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A STRUCTURE-PRESERVING ALGORITHM FOR SEMI-STABILIZING SOLUTIONS OF GENERALIZED ALGEBRAIC RICCATI EQUATIONS*

TIEXIANG LI† AND DELIN CHU‡

Abstract. In this paper, a structure-preserving algorithm is developed for the computation of a semi-stabilizing solution of a Generalized Algebraic Riccati Equation (GARE). The semi-stabilizing solution of GAREs has been used to characterize the solvability of the (J, J')-spectral factorization problem in control theory for general rational matrices which may have poles and zeros on the extended imaginary axis. The main difficulty in solving such a GARE lies in the fact that its associated Hamiltonian/skew-Hamiltonian pencil has eigenvalues on the extended imaginary axis. Consequently, it is not clear which eigenspace of the associated Hamiltonian/skew-Hamiltonian pencil can characterize the desired semi-stabilizing solution. That is, it is not clear which eigenvectors and principal vectors corresponding to the eigenvalues on the extended imaginary axis should be contained in the eigenspace that we wish to compute. Hence, the well-known generalized eigenspace approach for the classical algebraic Riccati equations cannot be employed directly. The proposed algorithm consists of a structure-preserving doubling algorithm (SDA) and a postprocessing procedure to determine the desired eigenvectors and principal vectors corresponding to the purely imaginary and infinite eigenvalues. Under mild assumptions, linear convergence of rate $1/2$ for the SDA is proved. Numerical experiments illustrate that the proposed algorithm performs efficiently and reliably.

Key words. Generalized Algebraic Riccati Equation, structure-preserving doubling algorithm, semi-stabilizing solution

AMS subject classifications. 15A15, 15A09, 15A23

1. Introduction. Throughout this paper, the sets of $m \times n$ complex and real matrices are denoted by $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$, respectively. For convenience, we identify $\mathbb{C}^n = \mathbb{C}^{n\times 1}$, $\mathbb{C} = \mathbb{C}^1$, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, and $\mathbb{R} = \mathbb{R}^1$. The open left-half complex plane and the imaginary axis are denoted by $\mathbb{C}_-$ and $\mathbb{C}_0$, respectively. The open unit disk and the unit circle are denoted by $\mathbb{D}_-$ and $\mathbb{D}_1$, respectively. The notations $0_{m\times n}(0_m)$ and I_m stand for the $m \times n$ $(m \times m)$ zero matrix and the $m \times m$ identity matrix, respectively. The spectra of the matrix A and the matrix pair (A, B) are denoted by $\sigma(A)$ and $\sigma(A, B)$, respectively.

In this paper, we consider the semi-stabilizing solution of the Generalized Algebraic Riccati Equation (GARE) of the form

$$\begin{aligned} &A_a^\top X_a + X_a^\top A_a + \left(C_a^\top J C_a - B_a J' B_a^\top\right) - X_a^\top B_a J'^{-1} B_a^\top X_a = 0, \\ &E_a^\top X_a = X_a^\top E_a, \end{aligned} \tag{1.1a}$$

where

$$E_a = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},\ A_a = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix},\ C_a = \begin{bmatrix} C & D \end{bmatrix},\ B_a = \begin{bmatrix} 0 \\ -I_m \end{bmatrix}, \tag{1.1b}$$

in which $E, A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m},\ J \in \mathbb{R}^{p\times p},\ J' \in \mathbb{R}^{m\times m}$, and $p \geq m$. Furthermore, it is assumed that the pencil $-\lambda E + A$ is regular with E being

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†Department of Mathematics, Southeast University, Nanjing, 211189, People's Republic of China (txli@seu.edu.cn).

‡Department of Mathematics, National University of Singapore S17, 10 Lower Kent Ridge Road, Singapore 119076 (matchudl@nus.edu.sg).

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singular and that J and J' are symmetric and nonsingular. A semi-stabilizing solution of the GARE (1.1) is defined as follows.

DEFINITION 1.1 ([12, 13]). *A solution $X_a \in \mathbb{R}^{(n+m)\times(n+m)}$ of the GARE (1.1) is called a semi-stabilizing solution if*

(i) the pencil $A_a - B_a {J'}^{-1} B_a^\top X_a - \lambda E_a$ is regular and its eigenvalues lie in $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$,

(ii) the matrix pair $(C_a,\ A_a - B_a {J'}^{-1} B_a^\top X_a - \lambda E_a)$ has neither observable finite poles on $\mathbb{C}_0$ nor observable impulsive poles.

The GARE (1.1) plays an important role in the (J, J')-spectral factorization problem in control theory, which has found many important applications in optimal Hankel-norm model reduction [1], H_∞-optimization [9], transport theory [10], and stochastic filtering [15].

DEFINITION 1.2 ([12, 13]). *Let all finite generalized eigenvalues of the pencil $-\lambda E + A$ be in $\mathbb{C}_- \cup \mathbb{C}_0$. The (J, J')-spectral factorization problem for the descriptor system*

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{1.2}$$

is solvable if $G(\lambda) = D + C(\lambda E - A)^{-1}B$ has a (J, J')-spectral factorization, i.e., there exists an invertible matrix $\Xi(\lambda) \in \mathbb{R}^{m\times m}(\lambda)$ such that

(i) $G^T(-\lambda) J G(\lambda) = \Xi^T(-\lambda) J' \, \Xi(\lambda)$,

(ii) all poles and zeros of $\Xi(\lambda)$ lie in $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$,

(iii) $G(s)\Xi^{-1}(\lambda) \in \mathbb{RL}_\infty^{p\times m}(\lambda)$, where $\mathbb{RL}_\infty^{p\times m}(\lambda)$ denotes the set of $p \times m$ proper rational matrices without poles on $\mathbb{C}_0$.

THEOREM 1.3 ([12, 13]). *Assume that all the finite generalized eigenvalues of the pencil $-\lambda E + A$ lie in $\mathbb{C}_- \cup \mathbb{C}_0$, and*

(i) (E, A, B) is finite dynamics stabilizable and impulse controllable, i.e., $\operatorname{rank}[-\lambda E + A\ B] = n$, for all $\lambda \in \mathbb{C}\backslash\mathbb{C}_-$, and $\operatorname{rank}\begin{bmatrix} E & A & B \\ 0 & E & 0 \end{bmatrix} = n + \operatorname{rank}(E)$,

(ii) $\max_{\lambda\in\mathbb{C}} \left\{ \operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C & D \end{bmatrix} \right\} = n + m.$

Then the (J, J')-spectral factorization problem for the descriptor system (1.2) is solvable if and only if the GARE (1.1) has a semi-stabilizing solution X_a, where

$$X_a = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \qquad X_{11} \in \mathbb{R}^{n\times n}, X_{22} \in \mathbb{R}^{m\times m}.$$

Furthermore, in this case, a (J, J')-spectral factor $\Xi(\lambda)$ is given by

$$\Xi(\lambda) = (I - {J'}^{-1} X_{22}) - {J'}^{-1} X_{21} (\lambda E - A)^{-1} B.$$

A numerical method involving a key step by seeking a nonsingular solution of a nonsymmetric ARE was proposed in [12, 13]. Indeed, there are few numerically reliable methods for solving such a nonsymmetric ARE. Recently, numerically verifiable necessary and sufficient conditions for the existence of the semi-stabilizing solution of the GARE (1.1) and a numerically reliable method for computing such a semi-stabilizing solution were proposed in [5]. The main idea in [5] for solving the GARE (1.1) is to find a suitable semi-stable eigenspace corresponding to all eigenvalues in $\mathbb{C}_-$ and some part of the eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ of the augmented matrix pencil associated with (1.1),

$$\mathcal{H}_a - \lambda \mathcal{E}_a \equiv \begin{bmatrix} A_a & -G_a \\ -H_a & -A_a^\top \end{bmatrix} - \lambda \begin{bmatrix} E_a & 0 \\ 0 & E_a^\top \end{bmatrix}, \tag{1.3}$$

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where

$$H_a = C_a^\top J C_a - B_a J' B_a^\top, \qquad G_a = B_a {J'}^{-1} B_a^\top.$$

It is easily seen that $(\mathcal{H}_a \mathcal{J})^T = \mathcal{H}_a \mathcal{J}$ and $(\mathcal{E}_a \mathcal{J})^T = -\mathcal{E}_a \mathcal{J}$ with $\mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Consequently, $\mathcal{H}_a - \lambda\mathcal{E}_a$ forms a Hamiltonian/skew-Hamiltonian pencil, and its eigenvalues occur in quadruples $\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$ (including $\pm\infty$). Note that the eigenstructure of the pencil $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to the eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ is much more complicated than the structure of its stable eigenspace. Hence, we must analyze whether such an eigenspace characterizes the existence of the semi-stabilizing solution of the GARE (1.1). The following connection between the semi-stabilizing solution of (1.1) and the eigenspace of the matrix pair $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to eigenvalues on $\mathbb{C}_-$ and $\mathbb{C}_0 \cup \{\infty\}$ can be obtained easily.

THEOREM 1.4 ([5]).

(i) X_a *is a solution of the GARE* (1.1) *if and only if*

$$(\mathcal{H}_a - \lambda\mathcal{E}_a) \begin{bmatrix} I \\ X_a \end{bmatrix} = \begin{bmatrix} I \\ X_a^\top \end{bmatrix} (A_a - G_a X_a - \lambda E_a).$$

(ii) The GARE (1.1) *has a solution* X_a *such that the pencil* $(A_a - G_a X_a) - \lambda E_a$ *is regular and all its eigenvalues are on* $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$ *if and only if there exist matrices* $[\Phi_1^\top, \Phi_2^\top]^\top$ *and* $[\Psi_1^\top, \Psi_2^\top]^\top$ *with* $\Phi_i, \Psi_i \in \mathbb{R}^{(n+m)\times(n+m)}$ $(i = 1, 2)$ *and* $\operatorname{rank}(\Phi_1) = \operatorname{rank}(\Psi_1) = n + m$ *such that*

$$(\mathcal{H}_a - \lambda\mathcal{E}_a) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} (S_a - \lambda T_a), \qquad \Phi_1^\top \Psi_2 = \Phi_2^\top \Psi_1, \tag{1.4}$$

where $S_a - \lambda T_a \in \mathbb{R}^{(n+m)\times(n+m)}$ *is regular and all its eigenvalues are on* $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$. *In this case,* $X_a = \Phi_2 \Phi_1^{-1}$.

Furthermore, from Weierstrass Theorem [7, Chapter 12], there exists a regular pair $(\widehat{S}_a, \widehat{T}_a)$ which is equivalent to (S_a, T_a) such that (1.4) can be expressed as

$$\mathcal{H}_a \begin{bmatrix} I \\ X_a \end{bmatrix} \widehat{T}_a = \mathcal{E}_a \begin{bmatrix} I \\ X_a \end{bmatrix} \widehat{S}_a. \tag{1.5}$$

However, the relation in (1.5) is only a necessary condition for (1.4).

Theorem 1.4 reveals the relationship between the semi-stabilizing solution of the GARE (1.1) and the eigenspace of $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$. As mentioned above, the eigenstructure of $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ is much more complicated than the stable eigenstructure. This issue can be understood as follows: let τ_1 and τ_2 denote the dimensions of the eigenspaces of the pencil $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to the eigenvalues on $\mathbb{C}_-$ and $\mathbb{C}_0 \cup \{\infty\}$, respectively. Since E is singular, we have

$$\tau_1 < n, \qquad \tau_1 + \frac{1}{2}\tau_2 = n + m$$

provided that $\mathcal{H}_a - \lambda\mathcal{E}_a$ is regular. So there are many different eigenspaces with dimension $n + m$ corresponding to the relevant part of the eigenvalues on $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$. Hence, it is not possible to check whether one of these eigenspaces characterizes the existence of the semi-stabilizing solution of the GARE (1.1) without having some extra insight. Consequently, it is not clear which eigenvectors and principal vectors corresponding to the eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ should be contained in the eigenspace that we wish to compute. Therefore, it is a challenge to develop a structure-preserving algorithm for the computation of a semi-stabilizing solution of the GARE (1.1).

The main contribution of this paper is to propose a structure-preserving algorithm for the computation of a semi-stabilizing solution of the GARE (1.1). The main ingredients of our method include (i) computing the stable eigenspace of the Hamiltonian/skew-Hamiltonian pencil $\mathcal{H}_a - \lambda\mathcal{E}_a$ by a structure-preserving doubling algorithm and (ii) computing a suitable semi-stable eigenspace of $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to the relevant part of the eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ by the eigenstructure decomposition.

2. The Cayley transform of $(\mathcal{H}_a, \mathcal{E}_a)$**.** Let $(\mathcal{H}_a, \mathcal{E}_a)$ be the Hamiltonian/skew-Hamiltonian pair defined in (1.3). By the Cayley transform with an appropriate parameter $\gamma > 0$, the pair $(\mathcal{H}_a, \mathcal{E}_a)$ can be transformed into a new pair $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$. The eigenpairs of $(\mathcal{H}_a, \mathcal{E}_a)$ and $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$ satisfy the relation

$$\mathcal{H}_a x = \lambda\mathcal{E}_a x \quad \Longleftrightarrow \quad (\mathcal{H}_a + \gamma\mathcal{E}_a)x = \mu(\mathcal{H}_a - \gamma\mathcal{E}_a)x, \tag{2.1}$$

where $\mu = (\lambda + \gamma)/(\lambda - \gamma)$ and $\lambda = \gamma(\mu + 1)/(\mu - 1)$. The relation (2.1) implies the following results immediately.

PROPOSITION 2.1. *Let λ and μ be eigenvalues of $(\mathcal{H}_a, \mathcal{E}_a)$ and $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$, respectively, satisfying (2.1). Then*

(i) $|\lambda| = \infty$ if and only if $\mu = 1$,
(ii) $\lambda = 0$ if and only if $\mu = -1$,
(iii) $\lambda = i\beta$ with $\beta \in \mathbb{R}$ if and only if $|\mu| = 1$,
(iv) $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < 0$ $(\alpha > 0)$ if and only if $|\mu| < 1$ $(|\mu| > 1)$.

Since $\mathcal{H}_a - \lambda\mathcal{E}_a$ is regular, there is a $\gamma > 0$ such that $\mathcal{H}_a - \gamma\mathcal{E}_a$ is invertible. We choose a suitable parameter $\gamma > 0$ so that the matrices

$$A_\gamma \equiv A_a - \gamma E_a, \qquad W_\gamma \equiv A_\gamma^\top + H_a A_\gamma^{-1} G_a \tag{2.2}$$

are invertible. Let

$$\mathcal{T}_1 = \begin{bmatrix} A_\gamma^{-1} & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{T}_2 = \begin{bmatrix} I & 0 \\ H_a & I \end{bmatrix}, \quad \mathcal{T}_3 = \begin{bmatrix} I & 0 \\ 0 & -W_\gamma^{-1} \end{bmatrix}, \quad \mathcal{T}_4 = \begin{bmatrix} I & A_\gamma^{-1} G_a \\ 0 & I \end{bmatrix}.$$

Then the matrix pair $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$ can be transformed into the matrix pair $(\mathcal{M}, \mathcal{L})$ with

$$\mathcal{M} \equiv \mathcal{T}_4\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1(\mathcal{H}_a + \gamma\mathcal{E}_a) = \begin{bmatrix} I + 2\gamma A_\gamma^{-1}E_a - 2\gamma A_\gamma^{-1}G_a W_\gamma^{-1} H_a A_\gamma^{-1} E_a & 0 \\ -2\gamma W_\gamma^{-1} H_a A_\gamma^{-1} E_a & I \end{bmatrix},$$

$$\mathcal{L} \equiv \mathcal{T}_4\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1(\mathcal{H}_a - \gamma\mathcal{E}_a) = \begin{bmatrix} I_{n+m} & 2\gamma A_\gamma^{-1} G_a W_\gamma^{-1} E_a^\top \\ 0_{n+m} & I + 2\gamma W_\gamma^{-1} E_a^\top \end{bmatrix}. \tag{2.3}$$

The Sherman-Morrison-Woodbury Formula (SMWF) gives

$$\begin{aligned} I &+ \left(2\gamma A_\gamma^{-1} - 2\gamma A_\gamma^{-1} G_a W_\gamma^{-1} H_a A_\gamma^{-1}\right) E_a \\ &= I + 2\gamma \left[I - A_\gamma^{-1} G_a \left(A_\gamma^\top + H_a A_\gamma^{-1} G_a\right)^{-1} H_a\right] A_\gamma^{-1} E_a \\ &= I + 2\gamma \left(A_\gamma + G_a A_\gamma^{-\top} H_a\right)^{-1} E_a \\ &= I + 2\gamma W_\gamma^{-\top} E_a \equiv I + \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix} E_a. \end{aligned}$$

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Moreover, from (2.2) and the SMWF again, it follows that $G_\gamma := 2\gamma A_\gamma^{-1} G_a W_\gamma^{-1}$ and $H_\gamma := 2\gamma W_\gamma^{-1} H_a A_\gamma^{-1}$ are symmetric. Partition G_γ and H_γ as

$$G_\gamma \equiv \begin{bmatrix} G_1 & G_2^\top \\ G_2 & G_4 \end{bmatrix}, \qquad H_\gamma \equiv \begin{bmatrix} H_1 & H_2^\top \\ H_2 & H_4 \end{bmatrix},$$

where $G_1 = G_1^\top, H_1 = H_1^\top \in \mathbb{R}^{n\times n}$ and $G_4 = G_4^\top, H_4 = H_4^\top \in \mathbb{R}^{m\times m}$. Then $\mathcal{M}$ and $\mathcal{L}$ in (2.3) can be rewritten as

$$\mathcal{M} = \left[\begin{array}{cc|cc} I_n + A_1E & 0 & 0 & 0 \\ A_2E & I_m & 0 & 0 \\ \hline -H_1E & 0 & I_n & 0 \\ -H_2E & 0 & 0 & I_m \end{array}\right], \quad \mathcal{L} = \left[\begin{array}{cc|cc} I_n & 0 & G_1E^\top & 0 \\ 0 & I_m & G_2E^\top & 0 \\ \hline 0 & 0 & I_n + A_1^\top E^\top & 0 \\ 0 & 0 & A_3^\top E^\top & I_m \end{array}\right]. \tag{2.4}$$

Note that from Proposition 2.1 it is easily seen that the eigenvalues of $(\mathcal{M}, \mathcal{L})$ occur in quadruples $\{\mu, \bar{\mu}, \frac{1}{\mu}, \frac{1}{\bar{\mu}}\}$.

By (2.2), $\mathcal{H}_a - \gamma\mathcal{E}_a$ is invertible and so is $S_a - \gamma T_a$, where S_a and T_a are given in (1.4). Thus, the relation in (1.4) is equivalent to

$$(\mathcal{M} - \lambda\mathcal{L}) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} (R_a - \lambda I), \qquad \Phi_1^\top \Psi_2 = \Phi_2^\top \Psi_1,$$

where R_a is similar to $(S_a + \gamma T_a)(S_a - \gamma T_a)^{-1}$ with $\sigma(R_a) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$. That is, $(\mathcal{M}, \mathcal{L})$ and $(\mathcal{H}_a, \mathcal{E}_a)$ have the same invariant subspace corresponding to (R_a, I) and (S_a, T_a), respectively.

3. A structure-preserving algorithm for the GARE (1.1). In this section, we want to develop a structure-preserving algorithm for solving the GARE (1.1) efficiently. We first compute a basis for an auxiliary semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ in (2.4) of the form

$$\begin{aligned} &\left[\begin{array}{cccc} I_n + A_1E & 0 & 0 & 0 \\ A_2E & I_m & 0 & 0 \\ \hline -H_1E & 0 & I_n & 0 \\ -H_2E & 0 & 0 & I_m \end{array}\right] \left[\begin{array}{cc} I_n & 0 \\ 0 & I_m \\ \hline X_1 & 0 \\ X_2 & X_4 \end{array}\right] \\ &\quad = \left[\begin{array}{cccc} I_n & 0 & G_1E^\top & 0 \\ 0 & I_m & G_2E^\top & 0 \\ \hline 0 & 0 & I_n + A_1^\top E^\top & 0 \\ 0 & 0 & A_3^\top E^\top & I_m \end{array}\right] \left[\begin{array}{cc} I_n & 0 \\ 0 & I_m \\ \hline X_1 & 0 \\ X_2 & X_4 \end{array}\right] \begin{bmatrix} R_1 & 0 \\ R_2 & I_m \end{bmatrix}, \end{aligned} \tag{3.1}$$

where $\sigma(R_1) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$. The special basis in (3.1) spans a semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ with the second block columns consisting of m eigenvectors corresponding to the m trivial infinite eigenvalues of $(\mathcal{H}_a, \mathcal{E}_a)$. Using this special structure of the basis, we construct the basis $\left[I_{n+m}, X_a^\top\right]^\top$ for the desired semi-stable subspace of $(\mathcal{M}, \mathcal{L})$.

3.1. The Structure-preserving Doubling Algorithm (SDA) for $\mathbf{X_1}$. We denote by $\mathcal{M}_1$ and $\mathcal{L}_1$ the submatrices in (2.4) corresponding to the first and third block-rows and block-columns, respectively,

$$\mathcal{M}_1 = \begin{bmatrix} I_n + A_1E & 0_n \\ -H_1E & I_n \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} I_n & G_1E^\top \\ 0_n & I_n + A_1^\top E^\top \end{bmatrix}. \tag{3.2}$$

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It is easy to see from (3.1) and (3.2) that X_1 satisfies

$$\mathcal{M}_1 \begin{bmatrix} I \\ X_1 \end{bmatrix} = \mathcal{L}_1 \begin{bmatrix} I \\ X_1 \end{bmatrix} R_1. \tag{3.3}$$

In [2, 3], the matrix disk function method for computing X_1 was developed using a swapping technique built on the QR-factorization. As derived in [2, 3], for a given matrix pair $(\mathcal{M}_1, \mathcal{L}_1)$, we compute the QR-factorization of $[\mathcal{L}_1^\top, \mathcal{M}_1^\top]^\top$ by

$$\mathcal{Q} \begin{bmatrix} \mathcal{L}_1 \\ -\mathcal{M}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{L}_1 \\ -\mathcal{M}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{R} \\ 0 \end{bmatrix}, \tag{3.4}$$

where $\mathcal{Q}$ is orthogonal and $\mathcal{R}$ is upper triangular. Define

$$\widehat{\mathcal{M}}_1 \equiv \mathcal{Q}_{21}\mathcal{M}_1, \qquad \widehat{\mathcal{L}}_1 \equiv \mathcal{Q}_{22}\mathcal{L}_1. \tag{3.5}$$

It is easily verified that $(\widehat{\mathcal{M}}_1, \widehat{\mathcal{L}}_1)$ satisfies the doubling property [14], i.e., if $\mathcal{M}_1 x = \mu \mathcal{L}_1 x$, then $\widehat{\mathcal{M}}_1 x = \mu^2 \widehat{\mathcal{L}}_1 x$. Using (3.4)–(3.5), we propose the Doubling Algorithm (DA), Algorithm 3.1, for computing X_1 in (3.3).

Algorithm 3.1 Doubling Algorithm (DA) for X_1.

Require: A_1, E, G_1, H_1; τ (a small tolerance).
Ensure: An X_1 satisfying (3.3) with $X_1 = H_\infty E$ and H_∞ being symmetric.

1: Initialize $k \leftarrow 1$, $\mathcal{R}_1 \leftarrow 0_{2n}$, $\mathcal{M}_1 \leftarrow \begin{bmatrix} I + A_1 E & 0 \\ -H_1 E & I \end{bmatrix}$, $\mathcal{L}_1 \leftarrow \begin{bmatrix} I & G_1 E^\top \\ 0 & I + A_1^\top E^\top \end{bmatrix}$.
2: **repeat**
3: Compute the QR-factorization $\begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{L}_k \\ -\mathcal{M}_k \end{bmatrix} = \begin{bmatrix} \mathcal{R}_{k+1} \\ 0 \end{bmatrix}$;
4: **if** $\|\mathcal{R}_{k+1} - \mathcal{R}_k\| \leq \tau \|\mathcal{R}_{k+1}\|$, **then**
5: solve the least squares problem for
6: $X_1 : -\mathcal{M}_k(:, 1:n) = \mathcal{M}(:, n+1:2n) X_1$,
7: **else**
8: set $\mathcal{M}_{k+1} \leftarrow \mathcal{Q}_{21}\mathcal{M}_k$, $\mathcal{L}_{k+1} \leftarrow \mathcal{Q}_{22}\mathcal{L}_k$, $k \leftarrow k+1$,
9: **end if**
10: **until** there is a symmetric H_∞ such that $X_1 = H_\infty E$.

Algorithm 3.1 has the disadvantage of destroying the special block structure as given in (3.2). To remedy this shortcoming, we develop the Structure-preserving Doubling Algorithm (SDA) for solving (3.3).

Note that in [4, 6, 11, 14], some SDAs are proposed for the computation of a basis for the semi-stable subspace of a symplectic matrix pair of the form $(\mathcal{M}_1, \mathcal{L}_1)$ as in (3.2) with $E = I_n$. However, in general, the matrix pair $(\mathcal{M}_1, \mathcal{L}_1)$ in (3.2) is no longer symplectic. Nevertheless, in this section, we describe a new SDA algorithm for the computation of X_1 satisfying (3.3) with $\sigma(R_1) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$.

As derived in [11, 14], for the matrix pair $(\mathcal{M}_1, \mathcal{L}_1)$, we construct

$$\mathcal{M}_{1*} = \begin{bmatrix} T_1 & 0_n \\ -T_2 H_1 E & I_n \end{bmatrix}, \qquad \mathcal{L}_{1*} = \begin{bmatrix} I_n & T_1 G_1 E^\top \\ 0_n & T_2 \end{bmatrix} \tag{3.6}$$

with $T_1 = (I + A_1 E)(I + G_1 E^\top H_1 E)^{-1}$ and $T_2 = (I + A_1^\top E^\top)(I + H_1 E G_1 E^\top)^{-1}$ provided that $(I + G_1 E^\top H_1 E)^{-1}$ exists and deduce that $\mathcal{M}_{1*}\mathcal{L}_1 = \mathcal{L}_{1*}\mathcal{M}_1$. Note that

$I+G_1E^\top H_1E$ is invertible if and only if $I + H_1EG_1E^\top$ is invertible because of $\sigma(G_1(E^\top H_1E)) = \sigma(H_1(EG_1E^\top))$. Define

$$\widehat{\mathcal{M}}_1 \equiv \mathcal{M}_{1*}\mathcal{M}_1, \qquad \widehat{\mathcal{L}_1} \equiv \mathcal{L}_{1*}\mathcal{L}_1. \tag{3.7}$$

Then $(\widehat{\mathcal{M}}_1, \widehat{\mathcal{L}}_1)$ satisfies the doubling property. By a careful calculation, the pair $(\widehat{\mathcal{M}}_1, \widehat{\mathcal{L}}_1)$ in (3.7) can be simplified to the special form as given in (3.2) with

$$\widehat{\mathcal{M}}_1 = \begin{bmatrix} I+\widehat{A}_1E & 0 \\ -\widehat{H}_1E & I \end{bmatrix}, \qquad \widehat{\mathcal{L}}_1 = \begin{bmatrix} 0 & \widehat{G}_1E^\top \\ I & I+\widehat{A}_1^\top E^\top \end{bmatrix},$$

where

$$\begin{aligned} I+\hat{A}_1E &= (I+A_1E)(I+G_1E^\top H_1E)^{-1}(I+A_1E) \\ &= (I+A_1E)\left\{I-G_1(I+E^\top H_1EG_1)^{-1}E^\top H_1E\right\}(I+A_1E) \\ &\equiv I+\left[A_1+(A_1-G_1E^\top H_1(I+EG_1E^\top H_1)^{-1}(I+EA_1)\right]E, \end{aligned} \tag{3.8a}$$

$$\begin{aligned} \hat{H}_1E &= H_1E+(I+A_1^\top E^\top)(I+H_1EG_1E^\top)^{-1}H_1E(I+A_1E) \\ &\equiv \left[H_1+(I+A_1^\top E^\top)(I+H_1EG_1E^\top)^{-1}H_1(I+EA_1)\right]E, \end{aligned} \tag{3.8b}$$

$$\begin{aligned} \hat{G}_1E^\top &= G_1E^\top+(I+A_1E)(I+G_1E^\top H_1E)^{-1}G_1E^\top(I+A_1^\top E^\top) \\ &\equiv \left[G_1+(I+A_1E)(I+G_1E^\top H_1E)^{-1}G_1(I+E^\top A_1^\top)\right]E^\top, \end{aligned} \tag{3.8c}$$

$$\begin{aligned} I+\hat{A}_1^\top E^\top &= (I+A_1^\top E^\top)(I+H_1EG_1E^\top)^{-1}(I+A_1^\top E^\top) \\ &= (I+A_1^\top E^\top)\left\{I-H_1(I+EG_1E^\top H_1)^{-1}EG_1E^\top\right\}(I+A_1^\top E^\top) \\ &\equiv I+\left[A_1^\top+(I+A_1^\top E^\top)(A_1^\top-(I+H_1EG_1E^\top)^{-1}H_1EG_1)\right]E^\top. \end{aligned}$$

Since $H_1(I+EG_1E^\top H_1) = (I+H_1EG_1E^\top)H_1$, the matrix $\hat{H}_1$ in (3.8b) is symmetric. Similarly, $\hat{G}_1$ in (3.8c) can also be shown to be symmetric. Note that the matrix $(I+EG_1E^\top H_1)$ in (3.8) should be assumed to be invertible so that the structure-preserving doubling process can continue. Hence, for the case $(I+E^\top H_1EG_1)$ being singular, the doubling process should be switched back to Algorithm 3.1. Using (3.7)–(3.8), the new SDA algorithm for computing X_1 is summarized in Algorithm 3.2.

Under Assumption 4.2 in Section 4, convergence of the DA (Algorithm 3.1) can be shown in a similar way as in [11] and convergence of the SDA (Algorithm 3.2) will be proved in Theorem 4.7 in detail. In practice, the matrix $I+EG_{1,k}E^\top H_{1,k}$ in the SDA is often invertible. Thus, it is extremely rare to switch from SDA to DA.

When Algorithm 3.2 converges, X_1 satisfies (3.3) with some suitable matrix $R_1 \in \mathbb{R}^{n\times n}$ with $\sigma(R_1) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$. That is, span $\left\{\left[I_n, X_1^\top\right]^\top\right\}$ forms a semi-stable subspace of $(\mathcal{M}_1, \mathcal{L}_1)$. In the next section, we use this result to compute the unknown submatrices R_2, X_2, and X_4 in (3.1).

3.2. The computation of $\mathbf{X_2}$ and $\mathbf{X_4}$. Once X_1 is obtained by Algorithm 3.2, from (3.2)–(3.3), the matrix $\left[R_1^\top, R_2^\top\right]^\top$ in (3.1) can be computed by

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} (I+G_1E^\top X_1)^{-1}(I+A_1E) \\ A_2E-G_2E^\top X_1(I+G_1E^\top X_1)^{-1}(I+A_1E) \end{bmatrix}.$$

Subsequently, we compare the $(4,1)$–block of (3.1) and obtain

$$-H_2E+X_2 = (A_3^\top E^\top X_1+X_2)R_1+X_4R_2.$$

Algorithm 3.2 Structure-preserving Doubling Algorithm (SDA) for X_1.

Require: $A_1, E, G_1, H_1; \tau$ (a small tolerance).
Ensure: An X_1 satisfying (3.3) with $X_1 = H_\infty E$ (see (4.19) of Theorem 4.7 for details) and H_∞ being symmetric.

Initialize $k \leftarrow 1,\ A_{1,1} \leftarrow A_1, G_{1,1} \leftarrow G_1,\ H_{1,1} \leftarrow H_1$.
repeat
  if $(I + EG_{1,k}E^\top H_{1,k})$ is nearly singular or singular, **then**
    $A_1 \leftarrow A_{1,k}, G_1 \leftarrow G_{1,k}, H_1 \leftarrow H_{1,k}$ and call Algorithm 3.1,
  else
    $A_{1,k+1} \leftarrow A_{1,k} + (A_{1,k} - G_{1,k}E^\top H_{1,k}(I + EG_{1,k}E^\top H_{1,k})^{-1})(I + EA_{1,k}),$
    $G_{1,k+1} \leftarrow G_{1,k} + (I + A_{1,k}E)(I + G_{1,k}E^\top H_{1,k}E)^{-1}G_{1,k}(I + E^\top A_{1,k}^\top),$
    $H_{1,k+1} \leftarrow H_{1,k} + (I + A_{1,k}^\top E^\top)H_{1,k}(I + EG_{1,k}E^\top H_{1,k})^{-1}(I + EA_{1,k}),$
  end if
  $k \leftarrow k + 1,$
until $\|H_{1,k+1}E - H_{1,k}E\| \le \tau \|H_{1,k+1}E\|$.
$X_1 \leftarrow H_{1,k+1}E \equiv H_\infty E$.

Thus, the matrix $[X_2, X_4]$ can be computed by solving the underdetermined equation

$$[X_2, X_4] \begin{bmatrix} I - R_1 \\ -R_2 \end{bmatrix} = H_2 E + A_3^\top E^\top X_1 R_1. \tag{3.9}$$

REMARK 3.1. A number of methods can be applied to solve (3.9), and any solution of (3.9) can be chosen as $[X_2, X_4]$.

In the following section, we want to use the auxiliary basis in (3.1) to construct bases V_s and V_∞ for the semi-stable subspaces corresponding to $\lambda \in \mathbb{D}_- \cup \mathbb{D}_1 \backslash \{1\}$ and $\lambda \in \{1\}$, respectively, which are essential for the computation of the desired X_a.

3.3. The computation of $\mathbf{V_s}$ and $\mathbf{V_\infty}$. From (3.1) we see that the matrix $R \equiv \begin{bmatrix} R_1 & 0 \\ R_2 & I_m \end{bmatrix}$ has the same eigenvalues as the matrix pair

$$(\mathcal{A}, \mathcal{B}) \equiv \left(\begin{bmatrix} I + A_1 E & 0 \\ A_2 E & I_m \end{bmatrix}, \begin{bmatrix} I + G_1 E^\top X_1 & 0 \\ G_2 E^\top X_1 & I_m \end{bmatrix} \right).$$

Let

$$E\,[V_0, V_r] = [U_0, U_r] \begin{bmatrix} 0_e & 0 \\ 0 & \Delta \end{bmatrix} \tag{3.10}$$

be the singular value decomposition of E, where $\Delta =$ diagonal > 0 and $[V_0, V_r]$ and $[U_0, U_r]$ are orthogonal with $V_0, U_0 \in \mathbb{R}^{n \times e}$. Then it holds that

$$\begin{aligned} &\begin{bmatrix} V_0^\top & 0 \\ 0 & I_m \\ V_r^\top & 0 \end{bmatrix} \begin{bmatrix} I + A_1 E & 0 \\ A_2 E & I_m \end{bmatrix} \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix} \\ &= \left[\begin{array}{cc|c} I_e & 0 & V_0^\top A_1 U_r \Delta \\ 0 & I_m & A_2 U_r \Delta \\ \hline 0 & 0 & I + V_r^\top A_1 U_r \Delta \end{array}\right] \equiv \left[\begin{array}{c|c} I_{e+m} & \star \\ \hline 0 & \mathcal{C} \end{array}\right], \end{aligned} \tag{3.11a}$$

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$$
\begin{aligned}
&\begin{bmatrix} V_0^\top & 0 \\ 0 & I_m \\ V_r^\top & 0 \end{bmatrix} \begin{bmatrix} I + G_1 E^\top X_1 & 0 \\ G_2 E^\top X_1 & I_m \end{bmatrix} \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix} \\
&= \left[\begin{array}{cc|c} I_e & 0 & V_0^\top G_1 E^\top X_1 V_r \\ 0 & I_m & G_2 E^\top X_1 V_r \\ \hline 0 & 0 & I + V_r^\top G_1 E^\top X_1 V_r \end{array}\right] \equiv \left[\begin{array}{c|c} I_{e+m} & \star \\ \hline 0 & \mathcal{D} \end{array}\right].
\end{aligned} \tag{3.11b}
$$

Now, we want to separate the eigenvalue 1 from the other semi-stable eigenvalues of $(\mathcal{A}, \mathcal{B})$. Using the backward stable numerical algorithm [17] to compute the Kronecker structure of the eigenvalue 1 of $(\mathcal{C}, \mathcal{D})$, there are orthogonal matrices Q and $Y \equiv [Y_1, Y_2]$ such that

$$
Q^\top (\mathcal{C}, \mathcal{D})\, Y = \left(\begin{bmatrix} \hat{C}_1 & \star \\ 0 & C_2 \end{bmatrix}, \begin{bmatrix} \hat{D}_1 & \star \\ 0 & D_2 \end{bmatrix} \right),
$$

where $Y_1 \in \mathbb{R}^{(n-e)\times f}$, $\hat{C}_1$ and $\hat{D}_1 \in \mathbb{R}^{f\times f}$ are upper triangular with diagonal elements being one, and $1 \notin \sigma(C_2, D_2)$.

Let

$$
V = \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix}, \qquad \widetilde{Y} = \begin{bmatrix} I_{e+m} & 0 \\ 0 & Y \end{bmatrix}, \qquad \widetilde{Q}^\top = \begin{bmatrix} I_{e+m} & 0 \\ 0 & Q^\top \end{bmatrix}. \tag{3.12}
$$

Then from (3.11)–(3.12), we have

$$
\widetilde{Q} V^\top (\mathcal{A}, \mathcal{B})\, V \widetilde{Y} = \left(\begin{bmatrix} C_1 & C_3 \\ 0 & C_2 \end{bmatrix}, \begin{bmatrix} D_1 & D_3 \\ 0 & D_2 \end{bmatrix} \right) \tag{3.13}
$$

with

$$
(C_1, D_1) = \left(\begin{bmatrix} I_{e+m} & \star \\ 0 & \hat{C}_1 \end{bmatrix}, \begin{bmatrix} I_{e+m} & \star \\ 0 & \hat{D}_1 \end{bmatrix} \right).
$$

Since $\sigma(C_1, D_1) \cap \sigma(C_2, D_2) = \emptyset$ with $m' = e + m + f$ and $n' = n - e - f$, there are matrices W_1 and W_2 such that

$$
\begin{aligned}
\begin{bmatrix} I_{m'} & W_2 \\ 0 & I_{n'} \end{bmatrix} \begin{bmatrix} C_1 & C_3 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} I_{m'} & W_1 \\ 0 & I_{n'} \end{bmatrix} &= C_1 \oplus C_2, \\
\begin{bmatrix} I_{m'} & W_2 \\ 0 & I_{n'} \end{bmatrix} \begin{bmatrix} D_1 & D_3 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} I_{m'} & W_1 \\ 0 & I_{n'} \end{bmatrix} &= D_1 \oplus D_2,
\end{aligned}
$$

where W_1 and W_2 solve the generalized Sylvester equations $C_1 W_1 + W_2 C_2 = -C_3$ and $D_1 W_1 + W_2 D_2 = -D_3$. Here and hereafter "$\oplus$" denotes the direct sum of two matrices. Let

$$
\mathcal{V} = V \widetilde{Y} \begin{bmatrix} I_{m'} & W_1 \\ 0 & I_{n'} \end{bmatrix}. \tag{3.14}
$$

Then from (3.1) and (3.13)–(3.14), we have the matrix

$$
V_s = \begin{bmatrix} V_{s,1} \\ \hline V_{s,2} \end{bmatrix} \equiv \left[\begin{array}{cc} I_n & 0 \\ 0 & I_m \\ \hline X_1 & 0 \\ X_2 & X_4 \end{array}\right] \mathcal{V}(:, m'+1 : n+m) \in \mathbb{R}^{2(n+m)\times(n-(e+f))}, \tag{3.15}
$$

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whose columns span the semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ corresponding to (C_2, D_2). Note that $1 \notin \sigma(C_2, D_2)$.

On the other hand, using (3.12), (3.14), and $Y = [Y_1, Y_2]$, we get the generalized eigenvectors

$$W_\infty \equiv \begin{bmatrix} I_n & 0 \\ 0 & I_m \\ X_1 & 0 \\ X_2 & X_4 \end{bmatrix} \mathcal{V}(:, e+m+1 : m') = \begin{bmatrix} V_r \\ 0 \\ X_1 V_r \\ X_2 V_r \end{bmatrix} Y_1 \tag{3.16}$$

corresponding to $(\hat{C}_1, \hat{D}_1)$. Then from (3.1) and (3.10), we have

$$V_\infty = \left[\begin{array}{c} V_{\infty,1} \\ \hline V_{\infty,2} \end{array}\right] \equiv \left[\begin{array}{cccc|c} V_0 & 0 & 0 & 0 & \\ 0 & I_m & 0 & 0 & \\ 0 & 0 & U_0 & 0 & W_\infty \\ 0 & 0 & 0 & I_m & \end{array} \right] \in \mathbb{R}^{2(n+m)\times\nu} \tag{3.17}$$

spanning the semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ corresponding to $(I_{e+m} \oplus C_1, I_{e+m} \oplus D_1)$ with $\nu = 2(e+m) + f$. Note that $\sigma(C_1, D_1) = \{1\}$. Moreover, we have the following lemma.

LEMMA 3.2. *V_∞ in (3.17) satisfies $V_\infty^\top \mathcal{J}\mathcal{E}_a V_\infty = 0$.*

Proof. From (3.17) we have $V_\infty^\top \mathcal{J}\mathcal{E}_a V_\infty = 0_{2(e+m)} \oplus (W_\infty^\top \mathcal{J}\mathcal{E}_a W_\infty)$. It suffices to show that $W_\infty^\top \mathcal{J}\mathcal{E}_a W_\infty = 0$. Since $X_1 = H_\infty E$ and $H_\infty^\top = H_\infty$, from (3.16) we conclude that

$$\begin{aligned} W_\infty^\top \mathcal{J}\mathcal{E}_a W_\infty &= Y_1^\top \left[-V_r^\top X_1^\top E, 0, V_r^\top E^\top, 0 \right] \begin{bmatrix} V_r \\ 0 \\ X_1 V_r \\ X_2 V_r \end{bmatrix} Y_1 \\ &= Y_1^\top (-V_r^\top X_1^\top E V_r + V_r^\top E^\top X_1 V_r) Y_1 = 0. \qquad \square \end{aligned}$$

Furthermore, from (2.1), (3.15), and (3.17), there exist matrices $R_s \in \mathbb{R}^{n'\times n'}$ and $N_\infty \in \mathbb{R}^{\nu\times\nu}$ such that

$$\mathcal{H}_a \begin{bmatrix} V_{s,1} \\ V_{s,2} \end{bmatrix} = \begin{bmatrix} U_{s,1} \\ U_{s,2} \end{bmatrix} R_s, \qquad \mathcal{E}_a \begin{bmatrix} V_{s,1} \\ V_{s,2} \end{bmatrix} = \begin{bmatrix} U_{s,1} \\ U_{s,2} \end{bmatrix}, \tag{3.18}$$

$$\mathcal{H}_a \begin{bmatrix} V_{\infty,1} \\ V_{\infty,2} \end{bmatrix} = \begin{bmatrix} U_{\infty,1} \\ U_{\infty,2} \end{bmatrix}, \qquad \mathcal{E}_a \begin{bmatrix} V_{\infty,1} \\ V_{\infty,2} \end{bmatrix} = \begin{bmatrix} U_{\infty,1} \\ U_{\infty,2} \end{bmatrix} N_\infty, \tag{3.19}$$

where R_s is equivalent to $\gamma(C_2 + D_2)(C_2 - D_2)^{-1}$ with $\sigma(R_s) \subseteq \mathbb{C}_- \cup \mathbb{C}_0$ and N_∞ is equivalent to $\left(0_{2(e+m)-1} \oplus K_{0,f+1}\right)$ with $K_{0,f+1}$ being the nilpotent matrix of size $f+1$. (This coincides with Assumption 4.2(ii) in Section 4.)

3.4. The computation of $\mathbf{X_a}$. From the identities (3.15) and (3.17), we observe that $\dim(\operatorname{span}\{[V_s, V_\infty]\}) = (n+m) + (e+m) > n+m$. According to the second condition in (1.4), we find a compression matrix $Z_\infty \in \mathbb{R}^{\nu\times m'}$ for V_∞ and U_∞ such that

$$\begin{bmatrix} V_{s,1}^\top \\ Z_\infty^\top V_{\infty,1}^\top \end{bmatrix} [U_{s,2}, U_{\infty,2} Z_\infty] = \begin{bmatrix} V_{s,2}^\top \\ Z_\infty^\top V_{\infty,2}^\top \end{bmatrix} [U_{s,1}, U_{\infty,1} Z_\infty] \tag{3.20}$$

and $\mathcal{E}_a V_\infty Z_\infty = U_\infty Z_\infty \hat{N}_\infty$ for some appropriate nilpotent matrix $\hat{N}_\infty$. The latter statement will be proved in Theorem 3.5 below.

From (3.14) and (3.15), we have

$$V_s \equiv \begin{bmatrix} V_{s,1} \\ V_{s,2} \end{bmatrix} = \left[\begin{array}{ccc} V_0 & 0 & V_r \\ 0 & I_m & 0 \\ \hline 0 & 0 & X_1V_r \\ X_2V_0 & X_4 & X_1V_r \end{array}\right] \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \tag{3.21}$$

and

$$U_s \equiv \begin{bmatrix} U_{s,1} \\ U_{s,2} \end{bmatrix} = \mathcal{E}_a V_s = \left[\begin{array}{ccc} 0 & 0 & EV_r \\ 0 & 0 & 0 \\ \hline 0 & 0 & E^\top X_1 V_r \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix},$$

where

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \equiv \begin{bmatrix} I_{e+m} & 0 \\ 0 & Y_1 \end{bmatrix} W_1 + \begin{bmatrix} 0 \\ Y_2 \end{bmatrix}$$

with $\Gamma_1 \in \mathbb{R}^{(e+m)\times n'}$ and $\Gamma_2 \in \mathbb{R}^{f\times n'}$. Since $X_1 = H_\infty E$ and $H_\infty^\top = H_\infty$, it holds that

$$V_{s,1}^\top U_{s,2} = \left[\Gamma_1^\top, \Gamma_2^\top\right] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & V_r^\top E^\top X_1 V_r \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = V_{s,2}^\top U_{s,1}.$$

From (3.18) and (3.19), it follows that

$$\begin{aligned} N_\infty^\top (V_\infty^\top \mathcal{J} U_s) R_s &= N_\infty^\top V_\infty^\top \mathcal{J} \mathcal{H}_a V_s = -N_\infty^\top V_\infty^\top \mathcal{H}_a^\top \mathcal{J} V_s \\ &= -V_\infty^\top \mathcal{E}_a^\top \mathcal{J} V_s = -V_\infty^\top \mathcal{J} \mathcal{E}_a V_s = -V_\infty^\top \mathcal{J} U_s \end{aligned} \tag{3.22}$$

and

$$N_\infty^\top (U_\infty^\top \mathcal{J} V_s) R_s = V_\infty^\top \mathcal{J} \mathcal{E}_a V_s R_s = (V_\infty^\top \mathcal{J} U_s) R_s = -V_\infty^\top \mathcal{H}_a^\top \mathcal{J} V_s = -U_\infty^\top \mathcal{J} V_s. \tag{3.23}$$

Since $\sigma(N_\infty) = \{0\}$, the Stein equations (3.22) and (3.23) (after ignoring all intermediate terms) have only trivial solutions, i.e., $V_\infty^\top \mathcal{J} U_s = 0$ and $U_\infty^\top \mathcal{J} V_s = 0$.

To show (3.20), it remains to construct a matrix Z_∞ of full rank such that

$$Z_\infty^\top V_{\infty,1}^\top U_{\infty,2} Z_\infty = Z_\infty^\top V_{\infty,2}^\top U_{\infty,1} Z_\infty.$$

Let

$$\Upsilon \equiv V_{\infty,1}^\top U_{\infty,2} - V_{\infty,2}^\top U_{\infty,1} = V_\infty^\top \mathcal{J} \mathcal{H}_a V_\infty = V_\infty^\top \mathcal{J} U_\infty. \tag{3.24}$$

Since Υ is symmetric, we can compute its spectral decomposition

$$\Upsilon = Q^\top \Sigma Q, \tag{3.25}$$

where $\Sigma = \text{diagonal} \equiv \Sigma_1 \oplus (-\Sigma_2) \oplus 0_{\eta_0}$ with $\Sigma_1 > 0$ and $\Sigma_2 > 0$ of dimension η_1 and η_2, respectively, and $\eta_0 = \nu - (\eta_1 + \eta_2)$.

THEOREM 3.3. *With $m' = e + m + f$, there is a full rank matrix $Z_\infty \in \mathbb{R}^{\nu \times m'}$ with*

$$Z_\infty^\top \Upsilon Z_\infty = 0 \tag{3.26}$$

if and only if $\eta_0 + \min\{\eta_1, \eta_2\} \geq m'$.

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Proof. From (3.24) and (3.25), it follows that $Z_\infty^\top \Upsilon Z_\infty = 0$ holds if and only if $Z_\infty^\top Q^\top \Sigma Q Z_\infty = 0$. Let $\zeta = QZ_\infty \equiv [\zeta_1^\top, \zeta_2^\top, \zeta_3^\top]^\top$ have the same partition in Σ. Then $Z_\infty^\top \Upsilon Z_\infty = 0$ is equivalent to

$$\zeta_1^\top \Sigma_1 \zeta_1 - \zeta_2^\top \Sigma_2 \zeta_2 = 0. \tag{3.27}$$

We prove necessity of the condition. Without loss of generality, we assume that $\eta_1 = \min\{\eta_1, \eta_2\}$. Since $\eta_0 + \eta_1 \geq m'$ and $\eta_0 + \eta_1 + \eta_2 = \nu$, it implies that $\eta_1 \leq \eta_2 \leq \nu - m' = e + m$. We choose

$$\zeta_1 = (\Sigma_1)^{-\frac{1}{2}} \widehat{\zeta}, \qquad \zeta_2 = (\Sigma_2)^{-\frac{1}{2}} \begin{bmatrix} \widehat{\zeta} \\ 0 \end{bmatrix} \tag{3.28}$$

with $\widehat{\zeta} \in \mathbb{R}^{\eta_1 \times m'}$ being any matrix of full row rank and $\zeta_3 \in \mathbb{R}^{\eta_0 \times m'}$ such that $\begin{bmatrix} \widehat{\zeta}^\top & \zeta_3^\top \end{bmatrix}^\top$ is of full column rank. It is easily seen that ζ_1 and ζ_2 satisfy (3.27). Thus, we have a full column rank matrix

$$Z_\infty = Q^\top [\zeta_1^\top, \zeta_2^\top, \zeta_3^\top]^\top \tag{3.29}$$

satisfying $Z_\infty^\top \Upsilon Z_\infty = 0$.

We prove sufficiency of the condition. If $\eta_0 + \eta_1 < m'$, then from (3.27) we see that

$$\operatorname{rank} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \operatorname{rank}(\zeta_1) \leq \eta_1 \quad \text{and} \quad \operatorname{rank}(\zeta) = \operatorname{rank}\left([\zeta_1^\top, \zeta_2^\top, \zeta_3^\top] \right) \leq \eta_0 + \eta_1 < m'.$$

Thus, we have $\operatorname{rank}(Z_\infty) = \operatorname{rank}(Q^\top \zeta) < m'$. Therefore, there is no full rank matrix Z_∞ satisfying (3.26). □

REMARK 3.4. Note that $Z_\infty^\top \Upsilon Z_\infty = Z_\infty^\top (V_{\infty,1}^\top U_{\infty,2} - V_{\infty,2}^\top U_{\infty,1}) Z_\infty$, and such a matrix pair $(V_\infty Z_\infty, U_\infty Z_\infty)$ is called bi-isotropic. In Theorem 3.3 we gave a necessary and sufficient condition for the bi-isotropicity of $V_\infty Z_\infty$ and $U_\infty Z_\infty$. In the following theorem, we show that the matrix pair $(V_\infty Z_\infty, U_\infty Z_\infty)$ spans a deflating subspace pair of $(\mathcal{H}_a, \mathcal{E}_a)$ corresponding to $(I_{m'}, \hat{N}_\infty)$ with some suitable nilpotent matrix $\hat{N}_\infty$.

THEOREM 3.5.

(i) If $\eta_0 + \min\{\eta_1, \eta_2\} = m'$, then there is a nilpotent matrix $\hat{N}_\infty \in \mathbb{R}^{m' \times m'}$ such that

$$\mathcal{E}_a V_\infty Z_\infty = U_\infty Z_\infty \hat{N}_\infty, \tag{3.30}$$

where Z_∞ is given by (3.29).

(ii) If $\eta_0 + \min\{\eta_1, \eta_2\} > m'$, then, generically, there is a nilpotent matrix $\hat{N}_\infty \in \mathbb{R}^{m' \times m'}$ such that (3.30) holds, where Z_∞ is given by (3.40) below.

Proof. Without loss of generality, we assume that $\eta_1 = \min\{\eta_1, \eta_2\}$ and adopt the notations used in Theorem 3.3. From (3.17) and (3.19), there is a matrix $B_\infty \in \mathbb{R}^{\nu \times f}$ of full column rank such that $\mathcal{E}_a W_\infty = U_\infty B_\infty$. Let $N_\infty = [0_{\nu, 2(e+m)} \mid B_\infty]$. We then have

$$\mathcal{E}_a V_\infty = [0 \,|\, \mathcal{E}_a W_\infty] = [0 \,|\, U_\infty B_\infty] = U_\infty N_\infty, \tag{3.31}$$

where $0 = 0_{2(n+m), 2(e+m)}$. Partition Q and Z_∞ in (3.29) as

$$Q = [\underbrace{Q_1}_{e+m}, \underbrace{Q_2}_{e+m}, \underbrace{Q_3}_{f}] = \begin{bmatrix} Q' \\ Q'' \\ Q''' \end{bmatrix} \begin{matrix} \}\,\eta_1 \\ \}\,\eta_2 \\ \}\,\eta_0 \end{matrix}, \qquad Z_\infty = \begin{bmatrix} Z_{\infty,1} \\ Z_{\infty,2} \\ Z_{\infty,3} \end{bmatrix} \begin{matrix} \}\,e+m \\ \}\,e+m \\ \}\,f \end{matrix}. \tag{3.32}$$

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From Lemma 3.2 and (3.24), it follows that $V_\infty^\top \mathcal{J}\mathcal{E}_a V_\infty = (V_\infty^\top \mathcal{J} U_\infty) N_\infty = \Upsilon N_\infty = 0$. Therefore, $\Upsilon B_\infty = (Q^\top \Sigma Q) B_\infty = Q^\top [\Sigma_1 \oplus (-\Sigma_2) \oplus 0_{\eta_0}] Q B_\infty = 0$. From (3.32) we have

$$\begin{bmatrix} Q' \\ Q'' \end{bmatrix} B_\infty = 0. \tag{3.33}$$

By (3.28), ζ can be expressed by

$$\zeta = \begin{bmatrix} \Sigma_1' & 0 \\ \Sigma_2' & 0 \\ 0 & 0 \\ 0 & I_{\eta_0} \end{bmatrix} \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix}, \tag{3.34}$$

where $\Sigma_1' = \Sigma_1^{-\frac{1}{2}}$ and $\Sigma_2' = \Sigma_2^{-\frac{1}{2}}(1:\eta_1, 1:\eta_1)$. Since $\mathcal{E}_a V_\infty Z_\infty = U_\infty N_\infty Z_\infty$, it holds that $\mathcal{E}_a V_\infty Z_\infty \subseteq \mathcal{R}(U_\infty Z_\infty)$, $U_\infty B_\infty Z_{\infty,3} \subseteq \mathcal{R}(U_\infty Z_\infty)$ (by (3.31) and (3.32)), $B_\infty Z_{\infty,3} \subseteq \mathcal{R}(Z_\infty)$, and $B_\infty Q_3^\top \zeta \subseteq \mathcal{R}(Q^\top \zeta)$ (by (3.29) and (3.32)) or $Q B_\infty Q_3^\top \zeta \subseteq \mathcal{R}(\zeta)$. Equivalently, by (3.33) and (3.34), there is a $\Phi \in \mathbb{R}^{m' \times m'}$ such that

$$\begin{bmatrix} 0_{(\eta_1+\eta_2)\times m'} \\ F \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} \end{bmatrix} = \zeta \Phi \Longleftrightarrow \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} \Phi = \begin{bmatrix} 0_{\eta_1 \times m'} \\ F \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} \end{bmatrix}, \tag{3.35}$$

where

$$F := Q''' B_\infty Q_3^\top \begin{bmatrix} \Sigma_1' & 0 \\ \Sigma_2' & 0 \\ 0 & 0 \\ 0 & I_{\eta_0} \end{bmatrix}.$$

Thus, showing that $\mathcal{E}_a V_\infty Z_\infty \subseteq \mathcal{R}(U_\infty Z_\infty)$ is equivalent to show that (3.35) holds.

Case (i): For $\eta_0 + \eta_1 = m'$, $\left[\hat{\zeta}^\top, \zeta_3^\top\right]^\top$ is an $m' \times m'$-matrix. In this case, $\left[\hat{\zeta}^\top, \zeta_3^\top\right]^\top$ can be chosen to be invertible. So, (3.35) is always solvable for Φ. Hence, there is a nilpotent matrix $\hat{N}_\infty$ such that (3.30) holds, where Z_∞ is given by (3.29).

Case (ii): For $\eta_0 + \eta_1 > m'$, we partition ζ_3 in (3.34) and F in (3.35) as

$$\zeta_3 = \begin{bmatrix} \zeta_{3,0} \\ \zeta_{3,1} \end{bmatrix} \begin{matrix} \}\, l' \equiv m' - \eta_1 \\ \}\, d' \equiv \eta_0 - l' \end{matrix}, \qquad F = \begin{matrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \\ \begin{matrix} \underbrace{\hphantom{F_{11}}}_{m'} & \underbrace{\hphantom{F_{12}}}_{d'} \end{matrix} \end{matrix} \begin{matrix} \}\, l' \\ \}\, d' \end{matrix}.$$

Rewrite

$$\begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} \hat{\zeta}_0 \\ \hat{\zeta}_3 \end{bmatrix} \begin{matrix} \}\, m' \\ \}\, d' \end{matrix}, \tag{3.36}$$

where $\hat{\zeta}_0 := \begin{bmatrix} \hat{\zeta} \\ \zeta_{3,0} \end{bmatrix}$ and $\hat{\zeta}_3 := \zeta_{3,1}$, then equation (3.35) becomes

$$\begin{bmatrix} 0 \\ F_{11}\hat{\zeta}_0 + F_{12}\hat{\zeta}_3 \end{bmatrix} = \hat{\zeta}_0 \Phi, \quad F_{21}\hat{\zeta}_0 + F_{22}\hat{\zeta}_3 = \hat{\zeta}_3 \Phi. \tag{3.37}$$

Since $\hat{\zeta}_0$ can be chosen invertible, we partition $\hat{\zeta}_3\hat{\zeta}_0^{-1}$, F_{11}, and F_{21} as

$$\hat{\zeta}_3\hat{\zeta}_0^{-1} = [\underbrace{\Omega_1}_{\eta_1} \mid \underbrace{\Omega_2}_{l'}]\}d', \quad F_{11} = [\underbrace{F_{11}^a}_{\eta_1} \mid \underbrace{F_{11}^b}_{l'}]\}l', \quad F_{21} = [\underbrace{F_{21}^a}_{\eta_1} \mid \underbrace{F_{21}^b}_{l'}]\}d'. \tag{3.38}$$

With (3.38), equations (3.37) can be written as a Riccati equation for Ω_2 and a linear equation for Ω_1:

$$\Omega_2 F_{12}\Omega_2 + \Omega_2 F_{11}^b - F_{22}\Omega_2 - F_{21}^b = 0, \tag{3.39a}$$

$$(F_{22} - \Omega_2 F_{12})\Omega_1 = \Omega_2 F_{11}^a - F_{21}^a. \tag{3.39b}$$

Equation (3.39a) for Ω_2 is generically solvable by the Schur method. The same holds for (3.39b) and the equation (3.35) for Φ. By (3.38), (3.36), and (3.34), Z_∞ can be chosen as

$$Z_\infty = Q^\top \begin{bmatrix} \Sigma_1' & 0 \\ \Sigma_2' & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_{m'} \\ [\Omega_1, \Omega_2] \end{bmatrix} \hat{\zeta}_0. \tag{3.40}$$

Hence, there is a nilpotent $\hat{N}_\infty$ such that (3.30) holds. □

REMARK 3.6. In our test examples, Example 5.1 [12, 13] and Example 5.2 [16] in Section 5, we will check that $\eta_0 + \min\{\eta_1, \eta_2\} = m'$ holds, which coincides with the condition in case (i) in Theorem 3.5.

Finally, we let $\hat{V}_\infty = V_\infty Z_\infty$ and $V_a = \begin{bmatrix} V_s, \widehat{V}_\infty \end{bmatrix} \equiv \begin{bmatrix} V_{a,1} \\ V_{a,2} \end{bmatrix}$. If $V_{a,1}$ is invertible, then the solution X_a for (1.1) is given by $X_a = V_{a,2}V_{a,1}^{-1}$. Therefore, we have to ensure that $V_{a,1}$ is invertible. In fact, from (3.16), (3.17), and (3.21), we have

$$\begin{aligned} V_{a,1} = [\ V_{s,1} \mid V_{\infty,1}Z_\infty\ ] &= \left[\begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \middle| \begin{bmatrix} V_0 & 0 & 0 & 0 & V_r Y_1 \\ 0 & I_m & 0 & 0 & 0 \end{bmatrix} Z_\infty \right] \\ &= \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} \Gamma_1 & Z_{\infty,1} \\ \Gamma_2 & Y_1 Z_{\infty,3} \end{bmatrix} = \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} I_{e+m} & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} W_1 & \begin{bmatrix} Z_{\infty,1} \\ Z_{\infty,3} \end{bmatrix} \\ I_{e+m} & 0 \end{bmatrix}, \end{aligned}$$

where $Z_\infty = \begin{bmatrix} Z_{\infty,1}^\top, Z_{\infty,2}^\top, Z_{\infty,3}^\top \end{bmatrix}^\top$ is defined in (3.32). Therefore, $V_{a,1}$ is nonsingular if and only if $\begin{bmatrix} Z_{\infty,1}^\top, Z_{\infty,3}^\top \end{bmatrix}^\top$ is nonsingular. We summarize the above procedures for the computation of X_a in Algorithm 3.3.

REMARK 3.7. In Algorithm 3.3, step 1 is carried out iteratively and converges quadratically under mild assumptions as proved in Theorem 4.7. As for step 4-8, since $\begin{bmatrix} \hat{\zeta}^\top, \zeta_3^\top \end{bmatrix}^\top$ in (3.29) or $\widehat{\zeta}_0$ in (3.40) can be chosen as arbitrary nonsingular matrices, there are many degrees of freedom in obtaining an invertible matrix $\begin{bmatrix} Z_{\infty,1}^\top, Z_{\infty,3}^\top \end{bmatrix}^\top$ and a desirable matrix $V_{a,1}$. Thus, Algorithm 3.3 solves the GARE (1.1) efficiently and reliably in most cases as illustrated by the numerical experiments presented in Section 5.

Algorithm 3.3 Structure-preserving Algorithm (SA) for GARE (1.1).

Require: $A_1,\ A_2,\ A_3,\ H_1,\ H_2,\ G_1,\ G_2,\ E$ as in (3.1).
Ensure: An X_a for GARE (1.1).
Compute X_1 by Algorithm 3.2.
Compute X_2 and X_4 by (3.9).
Compute V_s and V_∞ by (3.15) and (3.17), respectively.
if the condition $\eta_0 + \min\{\eta_1, \eta_2\} \geq m'$ in Theorem 3.5 holds, **then**
    compute Z_∞ by (3.29) or (3.40)
else
    there is no solution.
end if
Compute $\begin{bmatrix} V_{a,1} \\ V_{a,2} \end{bmatrix} \equiv [V_s, V_\infty Z_\infty]$.
if $V_{a,1}$ is invertible, **then**
    $X_a = V_{a,2} V_{a,1}^{-1}$,
else
    fails.
end if

4. Convergence of the SDA. We denote the Jordan block of size p corresponding to a unimodular eigenvalue $\omega \equiv e^{i\theta}$ by

$$K_{\omega,p} = \begin{bmatrix} \omega & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \omega \end{bmatrix}_{p\times p}.$$

The Jordan block $K_{\omega,p}$ raised to the power of 2^k can be evaluated to (see, e.g., [8, p. 557])

$$K_{\omega,p}^{2^k} = \begin{bmatrix} \gamma_{1,k} & \gamma_{2,k} & \cdots & \gamma_{p,k} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \gamma_{2,k} \\ 0 & & & \gamma_{1,k} \end{bmatrix}, \tag{4.1}$$

where

$$\gamma_{i,k} = \frac{2^k(2^k-1)\cdots(2^k-i+2)}{(i-1)!}\omega^{2^k-i+1} = \mathcal{O}(2^{k(i-1)}),$$

for $i = 1, \ldots, p$. If $p = 2q$, let

$$L_{\omega,k} \equiv K_{\omega,p}^{2^k}(1:q:q+1:p). \tag{4.2}$$

We quote the useful lemma from [11].

LEMMA 4.1. *For $p = 2q$, the matrix in (4.2) is invertible and satisfies*

$$\|L_{\omega,k}^{-1} K_{\omega,q}^{2^k}\| = \mathcal{O}(2^{-k}), \qquad \|K_{\omega,q}^{2^k} L_{\omega,k}^{-1} K_{\omega,q}^{2^k}\| = \mathcal{O}(2^{-k}). \tag{4.3}$$

To show convergence of the SDA algorithm, we first assume that the original matrix pencil $\mathcal{H}_a - \lambda \mathcal{E}_a$ satisfies the following assumption.

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ASSUMPTION 4.2. *For the Hamiltonian/skew-Hamiltonian pair* $(\mathcal{H}_a, \mathcal{E}_a)$, *we assume that*

(i) the partial multiplicities of the eigenvalue 0 *are either one or even, and the number of partial multiplicities equal to one is* 2μ *with* $\operatorname{rank}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = n+m-\mu$,

(ii) the eigenvalue ∞ *has a Jordan structure* $\left(I_{2(e+f+m)}, 0_{2m} \oplus 0_{2e-1} \oplus K_{0,2f+1}\right)$ *with*

$$\text{(4.4a)} \qquad \operatorname{nullity}\left(\begin{bmatrix} \widehat{F}_a & E_a & & 0 & \\ & F_a & E_a & & \\ & & \ddots & \ddots & \\ 0 & & & F_a & \widehat{E}_a \end{bmatrix}_{g\times(g+e)}\right) \geq e+1,$$

where $g = (n+m)f$ *and*

$$\text{(4.4b)} \qquad \widehat{F}_a = \begin{bmatrix} AV_0 & B \\ CV_0 & D \end{bmatrix},\ F_a = \begin{bmatrix} A & B \\ C & D \end{bmatrix},\ E_a = \begin{bmatrix} E & 0 \\ 0 & 0_m \end{bmatrix},\ \widehat{E}_a = \begin{bmatrix} E \\ 0 \end{bmatrix},$$

in which V_0 *is given by* (3.10),

(iii) each nonzero purely imaginary eigenvalue has even partial multiplicity.

REMARK 4.3. The Jordan structure of the eigenvalue ∞ in case (ii) can also be considered being of the more general form $\left(I_{2(e+f+m)}, 0_{2m} \oplus 0_e \oplus 0_{e-d} \overset{d}{\underset{i=1}{\oplus}} K_{0,2f_i+1}\right)$ with $f = f_1 + \ldots + f_d$ and $f_d \geq \ldots \geq f_1 \geq 1$. Then the condition in (4.4a) should be generalized to

$$\operatorname{nullity}\left(\begin{bmatrix} \widehat{F}_a & E_a & & \\ & F_a & \ddots & \\ & & \ddots & E_a \\ & & & F_a & \widehat{E}_a \end{bmatrix}_{g_i\times(g_i+e)}\right) \geq e + (d-i+1),$$

for $i = 1, \ldots, d$ and $f_i > f_{i-1}$ ($f_0 \equiv 0$), where $g_i = (n+m)f_i$. Since the proof for convergence of the SDA in Theorem 4.7 has a straightforward extension to the case $d > 1$, we only consider the simple case with $d = 1$ as in (ii) for convenience.

LEMMA 4.4. *Let* $(\mathcal{H}_a, \mathcal{E}_a)$ *satisfy Assumption 4.2. Then*

(i) for $\mu > 0$, *the null space of* $\mathcal{H}_a$ *contains* μ *linearly independent vectors of the form* $\zeta \equiv \left[\zeta_1^\top, \zeta_2^\top, 0_{n,\mu}^\top, \zeta_4^\top\right]^\top \in \mathbb{R}^{2(n+m)\times\mu}$ *with* $\zeta_1 \in \mathbb{R}^n$, ζ_2, *and* $\zeta_4 \in \mathbb{R}^m$,

(ii) for $f \geq 1$, *the generalized eigenvectors of* $(\mathcal{H}_a, \mathcal{E}_a)$ *corresponding to* ∞ *of degree* j *are of the form* $\eta_j \equiv \left[\eta_{j1}^\top, \eta_{j2}^\top, 0_{n,1}^\top, \eta_{j4}^\top\right]^\top \in \mathbb{R}^{2(n+m)}$ *with* $\eta_{j1} \neq 0 \in \mathbb{R}^n$, η_{j2}, $\eta_{j4} \in \mathbb{R}^m$, *for* $j = 1, \ldots, f$, *i.e., a vector* $\eta_0 \equiv \left[(V_0\alpha)^\top, \beta^\top, (U_0\gamma)^\top, \delta^\top\right]^\top$ *with* $0 \neq \alpha, \gamma \in \mathbb{R}^e$ *and* $\beta, \delta \in \mathbb{R}^m$ *exists such that*

$$\text{(4.5)} \qquad \mathcal{E}_a\eta_j = \mathcal{H}_a\eta_{j-1}, \qquad j = 1, \ldots, f.$$

Proof. Since

$$\begin{bmatrix} A & B \\ -C^\top JC & -C^\top JD \\ -D^\top JC & -D^\top JD \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -C^\top J \\ 0 & -D^\top J \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq n+m-\mu,$$

from Assumption 4.2 (i), (1.1b), and (1.3), it follows that

$$
\begin{aligned}
&\text{nullity}\left(\mathcal{H}_a \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0_{n,m} \\ 0 & I_m \end{bmatrix}\right) \\
(4.6)\qquad &= \text{nullity}\left(\left[\begin{array}{cc|c} A & B & 0 \\ 0 & I_m & -(J')^{-1} \\ \hline -C^\top JC & -C^\top JD & 0 \\ -D^\top JC & -D^\top JD - J' & I_m \end{array}\right]\right) \geq \mu.
\end{aligned}
$$

This proves the first assertion.

Let

$$
(4.7)\qquad \widehat{\mathcal{H}}_a = \mathcal{H}_a \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0_{n,m} \\ 0 & I_m \end{bmatrix}, \qquad \widehat{\mathcal{E}}_a = \mathcal{E}_a \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0_{n,m} \\ 0 & I_m \end{bmatrix},
$$

$$
(4.8)\qquad \widehat{\mathcal{H}}_{a,1} = \widehat{\mathcal{H}}_a \begin{bmatrix} V_0 & 0 \\ 0 & I_{2m} \end{bmatrix}, \qquad \widehat{\mathcal{E}}_{a,1} = \widehat{\mathcal{E}}_a \begin{bmatrix} I_n \\ 0_{2m,n} \end{bmatrix}.
$$

From Assumption 4.2 (ii), (4.7), and the equality of matrices in (4.6), we have

$$
(4.9)\qquad \text{nullity}\left(\begin{bmatrix} \widehat{\mathcal{H}}_{a,1} & \widehat{\mathcal{E}}_a & & 0 & \\ & \widehat{\mathcal{H}}_a & \widehat{\mathcal{E}}_a & & \\ & & \ddots & \ddots & \\ 0 & & & \widehat{\mathcal{H}}_a & \widehat{\mathcal{E}}_{a,1} \end{bmatrix}_{g\times(g+e)}\right) \geq e+1.
$$

Since $\text{nullity}(E) = e$, the matrix $\widehat{\mathcal{H}}_{a,1}$ is of full column rank and $\mathcal{H}_a - \lambda\mathcal{E}_a$ is regular. From (4.9) there are vectors $\widehat{\eta}_0 = \left(\alpha^\top, \beta^\top, \delta^\top\right)^\top$, with $0 \neq \alpha \in \mathbb{R}^e$, $\beta, \delta \in \mathbb{R}^m$, and $\widehat{\eta}_j = \left(\eta_{j1}^\top, \eta_{j2}^\top, \eta_{j4}^\top\right)^\top$, $0 \neq \widehat{\eta}_f \in \mathbb{R}^n$, with $0 \neq \eta_{j1} \in \mathbb{R}^n$, $\eta_{j2}, \eta_{j4} \in \mathbb{R}^m$, $j = 1, \ldots, f-1$, such that

$$
(4.10)\qquad \widehat{\mathcal{E}}_a\widehat{\eta}_1 = \widehat{\mathcal{H}}_{a,1}\widehat{\eta}_0, \qquad \widehat{\mathcal{E}}_a\widehat{\eta}_j = \widehat{\mathcal{H}}_a\widehat{\eta}_{j-1}, \qquad \widehat{\mathcal{E}}_{a,1}\widehat{\eta}_f = \widehat{\mathcal{H}}_a\widehat{\eta}_{f-1},
$$

for $j = 2, \ldots, f-1$. By taking $\gamma = 0$ and η_{j2}, η_{j4} arbitrary, it follows from (4.10) that (4.5) holds. □

Let

$$
(4.11)\qquad \mathcal{M}_1 = \begin{bmatrix} I_n + A_1E & 0 \\ -H_1E & I_n \end{bmatrix}, \qquad \mathcal{L}_1 = \begin{bmatrix} I_n & G_1E^\top \\ 0 & I_n + A_1^\top E^\top \end{bmatrix}
$$

be the submatrices of $\mathcal{M}$ and $\mathcal{L}$ in (2.4), respectively. By (2.1) the matrix pair $(\mathcal{M}, \mathcal{L})$ is the Cayley transform of $(\mathcal{H}_a, \mathcal{E}_a)$. Therefore, Assumption 4.2 is adapted to $(\mathcal{M}_1, \mathcal{L}_1)$ as follows.

ASSUMPTION 4.5. *For $(\mathcal{M}_1, \mathcal{L}_1)$ we assume that*

(i) the partial multiplicities of the eigenvalue -1 are either one or even, and the number of partial multiplicities equal to one is 2μ with rank $\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = n + m - \mu$,

(ii) the eigenvalue 1 has a Jordan structure $\left(I_{2(e+f)}, I_e \oplus I_{e-1} \oplus K_{1,2f+1}\right)$ satisfying (4.4),

(iii) each unimodular eigenvalue ω_j with $\omega_j \neq -1$ and 1 has even partial multiplicity $2m_j$.

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LEMMA 4.6. *Let $(\mathcal{M}_1, \mathcal{L}_1)$ satisfy Assumption 4.5. Then*

(i) for $\mu > 0$, the null space of $\mathcal{M}_1 + \mathcal{L}_1$ contains μ linearly independent vectors of the form $[\zeta_1^\top, 0_{n,\mu}^\top]^\top \in \mathbb{R}^{2n\times\mu}$,

(ii) for $f \geq 1$, the generalized eigenvectors of $(\mathcal{M}_1, \mathcal{L}_1)$ corresponding to 1 of degree j are of the form $\eta_j \equiv \left(\eta_{j1}^\top, 0_{n,1}^\top\right)^\top \neq 0 \in \mathbb{R}^{2n}$, $j = 1, \ldots, f$, i.e., there exist a vector $\eta_0 \equiv \left[(V_0\alpha)^\top, 0_{n,1}^\top\right]^\top$ such that

$$(\mathcal{M}_1 - \mathcal{L}_1)\eta_j = \mathcal{L}_1\eta_{j-1}, \qquad j = 1, \ldots, f.$$

Proof. The assertions (i) and (ii) follow immediately from Lemma 4.4 and the Cayley transform. □

From Kronecker's Theorem [7, Chapter 12], there are nonsingular matrices $\mathcal{Q}$ and $\mathcal{Z}$ such that

$$\mathcal{Q}\mathcal{M}_1\mathcal{Z} = \begin{bmatrix} J_s \oplus (-I_\mu) \oplus J_\omega \oplus J_1 & 0 \oplus 0_\mu \oplus \Gamma_\omega \oplus \Gamma_1 \\ 0_n & I_s \oplus (-I_\mu) \oplus J_\omega \oplus \widehat{J}_1 \end{bmatrix} \equiv J_{\mathcal{M}_1},$$

$$\mathcal{Q}\mathcal{L}_1\mathcal{Z} = \begin{bmatrix} I_n & 0_n \\ 0_n & J_s \oplus I_\mu \oplus I_r \oplus I_{e+f} \end{bmatrix} \equiv J_{\mathcal{L}_1}, \tag{4.12}$$

where $J_s \in \mathbb{R}^{s\times s}$ consists of asymptotically stable blocks with $\rho(J_s) < 1$,

$$\begin{aligned} J_\omega &= K_{\omega_1,m_1} \oplus \cdots \oplus K_{\omega_l,m_l} \in \mathbb{R}^{r\times r} && \text{with } \omega_j \neq 1, \\ \Gamma_\omega &= \Gamma_{1,m_1} \oplus \cdots \oplus \Gamma_{l,m_l} && \text{with } \Gamma_{1,m_j} = e_{m_j}e_1^\top, \\ J_1 &= K_{1,f+1} \oplus I_{e-1}, \quad \widehat{J}_1 = K_{1,f} \oplus I_e, && \Gamma_1 = e_{f+1}e_1^\top. \end{aligned}$$

On the other hand, if we interchange the roles of $\mathcal{M}_1$ and $\mathcal{L}_1$ in (4.12) and consider the pair $(\mathcal{L}_1, \mathcal{M}_1)$, then there are nonsingular matrices $\mathcal{P}$ and $\mathcal{Y}$ such that

$$\mathcal{P}\mathcal{L}_1\mathcal{Y} = J_{\mathcal{M}_1}, \qquad \mathcal{P}\mathcal{M}_1\mathcal{Y} = J_{\mathcal{L}_1}. \tag{4.13}$$

Since $J_{\mathcal{M}_1}$ and $J_{\mathcal{L}_1}$ in (4.12) commute with each other and from (4.12) and (4.13), one can derive that

$$\mathcal{M}_1\mathcal{Z}J_{\mathcal{L}_1} = \mathcal{L}_1\mathcal{Z}J_{\mathcal{M}_1}, \qquad \mathcal{L}_1\mathcal{Y}J_{\mathcal{L}_1} = \mathcal{M}_1\mathcal{Y}J_{\mathcal{M}_1}. \tag{4.14}$$

Partition $\mathcal{Z}$ and $\mathcal{Y}$ in (4.14) as

$$\mathcal{Z} = \begin{bmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{bmatrix}, \qquad \mathcal{Y} = \begin{bmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{bmatrix}, \tag{4.15}$$

where $Z_i,\ Y_i \in \mathbb{R}^{n\times n},\ i = 1, \ldots, 4$. From Lemma 4.6, we see that

$$Z_2 \begin{bmatrix} 0_{s,\mu} \\ I_\mu \\ 0_{n-s-\mu,\mu} \end{bmatrix} = 0 \qquad \text{and} \qquad Z_2 \begin{bmatrix} 0_{n-e,e} \\ I_e \end{bmatrix} = 0. \tag{4.16}$$

Let $\{(\mathcal{M}_{1,k}, \mathcal{L}_{1,k})\}_{k=1}^\infty$ be the sequence generated by the SDA algorithm of the form

$$\mathcal{M}_{1,k} = \begin{bmatrix} I_n + A_{1,k}E & 0 \\ -H_{1,k}E & I_n \end{bmatrix}, \qquad \mathcal{L}_{1,k} = \begin{bmatrix} I_n & G_{1,k}E^\top \\ 0 & I_n + A_{1,k}^\top E^\top \end{bmatrix} \tag{4.17}$$

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with $\mathcal{M}_{1,1} = \mathcal{M}_1$ and $\mathcal{L}_{1,1} = \mathcal{L}_1$. From (3.6)–(3.7) and (4.14), we have that

$$\mathcal{M}_{1,k}\mathcal{Z}J_{\mathcal{L}_1}^{2^k} = \mathcal{L}_{1,k}\mathcal{Z}J_{\mathcal{M}_1}^{2^k}, \qquad \mathcal{L}_{1,k}\mathcal{Y}J_{\mathcal{L}_1}^{2^k} = \mathcal{M}_{1,k}\mathcal{Y}J_{\mathcal{M}_1}^{2^k}. \tag{4.18}$$

THEOREM 4.7. *Let $(\mathcal{M}_1, \mathcal{L}_1)$ be given in (4.11) and satisfy Assumption 4.5. Let E have the singular value decomposition in (3.10), and suppose that Z_1 and Y_2 in (4.15) are invertible. If the sequence $\{(A_{1,k}, G_{1,k}, H_{1,k})\}$ generated by the SDA is well-defined, then we have*

$$\|H_{1,k}E - Z_2Z_1^{-1}\| \leq \mathcal{O}(\rho(J_s)^{2^k}) + \mathcal{O}(2^{-k}) \to 0, \qquad \textit{as } k \to \infty. \tag{4.19}$$

Here and hereafter, $\|\cdot\|$ denotes any matrix norm.

Proof. Substituting $(\mathcal{M}_{1,k}, \mathcal{L}_{1,k})$ in (4.17), $\mathcal{Z}$ in (4.15), and $J_{\mathcal{M}_1}$ and $J_{\mathcal{L}_1}$ in (4.12) into the first equation of (4.18) and comparing both sides, we obtain

$$-H_{1,k}EZ_1 + Z_2 = (I_n + A_{1,k}^\top E^\top)Z_2(J_s^{2^k} \oplus I_\mu \oplus J_\omega^{2^k} \oplus J_1^{2^k}), \tag{4.20a}$$

$$\begin{aligned}&(-H_{1,k}EZ_3 + Z_4)(J_s^{2^k} \oplus I_\mu \oplus I_r \oplus I_{e+f})\\ &\quad = (I_n + A_{1,k}^\top E^\top)Z_2(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k} \oplus \Gamma_{1,k})\\ &\qquad + (I_n + A_{1,k}^\top E^\top)Z_4(I_s \oplus I_\mu \oplus J_\omega^{2^k} \oplus \widehat{J}_1^{2^k}),\end{aligned} \tag{4.20b}$$

where

$$\begin{aligned}\Gamma_{\omega,k} &= \bigoplus_{j=1}^{l} K_{\omega_j,2m_j}^{2^k}(1:m_j, m_{j+1}:2m_j),\\ \Gamma_{1,k} &= K_{1,2f+1}^{2^k}(1:f+1, f+2:2f+1) \oplus 0_{e-1,e},\\ J_\omega^{2^k} &= \bigoplus_{j=1}^{l} K_{\omega_j,m_j}^{2^k}, \qquad J_1^{2^k} = K_{1,f+1}^{2^k} \oplus I_{e-1}, \widehat{J}_1^{2^k} = K_{1,f}^{2^k} \oplus I_e.\end{aligned}$$

Define

$$\widehat{\Gamma}_{1,k} := K_{1,2f+1}^{2^k}(1:f, f+2:2f+1) \oplus 0_{e,e}$$

for $k = 0, 1, 2, \ldots$ Then, from (4.1), we have that

$$\Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+} = \begin{bmatrix} I_f & 0_{f,e} \\ \zeta_k & 0_e \end{bmatrix} \qquad \text{with} \quad \zeta_k = \left[\begin{array}{c} 2^{-fk}, \cdots, 2^{-k} \\ \hline 0_{e-1,f} \end{array}\right].$$

Consequently, from (4.3) in Lemma 4.1, we find that

$$\left(I_n - \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+}\right) J_1^{2^k} = \begin{bmatrix} 0_f & 0_{f,e} \\ \zeta_k & I_e \end{bmatrix} \begin{bmatrix} K_{1,f}^{2^k} & \zeta_k^{+} \\ 0_{e,f} & I_e \end{bmatrix} = \begin{bmatrix} 0_f & 0_{f,e} \\ \zeta_k & I_e \end{bmatrix} \tag{4.21}$$

and

$$\begin{aligned}\left\|\widehat{J}_1^{2^k}\widehat{\Gamma}_{1,k}^{+}J_1^{2^k}\right\| &= \left\|\left(K_{1,f}^{2^k} \oplus I_e\right)\widehat{\Gamma}_{1,k}^{+} \begin{bmatrix} K_{1,f}^{2^k} & \zeta_k^{+} \\ 0_{e,f} & I_e \end{bmatrix}\right\|\\ &= \left\|\left(K_{1,f}^{2^k} \oplus I_e\right)\widehat{\Gamma}_{1,k}^{+}\left(K_{1,f}^{2^k} \oplus I_e\right) + \begin{bmatrix} 2^{-k} \\ \vdots \\ 2^{-fk} \\ 0_{e,1} \end{bmatrix} e_{f+1}^\top\right\| = \mathcal{O}(2^{-k}) \to 0,\end{aligned} \tag{4.22}$$

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as $k \to \infty$. Postmultiplying (4.20b) by $\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)$ yields

$$
\begin{aligned}
&(-H_{1,k} E Z_3 + Z_4)\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\
&\quad = (I_n + A_{1,k}^\top E^\top) Z_2 \left(0_s \oplus 0_\mu \oplus J_\omega^{2^k} \oplus \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\
&\qquad + (I_n + A_{1,k}^\top E^\top) Z_4 \left(0_s \oplus 0_\mu \oplus J_\omega^{2^k}\Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_1^{2^k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right).
\end{aligned}
\tag{4.23}
$$

From (4.20a), it follows that

$$
\begin{aligned}
&(I_n + A_{1,k}^\top E^\top) Z_2 \left(0_s \oplus 0_\mu \oplus J_\omega^{2^k} \oplus \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\
&\quad = -H_{1,k} E Z_1 + Z_2 - (I_n + A_{1,k}^\top E^\top)\left(J_s^{2^k} \oplus I_\mu \oplus 0_r \oplus (I - \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+}) J_1^{2^k}\right).
\end{aligned}
\tag{4.24}
$$

Substituting (4.24) into (4.23), we get

$$
\begin{aligned}
&- H_{1,k} E\left[Z_1 - Z_3\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\right] + Z_2\\
&\quad = Z_4\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\
&\qquad - (I_n + A_{1,k}^\top E^\top) Z_4 \left(0_s \oplus 0_\mu \oplus J_\omega^{2^k}\Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{J_1}^{2^k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\
&\qquad + (I_n + A_{1,k}^\top E^\top) Z_2 \left(J_s^{2^k} \oplus I_\mu \oplus 0_r \oplus \begin{bmatrix} 0 & 0 \\ \zeta_k & I_e \end{bmatrix}\right).
\end{aligned}
\tag{4.25}
$$

On the other hand, substituting $(\mathcal{L}_{1,k}, \mathcal{M}_{1,k})$ from (4.17) and $\mathcal{Y}$ from (4.15) into the second equation of (4.18), we have

$$
\left(I_n + A_{1,k}^\top E^\top\right) Y_2 = (-H_{1,k} E Y_1 + Y_2)\left(J_s^{2^k} \oplus I_\mu \oplus J_\omega^{2^k} \oplus J_1^{2^k}\right),
\tag{4.26a}
$$

$$
\begin{aligned}
&\left(I_n + A_{1,k}^\top E^\top\right) Y_4 \left(J_s^{2^k} \oplus I_\mu \oplus I_r \oplus I_{e+f}\right)\\
&\quad = (-H_{1,k} E Y_1 + Y_2)(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k} \oplus \Gamma_{1,k})\\
&\qquad + (-H_{1,k} E Y_3 + Y_4)\left(I_s \oplus I_\mu \oplus J_\omega^{2^k} \oplus \widehat{J_1}^{2^k}\right).
\end{aligned}
\tag{4.26b}
$$

As above, postmultiplying (4.26b) by $\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)$ and using (4.26a), we get

$$
\begin{aligned}
&\left(I_n + A_{1,k}^\top E^\top\right)\left[Y_2 - Y_4\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_1^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\right]\\
&\quad = (-H_{1,k} E Y_1 + Y_2)\left[J_s^{2^k} \oplus I_\mu \oplus 0_r \oplus \begin{bmatrix} 0 & 0 \\ \zeta_k & I_e \end{bmatrix}\right]\\
&\qquad + (-H_{1,k} E Y_3 + Y_4)\left(0_s \oplus 0_\mu \oplus J_\omega^{2^k}\Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{J_1}^{2^k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right).
\end{aligned}
\tag{4.27}
$$

Then from (4.16), (4.21)–(4.22), and Lemma 4.1, (4.25) can be simplified by

$$
\begin{aligned}
&H_{1,k} E Z_1 \left(I_n + \mathcal{O}(2^{-k})\right)\\
&\quad = -Z_2 + \mathcal{O}(2^{-k}) + \left(I_n + A_{1,k}^\top E^\top\right)\left(\mathcal{O}(\rho(J_s^{2^k})) + \mathcal{O}(2^{-k})\right),
\end{aligned}
\tag{4.28}
$$

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as k is sufficiently large. Since Z_1 is invertible, substituting $H_{1k}E$ in (4.28) into (4.27), we conclude that

$$\left(I_n + A_{1,k}^\top E^\top\right)\left(Y_2 + \mathcal{O}(\rho(J_s^{2^k})) + \mathcal{O}(2^{-k})\right) = \mathcal{O}(1).$$

Since Y_2 is invertible, it holds that $\|I_n + A_{1,k}^\top E^\top\| \le \mathcal{O}(1)$ for all k. Again from (4.16), (4.21)–(4.22), and (4.3), it follows that

$$\left\|H_{1,k}E - Z_2Z_1^{-1}\right\| \le \mathcal{O}(\rho(J_s^{2^k})) + \mathcal{O}(2^{-k}),$$

as $k \to \infty$. $\square$

REMARK 4.8. In Theorem 4.7, we assume that the sequence $\{A_{1,k}, G_{1,k}, H_{1,k}\}$ is well-defined (or the SDA does not break down). How to guarantee the existence of the sequence is still an open problem and is under investigation.

5. Numerical results. In this section, we test the Structure-preserving Algorithm (SA) (Algorithm 3.3) for the GARE (1.1) on two numerical examples of [12, 13] and [16] under Assumptions 4.2 or 4.5 to illustrate the convergence behavior. All computations were performed in MATLAB R2008a on a PC with IEEE double-precision floating-point arithmetic (eps$\approx 2.22\times 10^{-16}$).

EXAMPLE 5.1 ([12, 13]). Given

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -9 & -6 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},\ B = \begin{bmatrix} 0 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\\ 0 & -1\end{bmatrix},\ E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & -2 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},\ D = \begin{bmatrix} 0 & 0\\ 0 & 2\\ 0 & 0\end{bmatrix},\ J = \begin{bmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\ J' = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}.$$

The Kronecker structure of $(\mathcal{H}_a, \mathcal{E}_a)$ is

$$\left(\begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix} \oplus \begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix} \oplus \begin{bmatrix} \pm 1.414i & 1\\ 0 & \pm 1.414i\end{bmatrix} \oplus I_8,\ I_8 \oplus 0_5 \oplus \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\right).$$

We choose $\gamma = 9$ to transform $(\mathcal{H}_a, \mathcal{E}_a)$ to $(\mathcal{M}, \mathcal{L})$ as in (2.4). More details on finding a parameter γ by a Fibonacci sequence so that the condition numbers of A_γ and W_γ in (2.2) are as small as possible can be found in [6].

The corresponding Kronecker structure of $(\mathcal{M}, \mathcal{L})$ becomes

$$\left(\begin{bmatrix} -1.25 & 0\\ 0 & -0.8\end{bmatrix} \oplus \begin{bmatrix} -1 & 1\\ 0 & -1\end{bmatrix} \oplus \begin{bmatrix} z & 1\\ 0 & z\end{bmatrix} \oplus I_5 \oplus \begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix},\ I_{16}\right),$$

with $z = -0.952 \pm 0.3067i$. The related quantities in (4.12) are given by $n = 6$, $m = 2$, $e \equiv \text{nullity}(E) = 1$, $f = 1$, $s = 1$, $r = 3$, $\mu = 0$. We compute $\eta_1 = \eta_2 = 3$, $\eta_0 = 1$, and $\eta_0 + \eta_1 = e + m + f = m' = 4$, which coincides with case (i) in Theorem 3.5. We verify that Assumption 4.2 holds as

$$\text{rank}\begin{bmatrix} A & B\\ C & D\end{bmatrix} = 7 < n + m - \mu = 8, \qquad \text{nullity}\begin{bmatrix} AV_0 & B & E\\ CV_0 & D & 0\end{bmatrix} = e + 1 = 2.$$

The SDA (Algorithm 3.2) converges to $X_1 \equiv H_{1,9}E$ in 9 iterations. Using Algorithm 3.3, we get

$$X_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8.0 & 0 & 4.0 & -1.0\text{e-}7 & 0 & 0 \\ 0 & 0 & 4.0 & 0 & 2.0 & -5.0\text{e-}8 & 0 & 0 \\ 0 & 0 & -1.0\text{e-}7 & 0 & -5.0\text{e-}8 & -1.5\text{e-}7 & 0 & 0 \\ 0 & -0.43 & -1.70 & -0.64 & -0.64 & 0.11 & 0 & -1.38 \\ 0 & 0.50 & -1.74 & -1.12 & -1.12 & 0.93 & 1 & -3.12 \\ 0 & -1.12 & -0.12 & 0.50 & 0.50 & -0.80 & 0 & 0.81 \end{bmatrix}$$

satisfying

$$\begin{aligned} &\|E_a^\top X_a - X_a^\top E_a\|_2 = 1.47 \times 10^{-15}, \\ &\text{Res} \equiv \|A_a^\top X_a + X_a^\top A_a + H_a - X_a^\top G_a X_a\|_2 = 4.71 \times 10^{-14}, \\ &\text{Rel\_Res} \equiv \text{Res}/\left(2\|A_a^\top X_a\|_2 + \|X_a^\top G_a X_a\|_2 + \|H_a\|_2\right) = 9.12 \times 10^{-16}. \end{aligned}$$

EXAMPLE 5.2 ([16]). Given

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\top,$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} -1 & -1 & 0.005 & -0.005 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.005 & -0.005 & 0 & 0 & 0 & 0 \\ -0.001 & 0 & 0 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 & 0.1 & -0.2 & -0.2 \end{bmatrix},$$

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ J' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Kronecker structure of $(\mathcal{H}_a, \mathcal{E}_a)$ is

$$\left(0_2 \oplus \begin{bmatrix} \pm 1 & & & \\ & \pm 4.998 & & \\ & & \pm 3.79 & \\ & & & \pm 0.211 \end{bmatrix} \oplus I_{10}, \ I_{10} \oplus 0_7 \oplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right).$$

We choose $\gamma = 9$ to transform $(\mathcal{H}_a, \mathcal{E}_a)$ to $(\mathcal{M}, \mathcal{L})$. The corresponding Kronecker structure

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of $(\mathcal{M}, \mathcal{L})$ becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} -0.8 & & & \\ & -0.407 & & \\ & & -0.954 & \\ & & & 0.996 \end{bmatrix} \oplus \begin{bmatrix} -1.25 & & & \\ & -2.45 & & \\ & & -1.05 & \\ & & & 1.004 \end{bmatrix} \oplus I_7 \oplus \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I_{20} \right).$$

The related quantities in (4.12) are given by $n = 8$, $m = 2$, $e \equiv \text{nullity}(E) = 2$, $f = 1$, $s = 4$, $r = 0$, $\mu = 1$. We compute $\eta_1 = \eta_2 = 4$, $\eta_0 = 1$, and $\eta_0 + \eta_1 = e + m + f = m' = 5$, which coincides with case (i) in Theorem 3.5. To verify Assumption 4.2, we calculate

$$\text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 9 < n + m - \mu = 10, \qquad \text{nullity} \begin{bmatrix} AV_0 & B & E \\ CV_0 & D & 0 \end{bmatrix} = e + 1 = 3.$$

The SDA (Algorithm 3.2) converges to $X_1 \equiv H_{1,16}E$ in 16 iterations. Then, using Algorithm 3.3, we get

$$X_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & -4.9\text{e-}4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.65 & -1.2 & 0.034 & -0.52 & -7.1\text{e-}4 & 0 & 0.037 & -0.038 & -1.24 & 0.72 \\ 0.26 & -0.17 & 0.16 & -0.58 & -3.4\text{e-}3 & 0 & 0.88 & -0.88 & -0.17 & -0.42 \\ 0 & 0 & 0.5 & 0.025 & -0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.47 & 0.025 & -0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.79 & 1.1 & -0.14 & 1.6 & 2.9\text{e-}3 & 0 & -0.28 & 0.27 & 0.15 & 0.51 \\ -0.37 & -1.3 & -0.14 & -2.0 & 2.9\text{e-}3 & 0 & -0.031 & 0.031 & -1.3 & 0.35 \end{bmatrix},$$

which satisfies

$$\|E_a^\top X_a - X_a^\top E_a\|_2 = 1.37 \times 10^{-16}, \quad \text{Res} = 5.09 \times 10^{-14}, \quad \text{Rel\_Res} = 2.99 \times 10^{-15}.$$

6. Conclusions. In this paper, we propose a structure-preserving algorithm (SDA+post-processing procedure) for a semi-stabilizing solution for the GARE (1.1). Under Assumptions 4.2 or 4.5, in Theorem 4.7 we prove that the SDA algorithm converges globally and linearly provided that it does not break down. The advantage of the SDA algorithm is evident in that the E-symmetric solution $X_1 \equiv H_{1,\infty}E$ with $H_{1,\infty}$ being symmetric is obtained by a structure-preserving doubling iterative process without performing any preprocessing for deflating the associated unimodular eigenvalues. The normalized residuals of the desired E_a-symmetric solution X_a for the tested examples computed by the structure-preserving algorithm are accurate to machine precision.

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