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EXPLICIT FORMULAS FOR HERMITE-TYPE INTERPOLATION ON THE CIRCLE AND APPLICATIONS*

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Abstract. In this paper we study two ways of obtaining Laurent polynomials of Hermite interpolation on the unit circle. The corresponding nodal system is constituted by the nth roots of a complex number with modulus one. One of the interpolation formulas is given in terms of an appropriate basis which yields coefficients computable by means of the fast Fourier transform (FFT). The other formula is of barycentric type. As a consequence, we illustrate some applications to the Hermite interpolation problem on $[-1, 1]$. Some numerical tests are presented to emphasize the numerical stability of these formulas.

Key words. Hermite interpolation, Laurent polynomials, barycentric formulas, unit circle, Chebyshev polynomials

AMS subject classifications. 65D05, 41A05, 33C45

1. Introduction. An important result in the study of Hermite interpolation problems on the unit circle $\mathbb{T}$ is the extension of the Hermite-Fejér theorem (cf. [7]) to the unit circle. This result appears in [6], where it is proved that the Laurent polynomials of Hermite-Fejér interpolation related to a continuous function on $\mathbb{T}$ uniformly converge to the function, with the nodal system constituted by the nth roots of some complex number with modulus one. The extension of the second Fejér theorem to the unit circle has been studied in [2]. In the same reference one also finds a convergence result for continuous functions in the case of Hermite interpolation, that is, with non-vanishing derivatives. The exact knowledge of the interpolants is the main tool to obtain the results in [2], and the sufficient condition given there cannot be improved.

A method to determine in an efficient way the Laurent polynomials of Hermite interpolation is presented in [1]. There the nodes are equally spaced on $\mathbb{T}$, and the method makes use of the fast Fourier transform (FFT). The formulas obtained in [1] are based on the construction of an orthogonal basis of the space of algebraic polynomials with respect to a Sobolev-type inner product related with the nodal system. Taking into account that continuous functions on $\mathbb{T}$ can be uniformly approximated by Laurent polynomials, these type of polynomials are very well suited for interpolation. Thus, the aim of this paper is to present two different methods for obtaining Laurent polynomials of Hermite interpolation on the unit circle with nodal systems constituted by the nth roots of some complex number with modulus one. One of them is based on a functional series whose coefficients can be computed efficiently by using the FFT. The other relies on a barycentric formulation which is new and very suitable for numerical evaluations. Therefore, the main objective of this work is to obtain such type of formulas as well as other expressions which have been shown to be adequate for numerical calculations. As an application, two different formulations are derived for Hermite interpolation polynomials in the interval $[-1, 1]$ with the zeros of the four families of Chebyshev polynomials as nodes. One formulation is given in terms of the Chebyshev basis of the first kind while the other is of barycentric type.

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The organization of the paper is as follows. In Section 2 we present two different expressions for the Laurent polynomials of Hermite interpolation whose nodes are the roots of complex unimodular numbers. The first form is derived by using an appropriate basis which in turn allows to obtain coefficients computable using the FFT. The second form is the so-called barycentric one. As an application of the previous results, in Section 3, we obtain Hermite interpolation formulas for nodal systems on $[-1, 1]$. These systems are constituted by the zeros of the four families of Chebyshev polynomials; cf. [11]. Some aspects of a computational implementation of these formulas are discussed in Section 4. Through the numerical results shown in that section, the exceptional numerical stability of the mathematical formulas derived is emphasized.

2. Hermite interpolation on the unit circle. First we present two ways of obtaining Hermite interpolation polynomials in the space of Laurent polynomials. The nodes we select are the nth roots $\{\alpha_j\}_{j=0}^{n-1}$ of a complex number λ with $|\lambda| = 1$.

We recall that the Hermite interpolation problem on the unit circle consists in obtaining a Laurent polynomial $\mathcal{H}_{-n,n-1}(z) \in \Lambda_{-n,n-1}[z] = \text{span}\{z^k : -n \leq k \leq n-1\}$ that satisfies the interpolation conditions

$$\mathcal{H}_{-n,n-1}(\alpha_j) = u_j \quad \text{and} \quad \mathcal{H}'_{-n,n-1}(\alpha_j) = v_j, \qquad \text{for } j = 0, \dots, n-1, \tag{2.1}$$

where $\{u_j\}_{j=0}^{n-1}$ and $\{v_j\}_{j=0}^{n-1}$ are fixed complex values. The situation corresponding to $v_j = 0$, for $j = 0, \dots, n-1$, is called Hermite-Fejér interpolation.

In a more general setting the problem can be posed as follows: if $p(n)$ and $q(n)$ are two nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = 2n - 1$, for $n = 1, 2, \dots$, find the unique Laurent polynomial

$$\mathcal{H}_{-p(n),q(n)}(z) \in \Lambda_{-p(n),q(n)} = \text{span}\{z^k : -p(n) \leq k \leq q(n)\}$$

such that

$$\mathcal{H}_{-p(n),q(n)}(\alpha_j) = u_j \quad \text{and} \quad \mathcal{H}'_{-p(n),q(n)}(\alpha_j) = v_j, \qquad \text{for } j = 0, \dots, n-1. \tag{2.2}$$

For simplicity and without loss of generality, only the problem posed in equation (2.1) will be considered. The other one stated in equation (2.2) can be solved in a similar way.

It is well-known that $\mathcal{H}_{-n,n-1}(z)$ can be computed by using the fundamental polynomials of Hermite interpolation (cf. [6, 15]); another useful formula is given in [1]. One of the advantages we observe in the latter is that the coefficients can be computed in an efficient way by the FFT.

In order to obtain another formula, we introduce the following auxiliary polynomials. We denote by $\mathcal{L}_{0,k}(z)$ the Laurent polynomial of Hermite-Fejér interpolation related to z^k for $k = 0, \dots, n-1$ and which is characterized by

$$\mathcal{L}_{0,k}(z) \in \Lambda_{-n,n-1}[z], \;\; \mathcal{L}_{0,k}(\alpha_j) = \alpha_j^k, \;\; \mathcal{L}'_{0,k}(\alpha_j) = 0, \quad \text{for } j = 0, \dots, n-1. \tag{2.3}$$

We also denote by $\mathcal{L}_{1,k}(z)$ the Laurent polynomial of Hermite interpolation given by $\mathcal{L}_{1,k}(z) \in \Lambda_{-n,n-1}[z]$ and characterized by

$$\begin{aligned} &\mathcal{L}_{1,k}(\alpha_j) = 0, \;\; \mathcal{L}'_{1,k}(\alpha_j) = k\alpha_j^{k-1}, && \text{for } j = 0, \dots, n-1, k = 1, \dots, n-1, \\ &\mathcal{L}_{1,0}(\alpha_j) = 0, \;\; \mathcal{L}'_{1,0}(\alpha_j) = \alpha_j^{-1}, && \text{for } j = 0, \dots, n-1. \end{aligned} \tag{2.4}$$

It is easy to obtain explicit expressions for the above polynomials as well as to deduce some properties that they satisfy. These results are summarized in the following proposition.

PROPOSITION 2.1. *Let* $\{\mathcal{L}_{0,k}(z)\}_{k=0}^{n-1}$ *and* $\{\mathcal{L}_{1,k}(z)\}_{k=0}^{n-1}$ *be the Laurent polynomials characterized by* (2.3) *and* (2.4)*, respectively. Then the following relations hold true.*

(i) The system $\{\mathcal{L}_{0,k}(z)\}_{k=0}^{n-1} \bigcup \{\mathcal{L}_{1,k}(z)\}_{k=0}^{n-1}$ *is an orthogonal basis of* $\Lambda_{-n,n-1}[z]$ *with respect to the discrete inner product defined by*

$$\langle \mathcal{P}, \mathcal{Q} \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \left[\mathcal{P}(\alpha_i)\overline{\mathcal{Q}(\alpha_i)} + \mathcal{P}'(\alpha_i)\overline{\mathcal{Q}'(\alpha_i)} \right], \qquad \text{for } \mathcal{P}, \mathcal{Q} \in \Lambda_{-n,n-1}[z].$$

(ii) The Laurent polynomials $\mathcal{L}_{0,k}(z)$ *are given by*

$$\mathcal{L}_{0,k}(z) = \frac{\lambda k}{n} z^{k-n} + (1 - \frac{k}{n}) z^k, \qquad \text{for } k = 0, \ldots, n-1, \tag{2.5}$$

and the Laurent polynomials $\mathcal{L}_{1,k}(z)$ *are given by*

$$\begin{aligned} &\mathcal{L}_{1,k}(z) = -\frac{\lambda k}{n} z^{k-n} + \frac{k}{n} z^k = \frac{k}{n} z^{k-n}(z^n - \lambda), \quad \text{for } k = 1, \ldots, n-1, \\ &\mathcal{L}_{1,0}(z) = \frac{1}{n} z^{-n}(z^n - \lambda). \end{aligned} \tag{2.6}$$

Proof. (i) By using the well-known properties of the roots of the unity, we have

$$\begin{aligned} \langle \mathcal{L}_{0,k}, \mathcal{L}_{0,l} \rangle &= \frac{1}{n} \sum_{i=0}^{n-1} \alpha_i^{k-l} = \delta_{k,l} && \text{for } k, l = 0, \ldots, n-1, \\ \langle \mathcal{L}_{1,k}, \mathcal{L}_{1,l} \rangle &= \frac{1}{n} \sum_{i=0}^{n-1} kl\alpha_i^{k-l} = k^2 \delta_{k,l} && \text{for } k, l = 1, \ldots, n-1, \\ \langle \mathcal{L}_{1,0}, \mathcal{L}_{1,l} \rangle &= \frac{l}{n} \sum_{i=0}^{n-1} \alpha_i^{-l} = 0 && \text{for } l = 1, \ldots, n-1, \\ \langle \mathcal{L}_{1,0}, \mathcal{L}_{1,0} \rangle &= 1, \\ \langle \mathcal{L}_{0,k}, \mathcal{L}_{1,l} \rangle &= 0 && \text{for } k, l = 0, \ldots, n-1, \end{aligned}$$

which proves (i).

(ii) For each $k = 0, \ldots, n-1$, it is clear that $\mathcal{L}_{0,k}(z)$ defined by (2.5) satisfies (2.3). Indeed,

$$\begin{aligned} \mathcal{L}_{0,k}(\alpha_i) &= \frac{\lambda k}{n} \alpha_i^{k-n} + \left(1 - \frac{k}{n}\right) \alpha_i^k = \frac{k}{n} \alpha_i^k + \left(1 - \frac{k}{n}\right) \alpha_i^k = \alpha_i^k, \\ \mathcal{L}'_{0,k}(\alpha_i) &= \frac{\lambda k}{n} (k-n) \alpha_i^{k-n-1} + \left(1 - \frac{k}{n}\right) k\alpha_i^{k-1} \\ &= \frac{k}{n} (k-n) \alpha_i^{k-1} + \left(1 - \frac{k}{n}\right) k\alpha_i^{k-1} = 0. \end{aligned}$$

The relation (2.6) can be obtained in the same way. ∎

We are now able to obtain the Laurent polynomial $\mathcal{H}_{-n,n-1}(z)$ satisfying (2.1).

PROPOSITION 2.2.

(i) The Laurent polynomial $\mathcal{HF}_{-n,n-1}(z) \in \Lambda_{-n,n-1}[z]$ *satisfying the conditions* $\mathcal{HF}_{-n,n-1}(\alpha_j) = u_j$ *and* $\mathcal{HF}'_{-n,n-1}(\alpha_j) = 0$*, for* $j = 0, \ldots, n-1$*, is*

$$\mathcal{HF}_{-n,n-1}(z) = \frac{1}{n^2} \sum_{k=0}^{n-1} \left(\sum_{i=0}^{n-1} u_i \overline{\alpha_i}^k \right) \left(\lambda k z^{k-n} + (n-k) z^k \right).$$

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(ii) The Laurent polynomial $\mathcal{HD}_{-n,n-1}(z) \in \Lambda_{-n,n-1}[z]$, which satisfies the conditions $\mathcal{HD}_{-n,n-1}(\alpha_j) = 0$ and $\mathcal{HD}'_{-n,n-1}(\alpha_j) = v_j$, for $j = 0, \dots, n-1$, can be written as

$$\mathcal{HD}_{-n,n-1}(z) = \frac{1}{n^2} \sum_{k=0}^{n-1} \left(\sum_{i=0}^{n-1} v_i \overline{\alpha_i}^{k-1} \right) z^{k-n}(z^n - \lambda).$$

(iii) The Laurent polynomial $\mathcal{H}_{-n,n-1}(z) \in \Lambda_{-n,n-1}[z]$ satisfying (2.1) is given by

$$\begin{aligned} \mathcal{H}_{-n,n-1}(z) = &\frac{1}{n^2} \sum_{k=0}^{n-1} \left[\left(\sum_{i=0}^{n-1} u_i \overline{\alpha_i}^{k} \right) \left(\lambda k z^{k-n} + (n-k) z^k \right) \right. \\ &\left. + \left(\sum_{i=0}^{n-1} v_i \overline{\alpha_i}^{k-1} \right) z^{k-n} \left(z^n - \lambda \right) \right]. \end{aligned} \tag{2.7}$$

Proof. (i) From Proposition 2.1 we have

$$\mathcal{HF}_{-n,n-1}(z) = \sum_{k=0}^{n-1} (a_{k,0} \mathcal{L}_{0,k}(z) + b_{k,0} \mathcal{L}_{1,k}(z)).$$

By using orthogonality properties, we have $0 = \langle \mathcal{HF}_{-n,n-1}, \mathcal{L}_{1,k} \rangle = b_{k,0} \|\mathcal{L}_{1,k}\|^2$, yielding the identity $b_{k,0} = 0$ for all $k = 0, \dots, n-1$. In the same way, taking into account that $\langle \mathcal{HF}_{-n,n-1}, \mathcal{L}_{0,k} \rangle = a_{k,0} \|\mathcal{L}_{0,k}\|^2$ and $\langle \mathcal{HF}_{-n,n-1}, \mathcal{L}_{0,k} \rangle = \frac{1}{n} \sum_{i=0}^{n-1} u_i \overline{\alpha_i}^k$, we obtain that $a_{k,0} = \frac{1}{n} \sum_{i=0}^{n-1} u_i \overline{\alpha_i}^k$.

(ii) Again, from Proposition 2.1, we have that

$$\mathcal{HD}_{-n,n-1}(z) = \sum_{k=0}^{n-1} (a_{k,1} \mathcal{L}_{0,k}(z) + b_{k,1} \mathcal{L}_{1,k}(z)).$$

Since $0 = \langle \mathcal{HD}_{-n,n-1}, \mathcal{L}_{0,k} \rangle = a_{k,1} \|\mathcal{L}_{0,k}\|^2$, we get that $a_{k,1} = 0$ for all $k = 0, \dots, n-1$. Moreover, $\langle \mathcal{HD}_{-n,n-1}, \mathcal{L}_{1,k} \rangle = b_{k,1} \|\mathcal{L}_{1,k}\|^2 = b_{k,1} k^2$ for $k = 1, \dots, n-1$, and since $\langle \mathcal{HD}_{-n,n-1}, \mathcal{L}_{1,k} \rangle = \frac{1}{n} \sum_{i=0}^{n-1} v_i k \overline{\alpha_i}^{k-1}$, we obtain that $b_{k,1} = \frac{1}{kn} \sum_{i=0}^{n-1} v_i \overline{\alpha_i}^{k-1}$ for $k = 1, \dots, n-1$. To obtain the coefficient $b_{0,1}$, we take into account that

$$\langle \mathcal{HD}_{-n,n-1}, \mathcal{L}_{1,0} \rangle = b_{0,1} \|\mathcal{L}_{1,0}\|^2 = b_{0,1} \quad \text{and} \quad \langle \mathcal{HD}_{-n,n-1}, \mathcal{L}_{1,0} \rangle = \frac{1}{n} \sum_{i=0}^{n-1} v_i \overline{\alpha_i}^{-1}.$$

Therefore $b_{0,1} = \frac{1}{n} \sum_{i=0}^{n-1} v_i \overline{\alpha_i}^{-1}$, and (ii) follows.

(iii) This is a consequence of the fact that $\mathcal{H}_{-n,n-1}(z) = \mathcal{HF}_{-n,n-1}(z) + \mathcal{HD}_{-n,n-1}(z)$. ∎

REMARK 2.3. The coefficients of the Laurent polynomials $\mathcal{HF}_{-n,n-1}(z), \mathcal{HD}_{-n,n-1}(z)$, and $\mathcal{H}_{-n,n-1}(z)$ given in the preceding Proposition 2.2 can be obtained by using the FFT as in [1]. These expressions can be deduced, after some tedious computations, from those given in [1]. Thus, the approach presented here is simpler and more natural than that given there.

COROLLARY 2.4. *In the Laurent space $\Lambda_{-n,n-1}[z]$, the fundamental polynomials of Hermite interpolation, $\mathcal{A}_{-n,n-1,j}(z)$ and $\mathcal{B}_{-n,n-1,j}(z)$, for $j = 0, \dots, n-1$, characterized by*

$$\begin{array}{lll} \mathcal{A}_{-n,n-1,j}(\alpha_i) = \delta_{i,j}, & \mathcal{A}'_{-n,n-1,j}(\alpha_i) = 0, & \forall i = 0, \dots, n-1, \\ \mathcal{B}_{-n,n-1,j}(\alpha_i) = 0, & \mathcal{B}'_{-n,n-1,j}(\alpha_i) = \delta_{i,j}, & \forall i = 0, \dots, n-1, \end{array}$$

are given by the following expressions

$$\mathcal{A}_{-n,n-1,j}(z) = \frac{1}{n^2} \sum_{k=0}^{n-1} \left[\lambda k \overline{\alpha_j}^k z^{k-n} + (n-k) \overline{\alpha_j}^k z^k \right], \tag{2.8}$$

$$\mathcal{B}_{-n,n-1,j}(z) = \frac{1}{n^2} \sum_{k=0}^{n-1} \overline{\alpha_j}^{k-1} z^{k-n} (z^n - \lambda). \tag{2.9}$$

Proof. It is an immediate consequence of Proposition 2.2. ◻

A useful way to express the Hermite interpolation polynomials are the so-called barycentric formulas. To obtain them, we first write the fundamental polynomials in the compact form given in [6].

From (2.9) we obtain

$$\begin{aligned} \mathcal{B}_{-n,n-1,j}(z) &= \frac{1}{n^2} \sum_{k=0}^{n-1} \overline{\alpha_j}^{k-1} z^{k-n} (z^n - \lambda) = \frac{(z^n - \lambda)}{n^2 \overline{\alpha_j} z^n} \sum_{k=0}^{n-1} (\overline{\alpha_j} z)^k \\ &= \frac{(z^n - \lambda)^2}{n^2 \lambda \overline{\alpha_j}^2 z^n (z - \alpha_j)}, \end{aligned} \tag{2.10}$$

and from (2.8) we get

$$\begin{aligned} \mathcal{A}_{-n,n-1,j}(z) &= \frac{1}{n} \sum_{k=0}^{n-1} (\overline{\alpha_j} z)^k + \frac{1}{n^2} \sum_{k=0}^{n-1} (\overline{\alpha_j})^k (\lambda k z^{k-n} - k z^k) \\ &= \frac{(z^n - \lambda)}{n \lambda \overline{\alpha_j} (z - \alpha_j)} - \frac{(z^n - \lambda)}{n^2 z^n} \sum_{k=0}^{n-1} k (z \overline{\alpha_j})^k \\ &= \frac{(z^n - \lambda)}{n^2 \lambda \overline{\alpha_j} (z - \alpha_j)} - \frac{(z^n - \lambda)}{n^2 \lambda \overline{\alpha_j}^2 z^n} \frac{(\overline{\alpha_j} \lambda z - z^n)}{(z - \alpha_j)^2} \\ &= \frac{\alpha_j (z^n - \lambda)^2}{n^2 \lambda z^n (z - \alpha_j)} + \frac{\alpha_j^2 (z^n - \lambda)^2}{n^2 \lambda z^n (z - \alpha_j)^2}. \end{aligned} \tag{2.11}$$

PROPOSITION 2.5. *The Laurent polynomial $\mathcal{H}_{-n,n-1}(z) \in \Lambda_{-n,n-1}[z]$ satisfying (2.1) may be written in barycentric formulation as*

$$\mathcal{H}_{-n,n-1}(z) = \frac{\sum_{j=0}^{n-1} \left[\left(\frac{\alpha_j^2}{(z-\alpha_j)^2} + \frac{\alpha_j}{z-\alpha_j} \right) u_j + \frac{\alpha_j^2}{z-\alpha_j} v_j \right]}{\sum_{j=0}^{n-1} \left(\frac{\alpha_j^2}{(z-\alpha_j)^2} + \frac{\alpha_j}{z-\alpha_j} \right)}, \tag{2.12}$$

or equivalently as

$$\mathcal{H}_{-n,n-1}(z) = \frac{\sum_{j=0}^{n-1} \left(\frac{\alpha_j z}{(z-\alpha_j)^2} u_j + \frac{\alpha_j^2}{z-\alpha_j} v_j \right)}{\sum_{j=0}^{n-1} \frac{\alpha_j z}{(z-\alpha_j)^2}}. \tag{2.13}$$

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Proof. Taking into account that

$$\mathcal{H}_{-n,n-1}(z) = \sum_{j=0}^{n-1} \left(\mathcal{A}_{-n,n-1,j}(z)u_j + \mathcal{B}_{-n,n-1,j}(z)v_j\right) \quad \text{and} \quad 1 = \sum_{j=0}^{n-1} \mathcal{A}_{-n,n-1,j}(z),$$

where $\mathcal{A}_{-n,n-1,j}(z)$ and $\mathcal{B}_{-n,n-1,j}(z)$ are given by (2.10) and (2.11), respectively, we obtain that

$$\mathcal{H}_{-n,n-1}(z) = \frac{\sum_{j=0}^{n-1} \left(\mathcal{A}_{-n,n-1,j}(z)u_j + \mathcal{B}_{-n,n-1,j}(z)v_j\right)}{\sum_{j=0}^{n-1} \mathcal{A}_{-n,n-1,j}(z)},$$

from which (2.12) follows. From (2.12), using $\frac{\alpha_j^2}{(z-\alpha_j)^2} + \frac{\alpha_j}{z-\alpha_j} = \frac{\alpha_j z}{(z-\alpha_j)^2}$, we obtain (2.13). ∎

COROLLARY 2.6. *The barycentric formulas of the Laurent polynomials $\mathcal{HF}_{-n,n-1}(z)$ and $\mathcal{HD}_{-n,n-1}(z)$ introduced in Proposition 2.2 are*

$$\mathcal{HF}_{-n,n-1}(z) = \frac{\sum_{j=0}^{n-1} \frac{\alpha_j}{(z-\alpha_j)^2} u_j}{\sum_{j=0}^{n-1} \frac{\alpha_j}{(z-\alpha_j)^2}}$$

and

$$\mathcal{HD}_{-n,n-1}(z) = \frac{\sum_{j=0}^{n-1} \frac{\alpha_j^2}{z-\alpha_j} v_j}{\sum_{j=0}^{n-1} \frac{\alpha_j z}{(z-\alpha_j)^2}} = \frac{\sum_{j=0}^{n-1} \left(-\frac{\alpha_j}{z} + \frac{\alpha_j}{z-\alpha_j}\right) v_j}{\sum_{j=0}^{n-1} \frac{\alpha_j}{(z-\alpha_j)^2}}.$$

Proof. These formulas follow immediately from (2.13). ∎

3. Applications to Hermite interpolation on $[-1, 1]$.

3.1. Formulation in terms of the Chebyshev basis. Using Proposition 2.2, we can deduce suitable expressions for the algebraic polynomials of Hermite interpolation related to the nodal systems formed by the zeros of the four families of Chebyshev polynomials. These formulas are given in terms of the Chebyshev basis of the first kind, and for evaluation, one can use the algorithm given in [5]. In this regard we have the following proposition.

PROPOSITION 3.1.

(i) Let $\{x_j\}_{j=0}^{n-1} = \left\{\cos\left(\frac{(2j+1)\pi}{2n}\right)\right\}_{j=0}^{n-1}$ be the zeros of the Chebyshev polynomial of the first kind $T_n(x)$. Let $\{\mathfrak{m}_j\}_{j=0}^{n-1}$ and $\{\mathfrak{n}_j\}_{j=0}^{n-1}$ be prefixed real values. The Hermite interpolation polynomial $h_{2n-1}(x) \in \mathbb{P}_{2n-1}[x]$ satisfying the conditions $h_{2n-1}(x_j) = \mathfrak{m}_j$, $h'_{2n-1}(x_j) = \mathfrak{n}_j$, for $j = 0, \dots, n-1$, is given by

$$\begin{aligned} h_{2n-1}(x) = {} & \frac{1}{n^2} \sum_{k=0}^{n-1} \Re \left(\sum_{j=0}^{n-1} \mathfrak{m}_j y_j^k \right) \left((2n-k)T_k(x) - kT_{2n-k}(x)\right) \\ & + \frac{1}{n^2} \sum_{k=1}^{n-1} \Im \left(\sum_{j=0}^{n-1} \sqrt{1-x_j^2}\, \mathfrak{n}_j y_j^k \right) \left(T_k(x) + T_{2n-k}(x)\right), \end{aligned} \tag{3.1}$$

where $\{y_j, \overline{y_j}\}_{j=0}^{n-1}$ are the $(2n)$th roots of -1, that is, $y_j = e^{\mathrm{i}\frac{(2j+1)\pi}{2n}}$, for $j = 0, \dots, n-1$.

(ii) *Let* $\{x_j\}_{j=1}^{n-1} = \left\{\cos(\frac{j\pi}{n})\right\}_{j=1}^{n-1}$ *be the zeros of the Chebyshev polynomial of the second kind* $U_{n-1}(x)$*, and given the endpoints* $x_0 = 1$ *and* $x_n = -1$*. Let* $\{\mathfrak{m}_j\}_{j=0}^{n}$ *and* $\{\mathfrak{n}_j\}_{j=1}^{n-1}$ *be prefixed real values. The Hermite-type interpolation polynomial* $k_{2n-1}(x) \in \mathbb{P}_{2n-1}[x]$ *satisfying the conditions* $k_{2n-1}(x_j) = \mathfrak{m}_j$*, for* $j = 0, \ldots, n$*, and* $k'_{2n-1}(x_j) = \mathfrak{n}_j$*, for* $j = 1, \ldots, n-1$*, is given by*

$$
\begin{aligned}
k_{2n-1}(x) = {} & \frac{1}{2n^2}\sum_{k=1}^{n-1}\left[\mathfrak{m}_0 + 2\Re\left(\sum_{j=1}^{n-1}\mathfrak{m}_j z_j^k\right) + (-1)^k\mathfrak{m}_n\right] \times \\
& \left(kT_{2n-k}(x) + (2n-k)T_k(x)\right) \\
& + \frac{1}{2n}\left(\mathfrak{m}_0 + 2\sum_{j=1}^{n-1}\mathfrak{m}_j + \mathfrak{m}_n\right) \\
& + \frac{1}{2n}\left(\mathfrak{m}_0 + 2\Re\left(\sum_{j=1}^{n-1}\mathfrak{m}_j z_j^n\right) + (-1)^n\mathfrak{m}_n\right) T_n(x) \\
& + \frac{1}{n^2}\sum_{k=1}^{n-1}\Im\left(\sum_{j=1}^{n-1}\sqrt{1-x_j^2}\,\mathfrak{n}_j z_j^k\right)\left(T_k(x) - T_{2n-k}(x)\right),
\end{aligned} \tag{3.2}
$$

where $\{1\} \cup \{z_j, \overline{z_j}\}_{j=1}^{n-1} \cup \{-1\}$ *are the* $(2n)$*th roots of* 1*, that is,* $z_j = e^{\mathrm{i}\frac{j\pi}{n}}$*, for* $j = 1, \ldots, n-1$*.*

(iii) *Let* $\{x_j\}_{j=1}^{n-1} = \left\{\cos(\frac{(2j-1)\pi}{2n-1})\right\}_{j=1}^{n-1}$ *be the zeros of the Chebyshev polynomial of the third kind* $V_{n-1}(x)$*, and given the point* $x_n = -1$*. Let* $\{\mathfrak{m}_j\}_{j=1}^{n}$ *and* $\{\mathfrak{n}_j\}_{j=1}^{n-1}$ *be prefixed real values. The Hermite-type interpolation polynomial* $j_{2n-2}(x) \in \mathbb{P}_{2n-2}[x]$ *satisfying the conditions* $j_{2n-2}(x_j) = \mathfrak{m}_j$*, for* $j = 1, \ldots, n$*, and* $j'_{2n-1}(x_j) = \mathfrak{n}_j$*, for* $j = 1, \ldots, n-1$*, is given by*

$$
\begin{aligned}
j_{2n-2}(x) = {} & \frac{2}{(2n-1)^2}\sum_{k=1}^{n-1}\left(2\Re(\sum_{j=1}^{n-1}\mathfrak{m}_j y_j^k) + \mathfrak{m}_n(-1)^k\right) \times \\
& \left(-kT_{2n-k-1}(x) + (2n-k-1)T_k(x)\right) \\
& + \frac{1}{(2n-1)}\left(2\sum_{j=1}^{n-1}\mathfrak{m}_j + \mathfrak{m}_n\right) \\
& + \frac{4}{(2n-1)^2}\sum_{k=1}^{n-1}\Im\left(\sum_{j=1}^{n-1}\sqrt{1-x_j^2}\,\mathfrak{n}_j y_j^k\right) \times \\
& \left(T_k(x) - T_{2n-k-1}(x)\right),
\end{aligned} \tag{3.3}
$$

where $\{-1\} \cup \{y_j, \overline{y_j}\}_{j=1}^{n-1}$ *are the* $(2n-1)$*st roots of* -1*, that is,* $y_j = e^{\mathrm{i}\frac{(2j-1)\pi}{2n-1}}$*, for* $j = 1, \ldots, n-1$*.*

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(iv) Let $\{x_j\}_{j=1}^{n-1} = \left\{\cos(\frac{2j\pi}{2n-1})\right\}_{j=1}^{n-1}$ *be the zeros of the Chebyshev polynomial of the fourth kind* $W_{n-1}(x)$, *and given the point* $x_0 = 1$. *Let* $\{\mathfrak{m}_j\}_{j=0}^{n-1}$ *and* $\{\mathfrak{n}_j\}_{j=1}^{n-1}$ *be prefixed real values. The Hermite-type interpolation polynomial* $l_{2n-2}(x) \in \mathbb{P}_{2n-2}[x]$ *satisfying the conditions* $l_{2n-2}(x_j) = \mathfrak{m}_j$, *for* $j = 0, \ldots, n-1$, *and* $l'_{2n-1}(x_j) = \mathfrak{n}_j$, *for* $j = 1, \ldots, n-1$, *is given by*

$$
\begin{aligned}
l_{2n-2}(x) = {} & \frac{2}{(2n-1)^2} \sum_{k=0}^{n-1} \left[\mathfrak{m}_0 + 2\Re\left(\sum_{j=1}^{n-1} \mathfrak{m}_j z_j^k\right)\right] \times \\
& \qquad \left(kT_{2n-k-1}(x) + (2n-k-1)T_k(x)\right) \\
& + \frac{4}{(2n-1)^2} \sum_{k=1}^{n-1} \Im\left(\sum_{j=1}^{n-1} \sqrt{1-x_j^2}\,\mathfrak{n}_j z_j^k\right) \times \\
& \qquad \left(T_k(x) - T_{2n-k-1}(x)\right),
\end{aligned} \tag{3.4}
$$

where $\{1\} \cup \{z_j, \overline{z_j}\}_{j=0}^{n-1}$ *are the* $(2n-1)$*st roots of* 1, *that is,* $z_j = e^{\mathrm{i}\frac{2j\pi}{2n-1}}$, *for* $j = 1, \ldots, n-1$.

Proof. For simplicity, we only prove equation (3.1). The proof of the remaining formulas is similar. For doing it we transform the problem to that of finding the Laurent polynomial of Hermite interpolation $\mathcal{H} \in \Lambda_{-2n,2n-1}[z]$ such that $\mathcal{H}(y_j) = \mathcal{H}(\overline{y_j}) = \mathfrak{m}_j$ and

$$
\mathcal{H}'(y_j) = \mathrm{i}\sqrt{1-x_j^2}\,\overline{y_j}\mathfrak{n}_j, \quad \mathcal{H}'(\overline{y_j}) = -\mathrm{i}\sqrt{1-x_j^2}\,y_j\mathfrak{n}_j, \qquad \text{for } j = 0, \ldots, n-1.
$$

Hence by applying (2.7) in Proposition 2.2, we obtain

$$
\begin{aligned}
\mathcal{H}(z) = {} & \frac{1}{4n^2} \sum_{k=0}^{2n-1} \left(\sum_{j=0}^{n-1} \mathfrak{m}_j (y_j^k + \overline{y_j}^k)\right) \left(-kz^{k-2n} + (2n-k)z^k\right) \\
& + \frac{1}{4n^2} \sum_{k=0}^{2n-1} \left(\sum_{j=0}^{n-1} \mathrm{i}\sqrt{1-x_j^2}(\overline{y_j}^k - y_j^k)\mathfrak{n}_j\right) \left(z^k + z^{k-2n}\right).
\end{aligned} \tag{3.5}
$$

After some calculations, we find the following expressions for the terms in (3.5):

$$
\begin{aligned}
& \sum_{k=0}^{2n-1} \left(\sum_{j=0}^{n-1} \mathfrak{m}_j (y_j^k + \overline{y_j}^k)\right) \left(-kz^{k-2n} + (2n-k)z^k\right) \\
& = 2 \sum_{k=0}^{2n-1} \Re\left(\sum_{j=0}^{n-1} \mathfrak{m}_j y_j^k\right) \left(-kz^{k-2n} + (2n-k)z^k\right) \\
& = 2 \left[\sum_{k=0}^{n-1} \Re\left(\sum_{j=0}^{n-1} \mathfrak{m}_j y_j^k\right) + \sum_{k=n}^{2n-1} \Re\left(\sum_{j=0}^{n-1} \mathfrak{m}_j y_j^k\right)\right] \left(-kz^{k-2n} + (2n-k)z^k\right)
\end{aligned}
$$

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$$
\begin{aligned}
&= 2\sum_{k=0}^{n-1}\Re\left(\sum_{j=0}^{n-1}\mathfrak{m}_j y_j^k\right)\left(-kz^{k-2n}+(2n-k)z^k\right)\\
&\qquad + 2\sum_{l=1}^{n}\Re\left(\sum_{j=0}^{n-1}\mathfrak{m}_j y_j^{2n-l}\right)\left((l-2n)z^{-l}+lz^{2n-l}\right)\\
&= 4n\sum_{j=0}^{n-1}\mathfrak{m}_j + 2\sum_{k=1}^{n-1}\Re\left(\sum_{j=0}^{n-1}\mathfrak{m}_j y_j^k\right)\left(-kz^{k-2n}+(2n-k)z^k+(2n-k)z^{-k}-kz^{2n-k}\right)\\
&= 4n\sum_{j=0}^{n-1}\mathfrak{m}_j + 2\sum_{k=1}^{n}\Re\left(\sum_{j=0}^{n-1}\mathfrak{m}_j y_j^k\right)\left[(2n-k)\left(z^k+\frac{1}{z^k}\right)-k\left(z^{2n-k}+\frac{1}{z^{2n-k}}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{2n-1}\left(\sum_{j=0}^{n-1}\mathfrak{i}\sqrt{1-x_j^2}(\overline{y_j}^k-y_j^k)\mathfrak{n}_j\right)\left(z^k+z^{k-2n}\right)\\
&\quad=\sum_{k=1}^{n}\left(\sum_{j=0}^{n-1}\mathfrak{i}\sqrt{1-x_j^2}(\overline{y_j}^k-y_j^k)\mathfrak{n}_j\right)\left(z^k+z^{k-2n}\right)\\
&\qquad\quad+\sum_{l=1}^{n}\left(\sum_{j=0}^{n-1}\mathfrak{i}\sqrt{1-x_j^2}(\overline{y_j}^{2n-l}-y_j^{2n-l})\mathfrak{n}_j\right)\left(z^{2n-l}+z^{-l}\right)\\
&\quad=\sum_{k=1}^{n}\left(\sum_{j=0}^{n-1}\mathfrak{i}\sqrt{1-x_j^2}(\overline{y_j}^k-y_j^k)\mathfrak{n}_j\right)\left(z^k+\frac{1}{z^k}+z^{2n-k}+\frac{1}{z^{2n-k}}\right)\\
&\quad=2\sum_{k=1}^{n}\Im\left(\sum_{j=0}^{n-1}\sqrt{1-x_j^2}y_j^k\mathfrak{n}_j\right)\left(z^k+\frac{1}{z^k}+z^{2n-k}+\frac{1}{z^{2n-k}}\right).
\end{aligned}
$$

The proof is finished by putting $h_{2n-1}(x)=\mathcal{H}(z)$ with $x=z+\frac{1}{z}$. ∎

REMARK 3.2.

(i) Although equations (3.1)–(3.4) can also be obtained from those given in [1, 2], the approach we have followed here is logically simpler.

(ii) The coefficients of these expressions can be computed by using the discrete Fourier transform of cosine and sine.

(iii) Taking into account the results due to Clenshaw in [5], one can evaluate the polynomials given in equations (3.1)–(3.4) at an arbitrary point x using only $\mathcal{O}(n)$ multiplications, and hence its algorithmic complexity is similar to that of Horner's rule for evaluating a polynomial as a sum of powers of x using nested multiplication.

(iv) Proceeding in a similar way as in Proposition 3.1, one can obtain a solution of the Hermite trigonometric problem.

3.2. Barycentric formulas. The connection, through the Joukowski transformation, between the unit circle $\mathbb{T}$ and the interval $[-1,1]$ is well known, whereby the zeros of the para-orthogonal polynomials with respect to the Lebesgue measure on the unit circle, that is, the nth roots of ± 1 (n even or odd), are transformed into the zeros of the four Chebyshev-type polynomials; cf. [9]. Therefore the preceding results can be applied to obtain the barycentric

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formulas for the algebraic polynomials of Hermite interpolation related to nodal systems constituted by the zeros of the Chebyshev polynomials of the first, second, third, and fourth kind; cf. [11]. The four expressions can be deduced from (2.13) in Proposition 2.5 after transforming the problems into Hermite interpolation problems on the unit circle as was done in the previous section. The expressions obtained for these polynomials are very suitable for numerical evaluations. Unlike what happens with Lagrange interpolation, the four barycentric formulas for Hermite interpolation possess the same coefficients. Notice that the barycentric formula for the Hermite interpolation polynomial related to the nodal system constituted by the zeros of the Chebyshev polynomials of the first kind is given in [8]. It is well-known that Hermite interpolation problems on the unit circle and Hermite trigonometric interpolation problems on $[0, 2\pi]$ are tightly connected. Two important references in this subject are [4, 10]. Barycentric expressions for trigonometric Hermite polynomials are presented in both papers. In Section 2 we have posed and solved a different problem with a different solution. We must point out that [4, Formula (7.2)] is different from our corresponding formula, but if we particularize our expression (2.13) for even functions, we should obtain the corresponding result, [4, Formula (7.2)].

4. Numerical tests. In this section we report some numerical results which are obtained when the algorithms are based upon formulas (2.13) and (3.2). For this we used Matlab and Mathematica.

Example 4.1 deals with the problem of estimating the maximum error which is obtained when a function $F(z)$ analytic on an annulus containing $\mathbb{T}$ is interpolated by Hermite-Fejér polynomials; cf. [3]. Related results for the bounded interval can be found in [12, 13, 14]. In [3], the first three authors of the present article consider the Laurent expansion around 0 of $F(z)$, say $F(z) = P(z) + Q(z)$, where $P(z)$ is the part corresponding to the positive powers of z and $Q(z)$ corresponds to the negative powers. The main result in [3] establishes that if K is a compact subset of $\mathbb{T}$ without isolated points and $\mathcal{HF}_{-n,n-1}(F(z), z)$ is the Hermite-Fejér polynomial corresponding to $F(z)$, then it holds that

$$\frac{\|n\Delta_n(F(z), z)\|_{\infty,K}}{\max\limits_{z\in K,\beta\in\mathbb{T}} |(\beta-1)(P'(z)+\beta Q'(z))|} \to 1, \qquad \text{as } n\to\infty,$$

where $\Delta_n(F(z), z) = \mathcal{HF}_{-n,n-1}(F(z), z) - F(z)$. Moreover, if $(z_0, \beta_0) \in K \times \mathbb{T}$ is a point where the maximum of $|(\beta-1)(P'(z)+\beta Q'(z))|$ is attained, then for every n sufficiently large, there exists z_n near z_0, $z_n \in K$, such that the value

$$\frac{|n\Delta_n(F(z), z_n)|}{\max\limits_{z\in K,\beta\in\mathbb{T}} |(\beta-1)(P'(z)+\beta Q'(z))|}$$

is close to 1. Besides, it was also proved in [3] that if $z_0 \notin K$, then

$$\lim_{n\to\infty} \frac{\|n\Delta_n(F(z), z)\|_{\infty,K}}{\max\limits_{z\in\mathbb{T},\beta\in\mathbb{T}} |(\beta-1)(P'(z)+\beta Q'(z))|} < 1.$$

Next we take into account these results by using formula (2.13) to compute the Hermite-Fejér interpolant.

EXAMPLE 4.1. Let $F(z) = \sin z$. Then $F(z) = P(z)$, and it is straightforward to see that the corresponding maximum, with $K = \mathbb{T}$, is attained at $(z_0, \beta_0) = (\mathrm{i}, -1)$ and $(z_0, \beta_0) = (-\mathrm{i}, -1)$, and the maximum value is $2|\cos \mathrm{i}|$. For $n = 2^p$, with $p = 4, 6, 8, 10, 12$,

TABLE 4.1
Maximum of $\frac{|n\Delta_n(F(z),z)|}{|P'(\mathrm{i})|}$ *detected in* K.

p	n	$K = K_1$	$K = K_2$
4	16	1.17108	0.928747
6	64	1.9972	0.984841
8	256	1.9996	1.05371
10	1024	1.99996	1.06864
12	4096	1.99999	1.06161

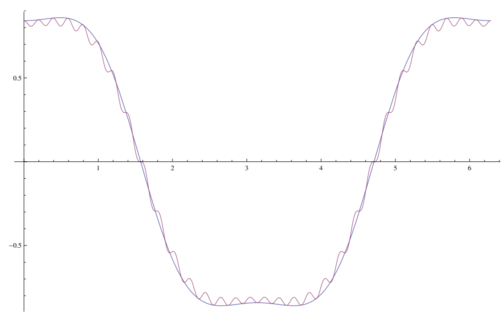

FIG. 4.1. $\Re(\mathcal{HF}_{-n,n-1}(\sin(z),z))$ *and* $\Re(\sin(z))$.

we obtain the corresponding Hermite-Fejér approximants (based on the nth roots of 1) by using formula (2.13), and we evaluate the quotient

$$\frac{|n\Delta_n(F(z),z)|}{|P'(\mathrm{i})|} = \frac{|n\Delta_n(F(z),z)|}{|\cos(\mathrm{i})|}$$

at 1000 random points on the arc $K_1 = [e^{0.995\frac{\pi}{2}\mathrm{i}}, e^{\frac{\pi}{2}\mathrm{i}}] \subset \mathbb{T}$. As said, the above expressions must converge to 2. Afterwards, these quotients are also evaluated at 1000 random points on the arc $K_2 = [1, e^{\frac{\pi}{6}\mathrm{i}}] \subset \mathbb{T}$. The latter sequence must converge to a number less than 2. Notice that the large number of evaluations should give an acceptable estimate of the uniform norm. We want to emphasize the good behavior of the algorithm based on formula (2.13). The numerical results are listed in Table 4.1.

Figure 4.1 depicts a graphical representation of the real part of $\mathcal{HF}_{-n,n-1}(\sin(z),z)$, with $n = 32$, and the real part of $\sin(z)$.

EXAMPLE 4.2. The main goal of this example is to emphasize the exceptional numerical stability of the Hermite interpolation formulas of Section 3. To this end, an algorithm is described below for implementing equation (3.2).

The discrepancies between a function f and its nth Hermite-type polynomial is estimated at 10001 equidistant points in $[-1, 1]$. Besides, the interpolation takes place at $n + 1$ nodes defined as the Chebyshev points of the second kind and the endpoints (practical abscissas).

In what follows, equation (3.2) is reformulated adequately for being encoded as a Matlab expression.

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Let $\{A_k\}_{j=0}^{n}$ and $\{B_k\}_{j=0}^{n}$ be defined as follows:

$$A_k = \Re\left(\frac{1}{2n}\sum_{j=0}^{2n-1} a_j e^{\imath\pi jk/n}\right), \qquad k = 0,\dots,n,$$

$$B_k = \Im\left(\frac{1}{2n}\sum_{j=0}^{2n-1} b_j e^{\imath\pi jk/n}\right), \qquad k = 1,\dots,n-1,$$

$$B_0 = B_n = 0,$$

where

$$a_j = \begin{cases} \mathfrak{m}_j, & 0 \le j \le n, \\ \mathfrak{m}_{2n-j}, & n+1 \le j \le 2n-1, \end{cases}$$

$$b_j = \begin{cases} 0, & j = 0, n, \\ \mathfrak{n}_j\sqrt{1-x_j^2}, & 1 \le j \le n-1, \\ \mathfrak{n}_{2n-j}\sqrt{1-x_{2n-j}^2}, & n+1 \le j \le 2n-1, \end{cases}$$

and $x_j = \cos(j\pi/n)$.

Both sequences $\{A_k\}$ and $\{B_k\}$ are the inverse discrete Fourier transform of $\{a_j\}$ and $\{b_j\}$, respectively. They further contain all interpolation data and can be calculated by means of the FFT algorithm.

Many terms x_j are close to one when n is quite large, so that $X_j = \sqrt{1-x_j^2}$ is definitely affected by a loss of digits. To avoid this drawback, the above expression of X_j should be replaced by $\sin(\pi j/n)$.

Let $(C_k)_{k=0}^{2n}$ be defined as

$$C_k = \begin{cases} nA_k & k = 0, n, \\ (2n-k)A_k + B_k, & 1 \le k \le n-1, \\ (2n-k)A_{2n-k} - B_{2n-k}, & n+1 \le k \le 2n. \end{cases}$$

Formula (3.2) can be expressed in the following form

$$k_{2n-1}(x) = \frac{1}{n}\sum_{k=0}^{2n} C_k T_k(x). \tag{4.1}$$

To design a flow chart, formulation (4.1) appears to be more suitable than (3.2). The computational implementation and performance of the remaining interpolation formulas of Section 3 are very similar to those shown in this section.

Table 4.2 lists the absolute errors produced by Hermite interpolation when the above algorithm is applied to the function $f(x) = 2 + \operatorname{sign}(x)x^2$, $-1 \le x \le 1$. Notice that the columns corresponding to B and D are the only ones which include $x = 0$ as node. It can be seen that these results suggest that the nth error behaves like $1/n^2$, a rate of convergence which can be considered as optimal in a certain sense. In effect, let $E_n(f) = \inf_p \sup_{x\in[-1,1]} |f(x) - p(x)|$, where the infimum is taken over all polynomials p of degree at most n. From $f'(x) = 2|x|$ and the general inequality

$$E_n(f) \le \frac{\pi}{2(n+1)} E_{n-1}(f'),$$

it follows immediately that the function f of this example satisfies $E_n(f) = \mathcal{O}(1/n^2)$.

TABLE 4.2
Errors when $f(x) = 2 + \mathrm{sign}(x)x^2$ is interpolated at $n+1$ points.

$n+1$	A	$n+1$	B	$n+1$	C	$n+1$	D
4	1.98e-02	5	3.18e-02	256	2.02e-06	257	7.39e-06
8	2.85e-03	9	7.67e-03	512	5.04e-07	513	1.84e-06
16	5.93e-04	17	1.90e-03	1024	1.26e-07	1025	4.61e-07
32	1.37e-04	33	4.74e-04	2048	3.05e-08	2049	1.13e-07
64	3.32e-05	65	1.18e-04	4096	7.44e-09	4097	2.71e-08
128	8.17e-06	129	2.96e-05	8192	1.86e-09	8193	6.76e-09

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