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DISCONTINUOUS GALERKIN METHODS FOR THE P-BIHARMONIC EQUATION FROM A DISCRETE VARIATIONAL PERSPECTIVE*

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Abstract. We study discontinuous Galerkin approximations of the p-biharmonic equation for $p \in (1, \infty)$ from a variational perspective. We propose a discrete variational formulation of the problem based on an appropriate definition of a finite element Hessian and study convergence of the method (without rates) using a semicontinuity argument. We also present numerical experiments aimed at testing the robustness of the method.

Key words. discontinuous Galerkin finite element method, discrete variational problem, p-biharmonic equation

AMS subject classifications. 65N30, 65K10, 35J40

1. Introduction, problem setup, and notation. The p-biharmonic equation is a fourth-order elliptic boundary value problem related to—in fact a nonlinear generalisation of—the biharmonic problem. Such problems typically arise in elasticity; in particular, the nonlinear case can be used as a model for travelling waves in suspension bridges [15, 19]. It is a fourth-order analog to its second-order sibling, the p-Laplacian, and, as such, is useful as a prototypical nonlinear fourth-order problem.

The efficient numerical simulation of general fourth-order problems has attracted recent interest. A conforming approach to this class of problems would require the use of C^1-finite elements, the Argyris element for example [7, Section 6]. From a practical point of view, this approach presents difficulties in that the C^1-finite elements are difficult to design and complicated to implement, especially when working in three spatial dimensions.

Discontinuous Galerkin (dG) methods form a class of nonconforming finite element methods. They are extremely popular due to their successful application to an ever expanding range of problems. A very accessible unification of these methods together with a detailed historical overview is presented in [1].

If $p = 2$, we have the special case that the (2–)biharmonic problem is linear. It has been well studied in the context of dG methods, for example, the papers [14, 22] study the use of h-p dG finite elements (where p here means the local polynomial degree) applied to the (2-)biharmonic problem. To the authors knowledge, there is currently no finite element method posed for the general p-biharmonic problem.

In this work we use discrete variational techniques to build a discontinuous Galerkin (dG) numerical scheme for the p-biharmonic operator with $p \in (1, \infty)$. We are interested in such a methodology due to its application to discrete symmetries, in particular, discrete versions of Noether's Theorem [24].

A key constituent to the numerical method for the p-biharmonic problem (and second-order variational problems in general) is an appropriate definition of the Hessian of a piecewise smooth function. To formulate the general dG scheme for this problem from a variational perspective, one must construct an appropriate notion of a Hessian of a piecewise smooth function. The *finite element Hessian* was first coined by [2] for use in the characterisation of discrete convex functions. Later in [20] it was employed in a method for nonvariational problems where the strong form of the PDE was approximated and put to use in the context of

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fully nonlinear problems in [21]. A generalisation of the finite element Hessian to incorporate the dG framework is given in [10], which we also summarise here for completeness.

Convergence of the method we propose is proved using the framework set out in [11], where some extremely useful discrete functional analysis results are given. Here, the authors use the framework to prove convergence of a dG approximation to the steady-state incompressible Navier-Stokes equations. A related but independent work containing similar results is given in [6], where the authors study dG approximations to generic first-order variational minimisation problems.

The rest of the paper is set out as follows: in the remaining part of this section, necessary notation and the model problem we consider are introduced. In Section 2 we give some properties of the continuous p-biharmonic problem. In Section 3 we give the methodology for the discretisation of the model problem. In Section 4 we detail solvability and convergence of the discrete problem. Finally, in Section 5 we study the discrete problem computationally and summarise numerical experiments.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$. We begin by introducing the Sobolev spaces [7, 13]

$$L_p(\Omega) = \left\{\phi : \int_\Omega |\phi|^p < \infty\right\} \text{ for } p \in [1,\infty) \quad \text{and} \quad L_\infty(\Omega) = \left\{\phi : \operatorname{ess\,sup}_\Omega |\phi| < \infty\right\},$$
$$W_p^l(\Omega) = \{\phi \in L_p(\Omega) : D^{\boldsymbol{\alpha}}\phi \in L_p(\Omega), \text{ for } |\boldsymbol{\alpha}| \leq l\} \quad \text{and} \quad H^l(\Omega) := W_2^l(\Omega),$$

which are equipped with the following norms and semi-norms:

$$\begin{aligned} \|v\|_{L_p(\Omega)}^p &:= \int_\Omega |v|^p, \\ \|v\|_{l,p}^p &:= \|v\|_{W_p^l(\Omega)}^p = \sum_{|\boldsymbol{\alpha}|\leq k} \|D^{\boldsymbol{\alpha}} v\|_{L_p(\Omega)}^p, \\ |v|_{l,p}^p &:= |v|_{W_p^l(\Omega)}^p = \sum_{|\boldsymbol{\alpha}|=k} \|D^{\boldsymbol{\alpha}} v\|_{L_p(\Omega)}^p, \\ \|v\|_l^2 &:= \|v\|_{H^l(\Omega)}^2 = \|v\|_{W_2^l(\Omega)}^2, \end{aligned}$$

where $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_d\}$ is a multi-index, $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i$, and the derivatives $D^{\boldsymbol{\alpha}}$ are understood in a weak sense. We pay particular attention to the cases $l = 1, 2$ and define

$$\mathring{W}_p^2(\Omega) := \left\{\phi \in W_p^2(\Omega) : \phi = (\nabla\phi)^\intercal \boldsymbol{n} = 0\right\}.$$

In this paper we use the convention that the derivative Du of a function $u : \Omega \to \mathbb{R}$ is a row vector, while the gradient of u, ∇u, is the derivatives transposed, i.e., $\nabla u = (Du)^\intercal$. We make use of the slight abuse of notation following a common practice whereby the Hessian of u is denoted as D^2u (instead of the correct ∇Du) and is represented by a $d \times d$ matrix.

Let $L = L\big(\boldsymbol{x}, u, \nabla u, D^2u\big)$ be the *Lagrangian*. We let

$$\begin{aligned} \mathscr{J}[\,\cdot\,;p] : \quad & \mathring{W}_p^2(\Omega) \to \mathbb{R} \\ & \phi \mapsto \mathscr{J}[\phi;p] := \int_\Omega L(\boldsymbol{x}, \phi, \nabla\phi, D^2\phi)\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

be known as the *action functional*. For the p-biharmonic problem, the action functional is given explicitly as

$$\mathscr{J}[u;p] := \int_\Omega L(\boldsymbol{x}, u, \nabla u, D^2u) = \int_\Omega \frac{1}{p}|\Delta u|^p - fu,$$

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where $\Delta u := \operatorname{trace}\big(D^2 u\big)$ is the Laplacian and $f \in L_q(\Omega)$ is a known source function. We then look to find a minimiser over the space $\mathring{W}^2_p(\Omega)$, that is, to find $u \in \mathring{W}^2_p(\Omega)$ such that

$$\mathscr{J}[u;p] = \min_{v \in \mathring{W}^2_p(\Omega)} \mathscr{J}[v;p].$$

If we assume temporarily that we have access to a smooth minimiser, i.e., $u \in C^4(\Omega)$, then, given that the Lagrangian is of second order, we have that the Euler-Lagrange equations are (in general) of fourth order.

Let $\boldsymbol{X}{:}\boldsymbol{Y} = \operatorname{trace}(\boldsymbol{X}^\intercal \boldsymbol{Y})$ be the Frobenius inner product between matrices. We then let

$$\boldsymbol{X} = \begin{bmatrix} x_1^1 & \dots & x_1^d \\ \vdots & \ddots & \vdots \\ x_d^1 & \dots & x_d^d \end{bmatrix}$$

and use

$$\frac{\partial L}{\partial(\boldsymbol{X})} := \begin{bmatrix} \partial L/\partial x_1^1 & \dots & \partial L/\partial x_1^d \\ \vdots & \ddots & \vdots \\ \partial L/\partial x_d^1 & \dots & \partial L/\partial x_d^d \end{bmatrix}.$$

The Euler-Lagrange equations for this problem now take the following form:

$$\mathscr{L}[u;p] := D^2 : \left(\frac{\partial L}{\partial(D^2 u)}\right) + \frac{\partial L}{\partial u} = 0.$$

These can then be calculated to be

$$\mathscr{L}[u;p] := \Delta\Big(|\Delta u|^{p-2}\,\Delta u\Big) - f = 0. \tag{1.1}$$

Note that for $p = 2$, the problem coincides with the biharmonic problem $\Delta^2 u = f$, which is well studied in the context of dG methods; see, e.g., [3, 14, 16, 25].

2. Properties of the continuous problem. To the authors knowledge, the numerical method described here is the first finite element method presented for the p-biharmonic problem. As such, we will state some simple properties of the problem which are well known for the problem's second-order counterpart, *the p-Laplacian* [4, 7].

PROPOSITION 2.1 (Equivalence of norms over $\mathring{W}^2_p(\Omega)$ [17, Corollary 9.10]). *Let Ω be a bounded domain with Lipschitz boundary. Then the norms $\|\cdot\|_{2,p}$ and $\left\|D^2\cdot\right\|_{L_p(\Omega)}$ are equivalent over $\mathring{W}^2_p(\Omega)$.*

PROPOSITION 2.2 (Coercivity of $\mathscr{J}$). *Let $u \in \mathring{W}^2_p(\Omega)$ and $f \in L_q(\Omega)$, where $\frac{1}{p}+\frac{1}{q}=1$. We have that the action functional $\mathscr{J}[\,\cdot\,;p]$ is coercive over $\mathring{W}^2_p(\Omega)$, that is,*

$$\mathscr{J}[u;p] \geq C\,|u|^p_{2,p} - \gamma\,,$$

for some $C > 0$ and $\gamma \geq 0$. Equivalently, let

$$\mathscr{A}(u,v;p) = \int_\Omega |\Delta u|^{p-2}\,\Delta u \Delta v,$$

then there exists a constant $C > 0$ such that

$$\mathscr{A}(v,v;p) \geq C\,|v|^p_{2,p} \qquad \forall\, v \in \mathring{W}^2_p(\Omega). \tag{2.1}$$

Proof. By definition of the $\overset{\circ}{W}{}_p^2(\Omega)$-norm and Proposition 2.1, we have that

$$\mathscr{J}[u;p] \geq C(p)\,|u|_{2,p}^p - fu\,.$$

Upon applying Hölder and Poincaré-Friedrichs inequalities, we see that

$$\begin{aligned}\mathscr{J}[u;p] &\geq C(p)\,|u|_{2,p}^p - \|f\|_{L_q(\Omega)}\,\|u\|_{L_p(\Omega)}\\ &\geq C(p)\,|u|_{2,p}^p - C\,\|f\|_{L_q(\Omega)}\,.\end{aligned}$$

The statement (2.1) is clear due to Proposition 2.1, which concludes the proof. ∎

PROPOSITION 2.3 (Convexity of L). *The Lagrangian of the p-biharmonic problem is convex with respect to its fourth argument.*

Proof. Using similar arguments to [7, Section 5.3] (also found in [5]), the convexity of the functional J is a consequence of the convexity of the mapping

$$\mathscr{F} : \xi \in \mathbb{R} \to \frac{1}{p}\,\|\xi\|^p\,. \qquad ∎$$

COROLLARY 2.4 (Weak lower semicontinuity). *The action functional $\mathscr{J}$ is weakly lower semicontinuous over $\overset{\circ}{W}{}_p^2(\Omega)$. That is, given a sequence of functions $\{u_j\}_{j\in\mathbb{N}}$ which has a weak limit $u \in \overset{\circ}{W}{}_p^2(\Omega)$, then*

$$\mathscr{J}[u;p] \leq \liminf_{j\to\infty} \mathscr{J}[u_j;p].$$

Proof. The proof of this statement is a straightforward extension of [13, Section 8.2, Theorem 1] to second-order Lagrangians noting that $\mathscr{J}$ is coercive (from Proposition 2.2) and that L is convex with respect to its fourth variable (from Proposition 2.3). We omit the full details for brevity. ∎

COROLLARY 2.5 (Existence and uniqueness). *There exists a unique minimiser to the p-biharmonic equation. Equivalently, there is a unique (weak) solution to the (weak) Euler-Lagrange equations: find $u \in \overset{\circ}{W}{}_p^2(\Omega)$ such that*

$$\int_\Omega |\Delta u|^{p-2}\,\Delta u \Delta\phi = \int_\Omega f\phi \qquad \forall\,\phi \in \overset{\circ}{W}{}_p^2(\Omega).$$

Proof. Again, the result can be deduced by extending the arguments in [13, Section 8.2] or [7, Theorem 5.3.1], noting the results of Propositions 2.2 and 2.3. The full argument is omitted for brevity. ∎

3. Discretisation. Let $\mathscr{T}$ be a conforming, shape regular triangulation of Ω, namely, $\mathscr{T}$ is a finite family of sets such that

1. $K \in \mathscr{T}$ implies K is an open simplex (segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$),
2. for any $K, J \in \mathscr{T}$ we have that $\overline{K} \cap \overline{J}$ is a full subsimplex (i.e., it is either $\emptyset$, a vertex, an edge, a face, or the whole of $\overline{K}$ and $\overline{J}$) of both $\overline{K}$ and $\overline{J}$ and
3. $\bigcup_{K\in\mathscr{T}} \overline{K} = \overline{\Omega}$.

The shape regularity of $\mathscr{T}$ is defined as the number

$$\mu(\mathscr{T}) := \inf_{K\in\mathscr{T}} \frac{\rho_K}{h_K},$$

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where ρ_K is the radius of the largest ball contained inside K and h_K is the diameter of K. An indexed family of triangulations $\{\mathscr{T}^n\}_n$ is called *shape regular* if

$$\mu := \inf_n \mu(\mathscr{T}^n) > 0.$$

We use the convention that $h : \Omega \to \mathbb{R}$ denotes the piecewise constant *meshsize function* of $\mathscr{T}$, i.e.,

$$h(\boldsymbol{x}) := \max_{\boldsymbol{x} \in \overline{K}} h_K,$$

which we shall commonly refer to as h.

Let $\mathscr{E}$ be the skeleton (set of common interfaces) of the triangulation $\mathscr{T}$, and we say that $e \in \mathscr{E}$ if e is on the interior of Ω and $e \in \partial\Omega$ if e lies on the boundary $\partial\Omega$ and set h_e to be the diameter of e.

We also make the assumption that the mesh is sufficiently shape regular such that for any $K \in \mathscr{T}$, we have the existence of a constant such that

$$\sum_{e \in \partial K} h_e \left|e\right| \leq C \left|K\right|, \tag{3.1}$$

where $|e|$ and $|K|$ denote the $(d-1)$- and d-dimensional measure of e and K, respectively.

Let $\mathbb{P}^k(\mathscr{T})$ denote the space of piecewise polynomials of degree k over the triangulation $\mathscr{T}$, i.e.,

$$\mathbb{P}^k(\mathscr{T}) = \left\{\phi \text{ such that } \phi|_K \in \mathbb{P}^k(K)\right\},$$

and introduce the *finite element space*

$$\mathbb{V} := \mathbb{DG}(\mathscr{T}, k) = \mathbb{P}^k(\mathscr{T})$$

to be the usual space of discontinuous piecewise polynomial functions.

DEFINITION 3.1 (Finite element sequence). *A finite element sequence* $\{v_h, \mathbb{V}\}$ *is a sequence of discrete objects indexed by the mesh parameter* h *and individually represented on a particular finite element space* $\mathbb{V}$, *which itself has a discretisation parameter* h, *that is, we have that* $\mathbb{V} = \mathbb{V}(h)$.

DEFINITION 3.2 (Broken Sobolev spaces, trace spaces). *We introduce the broken Sobolev space*

$$W_p^l(\mathscr{T}) := \left\{\phi : \ \phi|_K \in W_p^l(K), \text{ for each } K \in \mathscr{T}\right\}.$$

We also make use of functions defined in these broken spaces restricted to the skeleton of the triangulation. This requires an appropriate trace space

$$\mathcal{T}(\mathscr{E}) := \prod_{K \in \mathscr{T}} L_2(\partial K) \supset \prod_{K \in \mathscr{T}} W_p^{l-\frac{1}{2}}(K)$$

for $p \geq 2$, $l \geq 1$.

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DEFINITION 3.3 (Jumps, averages, and tensor jumps). *We may define average, jump, and tensor jump operators over $\mathcal{T}(\mathscr{E})$ for arbitrary scalar functions $v \in \mathcal{T}(\mathscr{E})$ and vectors $\boldsymbol{v} \in \mathcal{T}(\mathscr{E})^d$:*

$$
\begin{aligned}
\{\!\!\{\cdot\}\!\!\} : \quad & \mathcal{T}(\mathscr{E} \cup \partial\Omega) \to L_2(\mathscr{E} \cup \partial\Omega), \\
& v \mapsto \begin{cases} \frac{1}{2}(v|_{K_1} + v|_{K_2}) & \textit{over } \mathscr{E}, \\ v|_{\partial\Omega} & \textit{on } \partial\Omega\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\{\!\!\{\cdot\}\!\!\} : \quad & [\mathcal{T}(\mathscr{E} \cup \partial\Omega)]^d \to [L_2(\mathscr{E} \cup \partial\Omega)]^d, \\
& \boldsymbol{v} \mapsto \begin{cases} \frac{1}{2}(\boldsymbol{v}|_{K_1} + \boldsymbol{v}|_{K_2}) & \textit{over } \mathscr{E}, \\ \boldsymbol{v}|_{\partial\Omega} & \textit{on } \partial\Omega\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
[\![\cdot]\!] : \quad & \mathcal{T}(\mathscr{E} \cup \partial\Omega) \to [L_2(\mathscr{E} \cup \partial\Omega)]^d, \\
& v \mapsto \begin{cases} v|_{K_1}\boldsymbol{n}_{K_1} + v|_{K_2}\boldsymbol{n}_{K_2} & \textit{over } \mathscr{E}, \\ (v\boldsymbol{n})\,|_{\partial\Omega} & \textit{on } \partial\Omega\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
[\![\cdot]\!] : \quad & [\mathcal{T}(\mathscr{E} \cup \partial\Omega)]^d \to L_2(\mathscr{E} \cup \partial\Omega), \\
& \boldsymbol{v} \mapsto \begin{cases} (\boldsymbol{v}|_{K_1})^\intercal\boldsymbol{n}_{K_1} + (\boldsymbol{v}|_{K_2})^\intercal\boldsymbol{n}_{K_2} & \textit{over } \mathscr{E}, \\ (\boldsymbol{v}^\intercal\boldsymbol{n})\,|_{\partial\Omega} & \textit{on } \partial\Omega\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
[\![\cdot]\!]_\otimes : \quad & [\mathcal{T}(\mathscr{E} \cup \partial\Omega)]^d \to [L_2(\mathscr{E} \cup \partial\Omega)]^{d\times d}, \\
& \boldsymbol{v} \mapsto \begin{cases} \boldsymbol{v}|_{K_1} \otimes \boldsymbol{n}_{K_1} + \boldsymbol{v}|_{K_2} \otimes \boldsymbol{n}_{K_2} & \textit{over } \mathscr{E}, \\ (\boldsymbol{v} \otimes \boldsymbol{n})\,|_{\partial\Omega} & \textit{on } \partial\Omega\,. \end{cases}
\end{aligned}
$$

We will often use the following proposition, which we state in full for clarity but whose proof is merely using the identities in Definition 3.3.

PROPOSITION 3.4 (Elementwise integration). *For a generic vector-valued function $\boldsymbol{p}$ and scalar-valued function ϕ, we have*

$$
\sum_{K\in\mathscr{T}} \int_K \operatorname{div}(\boldsymbol{p})\, \phi \,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathscr{T}} \left(-\int_K \boldsymbol{p}^\intercal \nabla_h \phi \,\mathrm{d}\boldsymbol{x} + \int_{\partial K} \phi \boldsymbol{p}^\intercal \boldsymbol{n}_K \,\mathrm{d}s \right). \tag{3.2}
$$

In particular, if $\boldsymbol{p} \in \mathcal{T}(\mathscr{E} \cup \partial\Omega)^d$ and $\phi \in \mathcal{T}(\mathscr{E} \cup \partial\Omega)$, the following identity holds

$$
\begin{aligned}
\sum_{K\in\mathscr{T}} \int_{\partial K} \phi \boldsymbol{p}^\intercal \boldsymbol{n}_K \,\mathrm{d}s &= \int_{\mathscr{E}} [\![\boldsymbol{p}]\!]\, \{\!\!\{\phi\}\!\!\} \,\mathrm{d}s + \int_{\mathscr{E}\cup\partial\Omega} [\![\phi]\!]^\intercal \,\{\!\!\{\boldsymbol{p}\}\!\!\} \,\mathrm{d}s \\
&= \int_{\mathscr{E}\cup\partial\Omega} [\![\boldsymbol{p}\phi]\!] \,\mathrm{d}s.
\end{aligned} \tag{3.3}
$$

An equivalent tensor formulation of (3.2)–(3.3) *is*

$$
\sum_{K\in\mathscr{T}} \int_K D_h \boldsymbol{p} \phi \,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathscr{T}} \left(-\int_K \boldsymbol{p} \otimes \nabla_h \phi \,\mathrm{d}\boldsymbol{x} + \int_{\partial K} \phi \boldsymbol{p} \otimes \boldsymbol{n}_K \,\mathrm{d}s \right).
$$

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In particular, the following identity holds

$$\begin{aligned}(3.4)\qquad \sum_{K\in\mathscr{T}}\int_{\partial K}\phi\boldsymbol{p}\otimes\boldsymbol{n}_K\,\mathrm{d}s &= \int_{\mathscr{E}}[\![\boldsymbol{p}]\!]_\otimes\{\!\!\{\phi\}\!\!\}\ \mathrm{d}s+\int_{\mathscr{E}\cup\partial\Omega}[\![\phi]\!]\otimes\{\!\!\{\boldsymbol{p}\}\!\!\}\ \mathrm{d}s\\ &= \int_{\mathscr{E}\cup\partial\Omega}[\![\boldsymbol{p}\phi]\!]_\otimes\,\mathrm{d}s.\end{aligned}$$

The discrete problem we then propose is to minimise an appropriate discrete action functional, that is to seek $u_h\in\mathbb{V}$ such that

$$\mathscr{J}_h[u_h;p]=\inf_{v_h\in\mathbb{V}}\mathscr{J}_h[v_h;p].$$

REMARK 3.5. The choice of the discrete action functional is crucial. A naive choice would be to take the piecewise gradient and Hessian operators and to substitute them directly into the Lagrangian, i.e.,

$$\mathscr{J}_h[u_h;p]=\int_\Omega L\big(\boldsymbol{x},u_h,\nabla_h u_h,D_h^2u_h\big)\,.$$

This is, however, an inconsistent notion of derivative operators (as noted in [6]). Since for the biharmonic problem, the Lagrangian is only dependent on the Hessian of the sought function, we only need to construct an appropriate consistent notion of a discrete Hessian.

THEOREM 3.6 (dG Hessian [10]). *Let $v\in\mathring{W}_p^2(\mathscr{T})$, $\widehat{v}:H^1(\mathscr{T})\to\mathcal{T}(\mathscr{E}\cup\partial\Omega)$ be a linear form, and $\widehat{\boldsymbol{p}}:H^2(\mathscr{T})\times H^1(\mathscr{T})^d\to\mathcal{T}(\mathscr{E}\cup\partial\Omega)^d$ a bilinear form representing consistent numerical fluxes, i.e.,*

$$\widehat{v}(v)=v|_{\mathscr{E}\cup\partial\Omega}\qquad \widehat{p}(v,\nabla v)=\nabla v|_{\mathscr{E}\cup\partial\Omega},$$

*in the spirit of [1]. Then we define the dG Hessian, $\boldsymbol{H}[v]\in\mathbb{V}^{d\times d}$, to be the L_2-*Riesz representor *of the distributional Hessian of v. This has the general form*

$$\begin{aligned}\int_\Omega\boldsymbol{H}[v]\ \Phi=&-\int_\Omega\nabla_h v\otimes\nabla_h\Phi-\int_{\mathscr{E}\cup\partial\Omega}[\![\widehat{v}-v]\!]\otimes\{\!\!\{\nabla_h\Phi\}\!\!\}\\ &-\int_{\mathscr{E}}\{\!\!\{\widehat{v}-v\}\!\!\}\ [\![\nabla_h\Phi]\!]_\otimes+\int_{\mathscr{E}\cup\partial\Omega}[\![\Phi]\!]\otimes\{\!\!\{\widehat{\boldsymbol{p}}\}\!\!\}+\int_{\mathscr{E}}\{\!\!\{\Phi\}\!\!\}\ [\![\widehat{\boldsymbol{p}}]\!]_\otimes\\ &\qquad\forall\,\Phi\in\mathbb{V}.\end{aligned}$$

Proof. Note that in view of Green's Theorem, for smooth functions $w\in C^2(\Omega)\cap C^1(\overline{\Omega})$, we have

$$\int_\Omega D^2w\phi=-\int_\Omega\nabla w\otimes\nabla\phi+\int_{\partial\Omega}\nabla w\otimes\boldsymbol{n}\phi\qquad\forall\,\phi\in C^1(\Omega)\cap C^0(\overline{\Omega}).$$

As such for a broken function $v\in\mathring{W}_p^2(\mathscr{T})$, we introduce an auxiliary variable $\boldsymbol{p}=\nabla_h v$ and consider the following primal form of the representation of the Hessian of this function: for each $K\in\mathscr{T}$,

$$(3.5)\qquad \int_K\boldsymbol{H}[v]\ \Phi=-\int_K\boldsymbol{p}\otimes\nabla_h\Phi+\int_{\partial K}\widehat{\boldsymbol{p}}\otimes\boldsymbol{n}\ \Phi\qquad\forall\,\Phi\in\mathbb{V},$$

$$(3.6)\qquad \int_K\boldsymbol{p}\otimes\boldsymbol{q}=-\int_K v\ D\boldsymbol{q}+\int_{\partial K}\boldsymbol{q}\otimes\boldsymbol{n}\ \widehat{v}\qquad\forall\,\boldsymbol{q}\in\mathbb{V}^d,$$

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where $\nabla_h = (D_h)^{\intercal}$ is the elementwise spatial gradient. Noting the identity (3.4) and taking the sum of (3.5) over $K \in \mathscr{T}$, we observe that

$$\begin{aligned}\int_\Omega \boldsymbol{H}[v]\,\Phi &= \sum_{K\in\mathscr{T}} \int_K \boldsymbol{H}[v]\,\Phi = \sum_{K\in\mathscr{T}} \left(-\int_K \boldsymbol{p}\otimes\nabla_h\Phi + \int_{\partial K} \widehat{\boldsymbol{p}}\otimes\boldsymbol{n}\,\Phi \right)\\ &= -\int_\Omega \boldsymbol{p}\otimes\nabla_h\Phi + \int_{\mathscr{E}\cup\partial\Omega} [\![\Phi]\!]\otimes\{\!\{\widehat{\boldsymbol{p}}\}\!\} + \int_{\mathscr{E}} \{\!\{\Phi\}\!\}\,[\![\widehat{\boldsymbol{p}}]\!]_\otimes\,.\end{aligned}$$

Using the same argument for (3.6) yields

$$\begin{aligned}\int_\Omega \boldsymbol{p}\otimes\boldsymbol{q} &= \sum_{K\in\mathscr{T}} \int_K \boldsymbol{p}\otimes\boldsymbol{q} = \sum_{K\in\mathscr{T}} \left(-\int_K v\,D_h\boldsymbol{q} + \int_{\partial K} \boldsymbol{q}\otimes\boldsymbol{n}\,\widehat{v} \right)\\ &= -\int_\Omega v\,D_h\boldsymbol{q} + \int_{\mathscr{E}\cup\partial\Omega} [\![\widehat{v}]\!]\otimes\{\!\{\boldsymbol{q}\}\!\} + \int_{\mathscr{E}} \{\!\{\widehat{v}\}\!\}\,[\![\boldsymbol{q}]\!]_\otimes\,.\end{aligned}$$

Note that, again making use of (3.4), we have for each $\boldsymbol{q} \in H^1(\mathscr{T})^d$ and $w \in H^1(\mathscr{T})$ that

$$\int_\Omega \boldsymbol{q}\otimes\nabla_h w = -\int_\Omega D_h\boldsymbol{q}w + \int_{\mathscr{E}\cup\partial\Omega} \{\!\{\boldsymbol{q}\}\!\}\otimes[\![w]\!] + \int_{\mathscr{E}} [\![\boldsymbol{q}]\!]_\otimes\,\{\!\{w\}\!\}\,. \tag{3.7}$$

Taking $w = v$ in (3.7) and substituting into (3.6), we see that

$$\int_\Omega \boldsymbol{p}\otimes\boldsymbol{q} = \int_\Omega \boldsymbol{q}\otimes\nabla_h v + \int_{\mathscr{E}\cup\partial\Omega} [\![\widehat{v}-v]\!]\otimes\{\!\{\boldsymbol{q}\}\!\} + \int_{\mathscr{E}} \{\!\{\widehat{v}-v\}\!\}\,[\![\boldsymbol{q}]\!]_\otimes. \tag{3.8}$$

Now choosing $\boldsymbol{q} = \nabla_h\Phi$ and substituting (3.8) into (3.5) concludes the proof. ∎

EXAMPLE 3.7 ([10]). An example of a possible choice of fluxes is

$$\widehat{v} = \begin{cases} \{\!\{v\}\!\} & \text{over } \mathscr{E} \\ 0 & \text{on } \partial\Omega \end{cases}, \qquad \widehat{\boldsymbol{p}} = \{\!\{\nabla_h v\}\!\} \ \text{ on } \mathscr{E}\cup\partial\Omega.$$

The result is an interior penalty (IP) type method [9] applied to represent the finite element Hessian

$$\begin{aligned}\int_\Omega \boldsymbol{H}[v]\,\Phi &= -\int_\Omega \nabla_h v\otimes\nabla_h\Phi + \int_{\mathscr{E}\cup\partial\Omega} [\![v]\!]\otimes\{\!\{\nabla_h\Phi\}\!\} + \int_{\mathscr{E}\cup\partial\Omega} [\![\Phi]\!]\otimes\{\!\{\nabla_h v\}\!\}\\ &= \int_\Omega D_h^2 v\Phi - \int_{\mathscr{E}\cup\partial\Omega} [\![\nabla_h v]\!]_\otimes\,\{\!\{\Phi\}\!\} + \int_{\mathscr{E}\cup\partial\Omega} [\![v]\!]\otimes\{\!\{\nabla_h\Phi\}\!\}\,.\end{aligned}$$

This will be the form of the dG Hessian which we assume for the rest of this exposition.

DEFINITION 3.8 (lifting operators). *From the IP-Hessian defined in Example 3.7, we define the following lifting operator* $l_1, l_2 : \mathbb{V} \to \mathbb{V}^{d\times d}$ *such that*

$$\begin{aligned}\int_\Omega l_1[v_h]\Phi &= \int_{\mathscr{E}\cup\partial\Omega} [\![v_h]\!]\otimes\{\!\{\nabla_h\Phi\}\!\},\\ \int_\Omega l_2[v_h]\Phi &= -\int_{\mathscr{E}\cup\partial\Omega} [\![\nabla_h u_h]\!]_\otimes\,\{\!\{\Phi\}\!\}\,.\end{aligned} \tag{3.9}$$

As such, we may write the IP-Hessian as $\boldsymbol{H} : \mathbb{V} \to \mathbb{V}^{d\times d}$ *such that*

$$\int_\Omega \boldsymbol{H}[v_h]\Phi = \int_\Omega \left(D_h^2 v_h + l_1[v_h] + l_2[v_h]\right)\Phi \qquad \forall\,\Phi\in\mathbb{V}, \tag{3.10}$$

where D_h^2 *denotes the piecewise Hessian operator.*

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REMARK 3.9. When $\boldsymbol{H}[\cdot]$ is restricted to act on functions in $\mathbb{V} \cap H^1_0(\Omega)$, we have that

$$\int_\Omega \boldsymbol{H}[v_h]\Phi = \int_\Omega \left(D^2 v_h + l_2[v_h]\right)\Phi \qquad \forall\, \Phi \in \mathbb{V} \cap H^1_0(\Omega).$$

This definition coincides with the auxiliary variable introduced in [18] for Kirchhoff plate problems. In addition, it is the auxiliary variable used in [20, 21] for second-order nonvariational PDEs and fully nonlinear PDEs.

4. Convergence. In this section we use the discrete operators from Section 3 to build a consistent discrete variational problem and in addition prove convergence. To that end, we begin by defining the natural dG-norm for the problem.

DEFINITION 4.1 (dG-norm). *We define the dG-norm for this problem as*

$$\|v_h\|^p_{dG,p} := \left\|D^2_h v_h\right\|^p_{L_p(\Omega)} + h_e^{1-p} \left\|[\![\nabla_h v_h]\!]\right\|^p_{L_p(\mathscr{E}\cup\partial\Omega)} + h_e^{1-2p} \left\|[\![v_h]\!]\right\|^p_{L_p(\mathscr{E}\cup\partial\Omega)},$$

where $\|\cdot\|_{L_p(\mathscr{E}\cup\partial\Omega)}$ *is the* $(d-1)$*-dimensional* L_p*-norm over* $\mathscr{E}\cup\partial\Omega$*.*

To prove convergence for the p-biharmonic equation, we modify the arguments given in [11] to our problem. To keep the exposition clear, we use the same notation as in [11] wherever possible.

We state some basic propositions, i.e., a trace inequality and an inverse inequality in $L_p(\Omega)$, the proofs of which are readily available in, e.g., [7]. Henceforth, in this section and throughout the rest of the paper, we use C to denote an arbitrary positive constant which may depend upon μ, p, and Ω but is independent of h.

PROPOSITION 4.2 (Trace inequality). *Let* $v_h \in \mathbb{V}$ *be a finite element function, then for* $p \in (1,\infty)$ *there exists a constant* $C > 0$ *such that*

$$\|v_h\|_{L_p(\mathscr{E}\cup\partial\Omega)} \le C h^{-1/p} \|v_h\|_{L_p(\Omega)}.$$

PROPOSITION 4.3 (Inverse inequality). *Let* $v_h \in \mathbb{V}$ *be a finite element function, then for* $p \in (1,\infty)$ *there exists a constant* $C > 0$ *such that*

$$\|\nabla_h v_h\|^p_{L_p(\Omega)} \le C h^{-p} \|v_h\|^p_{L_p(\Omega)} \qquad \textit{and}$$
$$\|v_h\|^p_{L_p(\Omega)} \le C h^{p} \|\nabla_h v_h\|^p_{L_p(\Omega)}.$$

LEMMA 4.4 (relating $\|\cdot\|_{dG,s}$- and $\|\cdot\|_{dG,t}$-norms). *For* $s,t \in \mathbb{N}$ *with* $1 \le s < t < \infty$*, we have that there exists a constant* $C > 0$ *such that*

$$\|v_h\|_{dG,s} \le C \|v_h\|_{dG,t}.$$

Proof. The proof follows similar lines to [11, Lemma 6.1]. By definition of the $\|\cdot\|_{dG,s}$-norm, we have that

$$\|v_h\|^s_{dG,s} = \int_\Omega \left|D^2_h v_h\right|^s + h_e^{1-s} \int_{\mathscr{E}\cup\partial\Omega} \left|[\![\nabla_h v_h]\!]\right|^s + h_e^{1-2s} \int_{\mathscr{E}\cup\partial\Omega} \left|[\![v_h]\!]\right|^s.$$

Now let us denote $r = \frac{t}{s}$ and $q = \frac{r}{r-1}$, that is, we have $\frac{1}{r} + \frac{1}{q} = 1$. Hence, we may deduce

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that

$$\begin{aligned}\|v_h\|_{dG,s}^s &= \int_\Omega \left|D_h^2 v_h\right|^s + \int_{\mathscr{E}\cup\partial\Omega} h_e^{1/q} h_e^{(1-t)/r} \left|[\![\nabla_h v_h]\!]\right|^s + \int_{\mathscr{E}\cup\partial\Omega} h_e^{1/q} h_e^{(1-2t)/r} \left|[\![v_h]\!]\right|^s \\ &\le \left(\int_\Omega 1^q\right)^{1/q} \left(\int_\Omega \left|D_h^2 v_h\right|^t\right)^{1/r} + \left(h_e \int_{\mathscr{E}\cup\partial\Omega} 1^q\right)^{1/q} \left(\int_{\mathscr{E}\cup\partial\Omega} h_e^{1-t} \left|[\![\nabla_h v_h]\!]\right|^t\right)^{1/r} \\ &\quad + \left(h_e \int_{\mathscr{E}\cup\partial\Omega} 1^q\right)^{1/q} \left(\int_{\mathscr{E}\cup\partial\Omega} h_e^{1-2t} \left|[\![v_h]\!]\right|^t\right)^{1/r} \\ &\le C \|v_h\|_{dG,t}^s ,\end{aligned}$$

where we have used the Hölder inequality together with

$$1 - s = 1 - \tfrac{t}{r} = \tfrac{1}{q} + \tfrac{1-t}{r} \quad \text{and} \quad 1 - 2s = 1 - \tfrac{2t}{r} = \tfrac{1}{q} + \tfrac{1-2t}{r},$$

and the shape regularity of $\mathscr{T}$ given in (3.1). This concludes the proof. ∎

DEFINITION 4.5 (Bounded variation). *Let $\mathscr{V}[\cdot]$ denote the variation functional defined as*

$$\mathscr{V}[u] := \sup \left\{ \int_\Omega u \operatorname{div} \boldsymbol{\phi} : \ \boldsymbol{\phi} \in [C_0^1(\Omega)]^d, \ \|\boldsymbol{\phi}\|_{L_\infty(\Omega)} \le 1 \right\}.$$

The space of bounded variations, *denoted* BV, *is the space of functions with bounded variation functional,*

$$BV := \{\phi \in L_1(\Omega) : \ \mathscr{V}[\phi] < \infty\}.$$

Note that the variation functional defines a norm over BV; we set

$$\|u\|_{BV} = \mathscr{V}[u].$$

PROPOSITION 4.6 (Control of the $L_{\frac{d}{d-1}}(\Omega)$-norm [12]). *Let $u \in BV$. Then there exists a constant C such that*

$$\|u\|_{L_{\frac{d}{d-1}}(\Omega)} \le C \|u\|_{BV}.$$

PROPOSITION 4.7 (Broken Poincaré inequality [6]). *For $v_h \in \mathbb{V}$, we have that*

$$\|v_h\|_{L_1(\Omega)} \le C \left(\int_\Omega |\nabla_h v_h| + \int_{\mathscr{E}\cup\partial\Omega} \left|[\![v_h]\!]\right| \right).$$

LEMMA 4.8 (Control on the BV norm). *We have that for each $v_h \in \mathbb{V}$ and $p \in [1,\infty)$, there exists a constant $C > 0$ such that*

$$\|v_h\|_{BV} \le C \|v_h\|_{dG,p}.$$

Proof. Owing to [11, Lemma 6.2], we have that

$$\|v_h\|_{BV} \le \int_\Omega |\nabla_h v_h| + \int_{\mathscr{E}\cup\partial\Omega} \left|[\![v_h]\!]\right|. \tag{4.1}$$

Applying the broken Poincaré inequality given in Proposition 4.7 to the first term in (4.1) gives

$$\begin{aligned}\|v_h\|_{BV} &\leq C\left(\int_\Omega \left|D_h^2 v_h\right| + \int_{\mathscr{E}\cup\partial\Omega} |[\![\nabla_h v_h]\!]| + \int_{\mathscr{E}\cup\partial\Omega} |[\![v_h]\!]|\right)\\ &\leq C\left(\int_\Omega \left|D_h^2 v_h\right| + \int_{\mathscr{E}\cup\partial\Omega} |[\![\nabla_h v_h]\!]| + h_e^{-1}\int_{\mathscr{E}\cup\partial\Omega} |[\![v_h]\!]|\right)\\ &\leq C\,\|v_h\|_{dG,1}\,.\end{aligned}$$

Applying Lemma 4.4 concludes the proof. ∎

LEMMA 4.9 (Discrete Sobolev embeddings). *For $v_h \in \mathbb{V}$, there exists a constant $C > 0$ such that*

$$\|v_h\|_{L_p(\Omega)} \leq C\,\|v_h\|_{dG,p}\,.$$

Proof. The proof mimics that of the Gagliardo-Nirenberg-Sobolev inequality in [13, Theorem 1, p. 263]. We begin by noting that Proposition 4.6 together with Lemma 4.8 infers the result for $p = 1$, i.e.,

$$\|v_h\|_{L_1(\Omega)} \leq C\,\|v_h\|_{dG,1}\,.$$

Now, we divide the remaining cases into the two cases $p \in (1, d)$ and $p \in [d, \infty)$.

Step 1. We begin with $p \in (1, d)$. First note that the result of Proposition 4.6 together with Lemma 4.8 infer that

$$\|v_h\|_{L_{\frac{d}{d-1}}(\Omega)} \leq C\,\|v_h\|_{dG,1} \qquad \forall\, v_h \in \mathbb{V}.$$

Now taking $v_h = |w_h|^\gamma$, where $\gamma > 1$ is to be chosen later, we find that

$$\begin{aligned}\left(\int_\Omega |w_h|^{\frac{\gamma d}{d-1}}\right)^{\frac{d-1}{d}} \leq C\bigg(&\int_\Omega \left|D_h^2(|w_h|^\gamma)\right| + \int_{\mathscr{E}\cup\partial\Omega} |[\![\nabla_h(|w_h|^\gamma)]\!]|\\ &+ \int_{\mathscr{E}\cup\partial\Omega} h_e^{-1}\,|[\![|w_h|^\gamma]\!]|\bigg)\,.\end{aligned} \tag{4.2}$$

We proceed to bound each of these terms individually. Firstly, note that by the chain rule, we have that

$$\nabla_h(|w_h|^\gamma) = \gamma\,|w_h|^{\gamma-1}\,\nabla_h(|w_h|) = \gamma\,|w_h|^{\gamma-2}\,w_h\nabla_h w_h.$$

Hence, we see that

$$\begin{aligned}D_h^2(|w_h|^\gamma) &= D_h(\nabla_h|w_h|^\gamma) = D_h\Big(\gamma\,|w_h|^{\gamma-2}\,w_h\nabla_h w_h\Big)\\ &= \gamma\Big(D_h\Big(|w_h|^{\gamma-2}\Big)\,w_h\nabla_h w_h + |w_h|^{\gamma-2}\,D_h w_h\nabla_h w_h + |w_h|^{\gamma-2}\,w_h D_h^2 w_h\Big)\\ &= \gamma(\gamma-1)\,|w_h|^{\gamma-2}\,\nabla_h w_h\otimes\nabla_h w_h + \gamma\,|w_h|^{\gamma-2}\,w_h D_h^2 w_h.\end{aligned}$$

Using a triangle inequality, it follows that

$$\int_\Omega \left|D_h^2(|w_h|^\gamma)\right| \leq \gamma\int_\Omega \left||w_h|^{\gamma-1}\,D_h^2 w_h\right| + \gamma(\gamma-1)\int_\Omega \left||w_h|^{\gamma-2}\,\nabla_h w_h\otimes\nabla_h w_h\right|$$

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By the Hölder inequality, we have that

$$\int_\Omega |w_h|^{\gamma-1} \left|D_h^2 w_h\right| \leq \left(\int_\Omega |w_h|^{q(\gamma-1)}\right)^{\frac{1}{q}} \left(\int_\Omega \left|D_h^2 w_h\right|^p\right)^{\frac{1}{p}},$$

where $q = \frac{p}{p-1}$. In addition, we have

$$\int_\Omega \left||w_h|^{\gamma-2} \nabla_h w_h \otimes \nabla_h w_h\right| \leq \left(\int_\Omega \left||w_h|^{\gamma-2} \nabla_h w_h\right|^q\right)^{\frac{1}{q}} \left(\int_\Omega |\nabla_h w_h|^p\right)^{\frac{1}{p}}.$$

Noting that

$$\nabla_h \left(|w_h|^{\gamma-1}\right) = (\gamma-1) |w_h|^{\gamma-3} w_h \nabla_h w_h,$$

we observe that

$$\begin{aligned}\int_\Omega \left||w_h|^{\gamma-2} \nabla_h w_h \otimes \nabla_h w_h\right| &\leq \frac{1}{\gamma-1} \left(\int_\Omega \left|\nabla_h \left(|w_h|^{\gamma-1}\right)\right|^q\right)^{\frac{1}{q}} \left(\int_\Omega |\nabla_h w_h|^p\right)^{\frac{1}{p}} \\ &\leq \frac{C}{\gamma-1} \left(\int_\Omega |w_h|^{q(\gamma-1)}\right)^{\frac{1}{q}} \left(\int_\Omega \left|D_h^2 w_h\right|^p\right)^{\frac{1}{p}}\end{aligned}$$

by the inverse inequalities from Proposition 4.3. Hence, we have that

$$\int_\Omega \left|D_h^2(|w_h|^\gamma)\right| \leq C\gamma \left(\int_\Omega |w_h|^{q(\gamma-1)}\right)^{\frac{1}{q}} \left(\int_\Omega \left|D_h^2 w_h\right|^p\right)^{\frac{1}{p}}. \tag{4.3}$$

Now we must bound the skeletal terms appearing in (4.2). The jump terms here also act like derivatives in that they satisfy a “chain rule” inequality. Using the definition of the jump and average operators, it holds that

$$\begin{aligned}\int_{\mathscr{E}\cup\partial\Omega} |[\![\nabla_h |w_h|^\gamma]\!]| &\leq \int_{\mathscr{E}\cup\partial\Omega} 2\gamma \{\!\{|w_h|^{\gamma-1}\}\!\} [\![\nabla_h w_h]\!] \\ &\leq 2\gamma \left\|h_e^\alpha \{\!\{|w_h|^{\gamma-1}\}\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)} \left\|h_e^{-\alpha} [\![\nabla_h w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}\end{aligned} \tag{4.4}$$

by the Hölder inequality.

Focusing our attention on the average term, in view of the trace inequality in Proposition 4.2, it holds that

$$\begin{aligned}\left\|h_e^\alpha \{\!\{|w_h|^{\gamma-1}\}\!\}\right\|^q_{L_q(\mathscr{E}\cup\partial\Omega)} &\leq C \sum_{K\in\mathscr{T}} h_e^{q\alpha-1} \left\||w_h|^{\gamma-1}\right\|^q_{L_q(K)} \\ &\leq C h_e^{q\alpha-1} \left(\int_\Omega |w_h|^{q(\gamma-1)}\right).\end{aligned}$$

Upon taking the q-th root, we find

$$\left\|h_e^\alpha \{\!\{|w_h|^{\gamma-1}\}\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)} \leq C h_e^{\alpha-\frac{1}{q}} \left(\int_\Omega |w_h|^{q(\gamma-1)}\right)^{\frac{1}{q}}. \tag{4.5}$$

Choosing $\alpha = \frac{1}{q}$ such that the exponent of h vanishes and substituting into (4.4) gives

$$\int_{\mathscr{E}\cup\partial\Omega} |[\![\nabla_h |w_h|^\gamma]\!]| \leq C \left(\int_\Omega |w_h|^{q(\gamma-1)}\right)^{\frac{1}{q}} \left\|h_e^{-\frac{1}{q}} [\![\nabla_h w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}. \tag{4.6}$$

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The final term is dealt with in nearly the same way. Again, using the "chain rule" type inequality, we see that

$$\int_{\mathscr{E}\cup\partial\Omega} h_e^{-1}\left|[\![|w_h|^\gamma]\!]\right| \leq 2\gamma \int_{\mathscr{E}\cup\partial\Omega} h_e^{-1} \{\!\!\{|w_h|^{\gamma-1}\}\!\!\} \left|[\![w_h]\!]\right|$$
$$\leq 2\gamma \left\| h_e^{\alpha} \{\!\!\{|w_h|^{\gamma-1}\}\!\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)} \left\| h_e^{-\alpha-1} [\![w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)},$$

which in view of (4.5) gives again

$$\int_{\mathscr{E}\cup\partial\Omega} h_e^{-1}\left|[\![|w_h|^\gamma]\!]\right| \leq C\left(\int_\Omega |w_h|^{q(\gamma-1)}\right)^{\frac{1}{q}} \left\| h_e^{-\frac{1}{q}-1} [\![w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}, \tag{4.7}$$

where $\alpha = \frac{1}{q}$.

Collecting the three bounds (4.3), (4.6), and (4.7) and substituting into (4.2) yields

$$\left(\int_\Omega |w_h|^{\frac{\gamma d}{d-1}}\right)^{\frac{d-1}{d}} \leq \left(\int_\Omega |w_h|^{q(\gamma-1)}\right)^{\frac{1}{q}} \Bigg(\left\|D_h^2 w_h\right\|_{L_p(\Omega)} + \left\| h_e^{-\frac{1}{q}} [\![\nabla_h w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)} + \left\| h_e^{-\frac{1}{q}-1} [\![w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}\Bigg). \tag{4.8}$$

The main idea of the proof is to now choose γ such that $\frac{\gamma d}{d-1} = q(\gamma-1)$, i.e., $\gamma = \frac{p(d-1)}{d-p}$. Using this and dividing by the first term on the right hand side of (4.8) yields

$$\left(\int_\Omega |w_h|^{\frac{pd}{d-p}}\right)^{\frac{d-1}{d}-\frac{1}{q}} \leq \Bigg(\left\|D_h^2 w_h\right\|_{L_p(\Omega)} + \left\| h_e^{-\frac{1}{q}} [\![\nabla_h w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)} + \left\| h_e^{-\frac{1}{q}-1} [\![w_h]\!]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}\Bigg).$$

Now noting that

$$\frac{d-1}{d} - \frac{1}{q} = \frac{d-p}{dp}, \qquad h_e^{-\frac{p}{q}} = h_e^{1-p}, \qquad \text{and} \qquad h_e^{-\frac{p}{q}-p} = h_e^{1-2p}$$

yields

$$\left\|w_h\right\|_{L_{p^*}(\Omega)} \leq \left\|w_h\right\|_{dG,p},$$

where $p^* = \frac{pd}{p-d}$ is the *Sobolev conjugate* of p. This yields the desired result since $p^* > p$ for $p \in (1,d)$, and hence, we may use the embedding $L_{p^*}(\Omega) \subset\subset L_p(\Omega)$.

Step 2. For the case $p \in [d,\infty)$ we set $r = \frac{dp}{d+p}$. We note that $r < d$ and that the Sobolev conjugate of r, $r^* = \frac{dr}{d-r} > r$. Following the arguments given in Step 1, we arrive at

$$\left\|w_h\right\|_{L_{r^*}(\Omega)} \leq \left\|w_h\right\|_{dG,r}.$$

Note that

$$r^* = \frac{rd}{d-r} = \frac{\frac{d^2p}{d+p}}{d - \frac{dp}{d+p}} = p.$$

Hence, we see that

$$\|w_h\|_{L_p(\Omega)} = \|w_h\|_{L_{r^*}(\Omega)} \leq C\,\|w_h\|_{dG,r} \leq C\,\|w_h\|_{dG,p}\,,$$

where the final bound follows from Lemma 4.4 concluding the proof. ∎

ASSUMPTION 4.10 (Approximability of the finite element space). *Henceforth, we will assume the finite element space* $\mathbb{V}$ *to be chosen such that the* $L_2(\Omega)$ *orthogonal projection operator* $\mathrm{P}_{\mathbb{V}}$ *satisfies*

$$\begin{aligned}
&\lim_{h\to 0}\|v - \mathrm{P}_{\mathbb{V}}\, v\|_{L_p(\Omega)} = 0,\\
&\lim_{h\to 0}\|\nabla v - \nabla_h(\mathrm{P}_{\mathbb{V}}\, v)\|_{L_p(\Omega)} = 0, \qquad \textit{and}\\
&\lim_{h\to 0}\|v - \mathrm{P}_{\mathbb{V}}\, v\|_{dG,p} = 0.
\end{aligned}$$

A choice of $k \geq 2$ *satisfies these assumptions.*

THEOREM 4.11 (Stability). *Let* $\boldsymbol{H}[\cdot]$ *be defined as in Example 3.7. Then the dG Hessian is stable in the sense that*

$$\begin{aligned}
\left\|D_h^2 v_h - \boldsymbol{H}[v_h]\right\|^p_{L_p(\Omega)^{d\times d}} &\leq C\Big(\|l_1[v_h] + l_2[v_h]\|^p_{L_p(\Omega)^{d\times d}}\Big)\\
&\leq C\left(\int_{\mathscr{E}\cup\partial\Omega} h_e^{1-p}\,|[\![\nabla_h v_h]\!]|^p + h_e^{1-2p}\,|[\![v_h]\!]|^p\right).
\end{aligned} \tag{4.9}$$

Consequently, we have

$$\|\boldsymbol{H}[v_h]\|^p_{L_p(\Omega)^{d\times d}} \leq C\,\|v_h\|^p_{dG,p}\ .$$

Proof. We begin by bounding each of the lifting operators individually. Let $q = \frac{p}{p-1}$. Then by the definition of the $L_p(\Omega)$-norm, we have that

$$\|l_1[v_h]\|_{L_p(\Omega)} = \sup_{z\in L_q(\Omega)}\int_\Omega \frac{l_1[v_h]z}{\|z\|_{L_q(\Omega)}}.$$

Let $\mathrm{P}_{\mathbb{V}} : L_2(\Omega) \to \mathbb{V}$ denote the orthogonal projection operator. Then using the definition of $l_1[\cdot]$ in (3.9), we see that

$$\begin{aligned}
&\|l_1[v_h]\|_{L_p(\Omega)}\\
&\quad= \sup_{z\in L_q(\Omega)}\int_\Omega \frac{l_1[v_h]\,\mathrm{P}_{\mathbb{V}}\, z}{\|z\|_{L_q(\Omega)}}\\
&\quad= \sup_{z\in L_q(\Omega)}\int_{\mathscr{E}\cup\partial\Omega} \frac{[\![v_h]\!]\otimes\{\!\{\nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\}\!\}}{\|z\|_{L_q(\Omega)}}\\
&\quad\leq d^2 \sup_{z\in L_q(\Omega)} \frac{\|h_e^{-\alpha}\,[\![v_h]\!]\|_{L_p(\mathscr{E}\cup\partial\Omega)}\,\|\{\!\{h_e^{\alpha}\nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\}\!\}\|_{L_q(\mathscr{E}\cup\partial\Omega)}}{\|z\|_{L_q(\Omega)}}\\
&\quad\leq d^2 \sup_{z\in L_q(\Omega)} \frac{\Big(\|h_e^{-\alpha}\,[\![v_h]\!]\|^p_{L_p(\mathscr{E}\cup\partial\Omega)}\Big)^{1/p}\Big(\|\{\!\{h_e^{\alpha}\nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\}\!\}\|^q_{L_q(\mathscr{E}\cup\partial\Omega)}\Big)^{1/q}}{\|z\|_{L_q(\Omega)}},
\end{aligned} \tag{4.10}$$

using the Hölder inequality followed by a discrete Hölder inequality and where $\alpha \in \mathbb{R}$ is some parameter to be chosen.

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Using the definition of the average operator, we find that

$$\left\|\{\!\!\{h_e^\alpha \nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\}\!\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)}^q \leq \tfrac{1}{2}\sum_{K\in\mathscr{T}} \left\|h_e^\alpha \nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\right\|_{L_q(\partial K)}^q.$$

Now by the trace inequality in Proposition 4.2, we have that

$$\left\|\{\!\!\{h_e^\alpha \nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\}\!\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)}^q \leq C\sum_{K\in\mathscr{T}} h^{q\alpha-1}\left\|\nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\right\|_{L_q(K)}^q.$$

Making use of the inverse inequality given in Proposition 4.3, we have

$$\left\|\{\!\!\{h_e^\alpha \nabla_h(\mathrm{P}_{\mathbb{V}}\, z)\}\!\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)}^q \leq C\sum_{K\in\mathscr{T}} h^{q\alpha-1-q}\left\|\mathrm{P}_{\mathbb{V}}\, z\right\|_{L_q(K)}^q. \tag{4.11}$$

We choose $\alpha = 2-\frac{1}{p}$ such that the exponent of h in the final term of (4.11) is zero. Substituting this bound into (4.11) and making use of the stability of the $L_2(\Omega)$ orthogonal projection in $L_p(\Omega)$ [8], we conclude that

$$\left\|l_1[v_h]\right\|_{L_p(\Omega)}^p \leq C\left\|h_e^{\frac{1}{p}-2}\left[\!\left[v_h\right]\!\right]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}^p \leq Ch_e^{1-2p}\left\|\left[\!\left[v_h\right]\!\right]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}^p. \tag{4.12}$$

The bound on $l_2[\cdot]$ is achieved using similar arguments. Following the steps given in (4.10), it can be verified that

$$\begin{aligned}&\left\|l_2[v_h]\right\|_{L_p(\Omega)}\\ &\quad\leq d^2\sup_{z\in L_q(\Omega)}\frac{\left(\left\|h^{-\beta}\left[\!\left[\nabla_h v_h\right]\!\right]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}^p\right)^{1/p}\left(\left\|\{\!\!\{h^\beta \mathrm{P}_{\mathbb{V}}\, z\}\!\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)}^q\right)^{1/q}}{\left\|z\right\|_{L_q(\Omega)}}\end{aligned} \tag{4.13}$$

for some $\beta\in\mathbb{R}$. To bound the average term, we follow the same steps (without the inverse inequality), i.e.,

$$\left\|\{\!\!\{h_e^\beta \mathrm{P}_{\mathbb{V}}\, z\}\!\!\}\right\|_{L_q(\mathscr{E}\cup\partial\Omega)}^q \leq \tfrac{1}{2}\sum_{K\in\mathscr{T}}\left\|h^\beta \mathrm{P}_{\mathbb{V}}\, z\right\|_{L_q(\partial K)}^q \leq C\sum_{K\in\mathscr{T}} h^{q\beta-1}\left\|\mathrm{P}_{\mathbb{V}}\, z\right\|_{L_q(K)}^q.$$

We choose $\beta = 1-\frac{1}{p}$ such that the exponent of h vanishes and substitute into (4.13) to find

$$\left\|l_2[v_h]\right\|_{L_p(\Omega)}^p \leq C\left\|h_e^{\frac{1}{p}-1}\left[\!\left[v_h\right]\!\right]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}^p \leq Ch_e^{1-p}\left\|\left[\!\left[v_h\right]\!\right]\right\|_{L_p(\mathscr{E}\cup\partial\Omega)}^p. \tag{4.14}$$

The result (4.9) now follows by noting the definition of $\boldsymbol{H}$ given in (3.10), a Minkowski inequality, and the two results (4.12) and (4.14).

To see (4.11) it suffices to again use a Minkowski inequality together with (3.10) and the two results (4.12) and (4.14). $\square$

COROLLARY 4.12 (Strong convergence of the dG-Hessian). *Given a smooth function $v\in C_0^\infty(\Omega)$ with $\mathrm{P}_{\mathbb{V}} : L_2(\Omega)\to\mathbb{V}$ being the L_2 orthogonal projection operator, we have that*

$$\left\|D^2 v - \boldsymbol{H}[\mathrm{P}_{\mathbb{V}}\, v]\right\|_{L_p(\Omega)^{d\times d}} \leq C\left\|v-\mathrm{P}_{\mathbb{V}}\, v\right\|_{dG,p}.$$

Hence, using the approximation properties given in Assumption 4.10, we have the convergence result that $\boldsymbol{H}[\mathrm{P}_{\mathbb{V}}\, v]\to D^2 v$ strongly in $L_p(\Omega)^{d\times d}$.

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4.1. The numerical minimisation problem and discrete Euler-Lagrange equations. The properties of the IP-Hessian allow us to define the following numerical scheme: find $u_h \in \mathbb{V}$ such that

$$\mathscr{J}_h[u_h; p] = \inf_{v_h \in \mathbb{V}} \mathscr{J}_h[v_h; p]. \tag{4.15}$$

Let $\mathscr{D}[v_h] := \operatorname{trace} \boldsymbol{H}[v_h]$, then the discrete action functional $\mathscr{J}_h$ is given by

$$\mathscr{J}_h[v_h; p] := \int_\Omega \frac{1}{p} |\mathscr{D}[v_h]|^p + f v_h + \frac{\sigma}{p} \left(\int_{\mathscr{E} \cup \partial\Omega} h_e^{1-p} |[\![\nabla_h v_h]\!]|^p + h_e^{1-2p} |[\![v_h]\!]|^p \right),$$

where $\sigma > 0$ is a *penalisation parameter*.

Let

$$\begin{aligned} \mathscr{A}_h(u_h, \Phi; p) := \int_\Omega & |\mathscr{D}[u_h]|^{p-2} \mathscr{D}[u_h] \mathscr{D}[\Phi] \\ & + \sigma \left(\int_{\mathscr{E} \cup \partial\Omega} h_e^{1-p} |[\![\nabla_h u_h]\!]|^{p-2} [\![\nabla_h u_h]\!] [\![\nabla_h \Phi]\!] \right. \\ & \qquad \left. + h_e^{1-2p} |[\![v_h]\!]|^{p-2} [\![u_h]\!] [\![\Phi]\!] \right). \end{aligned} \tag{4.16}$$

The associated (weak) discrete Euler-Lagrange equations to the problem are to find $(u_h, \boldsymbol{H}[u_h]) \in \mathbb{V} \times \mathbb{V}^{d \times d}$ such that

$$\mathscr{A}_h(u_h, \Phi; p) = \int_\Omega f \Phi \qquad \forall\, \Phi \in \mathbb{V}, \tag{4.17}$$

where $\boldsymbol{H}$ is defined in Example 3.7.

THEOREM 4.13 (Coercivity). *Let $f \in L_q(\Omega)$ and $\{u_h, \mathbb{V}\}$ be the finite element sequence of solutions to the discrete minimisation problem* (4.15). *Then there exists constants $C = C(p) > 0$ and $\gamma \geq 0$ such that*

$$\mathscr{J}_h[u_h; p] \geq C \|u_h\|_{dG,p}^p - \gamma. \tag{4.18}$$

Equivalently, let $\mathscr{A}_h(\cdot, \cdot; p)$ be defined as in (4.16)*. Then*

$$\mathscr{A}_h(u_h, u_h; p) \geq C \|u_h\|_{dG,p}^p. \tag{4.19}$$

Proof. We have by the definition of $\|\cdot\|_{dG,p}$ that

$$\|u_h\|_{dG,p}^p = \left\| D_h^2 u_h \right\|_{L_p(\Omega)}^p + h_e^{1-p} \|[\![\nabla_h u_h]\!]\|_{L_p(\mathscr{E} \cup \partial\Omega)}^p + h_e^{1-2p} \|[\![u_h]\!]\|_{L_p(\mathscr{E} \cup \partial\Omega)}^p.$$

We conclude by a Minkowski inequality that

$$\begin{aligned} \|u_h\|_{dG,p}^p \leq & \left\| D_h^2 u_h - \boldsymbol{H}[u_h] \right\|_{L_p(\Omega)}^p + \|\boldsymbol{H}[u_h]\|_{L_p(\Omega)}^p \\ & + h_e^{1-p} \|[\![\nabla_h u_h]\!]\|_{L_p(\mathscr{E} \cup \partial\Omega)}^p + h_e^{1-2p} \|[\![u_h]\!]\|_{L_p(\mathscr{E} \cup \partial\Omega)}^p. \end{aligned}$$

Hence, using the stability of the discrete Hessian given in Theorem 4.11, we have that

$$\begin{aligned} \|u_h\|_{dG,p}^p & \leq \|\boldsymbol{H}[u_h]\|_{L_p(\Omega)}^p + (1 + C(p)) \left(h_e^{1-p} \|[\![\nabla_h u_h]\!]\|_{L_p(\mathscr{E} \cup \partial\Omega)}^p \right. \\ & \qquad \left. + h_e^{1-2p} \|[\![u_h]\!]\|_{L_p(\mathscr{E} \cup \partial\Omega)}^p \right) \\ & \leq C(p) \mathscr{A}_h(u_h, u_h; p), \end{aligned}$$

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where we have made use of a piecewise equivalent of Proposition 2.1, hence showing (4.19). The result (4.18) follows by a similar argument. ∎

LEMMA 4.14 (Relative compactness). *Let $\{v_h, \mathbb{V}\}$ be a finite element sequence that is bounded in the $\|\cdot\|_{dG,p}$-norm. Then the sequence is relatively compact in $L_p(\Omega)$.*

Proof. The proof is an application of Kolmogorov's Compactness Theorem noting the result of Lemma 4.9 which yields boundedness of the finite element sequence in $L_p(\Omega)$. ∎

LEMMA 4.15 (Limit). *Given a finite element sequence $\{v_h, \mathbb{V}\}$ that is bounded in the $\|\cdot\|_{dG,p}$-norm. Then there exists a function $v \in \mathring{W}^2_p(\Omega)$ such that as $h \to 0$, we have, up to a subsequence, $v_h \rightharpoonup v$ weakly in $L_p(\Omega)$. Moreover, $\boldsymbol{H}[v_h] \rightharpoonup D^2 v$ weakly in $L_p(\Omega)^{d\times d}$.*

Proof. Lemma 4.14 infers that we may find a $v \in L_p(\Omega)$ which is the limit of our finite element sequence. To prove that $v \in \mathring{W}^2_p(\Omega)$, we must show that our sequence of discrete Hessians converges to $D^2 v$.

Recall that Theorem 4.11 gives that

$$\|\boldsymbol{H}[v_h]\|_{L_p(\Omega)^{d\times d}} \leq C \|v_h\|_{dG,p}.$$

As such, we may infer that the (matrix-valued) finite element sequence $\{\boldsymbol{H}[v_h], \mathbb{V}^{d\times d}\}$ is bounded in $L_p(\Omega)^{d\times d}$. Hence, we have that $\boldsymbol{H}[v_h] \rightharpoonup \boldsymbol{X} \in L_p(\Omega)^{d\times d}$ weakly for some matrix-valued function $\boldsymbol{X}$.

Now we must verify that $\boldsymbol{X} = D^2 v$. For each $\phi \in C_0^\infty(\Omega)$ we have that

$$\int_\Omega \boldsymbol{H}[v_h] \mathrm{P}_{\mathbb{V}}\,\phi = \int_\Omega D_h^2 v_h \mathrm{P}_{\mathbb{V}}\,\phi - \int_{\mathscr{E}} [\![\nabla_h v_h]\!]_\otimes \{\!\{\mathrm{P}_{\mathbb{V}}\,\phi\}\!\} + \int_{\mathscr{E}\cup\partial\Omega} [\![v_h]\!] \otimes \{\!\{\nabla_h(\mathrm{P}_{\mathbb{V}}\,\phi)\}\!\}.$$

Note that

$$\begin{aligned}
\int_\Omega D_h^2 v_h \mathrm{P}_{\mathbb{V}}\,\phi &= -\int_\Omega \nabla_h v_h \otimes \nabla_h(\mathrm{P}_{\mathbb{V}}\,\phi) + \int_{\mathscr{E}} [\![\nabla_h v_h]\!]_\otimes \{\!\{\mathrm{P}_{\mathbb{V}}\,\phi\}\!\} \\
&\qquad + \int_{\mathscr{E}\cup\partial\Omega} [\![\mathrm{P}_{\mathbb{V}}\,\phi]\!] \otimes \{\!\{\nabla_h v_h\}\!\} \\
&= \int_\Omega v_h D_h^2(\mathrm{P}_{\mathbb{V}}\,\phi) + \int_{\mathscr{E}} [\![\nabla_h v_h]\!]_\otimes \{\!\{\mathrm{P}_{\mathbb{V}}\,\phi\}\!\} - [\![\nabla_h(\mathrm{P}_{\mathbb{V}}\,\phi)]\!]_\otimes \{\!\{v_h\}\!\} \\
&\qquad + \int_{\mathscr{E}\cup\partial\Omega} [\![\mathrm{P}_{\mathbb{V}}\,\phi]\!] \otimes \{\!\{\nabla_h v_h\}\!\} - [\![v_h]\!] \otimes \{\!\{\nabla_h(\mathrm{P}_{\mathbb{V}}\,\phi)\}\!\} \\
&= \int_\Omega v_h \boldsymbol{H}[\mathrm{P}_{\mathbb{V}}\,\phi] + \int_{\mathscr{E}} [\![\nabla_h v_h]\!]_\otimes \{\!\{\mathrm{P}_{\mathbb{V}}\,\phi\}\!\} - \int_{\mathscr{E}\cup\partial\Omega} [\![v_h]\!] \otimes \{\!\{\nabla_h(\mathrm{P}_{\mathbb{V}}\,\phi)\}\!\}.
\end{aligned}$$

As such, we have that

$$\int_\Omega \boldsymbol{X}\phi = \lim_{h\to 0} \int_\Omega \boldsymbol{H}[v_h]\mathrm{P}_{\mathbb{V}}\,\phi = \lim_{h\to 0}\int_\Omega v_h \boldsymbol{H}[\mathrm{P}_{\mathbb{V}}\,\phi] = \int_\Omega v D^2\phi$$

by the strong convergence of the dG Hessian in Corollary 4.12. Hence, we have that $\boldsymbol{X} = D^2 v$ in the distributional sense. ∎

LEMMA 4.16 (A priori bound). *Let $f \in L_q(\Omega)$ with $q = \frac{p}{p-1}$, and let $\{u_h, \mathbb{V}\}$ be the finite element sequence satisfying (4.15). Then we have the following a priori bound:*

$$\|u_h\|_{dG,p} \leq \left(C \|f\|_{L_q(\Omega)}\right)^{q/p}.$$

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Proof. Using the coercivity condition given in Theorem 4.13 and the definition of the weak Euler-Lagrange equations, we have that

$$\|u_h\|_{dG,p}^p \le C \mathscr{A}_h(u_h, u_h; p) \le C \int_\Omega f u_h.$$

Now using the Hölder inequality and the discrete Sobolev embedding given in Lemma 4.9 yields

$$\|u_h\|_{dG,p}^p \le C \|f\|_{L_q(\Omega)} \|u_h\|_{L_p(\Omega)} \le C \|f\|_{L_q(\Omega)} \|u_h\|_{dG,p}.$$

Upon simplifying, we obtain the desired result. ∎

THEOREM 4.17 (Convergence). *Let $f \in L_q(\Omega)$ with $q = \frac{p}{p-1}$, and suppose $\{u_h, \mathbb{V}\}$ is the finite element sequence generated by solving the nonlinear system (4.17). Then we have that*

$$\begin{aligned} u_h &\to u && \text{in } L_p(\Omega) && \text{and} \\ \boldsymbol{H}[u_h] &\to D^2 u && \text{in } L_p(\Omega)^{d\times d}, \end{aligned}$$

where $u \in \overset{\circ}{W}{}_p^2(\Omega)$ is the unique solution to the p-biharmonic problem (1.1).

Proof. Given $f \in L_q(\Omega)$ we have that, in view of Lemma 4.16, the finite element sequence $\{u_h, \mathbb{V}\}$ is bounded in the $\|\cdot\|_{dG,p}$-norm. As such we may apply Lemma 4.15 which shows that there exists a (weak) limit to the finite element sequence $\{u_h, \mathbb{V}\}$, which we call u^*. We must now show that $u^* = u$, the solution of the p-biharmonic problem.

By Corollary 2.4, $\mathscr{J}[\cdot]$ is weakly lower semicontinuous, hence we have that

$$\begin{aligned} \mathscr{J}[u^*] &\le \liminf_{h\to 0} \left[\frac{1}{p} \|\mathscr{D}[u_h]\|_{L_p(\Omega)}^p + \int_\Omega f u_h \right] \\ &\le \liminf_{h\to 0} \left[\frac{1}{p} \|\mathscr{D}[u_h]\|_{L_p(\Omega)}^p + \int_\Omega f u_h \right. \\ &\qquad \left. + \frac{\sigma}{p} \left(h_e^{1-p} \|[\![\nabla_h u_h]\!]\|_{L_p(\Omega)}^p + h_e^{1-2p} \|[\![u_h]\!]\|_{L_p(\Omega)}^p \right) \right]. \\ &= \liminf_{h\to 0} \mathscr{J}_h[u_h]. \end{aligned}$$

Now owing to Assumption 4.10, we have that for any $v \in C_0^\infty(\Omega)$,

$$\begin{aligned} \mathscr{J}[v] &= \liminf_{h\to 0} \left[\frac{1}{p} \|\mathscr{D}[\mathrm{P}_{\mathbb{V}}\, v]\|_{L_p(\Omega)}^p + \int_\Omega f\, \mathrm{P}_{\mathbb{V}}\, v \right. \\ &\qquad \left. + \frac{\sigma}{p} \left(h_e^{1-p} \|[\![\nabla_h (\mathrm{P}_{\mathbb{V}}\, v)]\!]\|_{L_p(\Omega)}^p + h_e^{1-2p} \|[\![\mathrm{P}_{\mathbb{V}}\, v]\!]\|_{L_p(\Omega)}^p \right) \right] \\ &= \liminf_{h\to 0} \mathscr{J}_h[\mathrm{P}_{\mathbb{V}}\, v]. \end{aligned}$$

By the definition of the discrete scheme, we arrive at

$$\mathscr{J}[u^*] \le \mathscr{J}_h[u_h] \le \mathscr{J}_h[\mathrm{P}_{\mathbb{V}}\, v] = \mathscr{J}[v].$$

Now, since v was a generic element, we may use the density of $C_0^\infty(\Omega)$ in $\overset{\circ}{W}{}_p^2(\Omega)$ and the fact that u is the unique minimiser to conclude that $u^* = u$. ∎

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REMARK 4.18. In the papers [14, 25], rates of convergence are given for the 2-biharmonic problem. These are

$$\|u - u_h\| = \begin{cases} \mathrm{O}(h^2) & \text{for } k = 2, \\ \mathrm{O}(h^{k+1}) & \text{for } k > 2, \end{cases}$$

$$\|u - u_h\|_{dG,p} = \mathrm{O}(h^{k-1}).$$

Note that for piecewise quadratic finite elements, this convergence rate is suboptimal in $L_2(\Omega)$.

5. Numerical experiments. In this section we summarise some numerical experiments conducted for the method presented in Section 3. The numerical experiments were conducted using the DOLFIN interface for FENICS [23]. The graphics were generated using GNUPLOT and PARAVIEW. For computational efficiency, we choose to represent $\mathscr{D}[u_h]$ by an auxiliary variable in the mixed formulation, which only requires one additional variable as opposed to the full discrete Hessian $\boldsymbol{H}[u_h]$, which would require d^2 ones (or $\frac{d^2+d}{2}$ if one uses the symmetry of $\boldsymbol{H}$). We note that this is only possible due to the structure of the problem, i.e., that $L = L(\boldsymbol{x}, u, \nabla u, \Delta u)$ and would not be possible in a general setting.

5.1. Benchmarking. The aims of this section are to investigate the robustness of the numerical method for a model test solution of the p-biharmonic problem. We show that the method achieves the provable rates for $p = 2$ (Figure 5.1) and numerically gauge the convergence rates for $p > 2$ (Figures 5.2 and 5.3). To that end, we take $\mathscr{T}$ to be an unstructured Delaunay triangulation of the square $\Omega = [0, 1]^2$. We fix $d = 2$, let $\boldsymbol{x} = (x, y)^{\intercal}$, and choose f such that

$$u(\boldsymbol{x}) := \sin(2\pi x)^2 \sin(2\pi y)^2 . \tag{5.1}$$

Note that this is comparable to the numerical experiment in [14, Section 6.1].

REMARK 5.1. Computationally, the convergence rates we observe are

$$\|u - u_h\|_{L_p(\Omega)} = \begin{cases} \mathrm{O}(h^2) & \text{when } k = 2, \\ \mathrm{O}(h^{k+1}) & \text{otherwise}, \end{cases}$$

and

$$\|\Delta u - \mathscr{D}[u_h]\|_{L_p(\Omega)} = \mathrm{O}(h^{k-1}).$$

REMARK 5.2. Note that the dG Hessian $\boldsymbol{H}$ may be represented in a finite element space with a different degree for $u_h \in \mathbb{V}$. Let $\mathbb{W} := \mathbb{P}^{k-1}(\mathscr{T})$. Then the proof of Theorem 3.6 infers that we may choose to represent $\boldsymbol{H}[u_h] \in \mathbb{W}^{d\times d}$. For clarity of exposition, we chose to use $\boldsymbol{H}[u_h] \in \mathbb{V}^{d\times d}$, however, we see no difficulty extending the arguments presented here to the lower-degree dG Hessian. Numerically, we observe the same convergence rates as in Remark 5.1 for the lower-degree dG Hessian.

6. Conclusion and outlook. In this work we presented a dG finite element method for the p-biharmonic problem. To do this, we introduced an auxiliary variable, the *finite element Hessian* and constructed a discrete variational problem.

We proved that the numerical solution of this discrete variational problem converges to the extrema of the continuous problem and that the finite element Hessian converges to the Hessian of the continuous extrema.

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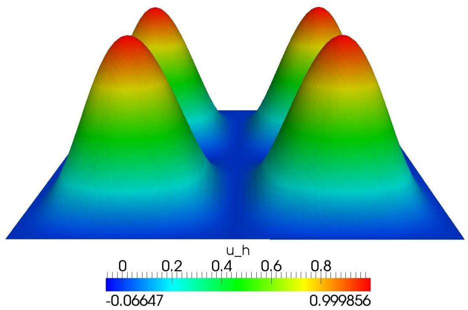
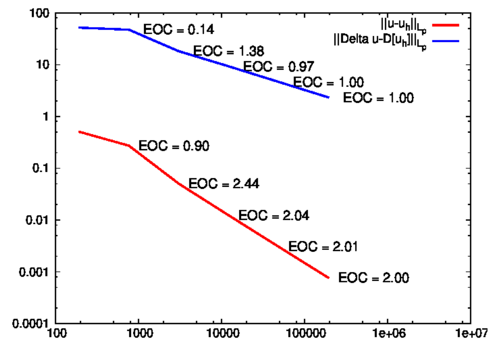
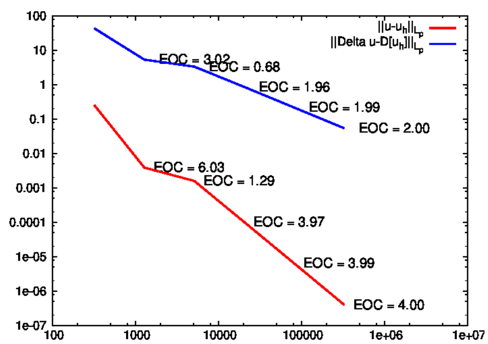


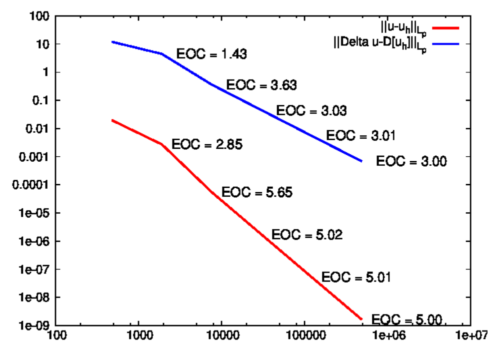
(a) *Finite element approximation to (5.1).*



(b) $k = 2$, *piecewise quadratic FEs.*

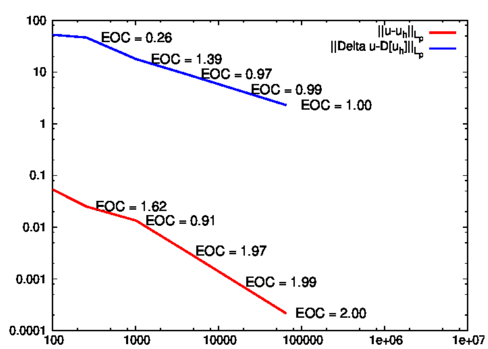


(c) $k = 3$, *piecewise cubic FEs.*

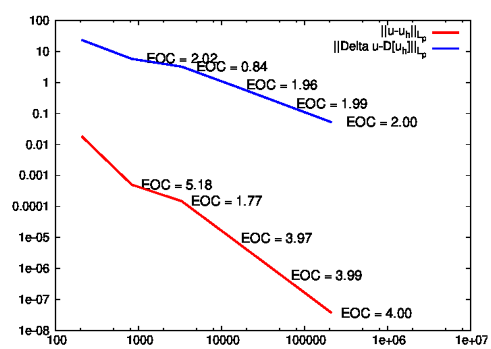


(d) $k = 4$, *piecewise quartic FEs.*

FIG. 5.1. *Numerical experiment benchmarking the numerical method for the 2-biharmonic problem. We fix f such that the solution u is given by (5.1). We plot the log of the error together with its estimated order of convergence. We study the $L_p(\Omega)$-norms of the error of the finite element solution u_h as well as the represented auxiliary variable $\mathscr{D}[u_h]$ for the dG method (4.17) with $k = 2, 3, 4$. We also give a solution plot. We observe that the method achieves the rates given in Remark 4.18.*



(a) $k = 2$, *piecewise quadratic FEs.*



(b) $k = 3$, *piecewise cubic FEs.*

FIG. 5.2. *The same test as in Figure 5.1 for the 2.1-biharmonic problem, i.e., $p = 2.1$ for $k = 2$ and 3.*

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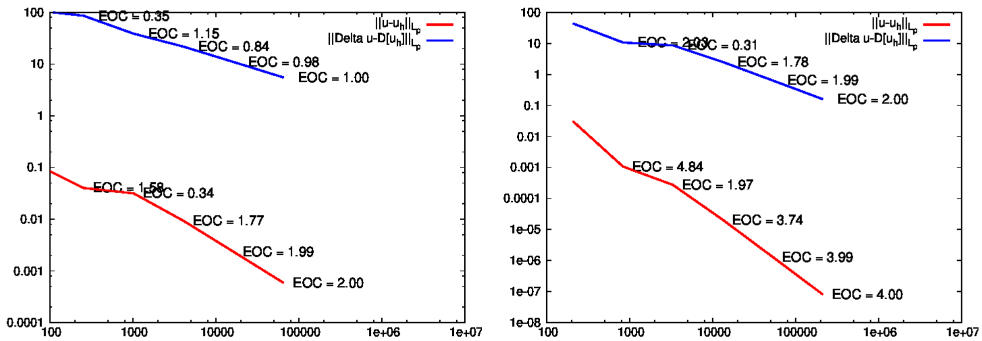

(a) $k = 2$, *piecewise quadratic FEs.* (b) $k = 3$, *piecewise cubic FEs.*

FIG. 5.3. *The same test as in Figure 5.2 for the* 10*-biharmonic problem, i.e.,* $p = 10$.

We foresee that this framework will prove useful when studying other (possibly more complicated) second-order variational problems such as discrete curvature problems like the affine maximal surface equation, which is the topic of ongoing research.

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