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INVERSION OF CENTROSYMMETRIC TOEPLITZ-PLUS-HANKEL BEZOUTIANS*

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Dedicated to Lothar Reichel on the occasion of his 60th birthday

Abstract. In this paper we discuss how to compute the inverse of a nonsingular, centrosymmetric Toeplitz-plus-Hankel Bezoutian B of order n and how to find a representation of B^{-1} as a sum of a Toeplitz and a Hankel matrix. Besides the known splitting property of B as a sum of two split-Bezoutians, the connection of the latter to Hankel Bezoutians of about half size is used. The fast inversion of the Hankel Bezoutians together with an inversion formula, which was the subject of a previous paper, leads us to an inversion formula for B^{-1} as a Toeplitz-plus-Hankel matrix. It also enables us to design an $O(n^2)$ inversion algorithm.

Key words. Bezoutian matrix, Toeplitz matrix, Hankel matrix, Toeplitz-plus-Hankel matrix, matrix inversion

AMS subject classifications. 15A09, 15B05, 65F05

1. Introduction. The present paper can be considered a continuation of [3] and [4]. In particular, in [4] we discussed how to invert a Hankel Bezoutian, and we presented a corresponding fast algorithm. Here, we will use these results for the inversion of a centrosymmetric Toeplitz-plus-Hankel Bezoutian (shortly, $T + H$ Bezoutian). A motivation to deal with the inversion of Bezoutians was recently given by Böttcher and Halwass; see [2].

Historically, Bezoutians were introduced in connection with elimination theory; see [19]. Much later, their importance for the inversion of Hankel and Toeplitz matrices was discovered by Lander [14]. In particular, he observed that the inverse of a nonsingular Hankel (Toeplitz) matrix is a Hankel (Toeplitz) Bezoutian and vice versa. A large amount of literature devoted to the inversion of Toeplitz and Hankel matrices has appeared. The starting points were the papers of Trench [18] and Gohberg and Semencul [6]. Later, in [9], it was discovered that the inverse of a nonsingular matrix which is the sum of a Toeplitz and a Hankel matrix (briefly $T + H$ matrix) possesses a generalized Bezoutian structure. Matrices with such a Bezoutian structure are referred to as $T + H$ Bezoutians. There is a number of papers dealing with the inversion of $T + H$ matrices; see, e.g., [5, 11, 12, 15, 16] and references therein.

The converse problem—the inversion of Bezoutians—has been given short shrift up to now. In [8, Part I, Subsection 3.8], the inverse of a Hankel Bezoutian was computed but only in the strongly nonsingular case. The motivation there was that such a procedure is of importance for solving matrix equations of Lyapunov-type. In [7] (see also [13]), a formula for the inverse of a Hankel or a Toeplitz Bezoutian was presented in the language of matrices generated by rational functions. A general approach to the inversion problem for Hankel or Toeplitz Bezoutians was given in [4].

As far as we are aware of, the question of how to obtain fast inversion algorithms or representations for inverses of $T + H$ Bezoutians is discussed here for the first time. We assume that the matrices under considerations possess an additional symmetry. The reason behind this is that on the one hand such additional symmetries of structured matrices are not unusual, but on the other hand this symmetry allows a splitting. The aim for the future is the

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fast inversion of general $T + H$ Bezoutians, but this requires new ideas and seems to be a big challenge.

We consider the setting where the entries of the matrices are taken from an arbitrary field $\mathbb{F}$ of characteristic different from two.

Let us now sketch the main ideas of the paper. $T + H$ Bezoutians are matrices of the form $B = [\, b_{ij}\,]_{i,j=0}^{n-1}$ for which there exist eight polynomials $\mathbf{g}_i$, $\mathbf{f}_i$ $(1 \le i \le 4)$ such that in polynomial language

$$(t-s)(1-ts)\sum_{i,j=0}^{n-1} b_{ij}t^i s^j = \sum_{i=1}^{4} \mathbf{g}_i(t)\mathbf{f}_i(s).$$

As already indicated above, $T + H$ Bezoutians arise as the inverses of nonsingular $T + H$ matrices, i.e., matrices of the form $[a_{i-j} + s_{i+j}]_{j,k=0}^{n-1}$. Conversely, the inverses of nonsingular $T + H$ Bezoutians are $T + H$ matrices [9].

In the present paper, we restrict ourselves to centrosymmetric $T + H$ Bezoutians whose inverses are centrosymmetric $T + H$ matrices. Recall that a matrix B is called centrosymmetric if $B = J_n B J_n$, where J_n is the flip matrix

$$J_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}. \tag{1.1}$$

Thus, given a nonsingular, centrosymmetric $T + H$ Bezoutian B, our aim is to compute the Toeplitz and Hankel parameters $\{a_i\}$ and $\{s_i\}$ of its inverse, a $T + H$ matrix represented by

$$B^{-1} = [\, a_{i-j} + s_{i+j}\,]_{i,j=0}^{n-1}\,.$$

This task is accomplished in several steps. Our starting point is the fact that the centrosymmetry of B leads to a splitting of the form $B = B_{++} + B_{--}$, where $B_{\pm\pm}$ are special $T + H$ Bezoutians, which are called split-Bezoutians of $(\pm)$ type. This splitting was discovered in [10, Section 8] (see also [12]) and arises from the property that both the space of all symmetric vectors ($\mathbf{x} = J_n\mathbf{x}$) and the space of all skewsymmetric vectors ($\mathbf{x} = -J_n\mathbf{x}$) are invariant subspaces of the centrosymmetric matrix B.

The second step consists in relating B_{++} and B_{--} to Hankel Bezoutians. In the case the matrix B being of odd order, say $n = 2\ell - 1$, we use a result of [12] to transform B_{++} into a nonsingular Hankel Bezoutian of order ℓ. Similarly, the matrix B_{--} can be transformed into a Hankel Bezoutian of size $\ell - 1$. In summary, we arrive at a representation of the form

$$B = W \begin{bmatrix} B_H^{(1)} & \mathbf{0} \\ \mathbf{0} & B_H^{(2)} \end{bmatrix} W^T\,,$$

where W is a certain explicit transformation (involving triangular matrices) and $B_H^{(1)}, B_H^{(2)}$ are the mentioned Hankel Bezoutians. A similar representation is derived in the case of even matrix order, $n = 2\ell$, with both Hankel Bezoutians of size ℓ.

Now we are in a position to use the formulas and algorithms established in [4] to compute the inverse of the Hankel Bezoutians, which are Hankel matrices. Consequently, the following structure of the inverse of the $T + H$ Bezoutian B is obtained,

$$B^{-1} = W^{-T} \begin{bmatrix} H_1 & \mathbf{0} \\ \mathbf{0} & H_2 \end{bmatrix} W^{-1}\,,$$

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where H_1, H_2 are Hankel matrices, the parameters of which are given by the solution of corresponding Bezout equations; see [4]. It remains to discover the Toeplitz-plus-Hankel structure behind this representation. In other words, we want to find a Toeplitz matrix T and a Hankel matrix H such that

$$B^{-1} = T + H\,.$$

This goal can be achieved utilizing finite versions of results given in [1]. These results are formulas between Hankel matrices and four kinds of particular symmetric $T+H$ matrices. It is perhaps interesting to note that these four types of matrices are related to the above Hankel matrices H_1 and H_2, where one has to distinguish between the case of even and odd order.

The paper is organized as follows. Starting with some preliminaries in Section 2, in Section 3, basic observations on $T + H$ matrices are made, which are useful to understand the final result and the structure of the formulas encountered. The issue that the symbols of $T + H$ matrices are not uniquely determined (since there are nonzero matrices which are both Toeplitz and Hankel) is also discussed.

In Section 4, known, but for us important, results on Toeplitz-, Hankel-, and $T + H$ Bezoutian are recalled. In particular, an answer is given to questions such as how to determine whether a matrix is a $T + H$ Bezoutian and what is specific if this matrix is centrosymmetric. Here also the structure of the splitting matrices $B_{\pm\pm}$ is investigated. In Section 5, we explicitly compute the inverses of certain triangular matrices which occur in our formulas. Section 6 highlights the connection between split-Bezoutians of $(+)$ type (of odd order) and Hankel Bezoutians. Inversion formulas for centrosymmetric $T + H$ Bezoutian of odd and even order are proved in Section 7 and Section 8, respectively.

In Section 9, we reinterpret the representations given for the inverses in the previous sections as a sum of a Toeplitz and a Hankel matrix and discuss how the corresponding parameters defining these matrices can be computed. Here the results of [1] are used.

Section 10 is not only meant as a summary but even more. Here we design an algorithm for the computation of the inverse of a centrosymmetric $T + H$ Bezoutian B of order n. The parameters in the $T + H$ matrix which occur in the representation of B^{-1} obtained in the previous section can be computed with $O(n^2)$ operations (additions and multiplications). Note that, as pointed out in Section 3, these parameters are not unique.

In Section 11 we present alternative representations of B^{-1} as a $T+H$ matrix. These representation are slightly more complicated, however, the issue of nonuniqueness is resolved. For matrix-vector multiplications they are equally suitable and have the advantage that the parameters are unique.

Besides the inversion of centrosymmetric $T + H$ Bezoutians, there is the related interesting problem of the inversion of centroskewsymmetric $T + H$ Bezoutians. Though similar, it is not completely analogous to the centrosymmetric case and will be the subject of a forthcoming paper.

2. Preliminaries. Throughout this paper, we consider vectors or matrices, the entries of which are taken from a field $\mathbb{F}$ with a characteristic not equal to 2. By $\mathbb{F}^n$ we denote the linear space of all vectors of length n, by $\mathbb{F}^{m\times n}$ the linear space of all $m \times n$ matrices, and I_n denotes the identity matrix in $\mathbb{F}^{n\times n}$.

We will often use polynomial language. We denote by $\mathbb{F}^n[t]$ the linear space of all polynomials in t of degree less than n, the coefficients of which are in $\mathbb{F}$. To each vector $\mathbf{x} = (x_j)_{j=0}^{n-1} \in \mathbb{F}^n$, we associate the polynomial

$$\mathbf{x}(t) := \boldsymbol{l}_n(t)^T\mathbf{x} = \sum_{j=0}^{n-1} x_j t^j \in \mathbb{F}^n[t],$$

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where

$$\boldsymbol{l}_n(t) := \left(1, t, t^2, \dots, t^{n-1}\right)^T . \tag{2.1}$$

Moreover, we associate to a matrix $A = [\, a_{ij} \,]_{i,j=0}^{n-1}$ the bivariate polynomial

$$A(t,s) = \boldsymbol{l}_n(t)^T A\, \boldsymbol{l}_n(s) = \sum_{i,j=0}^{n-1} a_{ij}\, t^i s^j$$

and call it the *generating polynomial of* A.

For a vector $\mathbf{x} \in \mathbb{F}^n$, we write

$$\mathbf{x}^J := J_n \mathbf{x} \,,$$

where J_n is introduced in (1.1). In polynomial language this means

$$\mathbf{x}^J(t) = \mathbf{x}(t^{-1}) t^{n-1} .$$

With this notation, a vector $\mathbf{x} \in \mathbb{F}^n$ is said to be *symmetric* if $\mathbf{x} = \mathbf{x}^J$ and *skewsymmetric* if $\mathbf{x} = -\mathbf{x}^J$. The matrices

$$P_\pm := \tfrac{1}{2}(I_n \pm J_n)$$

are the projections from $\mathbb{F}^n$ onto the subspaces $\mathbb{F}^n_\pm$ consisting of all symmetric, respectively skewsymmetric vectors, i.e.,

$$\mathbb{F}^n_\pm := \left\{\, \mathbf{x} \in \mathbb{F} \;:\; \mathbf{x}^J = \pm \mathbf{x} \,\right\} .$$

An $n \times n$ matrix A is called *centrosymmetric* if $A = J_n A J_n$. It is easy to see that a centrosymmetric matrix A maps $\mathbb{F}^n_\pm$ to $\mathbb{F}^n_\pm$, i.e., $AP_\pm = P_\pm A P_\pm$.

The various spaces $\mathbb{F}^n_\pm$ for n even or odd are related to each other. This can be most easily expressed in polynomial language. In fact, we have

$$\begin{aligned} \mathbb{F}^{2\ell}_+[t] &= \left\{\, (t+1)\,\mathbf{x}(t) \;:\; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_+[t] \,\right\}, \\ \mathbb{F}^{2\ell}_-[t] &= \left\{\, (t-1)\,\mathbf{x}(t) \;:\; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_+[t] \,\right\}, \\ \mathbb{F}^{2\ell+1}_-[t] &= \left\{\, (t^2-1)\,\mathbf{x}(t) \;:\; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_+[t] \,\right\}. \end{aligned} \tag{2.2}$$

These basic observations, which will be of importance in Lemma 4.7, Theorem 7.1, and Theorem 8.1, can be seen as follows. Let $\mathbf{x}_\pm \in \mathbb{F}^n_\pm$. Then in case n is even, we have $\mathbf{x}_-(1) = \mathbf{x}_+(-1) = 0$, while in case n is odd, we have $\mathbf{x}_-(1) = \mathbf{x}_-(-1) = 0$.

3. Basics on Toeplitz-plus-Hankel matrices. Let us first introduce Toeplitz matrices and Hankel matrices. To a given vector $\mathbf{a} = (a_i)_{i=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ we associate the $n \times n$ *Toeplitz matrix*

$$T_n(\mathbf{a}) = [a_{i-j}]_{i,j=0}^{n-1} ,$$

and to a vector $\mathbf{s} = (s_i)_{i=0}^{2n-2} \in \mathbb{F}^{2n-1}$ we associate the $n \times n$ *Hankel matrix*

$$H_n(\mathbf{s}) = [s_{i+j}]_{i,j=0}^{n-1} .$$

The vectors $\mathbf{a}$ and $\mathbf{s}$ are called the *symbol* of $T_n(\mathbf{a})$ and $H_n(\mathbf{s})$, respectively. For the symbol of the Hankel matrices, occasionally a different indexing will be useful. Note that a Hankel

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matrix multiplied (from the left or from the right) by the flip matrix J_n introduced in (1.1) is a Toeplitz matrix.

A matrix which is the sum of a Toeplitz and a Hankel matrix is called *Toeplitz-plus-Hankel matrix*, shortly $T+H$ matrix. For such matrices it is convenient to adopt the following notation. For $n = 2\ell - 1$ we write

$$T_n(\mathbf{a}) + H_n(\mathbf{s}) = [a_{i-j} + s_{i+j}]_{i,j=-\ell+1}^{\ell-1}\,, \tag{3.1}$$

while for $n = 2\ell$ we write

$$T_n(\mathbf{a}) + H_n(\mathbf{s}) = [a_{i-j} + s_{i+j+1}]_{i,j=-\ell}^{\ell-1}\,. \tag{3.2}$$

Alternatively, we could also write (in both cases)

$$T_n(\mathbf{a}) + H_n(\mathbf{s}) = [a_{i-j} + s_{i+j-n+1}]_{i,j=0}^{n-1}\,.$$

Therein, we set $\mathbf{s} = (s_i)_{i=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ in slight contrast to the above definition.

REMARK 3.1. Hankel matrices $H_n(\mathbf{s})$ are symmetric, and Toeplitz matrices $T_n(\mathbf{a})$ are persymmetric, i.e., $T_n(\mathbf{a})^T = J_n T_n(\mathbf{a}) J_n$. Thus, a Toeplitz matrix is symmetric if and only if it is centrosymmetric, while a Hankel matrix is persymmetric if and only if it centrosymmetric. Consequently, a $T+H$ matrix is symmetric if and only if its Toeplitz part is symmetric. A $T+H$ matrix is persymmetric if and only if its Hankel part is persymmetric.

A centrosymmetric $T+H$ matrix is always symmetric and persymmetric. Indeed, the centrosymmetry of $T_n(\mathbf{a}) + H_n(\mathbf{s})$ implies

$$T_n(\mathbf{a}) + H_n(\mathbf{s}) = T_n(\mathbf{a}^J) + H_n(\mathbf{s}^J).$$

Taking the transpose yields

$$T_n(\mathbf{a}^J) + H_n(\mathbf{s}) = T_n(\mathbf{a}) + H_n(\mathbf{s}^J).$$

Adding and subtracting these two equations and dividing by 2 implies $T_n(\mathbf{a}) = T_n(\mathbf{a}^J)$ and $H_n(\mathbf{s}) = H_n(\mathbf{s}^J)$. From here the symmetry and persymmetry of $T_n(\mathbf{a})$ and $H_n(\mathbf{s})$ and thus of the sum follows.

Let us continue with a few basic observations about $T+H$ matrices, which are motivated by our aim to construct $T+H$ matrices as inverses of $T+H$ Bezoutians. As we will see soon, a $T+H$ matrix given by (3.1) or (3.2) does not uniquely determine its symbols $\mathbf{a}$ and $\mathbf{s}$. Since this nonuniqueness issue will naturally reoccur in our construction, it is convenient to clarify the relationship between symbols and the matrix now. We first consider the general case, then the centrosymmetric case, and finally more specific cases that will also be encountered.

A general $T+H$ matrix of size n involves $4n-2$ parameters. However, matrices of "checkerboard pattern" are both Hankel and Toeplitz. Hence, the vectors $\mathbf{a}, \mathbf{s} \in \mathbb{F}^{2n-1}$ are not uniquely determined in the matrix $T_n(\mathbf{a}) + H_n(\mathbf{s})$. In fact, the linear space of all $T+H$ matrices of size n has dimension $4n-4$.

For centrosymmetric $T_n(\mathbf{a}) + H_n(\mathbf{s})$, the considerations in Remark 3.1 imply that $\mathbf{a} = \mathbf{a}^J$ and $\mathbf{s} = \mathbf{s}^J$. In other words, $\mathbf{a}, \mathbf{s} \in \mathbb{F}_+^{2n-1}$, and this is also sufficient for centrosymmetry. Thus, $2n$ parameters are involved. However, for the same reason as above, the linear space of all centrosymmetric $T+H$ matrices of size n has dimension $2n-2$.

We continue with a simple general observation; see [12, Lemma 5.1].

REMARK 3.2. Each centrosymmetric matrix A can be uniquely written as the sum of two (centrosymmetric) matrices $A = A_+ + A_-$ which possess the additional symmetries

$$A_\pm P_\pm = P_\pm A_\pm = \pm A_\pm. \tag{3.3}$$

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In fact, we can put $A_\pm := AP_\pm = P_\pm A$. In other words, all rows and columns of A_+ are symmetric vectors whereas all rows and columns of A_- are skewsymmetric vectors. Furthermore,

$$\operatorname{rank} A = \operatorname{rank} A_+ + \operatorname{rank} A_- \,.$$

Applying the previous remark to a centrosymmetric $T+H$ matrix A, it follows that A_+ and A_- are (centrosymmetric) $T+H$ matrices as well. What (3.3) means for $T+H$ matrices in terms of the symbol is stated next. We use the notation $(\pm)$ to distinguish the two cases.

PROPOSITION 3.3. *Let $A_\pm \in \mathbb{F}^{n\times n}$ be a $T+H$ matrix satisfying* (3.3). *Then there exists a vector $\mathbf{a}^{(\pm)} \in \mathbb{F}_+^{2n-1}$ such that*

$$A_\pm = T_n(\mathbf{a}^{(\pm)}) \pm H_n(\mathbf{a}^{(\pm)}). \tag{3.4}$$

Proof. Let $A_\pm = T_n(\mathbf{a}) + H_n(\mathbf{s})$ with $\mathbf{a}, \mathbf{s} \in \mathbb{F}_+^{2n+1}$. Then $A_\pm = \pm A_\pm P_\pm$ implies that

$$A_\pm = \pm\tfrac{1}{2}\left(T_n(\mathbf{a}\pm\mathbf{s}) \pm H_n(\mathbf{a}\pm\mathbf{s})\right).$$

Put $\mathbf{a}^{(\pm)} := \pm\frac{1}{2}(\mathbf{a}\pm\mathbf{s}) \in \mathbb{F}_+^{2n-1}$, and (3.4) follows. ∎

As mentioned above, the Toeplitz and Hankel symbols of a $T+H$ matrix are not uniquely determined. Notice that the proposition does not claim that every representation of $A_\pm$ as a sum of a Toeplitz and Hankel matrix is of the form (3.4). In fact, this is easily seen to be false in general.

Moreover, even if we restrict ourselves to representations (3.4), the vectors $\mathbf{a}^{(\pm)}$ need not be unique. In fact, assume $T_n(\mathbf{a}^{(\pm)}) \pm H_n(\mathbf{a}^{(\pm)}) = 0$ with $\mathbf{a}^{(\pm)} \in \mathbb{F}_+^{2n-1}$, and introduce

$$\mathbf{e}_{\alpha,\beta} := (\alpha, \beta, \alpha, \ldots, \beta, \alpha)^T \in \mathbb{F}_+^{2n-1}.$$

Then

$$\mathbf{a}^{(+)} = \begin{cases} 0 & \text{if } n \text{ odd,} \\ \mathbf{e}_{\alpha,-\alpha} & \text{if } n \text{ even,} \end{cases} \qquad \mathbf{a}^{(-)} = \begin{cases} \mathbf{e}_{\alpha,\beta} & \text{if } n \text{ odd,} \\ \mathbf{e}_{\alpha,\alpha} & \text{if } n \text{ even.} \end{cases}$$

For the dimension $d_n^\pm$ of the linear space of all $T+H$ matrices of order n satisfying (3.3), we obtain that $d_n^+ = n$ for n odd and $d_n^- = n-2$ for n odd, while $d_n^\pm = n-1$ for n even. Notice that $d_n^+ + d_n^- = 2n-2$ as observed earlier.

4. Bezoutians.

4.1. Displacement transformations. In order to define Hankel and Toeplitz Bezoutians, we use transformations ∇_H and ∇_T which transform a matrix $B = [b_{ij}]_{i,j=0}^{n-1} \in \mathbb{F}^{n\times n}$ into a matrix of $\mathbb{F}^{(n+1)\times(n+1)}$ according to the rule

$$\nabla_H(B) = [\, b_{i-1,j} - b_{i,j-1}\,]_{i,j=0}^{n} \,,$$
$$\nabla_T(B) = [\, b_{ij} - b_{i-1,j-1}\,]_{i,j=0}^{n} \,.$$

Here we put $b_{ij} = 0$ if i or j is not in the set $\{0, 1, \ldots, n-1\}$. In polynomial language, these transformations are given by

$$(\nabla_H(B))\,(t,s) = (t-s)B(t,s)\,,$$
$$(\nabla_T(B))\,(t,s) = (1-ts)B(t,s)\,.$$

For the definition of Toeplitz-plus-Hankel Bezoutians, we need the transformation

$$\nabla_{T+H} : \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}^{(n+2)\times(n+2)},$$

which sends a matrix $B = [b_{ij}]_{i,j=0}^{n-1}$ into

$$\nabla_{T+H}(B) = [\, b_{i-1,j} + b_{i-1,j-2} - b_{i,j-1} - b_{i-2,j-1}\,]_{i,j=0}^{n+1}.$$

Again, we put $b_{ij} = 0$ if i or j is not in the set $\{0, 1, \dots, n-1\}$. Notice that

$$(\nabla_{T+H}(B))\,(t,s) = (t-s)(1-ts)B(t,s)$$

in polynomial language. Moreover,

$$\nabla_{T+H}(B) = \nabla_T(\nabla_H(B)) = \nabla_H(\nabla_T(B)).$$

4.2. *H* Bezoutians and *T* Bezoutians. The *Hankel Bezoutian* (briefly H Bezoutian) of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{n+1}$ is, by definition, the $n \times n$ matrix $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$ with the generating polynomial

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s)}{t-s}.$$

Clearly, in case $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, B is the zero matrix.

PROPOSITION 4.1. *A nonzero matrix $B \in \mathbb{F}^{n\times n}$ is an H Bezoutian if and only if B is symmetric and*

$$\operatorname{rank} \nabla_H(B) = 2.$$

In this case there exists a rank decomposition of the form

$$\nabla_H(B) = [\mathbf{u}, \mathbf{v}] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\mathbf{u}, \mathbf{v}]^T = \mathbf{u}\mathbf{v}^T - \mathbf{v}\mathbf{u}^T,$$

with linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{n+1}$ and $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$.

REMARK 4.2. Different pairs of (linearly independent) vectors may produce the same nonzero H Bezoutian. In fact,

$$\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = \mathrm{Bez}_H(\hat{\mathbf{u}}, \hat{\mathbf{v}})$$

if and only if there is a 2×2 matrix φ such that $[\,\hat{\mathbf{u}}\ \hat{\mathbf{v}}\,] = [\,\mathbf{u}\ \mathbf{v}\,]\,\varphi$ with $\det \varphi = 1$.

REMARK 4.3. It is well-known (see, e.g., [8]) that $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$ is nonsingular if and only if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are *generalized coprime*, which means that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime in the usual sense and that $\deg \mathbf{u}(t) = n$ or $\deg \mathbf{v}(t) = n$.

The following connection between Hankel matrices and H Bezoutians is a classical result discovered by Lander in 1974 [14].

THEOREM 4.4. *A nonsingular matrix is an H Bezoutian if and only if its inverse is a Hankel matrix.*

The following question arises: given the H Bezoutian B of the generalized coprime polynomials $\mathbf{u}(t), \mathbf{v}(t)$, how can we compute the symbol $\mathbf{s}$ of its inverse, a Hankel matrix with $H_n(\mathbf{s}) = B^{-1}$? The answer is given in [4].

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THEOREM 4.5. *Assume* $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}[t]$ *to be generalized coprime, and let* $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$. *Then the Bezout equations*

$$\mathbf{u}(t)\boldsymbol{\alpha}(t) + \mathbf{v}(t)\boldsymbol{\beta}(t) = 1 \tag{4.1}$$

$$\mathbf{u}^J(t)\boldsymbol{\gamma}^J(t) + \mathbf{v}^J(t)\boldsymbol{\delta}^J(t) = 1 \tag{4.2}$$

have unique solutions $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t), \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t) \in \mathbb{F}^n[t]$, *and*

$$B^{-1} = H_n(\mathbf{s}),$$

with $\mathbf{s} = (s_i)_{i=0}^{2n-2} \in \mathbb{F}^{2n-1}$ *given by*

$$\mathbf{s}^J(t) = -\boldsymbol{\alpha}(t)\boldsymbol{\delta}(t) + \boldsymbol{\beta}(t)\boldsymbol{\gamma}(t).$$

Further possibilities for the computation of $\mathbf{s}$ are discussed in [4]. For instance, it is found that it suffices to solve only one of the Bezout equations (4.1) or (4.2).

Analogous results can be obtained for *Toeplitz Bezoutian* (briefly T Bezoutians), which are defined as matrices $B = \mathrm{Bez}_T(\mathbf{u}, \mathbf{v})$ of order n with the generating polynomial

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}^J(s) - \mathbf{v}(t)\mathbf{u}^J(s)}{1 - ts}.$$

Here, $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{n+1}$. A nonzero matrix $B \in \mathbb{F}^{n\times n}$ is a T Bezoutian if and only if B is persymmetric and

$$\mathrm{rank}\, \nabla_T B = 2\,.$$

There is a simple relation between H- and T Bezoutians,

$$\mathrm{Bez}_T(\mathbf{u}, \mathbf{v}) = -\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) J_n\,.$$

Thus, to all results about H Bezoutians, there are corresponding results for T Bezoutians.

4.3. $T + H$ Bezoutians. A matrix $B \in \mathbb{F}^{n\times n}$ is called a *Toeplitz-plus-Hankel Bezoutian* (briefly $T + H$ Bezoutian) if

$$\mathrm{rank}\, \nabla_{T+H}(B) \le 4\,.$$

This condition is equivalent to the existence of eight polynomials (vectors) $\mathbf{g}_i(t)$, $\mathbf{f}_i(t)$ $(i = 1, 2, 3, 4)$ in $\mathbb{F}^{n+2}[t]$ such that

$$(t - s)(1 - ts)B(t,s) = \sum_{i=1}^{4} \mathbf{g}_i(t)\mathbf{f}_i(s)\,.$$

The vectors $\mathbf{g}_i, \mathbf{f}_i$ $(i = 1, 2, 3, 4)$ are not uniquely determined by B. However, two different choices are related to each other by a simple transformation; see [17].

Clearly, T Bezoutians as well as H Bezoutians are also $T + H$ Bezoutians. But the sum of a T- and a H Bezoutian is, in general, not a $T + H$ Bezoutian.

The following important relationship was proved in [9].

THEOREM 4.6. *A nonsingular matrix* B *is a* $T + H$ *Bezoutian if and only if its inverse* B^{-1} *is a sum of a Toeplitz and a Hankel matrix.*

If B is a matrix of order $n \ge 2$ and $\mathrm{rank}\, \nabla_{T+H}(B) < 4$, then the first and the last column or the first and the last row of B are linearly dependent. Hence, for $T + H$ Bezoutian $(n \ge 2)$ to be nonsingular, it is necessary that $\mathrm{rank}\, \nabla_{T+H}(B) = 4$; see [9].

A simple criterion for the nonsingularity of a $T + H$ Bezoutian in terms of the eight vectors $\mathbf{g}_i, \mathbf{f}_i$ $(i = 1, 2, 3, 4)$ has not yet been discovered.

4.4. Centrosymmetric $T+H$ Bezoutians. Let us now specialize to $T+H$ Bezoutians B which are centrosymmetric, i.e., $J_nBJ_n = B$. The following decomposition result, proved in [12], characterizes *nonsingular*, centrosymmetric $T+H$ Bezoutians. The nonsingularity criterion is related to the greatest common divisors of two polynomials, henceforth denoted by $\gcd(\cdot,\cdot)$.

Hereafter, the subscripts $+$ or $-$ of a vector designate the symmetry or skewsymmetry of this vector.

LEMMA 4.7. *An $n\times n$ matrix B is a nonsingular, centrosymmetric $T+H$ Bezoutian if and only if $\nabla_{T+H}(B)$ admits a representation*

$$\nabla_{T+H}(B) = \mathbf{u}_+\mathbf{v}_+^T - \mathbf{v}_+\mathbf{u}_+^T + \mathbf{u}_-\mathbf{v}_-^T - \mathbf{v}_-\mathbf{u}_-^T\,, \tag{4.3}$$

where $\mathbf{u}_+,\mathbf{v}_+\in\mathbb{F}_+^{n+2}$ such that

$$\gcd(\mathbf{u}_+(t),\mathbf{v}_+(t)) = \begin{cases} 1 & \text{if } n \text{ odd},\\ t+1 & \text{if } n \text{ even},\end{cases} \tag{4.4}$$

and $\mathbf{u}_-,\mathbf{v}_-\in\mathbb{F}_-^{n+2}$ such that

$$\gcd(\mathbf{u}_-(t),\mathbf{v}_-(t)) = \begin{cases} t^2-1 & \text{if } n \text{ odd},\\ t-1 & \text{if } n \text{ even}.\end{cases} \tag{4.5}$$

Notice that if $\mathbf{u}_+,\mathbf{v}_+\in\mathbb{F}_+^{n+2}$ with n even, then $t+1$ is a common divisor of $\mathbf{u}_+(t)$ and $\mathbf{v}_+(t)$ due to (2.2). Similar statements hold also for $\mathbf{u}_-$ and $\mathbf{v}_-$. Thus, it is justified to call the greatest common divisor *minimal* in the above cases (4.4) and (4.5). Obviously, in view of the symmetries, $\mathbf{u}_\pm,\mathbf{v}_\pm\in\mathbb{F}_\pm^{n+2}$, if both $\deg\mathbf{u}_\pm(t)\le n$ and $\deg\mathbf{v}_\pm(t)\le n$ are true, then zero is a common root of $\mathbf{u}_\pm(t)$ and $\mathbf{v}_\pm(t)$. As a consequence, if (4.4) or (4.5) hold, then

$$\max\{\deg(\mathbf{u}_\pm(t)),\deg(\mathbf{v}_\pm(t))\} = n+1,$$

respectively.

On the other hand, every matrix B satisfying (4.3) with $\mathbf{u}_\pm,\mathbf{v}_\pm\in\mathbb{F}_\pm^{n+2}$ is a centrosymmetric $T+H$ Bezoutian even if the greatest common divisors are not minimal.

Moreover, since

$$(\nabla_{T+H}(B)(t,s) = -(\nabla_{T+H}(B))(s,t),$$

any nonsingular, centrosymmetric $T+H$ Bezoutian is a symmetric matrix. This follows, of course, also from Theorem 4.6 and Remark 3.1. Notice that the assumption of nonsingularity is essential. For instance, each $B\in\mathbb{F}^{3\times3}$ satisfying $J_3B=BJ_3=B$ is of the form

$$B = \begin{bmatrix} a & b & a\\ c & d & c\\ a & b & a\end{bmatrix}.$$

Such matrices are singular, centrosymmetric $T+H$ Bezoutians, which are symmetric only if $b=c$.

An immediate consequence of Lemma 4.7 is the following theorem. In fact, for its proof, it suffices to write down (4.3) in polynomial language.

THEOREM 4.8. *A nonsingular, centrosymmetric $T+H$ Bezoutian B of order n allows the following (unique) splitting*

$$B = B_{++} + B_{--}\,, \tag{4.6}$$

where $B_{++} = P_+ B$ and $B_{--} = P_- B$ are special centrosymmetric $T+H$ Bezoutians,

$$B_{\pm\pm}(t,s) = \frac{\mathbf{u}_\pm(t)\mathbf{v}_\pm(s) - \mathbf{v}_\pm(t)\mathbf{u}_\pm(s)}{(t-s)(1-ts)}\,,$$

and where $\mathbf{u}_\pm, \mathbf{v}_\pm \in \mathbb{F}_\pm^{n+2}$ satisfy (4.4) and (4.5).

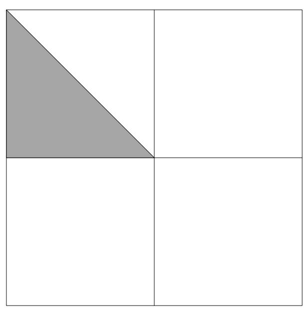
We call B_{++} or B_{--} *split-Bezoutian of* $(+)$ *or of* $(-)$ *type* and write

$$B_{\pm\pm} = \mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_\pm, \mathbf{v}_\pm). \tag{4.7}$$

The matrices B_{++} and B_{--}, besides being centrosymmetric, have the following additional symmetries,

$$B_{\pm\pm}P_\pm = P_\pm B_{\pm\pm} = \pm B_{\pm\pm}\,. \tag{4.8}$$

Thus, the splitting (4.6) is just that of Remark 3.2. This means that, as already stated there, all rows and columns of B_{++} are symmetric vectors, whereas all rows and columns of B_{--} are skewsymmetric vectors. Additionally, B_{++} and B_{--} are symmetric matrices. Hence, all entries of each of these matrices are determined by the entries in the highlighted triangle—which is about the eighth part of the matrix—in the following diagram.



Clearly, the entries of this triangle are given by the first $\lfloor \frac{n}{2} \rfloor + 1$ entries of $\mathbf{u}_+, \mathbf{v}_+$ in the $(+)$ case and by the first $\lfloor \frac{n}{2} \rfloor$ entries of $\mathbf{u}_-, \mathbf{v}_-$ in the $(-)$ case. Here $\lfloor \cdot \rfloor$ denotes the entire part.

For the split-Bezoutians, the following statement is analogous to Remark 4.2. Different pairs of (linearly independent) vectors may produce the same nonzero split-Bezoutian. In fact,

$$\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_\pm, \mathbf{v}_\pm) = \mathrm{Bez}_{\mathrm{sp}}(\hat{\mathbf{u}}_\pm, \hat{\mathbf{v}}_\pm) \tag{4.9}$$

if and only if there is a 2×2 matrix $\varphi_\pm$ such that $[\,\hat{\mathbf{u}}_\pm \;\; \hat{\mathbf{v}}_\pm\,] = [\,\mathbf{u}_\pm \;\; \mathbf{v}_\pm\,]\,\varphi_\pm$ with $\det \varphi_\pm = 1$.

REMARK 4.9. A centrosymmetric $T+H$ Bezoutian B which admits the splitting (4.6) has $\mathbb{F}_+^n$ and $\mathbb{F}_-^n$ as invariant subspaces with $B_{++}|\mathbb{F}_+^n$ and $B_{--}|\mathbb{F}_-^n$ being the corresponding restrictions. Therefore, B is similar to the direct sum of $B_{++}|\mathbb{F}_+^n$ and $B_{--}|\mathbb{F}_-^n$ and thus is nonsingular if and only if both $B_{++}|\mathbb{F}_+^n$ and $B_{--}|\mathbb{F}_-^n$ are invertible. The inverses can be identified with $T+H$ matrices A_+ and A_- which are of the form (3.4). This is due to the symmetries (3.3) and (4.8). In fact, as will be shown in more detail later, $B^{-1} = A_+ + A_-$.

REMARK 4.10. Given a centrosymmetric matrix B of order n, one can ask how to decide whether B is a nonsingular $T+H$ Bezoutian and how to determine vectors $\mathbf{u}_\pm$ and $\mathbf{v}_\pm$ occurring in (4.6) and (4.7). This can be done by the following procedure.

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1. Compute $B_{++} := P_+ B$ and $B_{--} := P_- B$.
2. Verify whether $\operatorname{rank} \nabla_{T+H}(B_{++}) = \operatorname{rank} \nabla_{T+H}(B_{--}) = 2$.
   (If this is not fulfilled, stop: B is not a $T+H$ Bezoutian or B is a singular Bezoutian.)
3. Determine bases $\{\mathbf{u}_\pm, \hat{\mathbf{v}}_\pm\}$ in the image of $\nabla_{T+H}(B_{\pm\pm})$.
   (Due to the assumption of centrosymmetry, we have $\mathbf{u}_\pm, \hat{\mathbf{v}}_\pm \in \mathbb{F}^{n+2}_\pm$.)
4. Verify whether the greatest common divisors of $\{\mathbf{u}_\pm(t), \hat{\mathbf{v}}_\pm(t)\}$ are minimal; cf. (4.4), (4.5).
   (If this is not fulfilled, stop: B is singular.)
5. Using the vectors $\mathbf{u}_\pm$ and $\hat{\mathbf{v}}_\pm$ (chosen in Step 3), compute the unique vectors $\mathbf{v}_\pm$ and $\hat{\mathbf{u}}_\pm$ such that
   $$\nabla_{T+H}(B_{\pm\pm}) = \mathbf{u}_\pm \mathbf{v}_\pm^T - \hat{\mathbf{v}}_\pm \hat{\mathbf{u}}_\pm^T.$$
   In fact, it is easy to see that $\mathbf{v}_\pm = \lambda_\pm \hat{\mathbf{v}}_\pm$ and $\mathbf{u}_\pm = \lambda_\pm^{-1} \hat{\mathbf{u}}_\pm$ with some $\lambda_\pm \in \mathbb{F} \setminus \{0\}$ so that we finally obtain
   $$\nabla_{T+H}(B_{\pm\pm}) = \mathbf{u}_\pm \mathbf{v}_\pm^T - \mathbf{v}_\pm \mathbf{u}_\pm^T.$$
   (To determine $\lambda_\pm$ it remains to compare a nonzero entry of $\nabla_{T+H}(B_{\pm\pm})$ with the corresponding entry of $\mathbf{u}_\pm \hat{\mathbf{v}}_\pm^T - \hat{\mathbf{v}}_\pm \mathbf{u}_\pm^T$.)
6. Now, $B = B_{++} + B_{--}$ is a nonsingular $T + H$ Bezoutian with
   $$B_{\pm\pm} = \operatorname{Bez}_{\rm sp}(\mathbf{u}_\pm, \mathbf{v}_\pm),$$
   where the two pairs $(\mathbf{u}_+, \mathbf{v}_+)$ and $(\mathbf{u}_-, \mathbf{u}_-)$ are unique up to transformations discussed in (4.9).

5. Inversion of certain matrices. In the following sections, certain upper triangular matrices and their inverses will occur, which are important for applying our algorithm to the inversion of $T+H$ Bezoutians in Section 10. As a preparation, we are now going to introduce these matrices and compute their inverses. One of the matrices is the $\ell \times \ell$ matrix

$$Q_\ell = \begin{bmatrix} \binom{0}{0} & 0 & \binom{2}{1} & 0 & \cdots & \\ & \binom{1}{0} & 0 & \binom{3}{1} & & \vdots \\ & & \binom{2}{0} & 0 & \ddots & 0 \\ & & & \binom{3}{0} & \ddots & \binom{\ell-1}{1} \\ & & & & \ddots & 0 \\ 0 & & & & & \binom{\ell-1}{0} \end{bmatrix}, \tag{5.1}$$

i.e.,

$$Q_\ell := [\, q_{ij} \,]_{i,j=0}^{\ell-1} \quad \text{with} \quad q_{ij} = \begin{cases} \binom{j}{\frac{j-i}{2}} & \text{if } j \geq i \text{ and } j-i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

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Upon defining the following $\ell \times \ell$ upper-triangular Toeplitz band matrices,

$$T_\ell := \begin{bmatrix} 1 & 0 & -1 & & 0 \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & 0 \\ 0 & & & & 1 \end{bmatrix}, \qquad T_\ell^{\pm} := \begin{bmatrix} 1 & \pm 1 & & & 0 \\ & 1 & \pm 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & \pm 1 \\ 0 & & & & 1 \end{bmatrix}, \tag{5.2}$$

we also introduce the following three $\ell \times \ell$ matrices

$$R_\ell := T_\ell Q_\ell \qquad \text{and} \qquad R_\ell^{\pm} := T_\ell^{\pm} Q_\ell. \tag{5.3}$$

Note that the entries of $R_\ell = [\, r_{ij} \,]_{i,j=0}^{\ell-1}$ are given by

$$r_{ij} = \begin{cases} \binom{j}{\frac{j-i}{2}} - \binom{j}{\frac{j-i}{2}-1} & \text{if } j > i \text{ and } j - i \text{ is even,} \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases}$$

and that the entries of $R_\ell^{\pm} = \left[r_{ij}^{\pm} \right]_{i,j=0}^{\ell-1}$ are given by

$$r_{ij}^{\pm} = \begin{cases} \binom{j}{\frac{j-i}{2}} & \text{if } j \geq i \text{ and } i - j \text{ is even,} \\ \pm\binom{j}{\frac{j-i-1}{2}} & \text{if } j > i \text{ and } i - j \text{ is odd,} \\ 0 & \text{if } i > j. \end{cases}$$

The inverses of Q_ℓ, R_ℓ, and $R_\ell^{\pm}$ can be described in terms of the matrix

$$U_\ell := [\, u_{ij} \,]_{i,j=0,}^{\ell-1} \qquad \text{with} \qquad u_{ij} = \begin{cases} \binom{-i-1}{\frac{j-i}{2}} & \text{if } j \geq i \text{ and } j - i \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Noting that $\binom{-i-1}{k} = (-1)^k \binom{i+k}{k}$, whence

$$\binom{-i-1}{\frac{j-i}{2}} = (-1)^{\frac{j-i}{2}} \binom{\frac{j+i}{2}}{\frac{j-i}{2}},$$

we observe that this matrix reads as

$$U_\ell = \begin{bmatrix} \binom{0}{0} & 0 & -\binom{1}{1} & 0 & \cdots & \\ & \binom{1}{0} & 0 & -\binom{2}{1} & & \vdots \\ & & \binom{2}{0} & 0 & \ddots & 0 \\ & & & \binom{3}{0} & \ddots & -\binom{\ell-2}{1} \\ & & & & \ddots & 0 \\ 0 & & & & & \binom{\ell-1}{0} \end{bmatrix}.$$

LEMMA 5.1. *We have* $U_\ell T_\ell Q_\ell = I_\ell$.

Proof. As usual, we let $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, for $k = 0, 1, \ldots,$ and $\binom{\alpha}{k} = 0$, for $k = -1, -2, \ldots$. The (i, k)-entry of $U_\ell T_\ell Q_\ell$ is given by

$$\sum_{j=0}^{\ell-1}(u_{ij} - u_{i,j-2})q_{jk} = \sum_{j\in I(i,k)} \left\{ \binom{-i-1}{\frac{j-i}{2}} - \binom{-i-1}{\frac{j-i}{2}-1} \right\} \binom{k}{\frac{k-j}{2}},$$

where $I(i, k)$ denotes the index set of all $0 \le j \le \ell - 1$ for which $j - i$ as well as $k - j$ are nonnegative and even. This index set is nonempty only if $k - i$ is nonnegative and even. In this case, using the familiar identity

$$\sum_{j+k=n} \binom{\alpha}{j}\binom{\beta}{k} = \binom{\alpha+\beta}{n},$$

we conclude that the above term equals

$$\binom{k-i-1}{\frac{k-i}{2}} - \binom{k-i-1}{\frac{k-i}{2}-1} = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

Thus, we have shown that $U_\ell T_\ell Q_\ell$ is the identity matrix. □

As a consequence, we obtain the explicit form of the inverses of Q_ℓ, R_ℓ, and $R_\ell^\pm$. Notice that $T_\ell = T_\ell^+ T_\ell^- = T_\ell^- T_\ell^+$.

PROPOSITION 5.2. *We have*

$$Q_\ell^{-1} = U_\ell T_\ell, \qquad R_\ell^{-1} = U_\ell, \qquad (R_\ell^\pm)^{-1} = U_\ell T_\ell^\mp. \tag{5.4}$$

6. Connections between H Bezoutians and split-Bezoutians of $(+)$ type of odd order. We are first going to show that a split-Bezoutian of $(+)$ type and of odd order is connected with an H Bezoutian of about half the size. This will be a main key for our further considerations.

Introduce a matrix S_ℓ of size $(2\ell - 1) \times \ell$ as the isomorphism defined by

$$S_\ell : \mathbb{F}^\ell \to \mathbb{F}_+^{2\ell-1}, \qquad (S_\ell \mathbf{x})(t) = \mathbf{x}(t + t^{-1})t^{\ell-1}, \quad \mathbf{x} \in \mathbb{F}^\ell.$$

The nonzero entries of the matrix S_ℓ are binomial coefficients arranged in the following triangular-type structure,

$$S_\ell = \begin{bmatrix} 0 & & & & \binom{\ell-1}{\ell-1} \\ & & & \iddots & 0 \\ & & \binom{2}{2} & \iddots & \binom{\ell-1}{\ell-2} \\ & \binom{1}{1} & 0 & \iddots & \vdots \\ \binom{0}{0} & 0 & \binom{2}{1} & & \vdots \\ & \binom{1}{0} & 0 & \ddots & \vdots \\ & & \binom{2}{0} & \ddots & \binom{\ell-1}{1} \\ & & & \ddots & 0 \\ 0 & & & & \binom{\ell-1}{0} \end{bmatrix}. \tag{6.1}$$

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Notice that the matrix Q_ℓ defined in (5.1) is (roughly) the lower half of S_ℓ. For later use, we observe that

$$S_\ell = Z_\ell^+ Q_\ell, \tag{6.2}$$

where Z_ℓ^+ is the following matrix of size $(2\ell-1)\times\ell$,

$$Z_\ell^+ = \begin{bmatrix} 0 & & & 1 \\ & & \iddots & 0 \\ & 1 & \iddots & \vdots \\ 1 & 0 & & \vdots \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix},$$

i.e., $Z_\ell^+ = [z_{ij}]_{i=0}^{2\ell-2}\,{}_{j=0}^{\ell-1}$ with

$$z_{ij} = \begin{cases} 1 & j = |\ell-1-i|, \\ 0 & \text{otherwise}. \end{cases}$$

Note that the first ℓ rows of this matrix are equal to the flip matrix J_ℓ introduced in (1.1) and the last ℓ rows are equal to the identity I_ℓ.

The main result of this section is the following.

THEOREM 6.1. [12] *Let* $\mathbf{u}_+, \mathbf{v}_+ \in \mathbb{F}_+^{n+2}$, $n = 2\ell-1$, *and let* $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{\ell+1}$ *be such that* $\mathbf{u}_+ = S_{\ell+1}\mathbf{u}$, $\mathbf{v}_+ = S_{\ell+1}\mathbf{v}$. *Then*

$$\operatorname{Bez}_{\mathrm{sp}}(\mathbf{u}_+, \mathbf{v}_+) = -S_\ell \operatorname{Bez}_H(\mathbf{u}, \mathbf{v})\, S_\ell^T. \tag{6.3}$$

Proof. Recall the definition of $\boldsymbol{l}_n(t)$ in (2.1) and observe that

$$S_\ell^T \boldsymbol{l}_{2\ell-1}(t) = \boldsymbol{l}_\ell(t+t^{-1})t^{\ell-1}.$$

Now we only need to consider

$$-\boldsymbol{l}_{2\ell-1}(t)^T \operatorname{Bez}_{\mathrm{sp}}(\mathbf{u}_+, \mathbf{v}_+)\, \boldsymbol{l}_{2\ell-1}(s) = -\frac{\mathbf{u}_+(t)\mathbf{v}_+(s) - \mathbf{v}_+(t)\mathbf{u}_+(s)}{(t-s)(1-ts)}$$

and

$$\begin{aligned} &\boldsymbol{l}_{2\ell-1}(t)^T S_\ell \operatorname{Bez}_H(\mathbf{u}, \mathbf{v})\, S_\ell^T \boldsymbol{l}_{2\ell-1}(s) \\ &\qquad = \boldsymbol{l}(t+t^{-1})^T t^{\ell-1} \operatorname{Bez}_H(\mathbf{u}, \mathbf{v})\, s^{\ell-1} \boldsymbol{l}(s+s^{-1}) \\ &\qquad = t^{\ell-1} \frac{\mathbf{u}(t+t^{-1})\mathbf{v}(s+s^{-1}) - \mathbf{v}(t+t^{-1})\mathbf{u}(s+s^{-1})}{(t+t^{-1}) - (s+s^{-1})} s^{\ell-1} \end{aligned}$$

to observe equality in (6.3). □

REMARK 6.2. Let $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{\ell+1}$ and $\mathbf{u}_+ = S_{\ell+1}\mathbf{u}$, $\mathbf{v}_+ = S_{\ell+1}\mathbf{v}$. Then the pair $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is generalized coprime if and only if the pair $\mathbf{u}_+(t)$ and $\mathbf{v}_+(t)$ is coprime. Hence, in view of Remark 4.3 and (6.3), it can be easily seen that $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+, \mathbf{v}_+)$ restricted to $\mathbb{F}_+^{2\ell-1}$ is invertible if and only if $\mathbf{u}_+(t)$ and $\mathbf{v}_+(t)$ are coprime.

REMARK 6.3. Since $S_{\ell+1}$ is an isomorphism from $\mathbb{F}^{\ell+1}$ to $\mathbb{F}_+^{2\ell+1}$, there is a one-to-one correspondence between $\mathbf{u}$ and $\mathbf{u}_+$ in Theorem 6.1 (and similarly between $\mathbf{v}$ and $\mathbf{v}_+$). In fact, the equation $\mathbf{u}_+ = S_{\ell+1}\mathbf{u}$ can be solved by using the inverse of $Q_{\ell+1}$; see Proposition 5.2. If $\mathbf{u}_+ = (u_i^+)_{i=-\ell}^{\ell} \in \mathbb{F}_+^{2\ell+1}$, then

$$\mathbf{u} = Q_{\ell+1}^{-1}(u_i^+)_{i=0}^{\ell} = U_{\ell+1}T_{\ell+1}(u_i^+)_{i=0}^{\ell}\,.$$

7. Inversion of centrosymmetric $T+H$ Bezoutians of odd order. In order to continue the discussion, let us come back to the splitting (4.6) for a nonsingular, centrosymmetric $T+H$ Bezoutian $B = B_{++} + B_{--}$ of order n.

If n is odd, we have just seen how to reduce its first term $B_{++} = \mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+, \mathbf{v}_+)$ to an H Bezoutian. We will now do the same for the second term B_{--} by first reducing it to a split-Bezoutian of $(+)$ type with odd order $n-2$. The resulting term can then be reduced to an H Bezoutian in the same way as B_{++}.

Thus, let $B_{--} = \mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_-, \mathbf{v}_-)$ with $\mathbf{u}_-, \mathbf{v}_- \in \mathbb{F}_-^{n+2}$ be given. In view of Lemma 4.7 (see also (2.2)), we can define $\mathbf{u}_+^n, \mathbf{v}_+^n \in \mathbb{F}_+^n$ by

$$\mathbf{u}_-(t) =: \big(t^2-1\big)\,\mathbf{u}_+^n(t), \qquad \mathbf{v}_-(t) =: (t^2-1)\mathbf{v}_+^n(t).$$

Here and in what follows, the superscript of a vector denotes its length. We obtain

$$\begin{aligned}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_-, \mathbf{v}_-)(t,s) &= (t^2-1)\frac{\mathbf{u}_+^n(t)\mathbf{v}_+^n(s) - \mathbf{v}_+^n(t)\mathbf{u}_+^n(s)}{(t-s)(1-ts)}(s^2-1)\\ &= (t^2-1)\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+^n, \mathbf{v}_+^n)(t,s)(s^2-1).\end{aligned}$$

In matrix language, this reads as

$$\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_-, \mathbf{v}_-) = M_{n-2}\,\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+^n, \mathbf{v}_+^n)\,M_{n-2}^T\,,$$

where M_{n-2} is the $n\times(n-2)$ matrix

$$M_{n-2} := \begin{bmatrix} -1 & & & 0 \\ 0 & -1 & & \\ 1 & 0 & \ddots & \\ & 1 & \ddots & -1 \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix}. \tag{7.1}$$

Since M_{n-2} is the matrix of the operator of multiplication by t^2-1 in the corresponding polynomial spaces (with respect to the canonical bases) and is an isomorphism from $\mathbb{F}_+^{n-2}$ to $\mathbb{F}_-^n$, whereas the transpose M_{n-2}^T is an isomorphism from $\mathbb{F}_-^n$ to $\mathbb{F}_+^{n-2}$, the split-Bezoutian $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_-, \mathbf{v}_-)$ restricted to $\mathbb{F}_-^n$ is invertible if and only if $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+^n, \mathbf{v}_+^n)$ restricted to $\mathbb{F}_+^{n-2}$ is invertible.

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Combining all this, we arrive at the following result. Therein, we rewrite $\mathbf{u}_+ =: \mathbf{u}_+^{n+2}$ and $\mathbf{v}_+ =: \mathbf{v}_+^{n+2}$.

THEOREM 7.1. [12] *Let n be odd. Then $B \in \mathbb{F}^{n\times n}$ is a nonsingular, centrosymmetric $T+H$ Bezoutian if and only if it can be represented in the form*

$$B = \mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+^{n+2}, \mathbf{v}_+^{n+2}) + M_{n-2}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+^{n}, \mathbf{v}_+^{n})M_{n-2}^T, \tag{7.2}$$

with $\mathbf{u}_+^{n+2i}(t), \mathbf{v}_+^{n+2i}(t) \in \mathbb{F}_+^{n+2i}[t]$ being coprime for $i=0,1$.

Let $n = 2\ell-1$. Taking into account Theorem 6.1 and Remark 6.2, we conclude that the split-Bezoutians occurring in (7.2) can be represented as

$$\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+^{n+2i}, \mathbf{v}_+^{n+2i}) = -S_{\ell+i-1}\mathrm{Bez}_H\left(\mathbf{u}^{\ell+i}, \mathbf{v}^{\ell+i}\right)S_{\ell+i-1}^T, \qquad i=0,1\,,$$

with

$$\mathbf{u}_+^{n+2i} = S_{\ell+i}\mathbf{u}^{\ell+i}, \quad \mathbf{v}_+^{n+2i} = S_{\ell+i}\mathbf{v}^{\ell+i} \tag{7.3}$$

and the pairs $(\mathbf{u}^{\ell+i}(t), \mathbf{v}^{\ell+i}(t))$ being generalized coprime. It follows that

$$B = W_n \begin{bmatrix} \mathrm{Bez}_H(\mathbf{v}^{\ell+1}, \mathbf{u}^{\ell+1}) & \mathbf{0} \\ \mathbf{0} & \mathrm{Bez}_H(\mathbf{v}^{\ell}, \mathbf{u}^{\ell}) \end{bmatrix} W_n^T\,, \tag{7.4}$$

where

$$W_n := [\,S_\ell \mid M_{n-2}S_{\ell-1}\,]\,. \tag{7.5}$$

Notice that the minus sign disappeared since we interchanged $\mathbf{u}^{\ell+i}$ and $\mathbf{v}^{\ell+i}$.

Due to (7.3), the vectors $\mathbf{u}^{\ell+i}$, $\mathbf{v}^{\ell+i}$ $(i=0,1)$ can be computed as indicated in Remark 6.3. We will come back to this in Section 10, where also another possibility is discussed; see Remark 10.1.

For the sake of simplicity, hereafter we write J for the matrix (1.1) of the corresponding order. Assuming B to be nonsingular and thus $(\mathbf{u}^{\ell+i}(t), \mathbf{v}^{\ell+i}(t))$ to be generalized coprime polynomials, the Bezout equations

$$\begin{array}{rcl} \mathbf{u}^{\ell+i}(t)\boldsymbol{\alpha}_i(t) + \mathbf{v}^{\ell+i}(t)\boldsymbol{\beta}_i(t) & = & 1 \\ \left(J\mathbf{u}^{\ell+i}\right)(t)\,(J\boldsymbol{\gamma}_i)\,(t) + \left(J\mathbf{v}^{\ell+i}\right)(t)\,(J\boldsymbol{\delta}_i)\,(t) & = & 1 \end{array} \tag{7.6}$$

have unique solution $\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_i, \boldsymbol{\delta}_i \in \mathbb{F}^{\ell+i-1}$ $(i=0,1)$. Defining vectors $\mathbf{s}^{2\ell-3}$ and $\mathbf{s}^{2\ell-1}$ by

$$\left(J\mathbf{s}^{2(\ell+i)-3}\right)(t) = \boldsymbol{\alpha}_i(t)\boldsymbol{\delta}_i(t) - \boldsymbol{\beta}_i(t)\boldsymbol{\gamma}_i(t), \tag{7.7}$$

we know from Theorem 4.5 that the inverse of $\mathrm{Bez}_H(\mathbf{v}^{\ell+i}, \mathbf{u}^{\ell+i})$ is the Hankel matrix

$$H_{\ell+i-1}\left(\mathbf{s}^{2(\ell+i)-3}\right) = \left(\mathrm{Bez}_H\left(\mathbf{v}^{\ell+i}, \mathbf{u}^{\ell+i}\right)\right)^{-1}, \quad i=0,1.$$

Taking into account (7.4), we obtain the following representation of B^{-1}.

THEOREM 7.2. *Let $n = 2\ell-1$, and let $B \in \mathbb{F}^{n\times n}$ be a nonsingular, centrosymmetric $T+H$ Bezoutian. Then, with the notation introduced above,*

$$B^{-1} = W_n^{-T}\begin{bmatrix} H_\ell(\mathbf{s}^{2\ell-1}) & \mathbf{0} \\ \mathbf{0} & H_{\ell-1}(\mathbf{s}^{2\ell-3}) \end{bmatrix} W_n^{-1}\,. \tag{7.8}$$

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As we will see, the matrix W_n is indeed invertible. For the purpose of representing its inverse, introduce the following $(2\ell-1)\times(\ell-1)$ matrix

$$Z_{\ell-1}^{-} := \begin{bmatrix} 0 & & -1 \\ & \iddots & 0 \\ -1 & \iddots & \vdots \\ 0 & & \vdots \\ 1 & \ddots & \vdots \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix}.$$

Recall (5.3) and (5.2), in particular, $R_{\ell-1} = T_{\ell-1}Q_{\ell-1}$. A simple but crucial computation yields

$$M_{n-2}S_{\ell-1} = Z_{\ell-1}^{-}R_{\ell-1}.$$

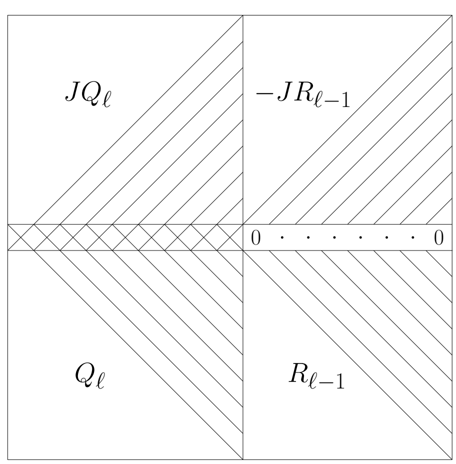
This together with (6.2) and (7.5) implies

$$W_n = [\, Z_\ell^{+}Q_\ell \mid Z_{\ell-1}^{-}R_{\ell-1} \,] = [\, Z_\ell^{+} \mid Z_{\ell-1}^{-} \,] \begin{bmatrix} Q_\ell & \mathbf{0} \\ \mathbf{0} & R_{\ell-1} \end{bmatrix}. \tag{7.9}$$

Therefore, W_n has the following structure,

$$W_n = \begin{bmatrix} J\widetilde{Q}_\ell & -JR_{\ell-1} \\ e_1^TQ_\ell & \mathbf{0}_{\ell-1} \\ \widetilde{Q}_\ell & R_{\ell-1} \end{bmatrix},$$

where $\widetilde{Q}_\ell$ is the matrix Q_ℓ with the first row deleted, e_1 is the first unit vector, and $\mathbf{0}_{\ell-1}$ is the row zero vector of appropriate length. The structure of W_n is displayed schematically in the following diagram.



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In view of (7.9), we see that W_n is invertible since so are the triangular matrices Q_ℓ and $R_{\ell-1}$, and the matrix

$$Z_n := [\ Z_\ell^+ \mid Z_{\ell-1}^-\ ] = \left[\begin{array}{cccc|ccc} 0 & & & 1 & 0 & & -1 \\ & & \iddots & 0 & & \iddots & 0 \\ & 1 & \iddots & \vdots & -1 & \iddots & \vdots \\ 1 & 0 & & \vdots & 0 & & \vdots \\ & 1 & \ddots & \vdots & 1 & \ddots & \vdots \\ & & \ddots & 0 & & \ddots & 0 \\ 0 & & & 1 & 0 & & 1 \end{array}\right].$$

In fact,

$$Z_n^{-1} = \tfrac{1}{2}\begin{bmatrix} D_\ell^{-1} & \mathbf{0} \\ \mathbf{0} & I_{\ell-1} \end{bmatrix} Z_n^T,$$

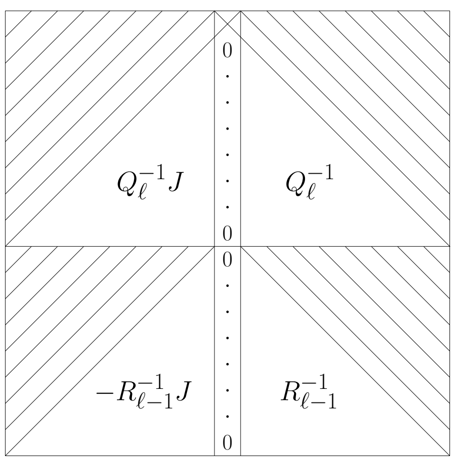
where

$$D_\ell := \operatorname{diag}(\tfrac{1}{2}, 1, \ldots, 1). \tag{7.10}$$

Hence,

$$W_n^{-1} = \tfrac{1}{2}\begin{bmatrix} Q_\ell^{-1} D_\ell^{-1} & \mathbf{0} \\ \mathbf{0} & R_{\ell-1}^{-1} \end{bmatrix} Z_n^T = \tfrac{1}{2}\begin{bmatrix} Q_\ell^{-1} D_\ell^{-1} (Z_\ell^+)^T \\ R_{\ell-1}^{-1} (Z_{\ell-1}^-)^T \end{bmatrix}.$$

As a consequence, the structure of W_n^{-1} is (up to a diagonal matrix) as indicated in the following diagram.



Collecting all results, we obtain the following reformulation of (7.8).

THEOREM 7.3. *Let $n = 2\ell - 1$, and let $B \in \mathbb{F}^{n\times n}$ be a nonsingular, centrosymmetric $T + H$ Bezoutian. Then, with the notation introduced above,*

$$\begin{aligned} B^{-1} &= \tfrac{1}{4} Z_n \begin{bmatrix} A_\ell^{(0)} & \mathbf{0} \\ \mathbf{0} & A_{\ell-1}^{(1)} \end{bmatrix} Z_n^T \\ &= \tfrac{1}{4} Z_\ell^+ A_\ell^{(0)} (Z_\ell^+)^T + \tfrac{1}{4} Z_{\ell-1}^- A_{\ell-1}^{(1)} (Z_{\ell-1}^-)^T, \end{aligned} \tag{7.11}$$

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where

$$A_\ell^{(0)} := D_\ell^{-1} Q_\ell^{-T} H_\ell(\mathbf{s}^{2\ell-1}) Q_\ell^{-1} D_\ell^{-1},$$
$$A_{\ell-1}^{(1)} := R_{\ell-1}^{-T} H_{\ell-1}(\mathbf{s}^{2\ell-3}) R_{\ell-1}^{-1}.$$

Let us remark that the matrices

$$A_+^o := \tfrac{1}{4} Z_\ell^+ A_\ell^{(0)} (Z_\ell^+)^T \quad \text{and} \quad A_-^o := \tfrac{1}{4} Z_{\ell-1}^- A_{\ell-1}^{(1)} (Z_{\ell-1}^-)^T \tag{7.12}$$

are precisely the inverses of B_{++} and B_{--}, restricted to $\mathbb{F}_+^n$ and $\mathbb{F}_-^n$, respectively; see Remark 4.9. As we know from Proposition 3.3, these are special centrosymmetric $T+H$ matrices (3.4) with symbols which we are going to compute in Section 9. There we also identify the matrices $A_\ell^{(0)}$ and $A_{\ell-1}^{(1)}$ as special symmetric (but in general not centrosymmetric) $T+H$ matrices.

8. Inversion of centrosymmetric $T+H$ Bezoutians of even order. We start again with the splitting formula (4.6) for B nonsingular, centrosymmetric but now of even order $n = 2\ell$. Remembering Lemma 4.7 and (2.2), we observe that there are vectors

$$\mathbf{u}_+^{n+1}, \mathbf{v}_+^{n+1}, \mathbf{y}_+^{n+1}, \mathbf{z}_+^{n+1} \in \mathbb{F}_+^{n+1}$$

such that

$$\mathbf{u}_+(t) =: (t+1)\mathbf{u}_+^{n+1}(t), \quad \mathbf{v}_+(t) =: (t+1)\mathbf{v}_+^{n+1}(t),$$
$$\mathbf{u}_-(t) =: (t-1)\mathbf{y}_+^{n+1}(t), \quad \mathbf{v}_-(t) =: (t-1)\mathbf{z}_+^{n+1}(t)\,.$$

We obtain

$$B_{++} = M_{n-1}^+ \mathrm{Bez}_{\mathrm{sp}}\left(\mathbf{u}_+^{n+1}, \mathbf{v}_+^{n+1}\right)(M_{n-1}^+)^T,$$
$$B_{--} = M_{n-1}^- \mathrm{Bez}_{\mathrm{sp}}\left(\mathbf{y}_+^{n+1}, \mathbf{z}_+^{n+1}\right)(M_{n-1}^-)^T,$$

where the matrices

$$M_{n-1}^{\pm} := \begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 1 & \pm 1 & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \pm 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{8.1}$$

are of size $n \times (n-1)$. We arrive at a similar assertion as in Theorem 7.1.

THEOREM 8.1. [12] *Let n be even. Then $B \in \mathbb{F}^{n\times n}$ is a nonsingular, centrosymmetric $T+H$ Bezoutian if and only if it can be represented in the form*

$$B = M_{n-1}^+ \mathrm{Bez}_{\mathrm{sp}}\left(\mathbf{u}_+^{n+1}, \mathbf{v}_+^{n+1}\right)(M_{n-1}^+)^T + M_{n-1}^- \mathrm{Bez}_{\mathrm{sp}}\left(\mathbf{y}_+^{n+1}, \mathbf{z}_+^{n+1}\right)(M_{n-1}^-)^T, \tag{8.2}$$

with $(\mathbf{u}_+^{n+1}(t), \mathbf{v}_+^{n+1}(t))$, $(\mathbf{y}_+^{n+1}(t), \mathbf{z}_+^{n+1}(t))$ being pairs of coprime polynomials in $\mathbb{F}_+^{n+1}[t]$.

Since the four vectors appearing in (8.2) belong to $\mathbb{F}_+^{n+1}$, the split-Bezoutians are of $(+)$ type and odd order $n-1 = 2\ell-1$. Taking into account (6.3), we obtain

$$\mathrm{Bez}_{\mathrm{sp}}\left(\mathbf{u}_+^{n+1}, \mathbf{v}_+^{n+1}\right) = S_\ell \mathrm{Bez}_H(\mathbf{v}, \mathbf{u}) S_\ell^T,$$
$$\mathrm{Bez}_{\mathrm{sp}}\left(\mathbf{y}_+^{n+1}, \mathbf{z}_+^{n+1}\right) = S_\ell \mathrm{Bez}_H(\mathbf{z}, \mathbf{y}) S_\ell^T,$$

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with $\mathbf{u}_+^{n+1} =: S_{\ell+1}\mathbf{u},\ \mathbf{v}_+^{n+1} =: S_{\ell+1}\mathbf{v},\ \mathbf{y}_+^{n+1} =: S_{\ell+1}\mathbf{y},\ \mathbf{z}_+^{n+1} =: S_{\ell+1}\mathbf{z}$. Hence,

$$B = \widetilde{W}_n \begin{bmatrix} \mathrm{Bez}_H(\mathbf{v},\mathbf{u}) & \mathbf{0} \\ \mathbf{0} & \mathrm{Bez}_H(\mathbf{z},\mathbf{y}) \end{bmatrix} \widetilde{W}_n^T,$$

with

$$\widetilde{W}_n := \left[\, M_{n-1}^+ S_\ell \mid M_{n-1}^- S_\ell \,\right].$$

Again a basic but crucial computation yields

$$M_{n-1}^{\pm} S_\ell = \begin{bmatrix} \pm J_\ell R_\ell^{\pm} \\ R_\ell^{\pm} \end{bmatrix},$$

with $R_\ell^{\pm}$ introduced in (5.3). Therefore,

$$\widetilde{W}_n = \begin{bmatrix} J_\ell & -J_\ell \\ I_\ell & I_\ell \end{bmatrix} \begin{bmatrix} R_\ell^+ & \mathbf{0} \\ \mathbf{0} & R_\ell^- \end{bmatrix}.$$

We obtain

$$\widetilde{W}_n^{-1} = \tfrac{1}{2} \begin{bmatrix} \left(R_\ell^+\right)^{-1} & \mathbf{0} \\ \mathbf{0} & \left(R_\ell^-\right)^{-1} \end{bmatrix} \begin{bmatrix} J_\ell & I_\ell \\ -J_\ell & I_\ell \end{bmatrix},$$

where $\left(R_\ell^{\pm}\right)^{-1}$ is given in Proposition 5.2.

The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{y}$, and $\mathbf{z}$ can be computed as indicated in Remark 6.3. This and alternative ways are outlined in Section 10.

In order to describe the inverses of $\mathrm{Bez}_H(\mathbf{v},\mathbf{u})$ and $\mathrm{Bez}_H(\mathbf{z},\mathbf{y})$, which are Hankel matrices, we consider the Bezout equations

$$\begin{array}{rcl} \mathbf{u}(t)\boldsymbol{\alpha}_1(t) + \mathbf{v}(t)\boldsymbol{\beta}_1(t) & = & 1 \\ \mathbf{u}^J(t)\boldsymbol{\gamma}_1^J(t) + \mathbf{v}^J(t)\boldsymbol{\delta}_1^J(t) & = & 1 \end{array} \tag{8.3}$$

and

$$\begin{array}{rcl} \mathbf{y}(t)\boldsymbol{\alpha}_2(t) + \mathbf{z}(t)\boldsymbol{\beta}_2(t) & = & 1 \\ \mathbf{y}^J(t)\boldsymbol{\gamma}_2^J(t) + \mathbf{z}^J(t)\boldsymbol{\delta}_2^J(t) & = & 1. \end{array} \tag{8.4}$$

These equations have unique solutions $\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_i, \boldsymbol{\delta}_i \in \mathbb{F}^\ell$ $(i = 1, 2)$ since $(\mathbf{u}(t), \mathbf{v}(t))$ and $(\mathbf{y}(t), \mathbf{z}(t))$ are pairs of generalized coprime polynomials. Finally, define $\mathbf{s}_1^{2\ell-1}$ and $\mathbf{s}_2^{2\ell-1}$ by

$$(J\mathbf{s}_i^{2\ell-1})(t) = \boldsymbol{\alpha}_i(t)\boldsymbol{\delta}_i(t) - \boldsymbol{\beta}_i(t)\boldsymbol{\gamma}_i(t), \qquad i = 1, 2. \tag{8.5}$$

Now we are able to represent B^{-1}.

THEOREM 8.2. *Let $B \in \mathbb{F}^{n\times n}$ be a nonsingular, centrosymmetric $T + H$ Bezoutian of order $n = 2\ell$. Then, with the notation introduced above,*

$$B^{-1} = \widetilde{W}_n^{-T} \begin{bmatrix} H_\ell(\mathbf{s}_1^{2\ell-1}) & \mathbf{0} \\ \mathbf{0} & H_\ell(\mathbf{s}_2^{2\ell-1}) \end{bmatrix} \widetilde{W}_n^{-1}.$$

Moreover,

$$\begin{aligned} B^{-1} &= \tfrac{1}{4} \begin{bmatrix} J_\ell & -J_\ell \\ I_\ell & I_\ell \end{bmatrix} \begin{bmatrix} A_\ell^+ & \mathbf{0} \\ \mathbf{0} & A_\ell^- \end{bmatrix} \begin{bmatrix} J_\ell & I_\ell \\ -J_\ell & I_\ell \end{bmatrix} \\ &= \tfrac{1}{4} \begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} A_\ell^+ \left[\, J_\ell \ \ I_\ell \,\right] + \tfrac{1}{4} \begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} A_\ell^- \left[\, -J_\ell \ \ I_\ell \,\right], \end{aligned} \tag{8.6}$$

where

$$A_\ell^+ := (R_\ell^+)^{-T} H_\ell(\mathbf{s}_1^{2\ell-1})(R_\ell^+)^{-1}, \quad A_\ell^- := (R_\ell^-)^{-T} H_\ell(\mathbf{s}_2^{2\ell-1})(R_\ell^-)^{-1}. \tag{8.7}$$

We remark that the first and the second summand of (8.6),

$$A_+^e := \tfrac{1}{4}\begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} A_\ell^+ \,[\, J_\ell \;\; I_\ell \,]\,, \quad A_-^e := \tfrac{1}{4}\begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} A_\ell^- \,[\, -J_\ell \;\; I_\ell \,]\,, \tag{8.8}$$

are the inverses of B_{++} and B_{--} restricted to $\mathbb{F}_+^n$ and $\mathbb{F}_-^n$, respectively; compare Remark 4.9. In fact, A_+^e and A_-^e are centrosymmetric $T+H$ matrices with the symmetry properties (3.3). Hence, they are of the form (3.4). The symbols of these $T+H$ matrices (which are not unique) are computed from $\mathbf{s}_i^{2\ell-1}$ in the next section. For this, we first identify $A_\ell^\pm$ as particular symmetric $T+H$ matrices.

9. Representations of inverses of $T+H$ Bezoutians as centrosymmetric $T+H$ matrices. In the previous two sections (see Theorems 7.3 and 8.2), we arrived at the matrices $A_\pm^o$ and $A_\pm^e$. As already promised there, we are going to identify these matrices as particular centrosymmetric $T+H$ matrices and compute their symbols. The starting point are the matrices $A_\ell^{(0)}$, $A_{\ell-1}^{(1)}$, and $A_\ell^\pm$, which will be identified as particular symmetric (not centrosymmetric) $T+H$ matrices based on auxiliary results established in the following subsection.

9.1. Relations between Hankel matrices and symmetric $T+H$ matrices. There exist identities between Hankel matrices and four kinds of particular *symmetric* $T+H$ matrices. In fact, these relationships hold for one-sided infinite matrices and were established in [1, Theorem 5]. The finite matrix versions, which are of interest to us, are immediate consequences. The identities involve the matrices $Q_\ell, R_\ell, R_\ell^\pm$, and D_ℓ, which were defined in (5.1), (5.3), and (7.10).

THEOREM 9.1. [1] *Let $\mathbf{a} = (a_k)_{k=-2\ell}^{2\ell}$, $\mathbf{a}^+ = (a_k^+)_{k=-2\ell+1}^{2\ell-1}$, $\mathbf{a}^- = (a_k^-)_{k=-2\ell+1}^{2\ell-1}$, and $\mathbf{a}^\# = (a_k^\#)_{k=-2\ell+2}^{2\ell-2}$ be four symmetric vectors,*

$$a_k = a_{-k}, \quad a_k^+ = a_{-k}^+, \quad a_k^- = a_{-k}^-, \quad \text{and} \quad a_k^\# = a_{-k}^\#,$$

where the first three vectors are related to the last one by

$$a_k^\# = 2a_k - a_{k-2} - a_{k+2} = 2a_k^+ + a_{k-1}^+ + a_{k+1}^+ = 2a_k^- - a_{k-1}^- - a_{k+1}^-. \tag{9.1}$$

Define the vector $\mathbf{s} = (s_k)_{k=0}^{2\ell-2}$ by

$$s_k = \frac{1}{2}\sum_{j=0}^{k} a_{k-2j}^\# \binom{k}{j} \tag{9.2}$$

as well as the $\ell \times \ell$ Toeplitz-plus-Hankel matrices

$$\begin{aligned} \mathrm{TH}_\ell(\mathbf{a}) &:= [\, a_{j-k} - a_{j+k+2} \,]_{j,k=0}^{\ell-1}, & \mathrm{TH}_\ell^+(\mathbf{a}^+) &:= [\, a_{j-k}^+ + a_{j+k+1}^+ \,]_{j,k=0}^{\ell-1}, \\ \mathrm{TH}_\ell^\#(\mathbf{a}^\#) &:= [\, a_{j-k}^\# + a_{j+k}^\# \,]_{j,k=0}^{\ell-1}, & \mathrm{TH}_\ell^-(\mathbf{a}^-) &:= [\, a_{j-k}^- - a_{j+k+1}^- \,]_{j,k=0}^{\ell-1}. \end{aligned}$$

Then the Hankel matrix $H_\ell(\mathbf{s}) = [\, s_{j+k} \,]_{j,k=0}^{\ell-1}$ can be represented as

$$\begin{aligned} H_\ell(\mathbf{s}) &= Q_\ell^T D_\ell \mathrm{TH}_\ell^\#(\mathbf{a}^\#) D_\ell Q_\ell = R_\ell^T \mathrm{TH}_\ell(\mathbf{a}) R_\ell \\ &= (R_\ell^+)^T \mathrm{TH}_\ell^+(\mathbf{a}^+) R_\ell^+ = (R_\ell^-)^T \mathrm{TH}_\ell^-(\mathbf{a}^-) R_\ell^-. \end{aligned} \tag{9.3}$$

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We can rephrase the relationships (9.1) and (9.2) between $\mathbf{s}$ and the symmetric vectors $\mathbf{a}^{\#}$, $\mathbf{a}$, $\mathbf{a}^{+}$, and $\mathbf{a}^{-}$ as follows. First of all, using the notation (6.1),

$$\mathbf{s} = \tfrac{1}{2}(S_{2\ell-1})^T \mathbf{a}^{\#} = Q_{2\ell-1}^T D_{2\ell-1} (a_k^{\#})_{k=0}^{2\ell-2}. \tag{9.4}$$

Thus, there is a one-to-one correspondence between $\mathbf{s}$ and $\mathbf{a}^{\#}$. In particular,

$$(a_k^{\#})_{k=0}^{2\ell-2} = D_{2\ell-1}^{-1} Q_{2\ell-1}^{-T} \mathbf{s}\,, \tag{9.5}$$

which can be written explicitly as

$$a_0^{\#} = 2s_0, \quad a_k^{\#} = a_{-k}^{\#} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j s_{k-2j} \left(\binom{k-j}{j} + \binom{k-j-1}{j-1} \right), \tag{9.6}$$

for $k = 1, 2, \dots, 2\ell - 2$. This can be derived using Proposition 5.2.

However, the correspondence (9.1) between the vectors $\mathbf{a}$, $\mathbf{a}^{\#}$, and $\mathbf{a}^{\pm}$ is not bijective. It can be expressed with the help of linear maps

$$\Lambda_{2\ell+1} : (a_k)_{k=0}^{2\ell} \mapsto (a_k^{\#})_{k=0}^{2\ell-2}, \qquad \Lambda_{2\ell}^{\pm} : (a_k^{\pm})_{k=0}^{2\ell-1} \mapsto (a_k^{\#})_{k=0}^{2\ell-2}$$

given by

$$\begin{bmatrix} 2 & 0 & -2 & & & & 0 \\ 0 & 1 & 0 & -1 & & & \\ -1 & 0 & 2 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & & -1 & 0 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2\ell} \end{bmatrix} = \begin{bmatrix} a_0^{\#} \\ a_1^{\#} \\ \vdots \\ a_{2\ell-2}^{\#} \end{bmatrix}, \tag{9.7}$$

$$\begin{bmatrix} 2 & \pm 2 & & & & 0 \\ \pm 1 & 2 & \pm 1 & & & \\ & \pm 1 & 2 & \pm 1 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & \pm 1 & 2 & \pm 1 \end{bmatrix} \begin{bmatrix} a_0^{\pm} \\ a_1^{\pm} \\ \vdots \\ a_{2\ell-1}^{\pm} \end{bmatrix} = \begin{bmatrix} a_0^{\#} \\ a_1^{\#} \\ \vdots \\ a_{2\ell-2}^{\#} \end{bmatrix}.$$

From this we observe that the kernel of $\Lambda_{2\ell+1}$ consists of all vectors with period two (i.e., $a_k = a_{k-2}$) and thus has dimension two, while the kernels of $\Lambda_{2\ell}^{+}$ and $\Lambda_{2\ell}^{-}$ consist of all vectors which are alternating ($a_k^{+} = -a_{k-1}^{+}$) or constant ($a_k^{-} = a_{k-1}^{-}$), respectively, and thus have dimension one. The maps $\Lambda_{2\ell+1}$ and $\Lambda_{2\ell}^{\pm}$ are surjective.

The fact that $\mathbf{a}$ and $\mathbf{a}^{\pm}$ are not uniquely determined by $\mathbf{a}^{\pm}$ corresponds to the fact that the matrices $\mathrm{TH}_\ell(\mathbf{a}), \mathrm{TH}_\ell^{+}(\mathbf{a}^{+})$, and $\mathrm{TH}_\ell^{-}(\mathbf{a}^{-})$ do not determine their symbols uniquely (as one can easily verify). Indeed, they are only unique up to the just mentioned vectors in the respective kernels. Still, as (9.3) requires, there is a one-to-one correspondence between $H_\ell(\mathbf{s})$, $\mathrm{TH}_\ell^{\#}(\mathbf{a}^{\#})$, $\mathrm{TH}_\ell(\mathbf{a})$, and $\mathrm{TH}_\ell^{\pm}(\mathbf{a}^{\pm})$.

It is worth to make the relationship between $\mathbf{a}$, $\mathbf{a}^{\pm}$, and $\mathbf{s}$ more explicit. Recall that T_ℓ and $T_\ell^{\pm}$ were introduced in (5.2) and that $M_{2\ell+1}$ and $M_{2\ell}^{\pm}$ were defined in (7.1) and (8.1). Clearly, (9.4) and (9.7) lead to

$$\Lambda_{2\ell+1}(a_k)_{k=0}^{2\ell} = D_{2\ell-1}^{-1} Q_{2\ell-1}^{-T} \mathbf{s} = D_{2\ell-1}^{-1} T_{2\ell-1}^{T} U_{2\ell-1}^{T} \mathbf{s},$$

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taking into account (5.4). By a straightforward computation, $T_{2\ell-1}^T M_{2\ell-1}^T = -D_{2\ell-1}\Lambda_{2\ell+1}$. Hence,

$$M_{2\ell-1}^T (a_k)_{k=0}^{2\ell} = -U_{2\ell-1}^T \mathbf{s} = -R_{2\ell-1}^{-T}\mathbf{s}. \tag{9.8}$$

Furthermore,

$$\Lambda_{2\ell}^{\pm}(a_k^{\pm})_{k=0}^{2\ell-1} = D_{2\ell-1}^{-1} Q_{2\ell-1}^{-T}\mathbf{s} = D_{2\ell-1}^{-1} T_{2\ell-1}^T U_{2\ell-1}^T \mathbf{s}.$$

Recall $T_{2\ell-1} = T_{2\ell-1}^{+} T_{2\ell-1}^{-} = T_{2\ell-1}^{-} T_{2\ell-1}^{+}$ and verify that $(T_{2\ell-1}^{\pm})^T (M_{2\ell}^{\pm})^T = \pm D_{2\ell-1}\Lambda_{2\ell}^{\pm}$ to conclude that

$$(M_{2\ell-1}^{\pm})^T (a_k^{\pm})_{k=0}^{2\ell-1} = \pm (T_{2\ell-1}^{\mp})^T U_{2\ell-1}^T \mathbf{s} = \pm (R_{2\ell-1}^{\pm})^{-T}\mathbf{s}. \tag{9.9}$$

The equations (9.5), (9.8), and (9.9) together with Proposition 5.2 can be used to determine vectors $\mathbf{a}^{\#}$, $\mathbf{a}$, and $\mathbf{a}^{\pm}$ satisfying the assumptions of Theorem 9.1 for each prescribed $\mathbf{s}$.

The formulas (9.8) and (9.9) can be written down explicitly as it was done in (9.6) for (9.5). This will be performed in Section 10.

9.2. Representation of B^{-1} as a centrosymmetric $T + H$ matrix. Let B be a nonsingular, centrosymmetric $T + H$ Bezoutian of order n. We continue the discussion of how to represent B^{-1} as $T + H$ matrix. Depending of whether n is odd or even, we arrived at representations of B^{-1} in the Theorems 7.3 and 8.2. On the other hand, we know from Theorem 4.6 that B^{-1} has to be a centrosymmetric $T + H$ matrix. Due to Remark 3.2 and Proposition 3.3, this $T + H$ matrix can be written as a sum of two particular $T + H$ matrices of the form (3.4). The formulas stated in the aforementioned theorems together with the results of the previous subsection now allow us to conclude this directly and to establish explicit formulas for the symbols of the $T + H$ matrices.

Let us start with considering a $T + H$ Bezoutian B of odd order $n = 2\ell - 1$. From Theorem 7.3 and Theorem 9.1 (see, in particular, the first line in (9.3)), it follows that

$$A_\ell^{(0)} = \mathrm{TH}_\ell^{\#}(\mathbf{a}^{(0)}) \quad \text{and} \quad A_{\ell-1}^{(1)} = \mathrm{TH}_{\ell-1}(\mathbf{a}^{(1)}),$$

where $\mathbf{a}^{(0)} = (a_k^{(0)})_{k=-2\ell+2}^{2\ell-2}$ and $\mathbf{a}^{(1)} = (a_k^{(1)})_{k=-2\ell+2}^{2\ell-2}$ are symmetric vectors of length $4\ell - 3 = 2n - 1$. In view of (9.5) and (9.8), we obtain

$$(a_k^{(0)})_{k=0}^{2\ell-2} = D_{2\ell-1}^{-1} Q_{2\ell-1}^{-T}\mathbf{s}^{2\ell-1} \quad \text{and} \quad M_{2\ell-1}^T (a_k^{(1)})_{k=0}^{2\ell-2} = -R_{2\ell-3}^{-T}\mathbf{s}^{2\ell-3}.$$

The vectors $\mathbf{s}^{2\ell-1}$ and $\mathbf{s}^{2\ell-3}$ are the ones featured in Theorem 7.3, which are determined by (7.7). Notice that $M_{2\ell-1}^T$ is surjective and has a two-dimensional kernel consisting of all vectors $\mathbf{e}_{\alpha,\beta} = (\alpha, \beta, \alpha, \beta, \dots)^T$.

Thus, in connection with (7.11), we conclude that

$$B^{-1} = \tfrac{1}{4} Z_\ell^{+} \mathrm{TH}_\ell^{\#}(\mathbf{a}^{(0)})(Z_\ell^{+})^T + \tfrac{1}{4} Z_{\ell-1}^{-} \mathrm{TH}_{\ell-1}(\mathbf{a}^{(1)})(Z_{\ell-1}^{-})^T.$$

Therein the first and the second term are A_+^o and A_-^o, respectively, introduced in (7.12). Taking into account the structure of Z_ℓ^{+} and $Z_{\ell-1}^{-}$, a straightforward computation yields

$$A_+^o = \tfrac{1}{4} Z_\ell^{+} \mathrm{TH}_\ell^{\#}(\mathbf{a}^{(0)})(Z_\ell^{+})^T = \tfrac{1}{4}\big[\, a_{j-k}^{(0)} + a_{j+k}^{(0)} \,\big]_{j,k=-\ell+1}^{\ell-1} \tag{9.10}$$

and

$$A_-^o = \tfrac{1}{4} Z_{\ell-1}^{-} \mathrm{TH}_{\ell-1}(\mathbf{a}^{(1)})(Z_{\ell-1}^{-})^T = \tfrac{1}{4}\big[\, a_{j-k}^{(1)} - a_{j+k}^{(1)} \,\big]_{j,k=-\ell+1}^{\ell-1}, \tag{9.11}$$

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which are precisely the two particular centrosymmetric $T+H$ matrices of the form (3.4); see also (3.1). Thus, we arrive at the following theorem.

THEOREM 9.2. *Let B be a nonsingular, centrosymmetric $T+H$ Bezoutian of odd order $n = 2\ell - 1$. Then with $\mathbf{a}^{(0)}$ and $\mathbf{a}^{(1)}$ as above, we have*

$$B^{-1} = \frac{1}{4}\left([\, a^{(0)}_{j-k} + a^{(1)}_{j-k}\,]_{j,k=-\ell+1}^{\ell-1} + [\, a^{(0)}_{j+k} - a^{(1)}_{j+k}\,]_{j,k=-\ell+1}^{\ell-1} \right). \tag{9.12}$$

Now, let B be a $T+H$ Bezoutian of even order, $n = 2\ell$. Consider in Theorem 9.1 the vectors $\mathbf{s}_i^{2\ell-1}$, for $i = 1, 2$, defined in (8.5) in place of $\mathbf{s}$. In view of Theorem 8.2 and the second line in (9.3), we obtain

$$A_\ell^+ = \mathrm{TH}_\ell^+(\mathbf{a}^+) \quad \text{and} \quad A_\ell^- = \mathrm{TH}_\ell^-(\mathbf{a}^-),$$

where the symmetric vectors $\mathbf{a}^\pm = (a_k^\pm)_{k=-2\ell+1}^{2\ell-1}$ of length $4\ell - 1 = 2n - 1$ are given by

$$\begin{aligned} (M_{2\ell}^+)^T (a_k^+)_{k=0}^{2\ell-1} &= (R_{2\ell-1}^+)^{-T} \mathbf{s}_1^{2\ell-1}, \\ (M_{2\ell}^-)^T (a_k^-)_{k=0}^{2\ell-1} &= -(R_{2\ell-1}^-)^{-T} \mathbf{s}_2^{2\ell-1}. \end{aligned}$$

Notice that the matrices $(M_{2\ell}^\pm)^T$ are surjective and have a one-dimensional kernel consisting of all vectors $\mathbf{e}_{\alpha,\mp\alpha}$.

Now, in view of (8.6), it follows that

$$B^{-1} = \tfrac{1}{4} \begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^+(\mathbf{a}^+) \, [\, J_\ell \;\; I_\ell \,] + \tfrac{1}{4} \begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^-(\mathbf{a}^-) \, [\, -J_\ell \;\; I_\ell \,] \, ,$$

where the first summand equals A_+^e and the second equals A_-^e; see (8.8). Since

$$\begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^+(\mathbf{a}^+) \, [\, J_\ell \;\; I_\ell \,] = \Big[\, a_{j-k}^+ + a_{j+k+1}^+ \,\Big]_{j,k=-\ell}^{\ell-1} \tag{9.13}$$

and

$$\begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^-(\mathbf{a}^-) \, [\, -J_\ell \;\; I_\ell \,] = \Big[a_{j-k}^- - a_{j+k+1}^- \Big]_{j,k=-\ell}^{\ell-1}, \tag{9.14}$$

which are $T+H$ matrices of the form (3.4), see also (3.2), we obtain the final result in the even case.

THEOREM 9.3. *Let B be a nonsingular, centrosymmetric $T+H$ Bezoutian of even order $n = 2\ell$. Then with $\mathbf{a}^+$ and $\mathbf{a}^-$ as above, we have*

$$B^{-1} = \frac{1}{4}\Big([\, a^+_{j-k} + a^-_{j-k}\,]_{j,k=-\ell}^{\ell-1} + [\, a^+_{j+k+1} - a^-_{j+k+1}\,]_{j,k=-\ell}^{\ell-1} \Big) \, . \tag{9.15}$$

10. Algorithms. Let an $n \times n$ matrix B be given. From Section 4.4 we know that if B is a nonsingular, centrosymmetric $T+H$ Bezoutian, then we can represent it in the form

$$B = \mathrm{Bez_{sp}}(\mathbf{u}_+, \mathbf{v}_+) + \mathrm{Bez_{sp}}(\mathbf{u}_-, \mathbf{v}_-),$$

where $(\mathbf{u}_\pm, \mathbf{v}_\pm)$ satisfy the conditions stated in Lemma 4.7. In Remark 4.10 we established a procedure to verify whether B is a nonsingular, centrosymmetric $T+H$ Bezoutian. The procedure also describes an algorithm for the computation of appropriate (nonunique) vectors $\mathbf{u}_\pm$, $\mathbf{v}_\pm$. Thus, we can consider these vectors to be the starting point for the computation of B^{-1}.

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First we discuss the case where n is odd, $n = 2\ell - 1$. We are going to present an $O(n^2)$ algorithm that computes the inverse of B based on the observations made in the previous sections.

(1o) Denote $\mathbf{u}_+^{n+2} := \mathbf{u}_+, \mathbf{v}_+^{n+2} := \mathbf{v}_+$. Compute (e.g., by using the Horner scheme)

$$\mathbf{u}_+^n(t) := \frac{\mathbf{u}_-(t)}{t^2-1}, \quad \mathbf{v}_+^n(t) := \frac{\mathbf{v}_-(t)}{t^2-1}.$$

(2o) Write these vectors as

$$\mathbf{u}_+^{n+2i} =: (u_{+,j}^{n+2i})_{j=-\ell-i+1}^{\ell+i-1}, \qquad \mathbf{v}_+^{n+2i} =: (v_{+,j}^{n+2i})_{j=-\ell-i+1}^{\ell+i-1}, \qquad i = 0, 1.$$

Recall Remark 6.3 and compute the vectors

$$\mathbf{u}^{\ell+i} := (Q_{\ell+i})^{-1} \left(u_{+,j}^{n+2i}\right)_{j=0}^{\ell+i-1}, \quad \mathbf{v}^{\ell+i} := (Q_{\ell+i})^{-1} \left(v_{+,j}^{n+2i}\right)_{j=0}^{\ell+i-1},$$

where $(Q_{\ell+i})^{-1}$ is given in Proposition 5.2.

(3o) Find the vectors $\mathbf{s}^{2(\ell+i)-3} = (s_j^{(i)})_{j=0}^{2(\ell+i)-3}$ $(i = 0, 1)$ from the solutions of the Bezout equations (7.6) and by polynomial multiplication (7.7). The Bezout equations can be solved via Euclid's algorithm.

(4o) It remains to compute the symmetric vectors $\mathbf{a}^{(i)} = (a_k^{(i)})_{k=-2\ell-2}^{2\ell-2}$ which determine the matrices $\mathrm{TH}_\ell^\#(\mathbf{a}^{(0)})$ and $\mathrm{TH}_{\ell-1}(\mathbf{a}^{(1)})$. From (9.6) and (9.8) we get

$$a_0^{(0)} = 2s_0^{(0)}, \quad a_k^{(0)} = a_{-k}^{(0)} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j s_{k-2j}^{(0)} \left(\binom{k-j}{j} + \binom{k-j-1}{j-1} \right),$$

for $k = 1, \ldots, 2\ell - 2$, and

$$a_k^{(1)} - a_{k+2}^{(1)} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j s_{k-2j}^{(1)} \binom{k-j}{j}, \qquad a_k^{(1)} = a_{-k}^{(1)},$$

for $k = 0, \ldots, 2\ell - 4$. Now B^{-1} is given by (9.12). Notice that, as expected, the vector $\mathbf{a}^{(1)}$ is not uniquely determined. However, this ambiguity does not affect $\mathrm{TH}_{\ell-1}(\mathbf{a}^{(1)})$ and the matrix (9.12).

REMARK 10.1. The steps (1o) and (2o) can be performed in an alternative way by using the inverse of $R_{\ell+i}$ (given in Proposition 5.2) and

$$\mathbf{u}_-^{n+2i+2}(t) := (t^2-1)\mathbf{u}_+^{n+2i}(t), \qquad \mathbf{v}_-^{n+2i+2}(t) := (t^2-1)\mathbf{v}_+^{n+2i}(t)$$

(belonging to $\mathbb{F}_-^{n+2i+2}[t]$) instead of $\mathbf{u}_+^{n+2i}$ and $\mathbf{v}_+^{n+2i}$. Clearly, in terms of the given vectors,

$$\begin{aligned} \mathbf{u}_-^{n+4}(t) &:= (t^2-1)\mathbf{u}_+(t), & \mathbf{u}_-^{n+2} &:= \mathbf{u}_-, \\ \mathbf{v}_-^{n+4}(t) &:= (t^2-1)\mathbf{v}_+(t), & \mathbf{v}_-^{n+2} &:= \mathbf{v}_-. \end{aligned}$$

Writing the vectors as

$$\mathbf{u}_-^{n+2i+2} =: (u_{-,j}^{n+2i+2})_{j=-\ell-i}^{\ell+i}, \qquad \mathbf{v}_-^{n+2i+2} =: (v_{-,j}^{n+2i+2})_{j=-\ell-i}^{\ell+i}, \qquad i = 0, 1,$$

it follows that

$$\mathbf{u}^{\ell+i} = (R_{\ell+i})^{-1}(u_{-,j}^{n+2i+2})_{j=1}^{\ell+i}, \qquad \mathbf{v}^{\ell+i} = (R_{\ell+i})^{-1}(v_{-,j}^{n+2i+2})_{j=1}^{\ell+i}.$$

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Secondly, we consider the even case $n = 2\ell$ and design an $O(n^2)$ inversion algorithm similar to the algorithm above for the odd case.

(1e) Compute (e.g., by using the Horner scheme)

$$\mathbf{u}_+^{n+1}(t) := \frac{\mathbf{u}_+(t)}{t+1},\ \mathbf{v}_+^{n+1}(t) := \frac{\mathbf{v}_+(t)}{t+1},\ \mathbf{y}_+^{n+1}(t) := \frac{\mathbf{u}_-(t)}{t-1},\ \mathbf{z}_+^{n+1}(t) := \frac{\mathbf{v}_-(t)}{t-1}.$$

(2e) Denote by $\left(u_j^{n+1}\right)_{j=0}^{\ell}, \left(v_j^{n+1}\right)_{j=0}^{\ell}, \left(y_j^{n+1}\right)_{j=0}^{\ell}, \left(z_j^{n+1}\right)_{j=0}^{\ell}$ the vectors of the last $\ell+1$ components of $\mathbf{u}_+^{n+1}, \mathbf{v}_+^{n+1}, \mathbf{y}_+^{n+1}, \mathbf{z}_+^{n+1}$, respectively. Compute the vectors

$$\begin{aligned} \mathbf{u} &= (Q_{\ell+1})^{-1}\left(u_j^{n+1}\right)_{j=0}^{\ell}, \quad & \mathbf{v} &= (Q_{\ell+1})^{-1}\left(v_j^{n+1}\right)_{j=0}^{\ell}, \\ \mathbf{y} &= (Q_{\ell+1})^{-1}\left(y_j^{n+1}\right)_{j=0}^{\ell}, \quad & \mathbf{z} &= (Q_{\ell+1})^{-1}\left(z_j^{n+1}\right)_{j=0}^{\ell}, \end{aligned}$$

where $(Q_{\ell+i})^{-1}$ is given in Proposition 5.2.

(3e) Find the vectors $\mathbf{s}_i^{2\ell-1} = (s_j^{(i)})_{j=0}^{2\ell-1}$ $(i = 1, 2)$ from the solutions of the Bezout equations (8.3) and (8.4) via Euclid's algorithm and by polynomial multiplications (8.5).

(4e) It remains to compute the symmetric vectors $\mathbf{a}^{\pm} = (a_k^{\pm})_{k=-2\ell+1}^{2\ell-1}$ which build the matrices $\mathrm{TH}_\ell^{\pm}(\mathbf{a}^{\pm})$. In view of (9.9) we obtain the following equations,

$$a_{k+1}^+ + a_k^+ = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k-j}{j} s_{k-2j}^{(1)} - \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{k-j-1}{j} s_{k-2j-1}^{(1)}$$

and

$$a_k^- - a_{k+1}^- = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k-j}{j} s_{k-2j}^{(2)} + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{k-j-1}{j} s_{k-2j-1}^{(2)},$$

for $k = 0, \dots, 2\ell-2$, along with $a_k^{\pm} = a_{-k}^{\pm}$.
Now B^{-1} is represented as $T+H$ matrix according to (9.15). As before, the vectors $\mathbf{a}^{\pm}$ are not uniquely determined. This does not affect $\mathrm{TH}_\ell^{\pm}(\mathbf{a}^{\pm})$ and the matrix (9.15).

REMARK 10.2. The steps (1e) and (2e) can be performed in one step using the inverse of $R_\ell^{\pm}$. Indeed, we have

$$\begin{aligned} \mathbf{u} &= (R_\ell^+)^{-1}(u_i^+)_{i=0}^{\ell-1}, \quad & \mathbf{u}_+ &= (u_i^+)_{i=-\ell}^{\ell-1} \in \mathbb{F}_+^n, \\ \mathbf{v} &= (R_\ell^+)^{-1}(v_i^+)_{i=0}^{\ell-1}, \quad & \mathbf{v}_+ &= (v_i^+)_{i=-\ell}^{\ell-1} \in \mathbb{F}_+^n, \\ \mathbf{y} &= (R_\ell^-)^{-1}(u_i^-)_{i=0}^{\ell-1}, \quad & \mathbf{u}_- &= (u_i^-)_{i=-\ell}^{\ell-1} \in \mathbb{F}_-^n, \\ \mathbf{z} &= (R_\ell^+)^{-1}(v_i^-)_{i=0}^{\ell-1}, \quad & \mathbf{v}_- &= (v_i^-)_{i=-\ell}^{\ell-1} \in \mathbb{F}_-^n. \end{aligned}$$

A formula for the inverse of $R_\ell^{\pm}$ is given in (5.4).

11. Alternative representations. Let us first consider the odd case $n = 2\ell-1$. We arrived at a representation of the inverse of the $T+H$ Bezoutian $B^{-1} = A_+^o + A_-^o$ as the sum of two centrosymmetric $T+H$ matrices with particular symmetries. These two matrices are represented in (9.10) and (9.11). While the first one,

$$A_+^o = \tfrac{1}{4} Z_\ell^+ \mathrm{TH}_\ell^{\#}(\mathbf{a}^{(0)})(Z_\ell^+)^T = \tfrac{1}{4}\left[\, a_{j-k}^{(0)} + a_{j+k}^{(0)} \,\right]_{j,k=-\ell+1}^{\ell-1},$$

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has a unique symbol, the symbol of the second one

$$A_-^o = \tfrac{1}{4} Z_{\ell-1}^- \mathrm{TH}_{\ell-1}(\mathbf{a}^{(1)})(Z_{\ell-1}^-)^T = \tfrac{1}{4}[\, a_{j-k}^{(1)} - a_{j+k}^{(1)}\,]_{j,k=-\ell+1}^{\ell-1}$$

is not unique. It is possible to remedy this situation and to obtain a representation that involves uniquely determined coefficients. It comes at the expense that the modified representation is a product of three matrices.

In order to derive this formula, we start again from

$$A_-^o = \tfrac{1}{4} Z_{\ell-1}^- A_{\ell-1}^{(1)} (Z_{\ell-1}^-)^T, \qquad A_{\ell-1}^{(1)} = R_{\ell-1}^{-T} H(\mathbf{s}^{2\ell-3}) R_{\ell-1}^{-1}$$

but apply the first relationship in (9.3) of Theorem 9.1, namely that between the Hankel matrix and the matrix $\mathrm{TH}_{\ell-1}^{\#}$ (rather than $\mathrm{TH}_{\ell-1}$). This yields

$$A_{\ell-1}^{(1)} = R_{\ell-1}^{-T} Q_{\ell-1}^T D_{\ell-1} \mathrm{TH}_{\ell-1}^{\#}(\hat{\mathbf{a}}^{(1)}) D_{\ell-1} Q_{\ell-1} R_{\ell-1}^{-1},$$

with

$$\hat{\mathbf{a}}^{(1)} = (\hat{a}_k^{(1)})_{k=-2\ell-4}^{2\ell-4} \in \mathbb{F}_+^{4\ell-7}, \quad (\hat{a}_k^{(1)})_{k=0}^{2\ell-4} = D_{2\ell-3}^{-1} (Q_{2\ell-3})^{-T} \mathbf{s}^{2\ell-3}. \tag{11.1}$$

Using (5.3) it follows that

$$A_{\ell-1}^{(1)} = T_{\ell-1}^{-T} D_{\ell-1} \mathrm{TH}_{\ell-1}^{\#}(\hat{\mathbf{a}}^{(1)}) D_{\ell-1} T_{\ell-1}^{-1}$$

and thus

$$A_-^o = \tfrac{1}{4} Z_{\ell-1}^- T_{\ell-1}^{-T} D_{\ell-1} \mathrm{TH}_{\ell-1}^{\#}(\hat{\mathbf{a}}^{(1)}) D_{\ell-1} T_{\ell-1}^{-1} (Z_{\ell-1}^-)^T.$$

Introduce the $(2\ell-3) \times (2\ell-1)$ matrix

$$\Sigma_{2\ell-1} = \frac{1}{2} \begin{bmatrix} -1 & & & & & & & & & & 1 \\ 0 & -1 & & & & & & & & 1 & 0 \\ -1 & 0 & -1 & & & & & & 1 & 0 & 1 \\ \vdots & & \ddots & \ddots & & & & \iddots & \iddots & & \vdots \\ \vdots & & & 0 & -1 & 0 & 1 & 0 & & & \vdots \\ \vdots & & \iddots & \iddots & & & & \ddots & \ddots & & \vdots \\ -1 & 0 & -1 & & & & & & 1 & 0 & 1 \\ 0 & -1 & & & & & & & & 1 & 0 \\ -1 & & & & & & & & & & 1 \end{bmatrix},$$

which has a triangular-type structure combined with a checkerboard pattern. Notice that the columns are even and the rows are odd vectors. The column in the middle is zero. Incidentally, $\Sigma_{2\ell-1}$ establishes an isomorphism between $\mathbb{F}_-^{2\ell-1}$ and $\mathbb{F}_+^{2\ell-3}$; compare (2.2). A straightforward computation gives that

$$Z_{\ell-1}^- = \Sigma_{2\ell-1}^T Z_{\ell-1}^+ D_{\ell-1}^{-1} T_{\ell-1}^T.$$

Therefore,

$$A_-^o = \tfrac{1}{4} \Sigma_{2\ell-1}^T Z_{\ell-1}^+ \mathrm{TH}_{\ell-1}^{\#}(\hat{\mathbf{a}}^{(1)}) (Z_{\ell-1}^+)^T \Sigma_{2\ell-1}\,,$$

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and the product in the middle computes to a particular centrosymmetric $T+H$ matrix of size $2\ell-3=n-2$.

We are now in the position to present the promised alternative formula, a representation as a product of three matrices.

PROPOSITION 11.1.

$$A_-^o = \tfrac{1}{4}\Sigma_{2\ell-1}^T \left[\hat{a}_{j-k}^{(1)} + \hat{a}_{j+k}^{(1)}\right]_{j,k=-\ell+2}^{\ell-2} \Sigma_{2\ell-1}\,,$$

with $\hat{\mathbf{a}}^{(1)}$ *being defined in* (11.1).

Although it is not immediately clear from this formula that A_-^o is a centrosymmetric $T+H$ matrix as well, this can be verified with little effort and follows also from (9.11). In particular, one can show that the relationship between the symbols $\mathbf{a}^{(1)} \in \mathbb{F}_+^{4\ell-3}$ and $\hat{\mathbf{a}}^{(1)} \in \mathbb{F}_+^{4\ell-7}$ is given by

$$\hat{a}_k^{(1)} = 2a_k^{(1)} - a_{k+2}^{(1)} - a_{k-2}^{(1)}\,;$$

see also (9.1).

The derived formula may be of advantage when one is interested in multiplying A_-^o with a vector since the multiplication with $\Sigma_{2\ell-1}$ and $\Sigma_{2\ell-1}^T$ can be performed fast.

Let us now turn to the even case $n=2\ell$, where we had the expressions (9.13) and (9.14) for A_+^e and A_-^e, both involving nonunique symbols $\mathbf{a}^\pm$. To obtain alternative formulas, start from (8.7) and (8.8). Again apply only the first identity in (9.3) in order to conclude that

$$A_\ell^\pm = (R_\ell^\pm)^{-T} Q_\ell^T D_\ell \mathrm{TH}_\ell^\#(\hat{\mathbf{a}}^\pm) D_\ell Q_\ell (R_\ell^\pm)^{-1}\,,$$

with a unique $\hat{\mathbf{a}}^\pm = (\hat{a}_k^\pm)_{k=-2\ell+2}^{2\ell-2} \in \mathbb{F}_+^{4\ell-3}$ given by

$$(\hat{a}_k^+)_{k=0}^{2\ell-2} = D_{2\ell-1}^{-1}(Q_{2\ell-1})^{-T}\mathbf{s}_1^{2\ell-1}\,, \quad (\hat{a}_k^-)_{k=0}^{2\ell-2} = D_{2\ell-1}^{-1}(Q_{2\ell-1})^{-T}\mathbf{s}_2^{2\ell-1}\,. \tag{11.2}$$

Use (5.3) to conclude

$$A_\ell^\pm = (T_\ell^\pm)^{-T} D_\ell \mathrm{TH}_\ell^\#(\hat{\mathbf{a}}^\pm) D_\ell (T_\ell^\pm)^{-1}$$

and thus

$$A_\pm^e = \tfrac{1}{4}\begin{bmatrix} \pm J_\ell \\ I_\ell \end{bmatrix} (T_\ell^\pm)^{-T} D_\ell \mathrm{TH}_\ell^\#(\hat{\mathbf{a}}^\pm) D_\ell (T_\ell^\pm)^{-1} \begin{bmatrix} \pm J_\ell & I_\ell \end{bmatrix}.$$

Introduce the $(2\ell-1)\times 2\ell$ matrices

$$\Sigma_{2\ell}^\pm = \frac{1}{2}\begin{bmatrix}
\pm1 & & & & & & & & & 1 \\
-1 & \pm1 & & & & & & & 1 & \mp1 \\
\pm1 & -1 & \pm1 & & & & & 1 & \mp1 & 1 \\
\vdots & & \ddots & \ddots & & & \iddots & \iddots & & \vdots \\
\vdots & & & -1 & \pm1 & 1 & \mp1 & & & \vdots \\
\vdots & & \iddots & \iddots & & & \ddots & \ddots & & \vdots \\
\pm1 & -1 & \pm1 & & & & & 1 & \mp1 & 1 \\
-1 & \pm1 & & & & & & & 1 & \mp1 \\
\pm1 & & & & & & & & & 1
\end{bmatrix}.$$

The columns are all even vectors, while the rows are even or odd depending on the $(+)$ or $(-)$ case. It is straightforwardly verified that

$$\begin{bmatrix} \pm J_\ell \\ I_\ell \end{bmatrix} = (\Sigma_{2\ell}^{\pm})^T Z_\ell^{+} D_\ell^{-1} (T_\ell^{\pm})^T .$$

It follows that

$$A_{\pm}^{e} = \tfrac{1}{4} (\Sigma_{2\ell}^{\pm})^T Z_\ell^{+} \mathrm{TH}_\ell^{\#}(\hat{\mathbf{a}}^{\pm}) (Z_\ell^{+})^T \Sigma_{2\ell}^{\pm} .$$

As before, the middle term becomes a $T+H$ matrix, and thus we obtain the desired alternative formula for $A_{\pm}^{e}$.

PROPOSITION 11.2.

$$A_{\pm}^{e} = \tfrac{1}{4} (\Sigma_{2\ell}^{\pm})^T \left[\hat{a}_{j-k}^{\pm} + \hat{a}_{j+k}^{\pm} \right]_{j,k=-\ell+1}^{\ell-1} \Sigma_{2\ell}^{\pm} ,$$

where $\hat{\mathbf{a}}^{\pm}$ *is defined in* (11.2).

The relationship to (9.13) and (9.14) in terms of the symbol is

$$\hat{a}_k^{\pm} = 2a_k^{\pm} \pm a_{k+1}^{\pm} \pm a_{k-1}^{\pm} ,$$

as one can easily check.

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