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AN IMPLICIT FINITE DIFFERENCE APPROXIMATION FOR THE SOLUTION OF THE DIFFUSION EQUATION WITH DISTRIBUTED ORDER IN TIME*

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Abstract. In this paper we are concerned with the numerical solution of a diffusion equation in which the time derivative is of non-integer order, i.e., in the interval $(0, 1)$. An implicit numerical method is presented and its unconditional stability and convergence are proved. Two numerical examples are provided to illustrate the obtained theoretical results.

Key words. Caputo derivative, fractional differential equation, subdiffusion, finite difference method, distributed order differential equation

AMS subject classifications. 35R11, 65M06, 65M12

1. Introduction. In the past decades, considerable attention has been devoted to the extension of the classical diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)$$

to the fractional setting. This extension can be obtained in several different ways, depending on the physical models we are interested in; see, for example, [18, 19, 22]. One can replace the first-order derivative in time by a derivative of non-integer order $\alpha > 0$ or the second-order derivative in space by a derivative of arbitrary real order $\beta > 1$, or one can replace both integer-order derivatives with non-integer ones. In each of these cases, we obtain a so-called fractional diffusion equation.

Here we are interested in the first case, that is, the case where the first-order time derivative is replaced by a derivative of real order α, and, as explained in [22], this generalisation may be given in two different forms if we consider the two most popular definitions of fractional derivative. We can obtain the time-fractional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{^{RL}\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left(\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \right), \quad t > 0,\ 0 < x < L,$$

where $\frac{^{RL}\partial^{\alpha}}{\partial t^{\alpha}}$ is the fractional Riemann-Liouville derivative of arbitrary real order α, or the following time-fractional diffusion equation (TFDE):

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad t > 0,\ 0 < x < L, \tag{1.1}$$

where $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the fractional Caputo derivative of arbitrary real order α.

The Riemann-Liouville and the Caputo derivatives of order α of a function $y(t)$ may be defined as follows [6, 25]. The Riemann-Liouville derivative is given by:

$$^{RL}D^{\alpha} := D^{\lceil \alpha \rceil} J^{\lceil \alpha \rceil - \alpha},$$

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with J^β being the Riemann-Liouville integral operator,

$$J^\beta y(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds, \quad t > 0,$$

and $D^{\lceil \alpha \rceil}$ is the classical integer order derivative, where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α. Analogously, $\lfloor \alpha \rfloor$ denotes the largest integer smaller than α.

The Caputo derivative is given by [6]

$$D^\alpha y(t) := {}^{RL}D^\alpha (y - T[y])(t), \quad t > 0,$$

where $T[y]$ is the Taylor polynomial of degree $\lfloor \alpha \rfloor$ for y centered at 0. The Caputo derivative has the advantage of dealing with initial value problems in which the initial conditions are given in terms of the field variables and their integer order derivatives, which is the case in most physical processes [6]. That is the reason why the Caputo derivative is more frequently used in applications, and therefore we choose to use this definition of fractional derivative in this paper. We will only consider the case $0 < \alpha < 1$. In this case and because we are dealing with a function of two variables, we have

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} := \frac{{}^{RL}\partial^\alpha}{\partial t^\alpha}(u(x,t) - u(x,0)).$$

Alternatively, we can also write [6]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s}\, ds.$$

These two generalisations of the diffusion equation have a physical meaning and are commonly used to describe anomalous diffusion processes. Basically, the fractional derivative represents a degree of memory in the diffusion material. If $0 < \alpha < 1$, then the time-fractional diffusion equation corresponds to a sub-diffusive model, if $1 < \alpha < 2$, then to a super-diffusive model, and if $\alpha = 1$, then we recover the classical diffusion model, in which it is assumed that the mean square displacement of the particles from the original starting site is linear in time. For the interested reader, a detailed physical interpretation of the time-fractional diffusion equation may be found in [11] and the references therein.

Concerning the numerical approximation to the solution of the time-fractional diffusion equation, several methods have been developed: finite element methods [14, 27], meshless collocation methods [12], collocation spectral methods [13], and finite difference methods; see, for example, [3, 4, 5, 9, 15, 16, 17, 23, 28, 29, 30]. We refer the reader to the recently published book [1] containing a survey on numerical methods for partial differential equations, where the TFDE is included.

A further generalisation of the classical diffusion equation may be obtained by using the time-fractional diffusion equation of distributed order (DODE):

$$\int_0^1 c(\alpha) \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\, d\alpha = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad t > 0,\ 0 < x < L. \tag{1.2}$$

This kind of equation has been less discussed than the TFDE. For the purpose of generalisation of (1.1), it is assumed that in (1.2) the function $c(\alpha)$ acting as a weight for the order of differentiation is a continuous function such that [10, 21]

$$c(\alpha) \geq 0 \text{ and } \int_0^1 c(\alpha)\, d\alpha = C > 0.$$

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While the fundamental solution for the Cauchy problem associated to (1.1) is interpreted as a probability density of a self-similar non-Markovian stochastic process related to the phenomenon of sub-diffusion (the variance grows sub-linearly in time), the fundamental solution of (1.2) is still a probability density of a non-Markovian process that, however, is no longer self-similar but exhibits a corresponding distribution of time-scales; see [22] for details. In [2, 26], both time and space distributed order diffusion equations were analysed. In [24], the diffusion equation of distributed order in time (between zero and one) was analysed for Dirichlet, Neumann, and Cauchy boundary conditions. The physical interpretation as well as some analytical aspects of the time-fractional diffusion equation of distributed order may also be found in [10, 20, 21] and the references therein.

As far as we know, numerical methods for this type of equation have not been reported yet, and these will be our concern in this paper. A first attempt to solve numerically a distributed-order differential equation was provided in [7]. In that paper the authors developed a numerical method for distributed-order linear equations of the form

$$\int_0^m \beta(r) D^r y(t) dr = f(t), \quad 0 \le t \le T, \tag{1.3}$$

for some positive real m, where $D^r y(t)$ is the derivative of $y(t)$ in the Caputo sense. They used a quadrature rule to approximate the integral term in (1.3), reducing (1.3) to a multi-term fractional ordinary differential equation, which could be solved with any available numerical method for ordinary fractional differential equations. The authors of that paper have presented several numerical examples in order to study the effects of the step sizes used in the quadrature rule and the step sizes in the fractional initial value problem solver. As they explained, in their approach, there were two sources for errors: the first one arises when the integral in the distributed-order equation is approximated by a finite sum, depending on the chosen quadrature rule, and the second one is due to the error related to the initial ordinary fractional value problem solver. Here we will use a similar approach, which we describe in detail in the next section. We, obviously, will have here another source for the error since we are dealing with a distributed-order partial differential operator instead of an ordinary differential operator as in [7]. We are interested in the numerical solution of (1.2) together with the initial condition

$$u(x, 0) = g(x), \tag{1.4}$$

and the boundary conditions

$$u(0, t) = u_0, u(L, t) = u_L, \tag{1.5}$$

where we assume that u_0 and u_L are constants, $g(x)$, $f(x, t)$, and the nonnegative function $c(\alpha)$ are continuous, and the fractional derivative is given in the Caputo sense.

The paper is organised in the following way: in Section 2 we provide an unconditionally stable and convergent numerical scheme for the approximation of the solution of (1.2), (1.4), (1.5), and in Section 3 we prove convergence and stability of the method. Finally, in Section 4 we illustrate the performance of the method and the obtained theoretical results with some numerical results obtained for examples whose analytical solutions are known. We end with some conclusions and plans for further investigation.

2. A numerical method.

In this section we present an implicit numerical method for the approximation to the solution of (1.2). Existence and uniqueness of the solution will not be addressed here (for theoretical aspects on this kind of problems, see [20, 21, 22]), and throughout the paper we always assume that the solution of (1.2), (1.4), (1.5) exists and is unique. As explained before, we first approximate the integral in (1.2) with a finite sum by using a quadrature rule and in this way obtain a multi-term equation (several orders for the time derivative will appear). Then, we need to approximate the derivatives with respect to t and x, and in order to do this, we must impose certain regularity assumptions on the solution $u(x,t)$.

REMARK 2.1. Throughout the next two sections we assume that the solution of (1.2) with initial condition (1.4) and boundary conditions (1.5) is of class C^2 with respect to the time variable t and is of class C^4 with respect to the variable x, and we assume that the function

$$H(\alpha) = c(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \in C^2([0,1]). \tag{2.1}$$

As we will see, this is needed for the convergence analysis.

Let us then consider a partition of the interval $[0,1]$, the interval where the order of the time derivative lies, into N subintervals, $[\beta_{j-1},\beta_j], j = 1,\dots,N$, of equal amplitude $h = 1/N$. Defining the midpoints of each one of these subintervals by

$$\alpha_j = \frac{\beta_{j-1}+\beta_j}{2}, \quad j = 1,\dots,N,$$

we can use the midpoint rule to approximate the integral in (1.2) to obtain

$$\int_0^1 c(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\,d\alpha = h\sum_{j=1}^{N} c(\alpha_j)\frac{\partial^{\alpha_j} u(x,t)}{\partial t^{\alpha_j}} - \frac{h^2}{24}H''(\nu), \quad \nu\in(0,1), \tag{2.2}$$

where H is defined by (2.1). Neglecting the $O(h^2)$ in the above inequality, (1.2) may be approximated by

$$h\sum_{j=1}^{N} c(\alpha_j)\frac{\partial^{\alpha_j} u(x,t)}{\partial t^{\alpha_j}} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t). \tag{2.3}$$

Next, we approximate the fractional derivatives $\frac{\partial^{\alpha_j} u(x,t)}{\partial t^{\alpha_j}}$ and $\frac{\partial^2 u(x,t)}{\partial x^2}$. In order to approximate the spatial derivative, we consider a uniform spatial mesh on the interval $[0,L]$, defined by the grid points $x_i = i\Delta x$, $i = 0,1,\dots,K$, where $\Delta x = \frac{L}{K}$, and we approximate the spatial derivative at $x = x_i$ by the second order finite difference:

$$\frac{\partial^2 u(x_i,t)}{\partial x^2} = \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t), \tag{2.4}$$

with $\xi_i \in (x_{i-1},x_{i+1})$.

For a fixed h, denoting by $U_i(t)$ the approximated value for $u(x_i,t)$, substituting (2.2) and (2.4), and neglecting the $O\left((\Delta x)^2\right)$ term in (2.3), we obtain the semi-discretised scheme

$$h\sum_{j=1}^{N} c(\alpha_j)\frac{\partial^{\alpha_j} U_i(t)}{\partial t^{\alpha_j}} = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{(\Delta x)^2} + f(x_i,t), \quad i = 1,\dots,K-1. \tag{2.5}$$

Note that from the boundary conditions (1.5), we have that

$$U_0(t) = u_0 \quad \text{and} \quad U_K(t) = u_L, \tag{2.6}$$

and from the initial condition (1.4) that

$$U_i(0) = g(x_i), \quad i = 1,\dots,K-1, \tag{2.7}$$

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holds. In order to approximate the fractional derivatives $\frac{\partial^{\alpha_j} u(x,t)}{\partial t^{\alpha_j}}$, we define the time grid points $t_l = l\Delta t$, $l = 0, 1, \ldots$, and use the backward finite difference formula provided by Diethelm (see [8]):

$$\frac{\partial^{\alpha_j} U_i(t_l)}{\partial t^{\alpha_j}} = \frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)} \sum_{m=0}^{l} a_{m,l}^{(\alpha_j)} \left(U_i\left(t_{l-m}\right) - U_i(0)\right) + c_{\alpha_j} (\Delta t)^{2-\alpha_j} \frac{\partial^2 u}{\partial t^2}(x_i, \eta_l), \quad \eta_l \in (0, t_l), \tag{2.8}$$

where the constants c_{α_j} do not depend on Δt and the coefficients $a_{m,l}^{(\alpha_j)}$ are given by

$$a_{m,l}^{(\alpha_j)} = \begin{cases} 1, & m = 0, \\ (m+1)^{1-\alpha_j} - 2m^{1-\alpha_j} + (m-1)^{1-\alpha_j}, & 0 < m < l, \\ (1-\alpha_j) l^{-\alpha_j} - l^{1-\alpha_j} + (l-1)^{1-\alpha_j}, & m = l. \end{cases} \tag{2.9}$$

Substituting in (2.5) and denoting $U_i^l \approx u(x_i, t_l)$, we obtain the finite difference scheme:

$$h \sum_{j=1}^{N} c(\alpha_j) \frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)} \sum_{m=0}^{l} a_{m,l}^{(\alpha_j)} \left(U_i^{l-m} - U_i^0\right) = \frac{U_{i+1}^l - 2U_i^l + U_{i-1}^l}{(\Delta x)^2} + f(x_i, t_l), \quad i = 1, \ldots, K-1, \ l = 1, 2, \ldots. \tag{2.10}$$

Hence, in order to obtain an approximation to the solution of (1.2) subject to the initial condition (1.4) and boundary conditions (1.5), we need to solve the linear systems of equations (2.10) taking into account (2.6) and (2.7):

$$\begin{aligned} &U_0^l = u_0,\ U_K^l = u_L, && l = 1, 2, \ldots, \\ &U_i^0 = g(x_i), && i = 1, \ldots, K-1. \end{aligned}$$

The numerical scheme may also be written equivalently in the following matrix form:

$$AU^l = \sum_{m=1}^{l-1} B_m U^{l-m} + C, \quad l = 1, 2, \ldots, \tag{2.11}$$

where

$$U^l = \begin{bmatrix} U_1^l \\ U_2^l \\ \vdots \\ U_{K-1}^l \end{bmatrix},$$

A and B_m are the diagonal matrices (we only write the non-zero entries):

$$A = \begin{bmatrix} \Lambda(h,\Delta t) + \frac{2}{(\Delta x)^2} & -\frac{1}{(\Delta x)^2} & & \\ -\frac{1}{(\Delta x)^2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{1}{(\Delta x)^2} \\ & & -\frac{1}{(\Delta x)^2} & \Lambda(h,\Delta t) + \frac{2}{(\Delta x)^2} \end{bmatrix},$$

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$$B_m = \begin{bmatrix} h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}a_{m,l}^{(\alpha_j)} & & & \\ & h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}a_{m,l}^{(\alpha_j)} & & \\ & & \ddots & \\ & & & h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}a_{m,l}^{(\alpha_j)} \end{bmatrix},$$

$$C = \begin{bmatrix} h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l} a_{m,l}^{(\alpha_j)}g(x_1) + f(x_1,t_l) + \frac{u_0}{(\Delta x)^2} \\ h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l} a_{m,l}^{(\alpha_j)}g(x_2) + f(x_2,t_l) \\ \vdots \\ h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l} a_{m,l}^{(\alpha_j)}g(x_{K-2}) + f(x_{K-2},t_l) \\ h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l} a_{m,l}^{(\alpha_j)}g(x_{K-1}) + f(x_{K-1},t_l) + \frac{u_L}{(\Delta x)^2} \end{bmatrix},$$

and $\Lambda(h,\Delta t) = h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)} > 0$ since the function c is nonnegative. As it can easily been seen, A is a strictly diagonal dominant matrix, and therefore A^{-1} exists, and we can conclude that for each $l = 1, 2, \ldots,$ equation (2.11) is solvable.

3. Stability and convergence of the numerical scheme. In this section we analyse stability and convergence of the implicit numerical scheme presented in the previous section. Our main results here are Theorems 3.3 and 3.6. Define

$$L_1\left(U_i^l\right) = h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}U_i^l - \frac{U_{i+1}^l - 2U_i^l + U_{i-1}^l}{(\Delta x)^2},$$

and

$$L_2\left(U_i^{l-1}\right) = -h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l} a_{m,l}^{(\alpha_j)}U_i^{l-m} + h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l} a_{m,l}^{(\alpha_j)}U_i^{0}.$$

It can be seen easily that the scheme (2.10) can be rewritten as (for $i = 1, 2, \ldots, K-1$)

$$L_1\left(U_i^l\right) = L_2\left(U_i^{l-1}\right) + f(x_i,t_l). \tag{3.1}$$

REMARK 3.1. Note that this is not a two-term recurrence relation as it happens for the classical diffusion equation case, where a relation between the time level t_l and t_{l-1} is established. Here, this recurrence relation is established between a time level t_l and all the previous time levels $t_0, \ldots, t_{l-1}$.

In order to prove that the scheme is unconditionally stable and convergent we will need the following auxiliary result:

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LEMMA 3.2. *The coefficients* $a_{m,l}^{(\alpha_j)}$, *defined by* (2.9) *satisfy the following conditions:*

$$\begin{aligned} &a_{m,l}^{(\alpha_j)} < 0, \qquad && m = 1,2,\dots,l-1,\\ &\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)} > 0, && l = 1,2,\dots. \end{aligned}$$

Proof. Let us prove that $a_{m,l}^{(\alpha_j)} < 0$, $m = 1,2,\dots,l-1$. For $1 < m < l$, the coefficients $a_{m,l}^{(\alpha_j)}$ are given by

$$a_{m,l}^{(\alpha_j)} = (m+1)^{1-\alpha_j} - 2m^{1-\alpha_j} + (m-1)^{1-\alpha_j}.$$

Therefore applying the mean value theorem to the function $g(x) = x^{1-\alpha_j}, 0 < \alpha_j < 1$, we obtain

$$\begin{aligned} a_{m,l}^{(\alpha_j)} &= \left((m+1)^{1-\alpha_j} - m^{1-\alpha_j}\right) + \left((m-1)^{1-\alpha_j} - m^{1-\alpha_j}\right)\\ &= (1-\alpha_j)\theta_1^{-\alpha_j} - (1-\alpha_j)\theta_2^{-\alpha_j}, \quad \theta_1 \in]m, m+1[, \quad \theta_2 \in]m-1, m[\\ &= (1-\alpha_j)\left(\theta_1^{-\alpha_j} - \theta_2^{-\alpha_j}\right). \end{aligned} \tag{3.2}$$

Using the fact that $\alpha_j \in]0,1[$, $j = 1,\dots,N$, and $\theta_1 > \theta_2$ (which implies $\theta_1^{-\alpha_j} < \theta_2^{-\alpha_j}$ and $\theta_1^{-\alpha_j} - \theta_2^{-\alpha_j} < 0$), from (3.2) it follows that $a_{m,l}^{(\alpha_j)} < 0$, $m = 1,2,\dots,l-1$.

With respect to the second inequality, note that

$$\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)} = l^{1-\alpha_j} - (l-1)^{1-\alpha_j}, \quad l = 1,2,\dots,$$

and therefore the result is proved. □

3.1. Stability analysis. Our first main result, concerning the stability of the numerical scheme, is presented in the following theorem.

THEOREM 3.3. *The implicit numerical scheme* (2.10) *is unconditionally stable.*

Proof. We assume that the initial data has error ε_i^0. Let $\widetilde{g}_i^0 = g(x_i) + \varepsilon_i^0$, $i = 1,\dots,K-1$, U_i^l and $\widetilde{U}_i^l$ $(i = 1,\dots,K-1)$ be the solutions of (2.10) corresponding to the initial data $g(x_i)$ and $\widetilde{g}_i^0$, respectively. Then, the error $\varepsilon_i^l = U_i^l - \widetilde{U}_i^l$ satisfies

$$L_1\left(\varepsilon_i^l\right) = L_2\left(\varepsilon_i^{l-1}\right), \quad l = 1,2,\dots, \quad i = 1,2,\dots,K-1.$$

Let $\mathbf{E}^l = [\varepsilon_1^l\, \varepsilon_2^l \,\dots\, \varepsilon_{K-1}^l]^T$, for $l = 0,1,\dots$. In order to prove stability of the proposed method, we must prove that

$$\|\mathbf{E}^l\|_\infty \leq \|\mathbf{E}^0\|_\infty, \quad l = 1,2,\dots. \tag{3.3}$$

We will verify (3.3) by mathematical induction. For $l = 1$, let $p \in \mathbb{N}$ such that $|\varepsilon_p^1| = \max_{1\leq i\leq K-1} |\varepsilon_i^1| = \|\mathbf{E}^1\|_\infty$.

Let $\Lambda(h,\Delta t) = h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}$. Since the function c is nonnegative, then $\Lambda(h,\Delta t) > 0$, and we have

$$\begin{aligned}
\Lambda(h,\Delta t)\|\mathbf{E}^1\|_\infty &= \Lambda(h,\Delta t)|\varepsilon_p^1| = \Lambda(h,\Delta t)|\varepsilon_p^1| + \frac{2|\varepsilon_p^1| - 2|\varepsilon_p^1|}{\Delta x^2}\\
&\le \Lambda(h,\Delta t)|\varepsilon_p^1| + \frac{2|\varepsilon_p^1| - |\varepsilon_{p-1}^1| - |\varepsilon_{p+1}^1|}{\Delta x^2}\\
&\le \left|\Lambda(h,\Delta t)\varepsilon_p^1 - \frac{\varepsilon_{p-1}^1 - 2\varepsilon_p^1 + \varepsilon_{p+1}^1}{\Delta x^2}\right|\\
&= |L_1(\varepsilon_p^1)| = |L_2(\varepsilon_p^0)| = |\Lambda(h,\Delta t)\varepsilon_p^0|\\
&= \Lambda(h,\Delta t)|\varepsilon_p^0| \le \Lambda(h,\Delta t)\|\mathbf{E}^0\|_\infty.
\end{aligned}$$

From the inequality above it follows that $\|\mathbf{E}^1\|_\infty \le \|\mathbf{E}^0\|_\infty$.

Suppose that $\|\mathbf{E}^j\|_\infty \le \|\mathbf{E}^0\|_\infty$, for $j = 1,2,\ldots,l-1$, and let $p \in \mathbb{N}$ be such that $|\varepsilon_p^l| = \max_{1\le i\le K-1}|\varepsilon_i^l| = \|\mathbf{E}^l\|_\infty$. Similarly to the case $l = 1$, using the induction argument and taking into account Lemma 3.2, we have that

$$\begin{aligned}
\Lambda(h,\Delta t)\|\mathbf{E}^l\|_\infty &\le \left|\Lambda(h,\Delta t)\varepsilon_p^l - \frac{\varepsilon_{p-1}^l - 2\varepsilon_p^l + \varepsilon_{p+1}^1}{\Delta x^2}\right| = |L_1(\varepsilon_p^l)| = |L_2(\varepsilon_p^{l-1})|\\
&= \left|-h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l} a_{m,l}^{(\alpha_j)}\varepsilon_p^{l-m} + h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l} a_{m,l}^{(\alpha_j)}\varepsilon_p^0\right|\\
&= \left|h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\left(\sum_{m=1}^{l-1}\left(-a_{m,l}^{(\alpha_j)}\right)\varepsilon_p^{l-m} + \left(\sum_{m=1}^{l-1} a_{m,l}^{(\alpha_j)} + 1\right)\varepsilon_p^0\right)\right|\\
&\le h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\left(\sum_{m=1}^{l-1}\left(-a_{m,l}^{(\alpha_j)}\right)|\varepsilon_p^{l-m}| + \left(\sum_{m=1}^{l-1} a_{m,l}^{(\alpha_j)} + 1\right)|\varepsilon_p^0|\right)\\
&\le h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\left(\sum_{m=1}^{l-1}\left(-a_{m,l}^{(\alpha_j)}\right)\|E^0\|_\infty + \left(\sum_{m=1}^{l-1} a_{m,l}^{(\alpha_j)} + 1\right)\|E^0\|_\infty\right)\\
&= h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\|E^0\|_\infty = \Lambda(h,\Delta t)\|\mathbf{E}^0\|_\infty,
\end{aligned}$$

and then it follows that $\|\mathbf{E}^l\|_\infty \le \|\mathbf{E}^0\|_\infty$, $l = 1,2,\ldots$. ☐

3.2. Convergence analysis. Let us define the error at each point of the mesh (x_i, t_l) by

$$e_i^l = u(x_i,t_l) - U_i^l, \quad l = 1,2,\ldots, \quad i = 1,\ldots,K-1,$$

where $u(x_i,t_l)$ is the exact solution of (1.2) with the initial condition (1.4) and boundary conditions (1.5), and U_i^l is the approximate solution of $u(x_i,t_l)$ obtained by the numerical scheme (2.10). Define $\mathbf{e}^l = \left[e_1^l, e_2^l, \ldots, e_{K-1}^l\right]$. From (1.4) and (2.7) it follows that $\mathbf{e}^0 = [0,0,\ldots,0]$.

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From (2.2), (2.4), and (2.8) it follows that the solution of equation (1.2) at $(x,t)=(x_i,t_l)$ satisfies

$$\begin{aligned} h\sum_{j=1}^{N}\Bigg(c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l}a_{m,l}^{(\alpha_j)}\left(u(x_i,t_{l-m})-u(x_i,0)\right) \\ +c_{\alpha_j}(\Delta t)^{2-\alpha_j}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_l)\Bigg)+\frac{h^2}{24}H''(\nu) \\ =\frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{(\Delta x)^2}-\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_l)+f(x_i,t_l), \end{aligned} \tag{3.4}$$

where H is the function defined by (2.1), $\xi_i \in]x_{i-1},x_{i+1}[$, $\eta_l \in]0,t_l[$, and $\nu \in]0,1[$. We can rewrite (3.4) as

$$\begin{aligned} h\sum_{j=1}^{N}c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}u(x_i,t_l)-\frac{u(x_{i+1},t_l)-2u(x_i,t_l)+u(x_{i-1},t_l)}{(\Delta x)^2} \\ =-h\sum_{j=1}^{N}c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l}a_{m,l}^{(\alpha_j)}u(x_i,t_{l-m}) \\ +h\sum_{j=1}^{N}c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l}a_{m,l}^{(\alpha_j)}u(x_i,t_0)-\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_l) \\ -h\sum_{j=1}^{N}c(\alpha_j)c_{\alpha_j}(\Delta t)^{2-\alpha_j}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_l)-\frac{h^2}{24}H''(\nu)+f(x_i,t_l). \end{aligned} \tag{3.5}$$

Therefore, from (3.5) and using the definition of L_1 and L_2, the solution of equation (1.2) at $(x,t)=(x_i,t_l)$ satisfies

$$\begin{aligned} L_1\left(u(x_i,t_l)\right)=L_2\left(u(x_i,t_l)\right)+f(x_i,t_l)-\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_l) \\ -h\sum_{j=1}^{N}c(\alpha_j)c_{\alpha_j}(\Delta t)^{2-\alpha_j}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_l)-\frac{h^2}{24}H''(\nu). \end{aligned} \tag{3.6}$$

Based on (3.1) and (3.6) we have that the errors e_i^l, $l=1,2,\ldots$, $i=1,\ldots,K-1$, satisfy

$$\begin{cases} e_i^0=0 & i=1,2,\ldots,K-1, \\ L_1(e_i^{l+1})=L_2(e_i^l)+R_i^{l+1} & l=0,1,\ldots,\ i=1,2,\ldots,K-1, \end{cases}$$

where

$$R_i^{l+1}=-\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_l)-h\sum_{j=1}^{N}c(\alpha_j)c_{\alpha_j}(\Delta t)^{2-\alpha_j}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_l)-\frac{h^2}{24}H''(\nu)$$
$$l=0,1,\ldots,\ i=1,\ldots,K-1.$$

Define $\mathbf{R}^{l+1}=[R_1^{l+1},\ R_2^{l+1},\ldots\ R_{K-1}^{l+1}]$, $\quad l=0,1,\ldots.$

LEMMA 3.4. *There exists a positive constant $C_1>0$ that does not depend on Δx, Δt and h such that*

$$\|R^{l+1}\|_\infty \le C_1\left((\Delta x)^2+h^2+(\Delta t)^{1+\frac{h}{2}}\right),\quad l=0,1,2,\ldots. \tag{3.7}$$

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Proof. Using the regularity assumptions on the solution of equation (1.2) and the function H, and because the function $c(\alpha)$ is positive, we have

$$\begin{aligned} |R_i^{l+1}| &\le M_1(\Delta x)^2 + M_2(\Delta t)^{2-\alpha_N} h \sum_{j=1}^{N} c(\alpha_j) + M_3 h^2 \\ &\le M_1(\Delta x)^2 + M_2(\Delta t)^{2-\alpha_N} hN \max_{\alpha\in[0,1]} c(\alpha) + M_3 h^2 \\ &= M_1(\Delta x)^2 + M_4 \Delta t^{2-\alpha_N} + M_3 h^2, \end{aligned} \tag{3.8}$$

where $M_1 = \frac{1}{12}\max_{x\in[0,L]}\left|\frac{\partial^4 u}{\partial x^4}(x,t_l)\right|$, $M_2 = \bar{c}\max_{t\in[0,t_l]}\left|\frac{\partial^2 u}{\partial t^2}(x_i,t)\right|$, $\bar{c} = \max_j\left\{c_{\alpha_j}\right\}$, $M_3 = \frac{1}{24}\max_{\alpha\in[0,1]}|H''(\alpha)|$, and $M_4 = M_2 \max_{\alpha\in[0,1]} c(\alpha)$. From (3.8) we obtain

$$\|R^{l+1}\|_\infty \le C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{2-\alpha_N}\right), \quad l = 1,2,\ldots, \tag{3.9}$$

with $C_1 = \max\{M_1,\, M_2,\, M_3,\, M_4\}$. Note that $\alpha_N = 1 + \frac{h}{2}$. Then, from (3.9), inequality (3.7) follows. $\square$

LEMMA 3.5. *There exists a positive constant $C_1 > 0$ not depending on Δx, Δt, and h such that*

$$\|\boldsymbol{e}^l\|_\infty \le \frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)}}, \quad l = 1,2,\ldots. \tag{3.10}$$

Proof. The proof is similar to the proof of Theorem 3.3. We use mathematical induction to prove (3.10). For $l = 1$, let $\|\mathbf{e}^1\|_\infty = \max_{1\le i\le K-1}|e_i^1| = |e_p^1|$. Then we have

$$\begin{aligned} \Lambda(h,\Delta t)\|\mathbf{e}^1\|_\infty &= \Lambda(h,\Delta t)|e_p^1| = \Lambda(h,\Delta t)|e_p^1| + \frac{2|e_p^1| - 2|e_p^1|}{\Delta x^2} \\ &\le |L_1(e_p^1)| = |L_2(e_p^0) + R_p^1| = |\Lambda(h,\Delta t)e_p^0 + R_p^1| \\ &\le \Lambda(h,\Delta t)\underbrace{|\varepsilon_p^0|}_{=0} + |R_p^1| \le \|\mathbf{R}^1\|_\infty. \end{aligned}$$

From the inequality above it follows that

$$\|\mathbf{e}^1\|_\infty \le \frac{\|\mathbf{R}^1\|_\infty}{\Lambda(h,\Delta t)}.$$

Therefore, from Lemma 3.4 it follows that

$$\|\mathbf{e}^1\|_\infty \le \frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{\Lambda(h,\Delta t)} = \frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{0} a_{m,1}^{(\alpha_j)}}.$$

Suppose that

$$\|\mathbf{e}^k\| \le \frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{k-1} a_{m,k}^{(\alpha_j)}}, \quad k = 1,2,\ldots,l-1,$$

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and let $\|\mathbf{e}^l\|_\infty = |e_p^l|$. Then

$$\begin{aligned}
\Lambda(h,\Delta t)\|\mathbf{e}^l\|_\infty &\le \left|\Lambda(h,\Delta t)e_p^l - \frac{e_{p-1}^l - 2e_p^l + e_{p+1}^l}{\Delta x^2}\right| = |L_1(e_p^l)| = |L_2(e_p^{l-1}) + R_p^l| \\
&= \left| -h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l} a_{m,l}^{(\alpha_j)} e_p^{l-m} + h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l} a_{m,l}^{(\alpha_j)} e_p^0 + R_p^l\right| \\
&= \left| -h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l-1} a_{m,l}^{(\alpha_j)} e_p^{l-m} + R_p^l\right| \\
&\le h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l-1} (-a_{m,l}^{(\alpha_j)})\left|e_p^{l-m}\right| + \left\|\mathbf{R}^l\right\|_\infty \\
&\le h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l-1} (-a_{m,l}^{(\alpha_j)})\frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\displaystyle\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{s=0}^{l-m-1} a_{s,l-m}^{(\alpha_j)}} \\
&\quad + C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right).
\end{aligned}$$

Because $0 < \sum_{s=0}^{l-m-1} a_{s,l-m}^{(\alpha_j)} = 1 + \sum_{s=1}^{l-m-1} a_{s,l-m}^{(\alpha_j)}$ and since the coefficients $a_{m,l}^{(\alpha_j)} < 0$, $m = 1,2,\ldots,l-1$, and hence

$$\sum_{s=0}^{l-m-1} a_{s,l-m}^{(\alpha_j)} > \sum_{s=0}^{l-1} a_{s,l}^{(\alpha_j)},$$

we have

$$\begin{aligned}
\Lambda(h,\Delta t)\|\mathbf{e}^l\|_\infty &\le h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l-1} (-a_{m,l}^{(\alpha_j)})\frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{s=0}^{l-1} a_{s,l}^{(\alpha_j)}} \\
&\qquad + C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right) \\
&= \frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\displaystyle\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)}}\left(h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=1}^{l-1} (-a_{m,l}^{(\alpha_j)})\right. \\
&\qquad \left. + h\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)}\right) \\
&= \frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\displaystyle\sum_{j=1}^N c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)}}\Lambda(h,\Delta t).
\end{aligned}$$

Thus, the proof is complete. $\square$

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Finally, we present the main result in this subsection.

THEOREM 3.6. *If the solution of (1.2) is of class C^2 with respect to the time variable t, is of class C^4 with respect to the variable x, and the function $H(\alpha) = c(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \in C^2([0,1])$, then there exists a positive constant C independent of h, Δx, and Δt such that*

$$\|\boldsymbol{e}^l\|_\infty \le C\left((\Delta x)^2 + (\Delta t)^{1+\frac{h}{2}} + h^2\right). \tag{3.11}$$

Proof. From Lemma 3.2 we have

$$\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)} = l^{1-\alpha_j} - (l-1)^{1-\alpha_j}.$$

On the other hand,

$$\lim_{l\to\infty} \frac{l^{-\alpha_j}}{l^{1-\alpha_j} - (l-1)^{1-\alpha_j}} = \lim_{l\to\infty} \frac{l^{-1}}{1-\left(\frac{l-1}{l}\right)^{1-\alpha_j}} = \lim_{l\to\infty} \frac{1}{1-\alpha_j}\left(1-\frac{1}{l}\right)^{\alpha_j} = \frac{1}{1-\alpha_j}.$$

Therefore, there exist a constant C_2, independent of h, Δx, and Δt such that

$$\begin{aligned}
\frac{C_1\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{m=0}^{l-1} a_{m,l}^{(\alpha_j)}} &\le \frac{C_1C_2\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\sum_{j=1}^{N} c(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)} l^{-\alpha_j}} \\
&= \frac{C_1C_2\left((\Delta x)^2 + h^2 + (\Delta t)^{1+\frac{h}{2}}\right)}{h\sum_{j=1}^{N} c(\alpha_j)\frac{(l\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}}.
\end{aligned}$$

Since $c(\alpha)$ is a continuous function and $l\Delta t$ is finite, we obtain

$$h\sum_{j=1}^{N} c(\alpha_j)\frac{(l\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)} \ge C_3 N h \min_{\alpha\in[0,1]}\left(\frac{c(\alpha)}{\Gamma(2-\alpha)}\right) = C_3 L \min_{\alpha\in[0,1]}\left(\frac{c(\alpha)}{\Gamma(2-\alpha)}\right).$$

Taking into account Lemma 3.5 we can then conclude that there must exist a constant C independent of h, Δx, and Δt such that (3.11) holds. ∎

4. Numerical results. In this section we present some numerical results. In order to show the performance of the proposed algorithm we consider the following two examples:

EXAMPLE 4.1.

$$\int_0^1 \Gamma(3-\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\,d\alpha = \frac{\partial^2 u(x,t)}{\partial x^2} + 2t^2 + \frac{2(t-1)t(2-x)x}{\ln t},$$
$$0 < t < 1, 0 < x < 2,$$
$$u(x,0) = 0,$$
$$u(0,t) = u(2,t) = 0.$$

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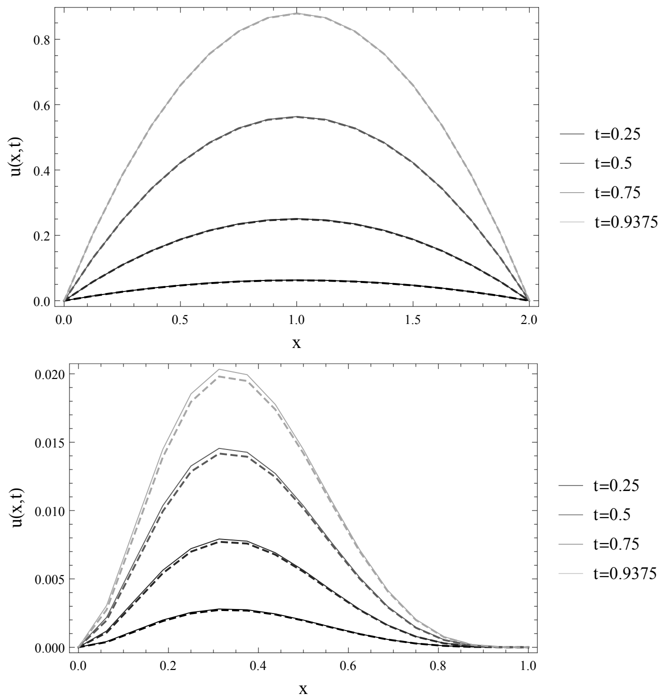

FIG. 4.1. *Exact (dashed line) and approximate (solid line) solutions obtained with* $\Delta t = 0.015625$ *and* $h = \Delta x = 0.125$*. Top: Example 4.1. Bottom: Example 4.2.*

EXAMPLE 4.2.

$$\int_0^1 \Gamma(\frac{5}{2}-\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\,d\alpha$$
$$= \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\sqrt{t}(x-1)^2}{4\log(t)}\left(3\sqrt{\pi}(t-1)(x-1)^2x^2 - 8t(5x(3x-2)+1)\log(t)\right),$$
$$0 < t < 1, 0 < x < 1,$$
$$u(x,0) = 0,$$
$$u(0,t) = u(1,t) = 0.$$

For both examples, analytical solutions are known and are given by $u(x,t) = t^2x(2-x)$ and $u(x,t) = x^2(1-x)^4t^{3/2}$, respectively. In order to obtain approximate solutions of the above examples, we use the proposed method (2.10) for several step sizes h, Δx, and Δt.

In Figure 4.1 we present a comparison of the exact and numerical solutions for the Examples 4.1 and 4.2 at several points $t \in (0,1)$ of the mesh. In both cases, we can see that the numerical solutions are in good agreement with the exact solutions.

In Figures 4.2 and 4.3 we compare the absolute errors, at the points $(x, 0.25)$ and $(x, 0.75)$ obtained for several meshes. These figures illustrate the convergence of the algorithm (2.10) applied to Example 4.1 and Example 4.2.

In Tables 4.1 and 4.2 we list the maximum of the errors

$$\|E\| = \max_{1\le i\le K-1, l=1,2,\dots} \left|U_i^l - u(x_i,t_l)\right|,$$

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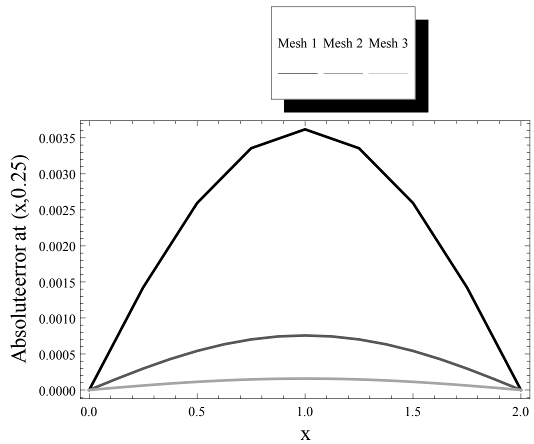

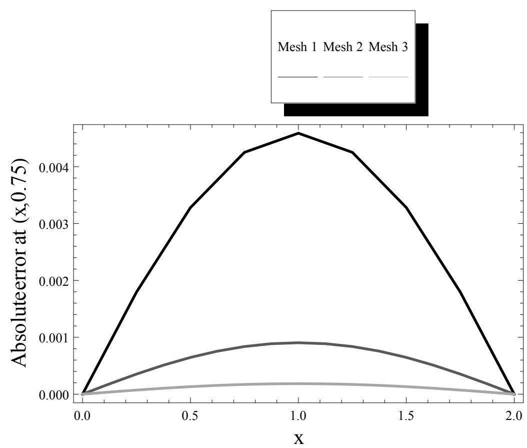

FIG. 4.2. *Example 4.1: Pointwise absolute error at the points* $(x, 0.25)$, $x \in [0, 2]$ *(top) and* $(x, 0.75)$, $x \in [0, 2]$ *(bottom), obtained by the algorithm (2.10) with several meshes (Mesh 1:*$\Delta x = h = 0.25, \Delta t = 0.0625$ *Mesh 2:*$\Delta x = h = 0.125, \Delta t = 0.015625$ *and Mesh 3:*$\Delta x = h = 0.0625, \Delta t = 0.00390625$*).*

TABLE 4.1
Example 4.1: Maximum of errors and experimental convergence orders.

Δt	$h = \Delta x$	$\|E\|$	$p_x = p_h$	p_t
0.25	0.5	$2.39 \cdot 10^{-2}$	-	-
0.0625	0.25	$4.66 \cdot 10^{-3}$	2.36	1.18
0.015625	0.125	$9.10 \cdot 10^{-4}$	2.36	1.18
0.00390625	0.0625	$1.84 \cdot 10^{-4}$	2.30	1.15

for several values of h, Δx, and Δt and the experimental spatial, temporal, and numerical integration convergence orders that we denote by p_x, p_t, and p_h, respectively. The results of Tables 4.1 and 4.2 indicate that the experimental order of convergence with respect to the time variable is approximately 1 and the spacial and numerical integration order is approximately 2, confirming the theoretical result (3.11) of Theorem 3.6.

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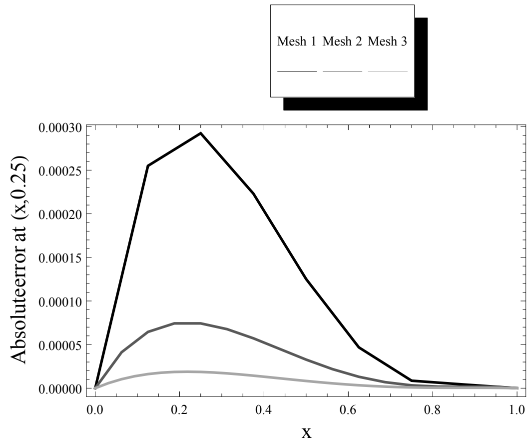

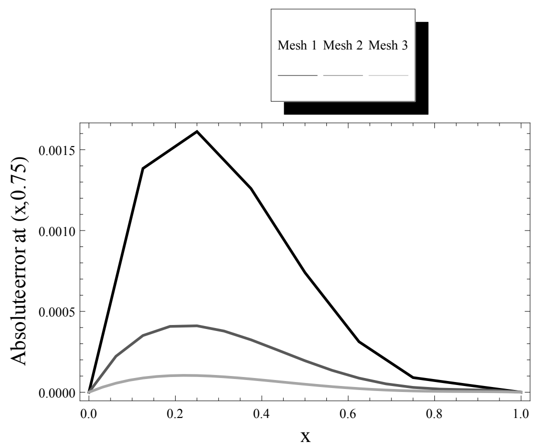

FIG. 4.3. *Example 4.2: Pointwise absolute error at the points* $(x, 0.25)$, $x \in [0, 1]$ *(top) and* $(x, 0.75)$, $x \in [0, 1]$ *(bottom), obtained by the algorithm (2.10) with several meshes (Mesh 1:*$\Delta x = h = 0.125$, $\Delta t = 0.015625$*, Mesh 2:*$\Delta x = h = 0.0625, \Delta t = 0.00390625$ *and Mesh 3:*$\Delta x = h = 0.03125$, $\Delta t = 0.000976563$*).*

TABLE 4.2
Example 4.2: Maximum of errors and experimental convergence orders.

Δt	$h = \Delta x$	$\|E\|$	$p_x = p_h$	p_t
0.25	0.5	$8.40 \cdot 10^{-3}$	-	-
0.0625	0.25	$2.45 \cdot 10^{-3}$	1.78	0.89
0.015625	0.125	$6.36 \cdot 10^{-4}$	1.95	0.97
0.00390625	0.0625	$1.62 \cdot 10^{-4}$	1.98	0.99

REMARK 4.3. From Table 4.2, it can be seen that the method presented here yields convergence of order of $p_t \sim 1$ in time and of $p_x = p_h \sim 2$ in space and numerical integration, which is in agreement with Theorem 3.6. Although the regularity assumptions in Theorem 3.6 are not satisfied, still the method performs well. Actually, in Example 4.2, the solution $u(x, t)$ is not in $C^2([0, 1])$ with respect to the time variable t as the solution is not a twice continuously differentiable function at $t = 0$.

5. Conclusions. In this work, an implicit difference method for the spatially one-dimensional diffusion equation with distributed-order of derivative in time has been presented, and its unconditional stability and convergence were proved. As far as we know, this is the first attempt to solve this kind of equation numerically. Some numerical examples are considered in order to illustrate the performance of the method. In the future, we intend to explore other approaches to the approximation of the time derivatives since it will be convenient to use approximations of higher order and ones that are independent of the order of the derivatives and in the discretization of the integral term as well as in the approximation of the space derivative. We also intend to use this method for problems with higher spacial dimension, which is a straightforward extension in view of potential applications. Finally, for further investigation, we also intend to analyse the super-diffusive case, i.e., where the order of the time-derivative is distributed over the interval $[0, 2]$. Note that in this case, the approximation (2.8) is no longer appropriate since if $\alpha_j \in (0, 2)$ because then the order of convergence of this approximation may become extremely low.

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