|_{\alpha,\beta} \leq C \left(N^{-m} \|u\|_{m,\alpha,\beta} + \log(N) N^{-\gamma_1} \left\|e_N^{\alpha,\beta}(u)\right\|_\infty \right),$$

for $\gamma_1 \in (0, 1 - \frac{1}{\mu})$, $-1 < \alpha, \beta \leq -\frac{1}{2}$, and

$$\left\|e_N^{\alpha,\beta}(u)\right\|_{\alpha,\beta} \leq C \left(N^{-m} \|u\|_{m,\alpha,\beta} + N^{\frac{1}{2}+s-\gamma_1} \left\|e_N^{\alpha,\beta}(u)\right\|_\infty \right),$$

for $\gamma_1 \in (\frac{1}{2} + s, 1 - \frac{1}{\mu})$, $s = \max\{\alpha, \beta\} < 0$, where $e_N^{\alpha,\beta}(u) = u(x) - u_N^{\alpha,\beta}(x)$ is defined as error function.

Proof. Considering equation (2.10), according to the proposed method we have

$$u_N^{\alpha,\beta}(x) = \mathcal{P}_N^{\alpha,\beta} f + \lambda \mathcal{P}_N^{\alpha,\beta} \int_0^x \frac{u_N^{\alpha,\beta}(t)}{\sqrt[\mu]{x-t}} dt. \tag{2.17}$$

Subtracting (2.10) from (2.17), we have

$$e_N^{\alpha,\beta}(u) = e_{\mathcal{P}_N}^{\alpha,\beta}(f) + \lambda \left(\int_0^x \frac{u(t)}{\sqrt[\mu]{x-t}} dt - \mathcal{P}_N^{\alpha,\beta} \int_0^x \frac{u_N^{\alpha,\beta}(t)}{\sqrt[\mu]{x-t}} dt \right), \tag{2.18}$$

where $e_{\mathcal{P}_N}^{\alpha,\beta}(v) = v - \mathcal{P}_N^{\alpha,\beta} v$. By some simple manipulations we can rewrite (2.18) as follows

$$\begin{aligned} e_N^{\alpha,\beta}(u) = e_{\mathcal{P}_N}^{\alpha,\beta}(f) + \lambda \Bigg(\int_0^x \frac{u(t)}{\sqrt[\mu]{x-t}} dt \\ - \mathcal{P}_N^{\alpha,\beta} \int_0^x \frac{u(t)}{\sqrt[\mu]{x-t}} dt + \int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}} dt - e_{\mathcal{P}_N}^{\alpha,\beta} \int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}} dt \Bigg). \end{aligned}$$

From (2.10) we have $\lambda \int_0^x \frac{u(t)}{\sqrt[\mu]{x-t}} dt = u(x) - f(x)$ and can rewrite the above relation as

$$e_N^{\alpha,\beta}(u) = e_{\mathcal{P}_N}^{\alpha,\beta}(u) + \lambda \left(\int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}} dt - e_{\mathcal{P}_N}^{\alpha,\beta} \int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}} dt \right),$$

which yields

$$\left|e_N^{\alpha,\beta}(u)\right| \leq \left|e_{\mathcal{P}_N}^{\alpha,\beta}(u) - \lambda e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}}dt\right| + |\lambda|\int_0^x \frac{|e_N^{\alpha,\beta}(u)|}{\sqrt[\mu]{x-t}}dt. \tag{2.19}$$

Using Gronwall's inequality (Lemma 2.3) in (2.19), we can write

$$\left\|e_N^{\alpha,\beta}(u)\right\|_\infty \leq C\left(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(u)\right\|_\infty + \left\|e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}}dt\right\|_\infty\right). \tag{2.20}$$

By applying the relations (2.11) and (2.13) in (2.20), we obtain

$$\begin{aligned}\left\|e_N^{\alpha,\beta}(u)\right\|_\infty &\leq C\big(1+\sigma_p(N)\big)\left(N^{-(\gamma+k)}\|u\|_{k,\gamma} + N^{-\gamma_1}\left\|\int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}}dt\right\|_{0,\gamma_1}\right)\\ &\leq C\big(1+\sigma_p(N)\big)\left(N^{-(\gamma+k)}\|u\|_{k,\gamma} + N^{-\gamma_1}\left\|e_N^{\alpha,\beta}(u)\right\|_\infty\right),\end{aligned}$$

where $k \geq 0, \gamma \in [0,1]$ and $\gamma_1 \in (0, 1-\frac{1}{\mu})$. The first result of the theorem regarding the error estimate in the uniform norm can be obtained under the following conditions

$$\begin{aligned} &0 < \gamma_1 < 1 - \frac{1}{\mu} && -1 < \alpha,\beta \leq -\frac{1}{2},\\ &\frac{1}{2} + s < \gamma_1 < 1 - \frac{1}{\mu} && s = \max\{\alpha,\beta\} < 0.\end{aligned} \tag{2.21}$$

Now, we derive a suitable error bound for the proposed scheme in the L^2-norm. To this end, by applying again Gronwall's inequality (Lemma 2.3) in (2.19), we write

$$\begin{aligned}\left\|e_N^{\alpha,\beta}(u)\right\|_{\alpha,\beta} &\leq C\left(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(u)\right\|_{\alpha,\beta} + \left\|e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}}dt\right\|_{\alpha,\beta}\right)\\ &\leq C\left(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(u)\right\|_{\alpha,\beta} + \left\|e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^x \frac{e_N^{\alpha,\beta}(u)}{\sqrt[\mu]{x-t}}dt\right\|_{\infty}\right).\end{aligned}$$

Using relations (2.11) and (2.13) in the integral term of the above equation yields

$$\left\|e_N^{\alpha,\beta}(u)\right\|_{\alpha,\beta} \leq C\left(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(u)\right\|_{\alpha,\beta} + \big(1+\sigma_p(N)\big)N^{-\gamma_1}\left\|e_N^{\alpha,\beta}(u)\right\|_\infty\right), \tag{2.22}$$

where γ_1 satisfies (2.21).

Finally, the second result of the theorem can be deduced by adopting the relation (2.12) in (2.22). $\square$

In general, the exact solution $u(x)$ of the Abel integral equation (2.10) behaves like $x^{1-\frac{1}{\mu}}$. Thus, $u(x) \in C^{0,\gamma}(I)$ with $0 < \gamma \leq 1 - \frac{1}{\mu}$. In this case, in Theorem 2.5 we have $m = 1$, $k = 0$ and $\gamma \in (0, 1-\frac{1}{\mu}]$. As a result we can deduce convergence of the operational Tau method from Theorem 2.5 by choosing suitable values of α, β. But due to the low smoothness of the exact solution, this scheme leads to a low order numerical method. To overcome this drawback, in the next section we propose a regularization process in which the original equation (1.1) will be changed into a new equation which possesses a smooth solution by applying a variable transformation introduced by Tao Tang in [8]. Also for equations with high order of smoothness in the exact solution we can deduce the convergence of the proposed method with larger values of m, k, and γ and obtain a higher rate of convergence. It is trivial that in this case, we do not require the regularization process.

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3. Operational Tau method for regularized Abel integral equations. It is well known that the spectral Tau method is an efficient tool for solving the differential equations with smooth solutions. In order to make it efficient for the Abel integral equation (1.1), the original equation will be changed into a new integral equation which possesses a smooth solution, by applying a suitable variable transformation. Our approach to choosing the proper transformation is based upon the impressive paper [8]. Consider equation (1.1). We know that near $x = 0$ the first derivative of the solution $u(x)$ behaves like $u'(x) \simeq x^{-\frac{1}{\mu}}$. To overcome this difficulty, we apply the variable transformation

$$x = z^{\mu}, \quad z = \sqrt[\mu]{x}, \quad t = w^{\mu}, \quad w = \sqrt[\mu]{t},$$

and change the Abel integral equation (1.1) as follows

$$\bar{u}(z) = \bar{f}(z) + \lambda \int_0^z \frac{\bar{K}(z,w)}{\sqrt[\mu]{z-w}}\, \bar{u}(w)dw, \quad z \in I, \tag{3.1}$$

where

$$\bar{f}(z) = f(z^{\mu}), \quad \bar{K}(z,w) = \frac{\mu w^{\mu-1} K(z^{\mu}, w^{\mu})}{\sqrt[\mu]{\sum_{j=0}^{\mu-1} z^{\mu-1-j} w^j}},$$

and $\bar{u}(z) = u(z^{\mu})$ is the smooth solution of equation (3.1). Since the exact solution of (1.1) can be written as $u(x) = \bar{u}(z)$, we can define $\tilde{u}_N^{\alpha,\beta}(x) = \bar{u}_N^{\alpha,\beta}(z)$, $x, z \in I$, as the approximate solution of problem (1.1).

3.1. Numerical treatment. Assume that $\bar{u}_N^{\alpha,\beta}(z)$ is the spectral Tau approximation of degree N for (3.1) as

$$\bar{u}_N^{\alpha,\beta}(z) = \sum_{j=0}^{N} b_j v_j^{\alpha,\beta}(z) = \underline{b_N} V^{\alpha,\beta} \underline{Z} = \underline{b_N}\,\underline{V}^{\alpha,\beta}, \tag{3.2}$$

where $\underline{b_N} = [b_0, b_1, \ldots, b_N, 0, \ldots]$, $\underline{Z} = [1, z, z^2, \ldots, z^N, \ldots]^T$. Let us define

$$\bar{L}(\bar{u}(z)) = \bar{u}(z) - \lambda \int_0^z \frac{\bar{K}(z,w)}{\sqrt[\mu]{z-w}}\, \bar{u}(w)dw, \tag{3.3}$$

and assume that

$$K(z^{\mu}, w^{\mu}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{k}_{ij} v_i^{\alpha,\beta}(z) v_j^{\alpha,\beta}(w),$$

which we can rearrange as

$$K(z^{\mu}, w^{\mu}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{\bar{k}}_{ij} z^i w^j.$$

Now, we find the unknown vector $\underline{b_N}$ in (3.2) using the operational Tau method based on the Jacobi polynomials according to the following theorem.

THEOREM 3.1. *Let $v_j^{\alpha,\beta}(z)$ be the orthogonal Jacobi polynomials with respect to weight function $w^{\alpha,\beta}(z)$ in I. Assume that the approximate solution $\bar{u}_N^{\alpha,\beta}(z)$ is given by (3.2). Then we have*

$$\bar{L}(\bar{u}_N^{\alpha,\beta}(z)) = \underline{b_N}\Big(\mathrm{id} - \lambda V^{\alpha,\beta}\bar{\Pi}^{\alpha,\beta}\left(V^{\alpha,\beta}\right)^{-1}\Big)\underline{V}^{\alpha,\beta},$$

where id *is the infinite identity matrix,*

$$\bar{\Pi}^{\alpha,\beta} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\tilde{k}}_{ij}\mathcal{T}^{i,j},$$

and $\mathcal{T}^{i,j}$ *for* $i,j = 0,1,2,\ldots,$ *is the infinite diagonal matrix with diagonal entries*

$$\mathcal{T}^{i,j}_{ss} = \begin{cases} 0, & s \le i+j+\mu-2, \\ \frac{\pi \csc\left(\frac{\pi}{\mu}\right)\Gamma\left(1+\frac{s-(i+\mu-1)}{\mu}\right)}{\Gamma\left(\frac{1}{\mu}\right)\Gamma\left(2+\frac{s-(i+\mu)}{\mu}\right)}, & s \ge i+j+\mu-1. \end{cases}$$

Proof. From the relations (3.3) and (3.2) we can write

$$\begin{aligned} \bar{L}(\bar{u}_N^{\alpha,\beta}(z)) &= \underline{b_N}\Big(\underline{V}^{\alpha,\beta} - \lambda V^{\alpha,\beta}\int_0^z \frac{\bar{K}(z,w)}{\sqrt[\mu]{z-w}}\,\underline{W}dw\Big) \\ &= \underline{b_N}\Big(\underline{V}^{\alpha,\beta} - \lambda V^{\alpha,\beta}\int_0^z \frac{\mu w^{\mu-1}K(z^\mu,w^\mu)}{\sqrt[\mu]{z^\mu-w^\mu}}\,\underline{W}dw\Big), \end{aligned} \tag{3.4}$$

where $\underline{W} = [1, w, w^2, ..., w^N, ...]^T$. Thus, the integral term of (3.4) can be written as

$$\int_0^z \frac{\mu w^{\mu-1}K(z^\mu,w^\mu)}{\sqrt[\mu]{z^\mu-w^\mu}}\,\underline{W}dw = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\tilde{k}}_{ij}z^i\left[\int_0^z \frac{\mu w^{j+l+\mu-1}}{\sqrt[\mu]{z^\mu-w^\mu}}dw\right]_{l=0}^{\infty}. \tag{3.5}$$

Using the relation

$$\int_0^z \frac{\mu w^{j+l+\mu-1}}{\sqrt[\mu]{z^\mu-w^\mu}}dw = z^{j+l+\mu-1}\Big(\frac{\pi\csc\left(\frac{\pi}{\mu}\right)\Gamma\left(1+\frac{j+l}{\mu}\right)}{\Gamma\left(\frac{1}{\mu}\right)\Gamma\left(2+\frac{j+l-1}{\mu}\right)}\Big),$$

we can rewrite (3.5) as

$$\begin{aligned} \int_0^z \frac{\mu w^{\mu-1}K(z^\mu,w^\mu)}{\sqrt[\mu]{z^\mu-w^\mu}}\,\underline{W}dw &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\tilde{k}}_{ij}\left[z^{i+j+l+\mu-1}\frac{\pi\csc\left(\frac{\pi}{\mu}\right)\Gamma\left(1+\frac{j+l}{\mu}\right)}{\Gamma\left(\frac{1}{\mu}\right)\Gamma\left(2+\frac{j+l-1}{\mu}\right)}\right]_{l=0}^{\infty} \\ &= \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\tilde{k}}_{ij}\mathcal{T}^{i,j}\right)\underline{Z} = \bar{\Pi}^{\alpha,\beta}\underline{Z} = \bar{\Pi}^{\alpha,\beta}\left(V^{\alpha,\beta}\right)^{-1}\underline{V}^{\alpha,\beta}, \end{aligned} \tag{3.6}$$

where $\mathcal{T}^{i,j}$ is the infinite diagonal matrix with the aforementioned diagonal entries, and

$$\bar{\Pi}^{\alpha,\beta} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\tilde{k}}_{ij}\mathcal{T}^{i,j}, \qquad \underline{Z} = [1, z, z^2, \ldots, z^N, \ldots]^T.$$

Substituting the relation (3.6) into (3.4) we deduce the result. □

Now, consider the transformed equation (3.1). Assume that

$$\bar{f}(z) = \sum_{i=0}^{N_{\bar{f}}}\bar{f}_i v_i^{\alpha,\beta}(z) = \underline{\bar{f}}\underline{V}^{\alpha,\beta},$$

where $\underline{\bar{f}} = [\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{N_{\bar{f}}}, 0, \ldots]$. From Theorem 3.1 and the above relation, we have

$$\underline{b_N}\left(\mathrm{id} - \lambda V^{\alpha,\beta}\bar{\Pi}^{\alpha,\beta}\left(V^{\alpha,\beta}\right)^{-1}\right)\underline{V}^{\alpha,\beta} = \underline{\bar{f}}\underline{V}^{\alpha,\beta}. \tag{3.7}$$

We let $\mathcal{M}^{\alpha,\beta} = \mathrm{id} - \lambda V^{\alpha,\beta}\bar{\Pi}^{\alpha,\beta}\left(V^{\alpha,\beta}\right)^{-1}$. Due to the orthogonality of $\{v_k^{\alpha,\beta}(z)\}_{k=0}^{\infty}$, projecting (3.7) onto the functions $\{v_k^{\alpha,\beta}(z)\}_{k=0}^{N}$ yields

$$\underline{b_N}\mathcal{M}_i^{\alpha,\beta} = \bar{f}_i,$$

where $\mathcal{M}_i^{\alpha,\beta}$ is the i-th column of $\mathcal{M}^{\alpha,\beta}$. By setting

$$\tilde{\mathcal{M}}_N^{\alpha,\beta} = [\mathcal{M}_0^{\alpha,\beta}, \mathcal{M}_1^{\alpha,\beta}, ..., \mathcal{M}_N^{\alpha,\beta}], \quad \tilde{\bar{f}}_N = [\bar{f}_0, \bar{f}_1, ..., \bar{f}_N],$$

we obtain $\underline{b_N}\tilde{\mathcal{M}}_N^{\alpha,\beta} = \tilde{\bar{f}}_N$ which gives the unknown vector $(b_0, b_1, ..., b_N)$.

3.2. Convergence analysis. In this section we provide an efficient error analysis, which theoretically confirms the exponential rate of convergence of the proposed method when applied to the regularized Abel integral equation (3.1) with $K(z^\mu, w^\mu) = 1$, i.e.,

$$\bar{u}(z) = \bar{f}(z) + \lambda \int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,\bar{u}(w)dw, \quad z \in I, \tag{3.8}$$

where $\tilde{\bar{K}}(z,w) = \frac{\mu w^{\mu-1}}{\sqrt[\mu]{\sum_{j=0}^{\mu-1} z^{\mu-1-j}w^j}}$. It is trivial to see that this form of the equation is obtained when we apply the proposed regularization process to the Abel integral equation (2.10).

Stability of the proposed scheme for the numerical solution of the regularized equation (3.8) can be directly concluded by adopting the same idea as in the proof of Theorem 2.4. Now, in the following theorem we will prove exponential convergence of the operational Tau method when applied to the regularized equation (3.8).

THEOREM 3.2 (Convergence). *Let $\bar{u}_N^{\alpha,\beta}(z)$ be the Tau approximation (3.2) to the exact solution $\bar{u}(z)$ of the regularized Abel integral equation (3.8). If $\bar{u}(z) \in C^{k,\gamma}(I)\bigcap B^m_{\alpha,\beta}(I)$ with $k \geq 0$, $\gamma \in (0,1]$ and $m \geq 0$, then we have*

$$\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_\infty \leq \begin{cases} C\log(N)N^{-(\gamma+k)}\|\bar{u}\|_{k,\gamma}, & \text{for } -1 < \alpha,\beta \leq -\frac{1}{2}, \\ CN^{\frac{1}{2}+s-\gamma-k}\|\bar{u}\|_{k,\gamma}, & \text{for } \gamma+k > \frac{1}{2},\ s = \max\{\alpha,\beta\} < 0, \end{cases}$$

as well as

$$\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_{\alpha,\beta} \leq C\left(N^{-m}\|\bar{u}\|_{m,\alpha,\beta} + \log(N)N^{-\gamma_1}\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_\infty\right)$$

for $\gamma_1 \in (0, 1-\frac{1}{\mu})$, $-1 < \alpha,\beta \leq -\frac{1}{2}$, and

$$\left\|e_N^{\alpha,\beta}(u)\right\|_{\alpha,\beta} \leq C\left(N^{-m}\|\bar{u}\|_{m,\alpha,\beta} + N^{\frac{1}{2}+s-\gamma_1}\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_\infty\right)$$

for $\gamma_1 \in (\frac{1}{2}+s, 1-\frac{1}{\mu})$, $s = \max\{\alpha,\beta\} < 0$.

Proof. Consider (3.8). According to the proposed method, we have

$$\bar{u}_N^{\alpha,\beta}(z) = \mathcal{P}_N^{\alpha,\beta}\bar{f} + \lambda\mathcal{P}_N^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,\bar{u}_N^{\alpha,\beta}(w)dw. \tag{3.9}$$

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Subtracting (3.8) from (3.9) we have

$$e_N^{\alpha,\beta}(\bar{u}) = e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{f}) + \lambda\Big(\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,\bar{u}(w)dw - \mathcal{P}_N^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,\bar{u}_N^{\alpha,\beta}(w)dw\Big). \tag{3.10}$$

By some simple manipulations we can rewrite (3.10) as follows

$$\begin{aligned} e_N^{\alpha,\beta}(\bar{u}) = e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{f}) + \lambda\Bigg(&\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,\bar{u}(w)dw - \mathcal{P}_N^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,\bar{u}(w)dw \\ &+ \int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw - e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw\Bigg). \end{aligned}$$

From (3.8) we have $\lambda\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,\bar{u}(w)dw = \bar{u}(z) - \bar{f}(z)$. We can rewrite the above relation as

$$e_N^{\alpha,\beta}(\bar{u}) = e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{u}) + \lambda\Bigg(\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw - e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw\Bigg).$$

Thus,

$$|e_N^{\alpha,\beta}(\bar{u})| \le \left|e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{u}) - \lambda e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw\right| + \Lambda\int_0^z \frac{|e_N^{\alpha,\beta}(\bar{u})|}{\sqrt[\mu]{z-w}}dw, \tag{3.11}$$

where $\Lambda = |\lambda| \max\limits_{0\le w<z\le 1} |\tilde{\bar{K}}(z,w)|$.

By using Gronwall's inequality (Lemma 2.3) in (3.11), we can write

$$\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_\infty \le C\Bigg(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{u})\right\|_\infty + \left\|e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw\right\|_\infty\Bigg). \tag{3.12}$$

Applying the relations (2.11) and (2.13) in (3.12), we obtain

$$\begin{aligned} \left\|e_N^{\alpha,\beta}(\bar{u})\right\|_\infty &\le C\big(1+\sigma_p(N)\big)\Bigg(N^{-(\gamma+k)}\|\bar{u}\|_{k,\gamma} + N^{-\gamma_1}\left\|\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw\right\|_{0,\gamma_1}\Bigg) \\ &\le C\big(1+\sigma_p(N)\big)\left(N^{-(\gamma+k)}\|\bar{u}\|_{k,\gamma} + N^{-\gamma_1}\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_\infty\right), \end{aligned}$$

where $k \ge 0, \gamma \in [0,1]$ and $\gamma_1 \in (0, 1-\frac{1}{\mu})$. The first result of the theorem regarding the error estimate in the uniform norm can be concluded under the condition (2.21).

Now, we derive a suitable error bound for the proposed scheme in the L^2-norm. To this end, applying again Gronwall's inequality (Lemma 2.3) in (3.11), we can write

$$\begin{aligned} \left\|e_N^{\alpha,\beta}(\bar{u})\right\|_{\alpha,\beta} &\le C\Bigg(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{u})\right\|_{\alpha,\beta} + \left\|e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw\right\|_{\alpha,\beta}\Bigg) \\ &\le C\Bigg(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{u})\right\|_{\alpha,\beta} + \left\|e_{\mathcal{P}_N}^{\alpha,\beta}\int_0^z \frac{\tilde{\bar{K}}(z,w)}{\sqrt[\mu]{z-w}}\,e_N^{\alpha,\beta}(\bar{u})dw\right\|_\infty\Bigg). \end{aligned}$$

Using relations (2.11) and (2.13) in the integral term of the above equation yields

$$\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_{\alpha,\beta} \le C\left(\left\|e_{\mathcal{P}_N}^{\alpha,\beta}(\bar{u})\right\|_{\alpha,\beta} + \big(1+\sigma_p(N)\big)N^{-\gamma_1}\left\|e_N^{\alpha,\beta}(\bar{u})\right\|_\infty\right), \tag{3.13}$$

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where γ_1 satisfies (2.21). Finally, the second result of the theorem can be deduced by adopting the relation (2.12) in (3.13). □

Since solutions of the regularized Abel integral equation (3.8) are smooth, then in Theorem 3.2, m and k are sufficiently large numbers and we have an exponential rate of convergence for the obtained numerical results.

4. Numerical results. In this section we apply a program written in Mathematica to three numerical examples to demonstrate the accuracy of the method and effectiveness of applying the Chebyshev and Legendre polynomial bases. In this section "Numerical error" always refers to the weighted L^2-norm of the obtained error function.

EXAMPLE 4.1. Consider Abel integral equation

$$u(x) = f(x) - x \int_0^x \frac{\sin t}{\sqrt{x-t}} u(t)dt, \quad x \in I,$$

with

$$f(x) = \cos x + \frac{4}{3} x^{\frac{5}{2}} \; {}_1F_2\big[1, \{\frac{5}{4}, \frac{7}{4}\}, -x^2\big],$$

where ${}_pF_q\big[\{a_1, ...a_p\}, \{b_1, ...b_q\}, z\big]$ is the generalized Hypergeometric function.

This example has a smooth solution $u(x) = \cos x$. Firstly, we apply the Tau method proposed in Section 2. The numerical results obtained are given in Table 4.1 and Figure 4.1. The results show that the errors decay exponentially and that the approximate solutions are in good agreement with the exact ones. Due to the infinite smoothness of the exact solution, we do not need the regularization process.

TABLE 4.1
Tau approximation errors of Example 4.1.

	Numerical error before regularization	
N	Chebyshev bases	Legendre bases
2	3.25×10^{-2}	2.14×10^{-2}
6	2.36×10^{-5}	1.27×10^{-5}
10	1.02×10^{-9}	1.04×10^{-9}
14	2.67×10^{-14}	1.35×10^{-14}
18	3.34×10^{-19}	1.22×10^{-19}

EXAMPLE 4.2. Consider the following Abel integral equation

$$u(x) = \left(x^{\frac{4}{3}} + \frac{4\pi x^2 \; {}_1F_1[\frac{7}{3}, 3, x^2]}{9\sqrt{x}} \right) - \int_0^x \frac{e^{xt} u(t)}{\sqrt[3]{x-t}} dt, \quad x \in I,$$

with the exact solution $u(x) = x^{\frac{4}{3}}$.

Numerical results before regularization are given in Table 4.2 and as solid line curves in Figure 4.2. As we can see, the numerical results obtained show convergence of our numerical method, but the rate of convergence is slow. To this end, we apply the variable transformation

$$z = \sqrt[3]{x}, \; w = \sqrt[3]{t}, \; x = z^3, \; t = w^3,$$

according to the previous section, and we get a new Abel equation with the smooth exact solution $\bar{u}(z) = z^4$. The numerical results obtained by applying the proposed Tau method in Section 3 to the regularized Abel integral equation are also provided in Table 4.2 and as the

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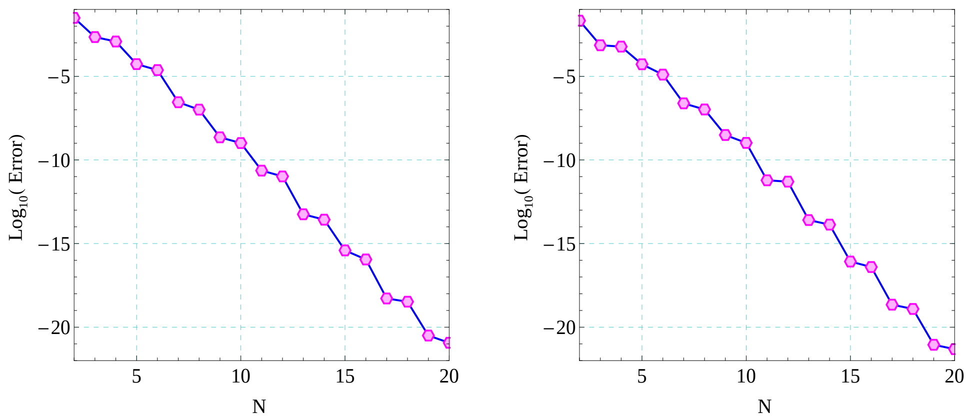

FIG. 4.1. *An illustration of the rate of convergence for the Tau method with various values of N before regularization. We display the errors of Example 4.1 using Chebyshev bases (left) and Legendre bases (right).*

TABLE 4.2
Tau approximation numerical errors of Example 4.2.

	Before regularization		After regularization	
N	Chebyshev bases	Legendre bases	Chebyshev bases	Legendre bases
2	3.13×10^{-2}	2.05×10^{-2}	3.74×10^{-2}	2.49×10^{-2}
6	1.18×10^{-4}	7.98×10^{-5}	1.15×10^{-10}	3.63×10^{-11}
10	1.64×10^{-5}	9.47×10^{-6}	4.79×10^{-12}	2.58×10^{-12}
14	4.28×10^{-6}	2.19×10^{-6}	3.16×10^{-14}	9.36×10^{-15}
18	1.54×10^{-6}	7.18×10^{-7}	1.65×10^{-15}	4.24×10^{-16}
22	6.78×10^{-7}	2.91×10^{-7}	1.02×10^{-16}	1.09×10^{-17}

dashed line curves in Figure 4.2. In general, the numerical results show that the regularization process increases the rate of convergence.

EXAMPLE 4.3. Consider the following Abel integral equation

$$u(x) = f(x) - \frac{1}{2}\int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \quad x \in I,$$

where $f(x) = \frac{\sin(x)}{\sqrt{x}} + \frac{\pi}{2}\sin\frac{x}{2}J_0(\frac{x}{2})$, $J_0(x)$ is the Bessel function, and the exact solution of the problem is $u(x) = \frac{\sin(x)}{\sqrt{x}}$.

This problem has the property stated at the beginning of this paper, i.e., $u'(x)$ is singular at $x = 0^+$. Our first attempt consists in a direct application of the Tau method that is proposed in Section 2 to this example. Numerical results before regularization are given in Table 4.3 and depicted by solid curves in Figure 4.3. Our obtained results before regularization show convergence of the method but with a very low rate of convergence. To overcome this difficulty, our main concern is the regularity of the transformed solution. To the present problem, we apply the variable transformation

$$z = \sqrt{x},\ w = \sqrt{t},\ x = z^2,\ t = w^2,$$

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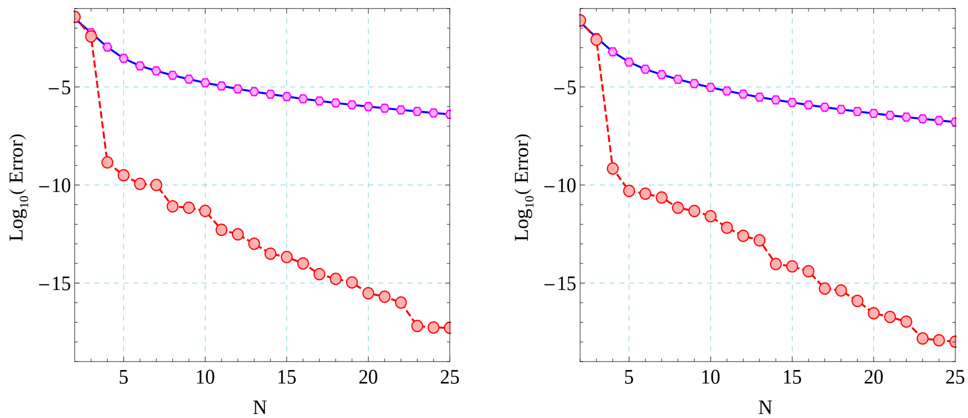

FIG. 4.2. *We display the errors of Example 4.2 for various values of N. The left and right hand figures show numerical errors concerning the shifted Chebyshev and Legendre Tau method on I respectively. In both cases the solid and dashed line curves indicate the numerical errors before and after regularization, respectively.*

TABLE 4.3
Tau approximation numerical errors of Example 4.3.

	Before regularization		After regularization	
N	Chebyshev bases	Legendre bases	Chebyshev bases	Legendre bases
2	5.36×10^{-3}	1.44×10^{-3}	3.42×10^{-3}	1.15×10^{-3}
6	7.23×10^{-4}	1.06×10^{-4}	1.37×10^{-6}	5.54×10^{-7}
10	2.41×10^{-4}	2.39×10^{-5}	1.91×10^{-10}	8.09×10^{-11}
14	1.14×10^{-4}	8.58×10^{-6}	1.32×10^{-14}	5.65×10^{-15}
18	6.36×10^{-5}	3.93×10^{-6}	5.23×10^{-19}	2.25×10^{-19}

and implement the Tau scheme proposed in Section 3 to the regularized Abel integral equation with the smooth exact solution $\bar{u}(z) = \frac{\sin(z^2)}{z}$. The results obtained are given in Table 4.3 and are shown by dashed curves in Figure 4.3. Comparing the results shows that we can reach an exponential rate of convergence after applying the regularization process to the original Abel integral equation.

Finally, in order to show the stability behavior of the proposed scheme that is proved in Theorem 2.4, we solve this problem by the Tau method before regularization with large values of N and give the results in Table 4.4. It can be seen that the results in Table 4.4 are in agreement with the theoretical result of Theorem 2.4. In principle, we can conclude the stability of the Tau method for the numerical solution of the regularized Abel integral equation in this example in the same manner as shown in Table 4.4, but since after regularization we reach an exponential rate of convergence, the numerical errors are almost zero already for moderate values of N. For example for $N = 30$, we obtain errors 1.23×10^{-33} with the Chebyshev bases and 4.92×10^{-34} with the Legendre bases. Then, in this case we do not need to examine larger values of N, and we do not present the stability results obtained for this example after regularization.

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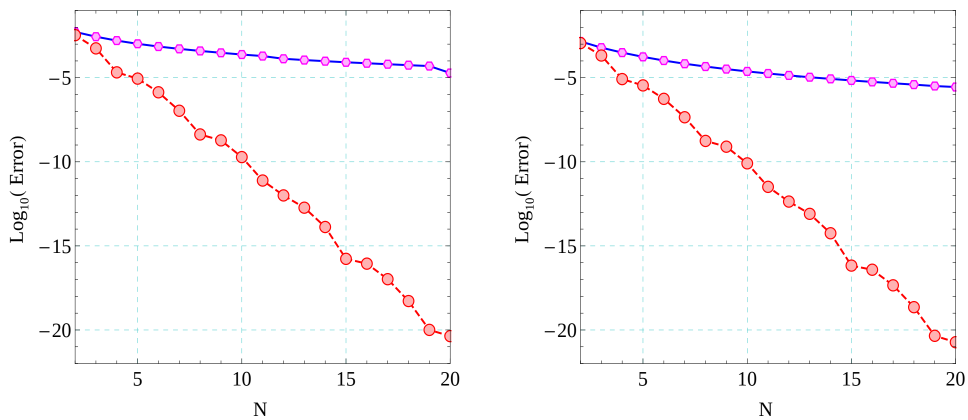

FIG. 4.3. *We observe the errors of Example 4.3 for various values of N. The left and right hand side figures show numerical errors concerning the shifted Chebyshev and Legendre Tau method on I, respectively. In both cases the solid and dashed line curves indicate the numerical errors before and after regularization, respectively.*

TABLE 4.4
Stability behavior of Example 4.3.

	Numerical results before regularization	
N	Chebyshev bases	Legendre bases
20	4.98×10^{-5}	2.83×10^{-6}
30	1.91×10^{-5}	7.98×10^{-7}
40	9.58×10^{-6}	3.27×10^{-7}
50	5.58×10^{-6}	1.65×10^{-7}
60	3.58×10^{-6}	9.1×10^{-8}
70	2.46×10^{-7}	5.91×10^{-8}
80	1.77×10^{-7}	3.95×10^{-8}
90	1.32×10^{-7}	2.76×10^{-8}
100	6.02×10^{-8}	2.014×10^{-8}

5. Conclusion. This work has been concerned with the operational Tau method and its convergence analysis for Abel-type Volterra integral equations in two stages. In the first step, the operational Tau method based upon Jacobi basis functions was introduced for the numerical solution of the original equation (1.1). In addition, in this step we also investigated the stability and convergence behavior of this method when $K(x,t) = 1$. We deduced convergence of the proposed method, but the fact that the derivative $u'(x)$ of the solution behaves like $x^{-\frac{1}{\mu}}$ near the origin is expected to cause a loss in the global convergence order of the Tau method. To overcome this drawback, the original equation was changed into a new Abel integral equation which possesses better regularity by applying a simple variable transformation that was introduced by Tao Tang in [8]. Next, we directly presented a new operational Tau scheme for the new Abel integral equation. We also proved the convergence of the method and obtained the error estimates in weighted L^2- and uniform norms of the approximated solution. These results were confirmed by some numerical examples.

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REFERENCES

[1] K. E. ATKINSON AND W. HAN, *Theoretical Numerical Analysis. A Framework,* Springer, New York, 2005.
[2] H. BRUNNER, *Polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels,* IMA J. Numer. Anal., 6 (1986), pp. 221–239.
[3] ———, *Collocation Methods for Volterra Integral and Related Functional Equations Methods,* Cambridge University Press, Cambridge, 2004.
[4] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI, AND A. T. ZANG, *Spectral Methods. Fundamentals in Single Domains,* Springer, Berlin, 2006.
[5] Y. CHEN AND T. TANG, *Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel,* Math. Comp., 79 (2010), pp. 147–167.
[6] T. DIOGO, S. MCKEE, AND T. TANG, *Collocation methods for second-kind Volterra integral equations with weakly singular kernels*, Proc. Roy. Soc. Edinburgh Sect. A, 124 (1994), pp. 199–210.
[7] A. KUFNER AND L. E. PERSSON, *Weighted Inequalities of Hardy Type,* World Scientific, River Edge, 2003.
[8] X. LI. AND T. TANG, *Convergence analysis of Jacobi spectral collocation methods for Abel-Volterra integral equations of second kind,* Front. Math. China, 7 (2012), pp. 69–84.
[9] CH. LUBICH, *Fractional linear multistep methods for Abel-Volterra integral equations of the second kind,* Math. Comp., 45 (1985), pp. 463–469.
[10] J. SHEN, T. TANG, AND L. WANG, *Spectral Methods: Algorithms, Analysis and Applications*, Springer, Berlin, 2011.
[11] T. TANG, *Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations,* Numer. Math., 61 (1992), pp. 373–382.