Electronic Transactions on Numerical Analysis.
Volume 44, pp. 462–471, 2015.

ISSN 1068–9613.

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HIGH-ORDER MODIFIED TAU METHOD FOR NON-SMOOTH SOLUTIONS OF ABEL INTEGRAL EQUATIONS*

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Abstract. In this paper, the spectral Tau method and generalized Jacobi functions are fruitfully combined to approximate Abel integral equations with solutions that may have singularities (non-smooth solutions) at the origin. In an earlier work of P. Mokhtary and F. Ghoreishi [Electron. Trans. Numer. Anal., 41 (2014), pp. 289–305], a regularization process was used to handle the high-order Tau method based on classical Jacobi polynomials for the numerical solution of Abel integral equations. However, it was found that this scheme makes the resulting equation and its Tau approximation more complicated. In this work, we introduce and analyze a new modified Tau method for the numerical solution of Abel integral equations with non-smooth solutions. The main advantage of this method is that it gains a high order of accuracy without adopting any regularization process. Illustrative examples are included to demonstrate the validity and applicability of the proposed technique.

Key words. modified Tau method, generalized Jacobi functions, Abel integral equations.

AMS subject classifications. 45E10, 41A25.

1. Introduction.

In this paper we introduce and analyze a modified Tau method for the numerical solution of the Abel integral equation

$$y(t) = f(t) + \lambda \int_0^t \frac{K(t,s)}{\sqrt{t-s}} y(s)\, ds, \quad t \in \Lambda = [0,1]. \tag{1.1}$$

Here, the sufficiently smooth functions $f(t)$ and $K(t,s)$ are given with $K(t,t) \neq 0$ for $t \in \Lambda$. The unknown function $y(t)$ is non-smooth, and λ is a generic constant. In the following lemma we give a regularity result for (1.1).

LEMMA 1.1 ([2]). *Assume that $f(t) \in C^m(\Lambda)$ and $K(t,s) \in C^m(\Lambda \times \Lambda)$ with $K(t,t) \neq 0$ and $m \geq 1$. The regularity of the unique solution of (1.1) is described by*

$$y(t) \in C^m(0,1] \bigcap C(\Lambda) \quad \text{with} \quad |y'(t)| \leq \frac{C}{\sqrt{t}} \qquad \text{for} \quad t \in (0,1],$$

and the solution $y(t)$ can be written in the form

$$y(t) = \sum_{(j,k)} \gamma_{j,k} t^{j+\frac{k}{2}} + Y_m(t), \quad t \in \Lambda,$$

where $(j,k) := \left\{(j,k); j,k \in \mathbb{N}_0,\ j + \frac{k}{2} < m\right\}$ and $Y_m(.) \in C^m(\Lambda)$. The coefficients $\gamma_{j,k}$ are known constants, and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, where $\mathbb{N}$ is the set of all natural numbers.

The above lemma concludes that the Abel integral equation (1.1) typically has a solution whose first derivative is unbounded at the origin and behaves like $y'(t) \simeq \frac{1}{\sqrt{t}}$.

Spectral methods have been studied intensively in the last two decades because of their good approximation properties. Global spectral methods use a representation of the function $u(t)$ throughout the domain via a truncated series expansion with suitable basis functions. This series is then substituted into a functional equation, and upon the minimization of the residual function, the unknown coefficients are computed. Spectral methods can be broadly

*Received December 24, 2014. Accepted June 9, 2015. Published online on August 21, 2015. Recommended by Frank Stenger.
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classified into three categories, pseudospectral or collocation, Galerkin, and Tau methods. The Tau method has found extensive application in the numerical solution of many operator equations in recent years. It involves the projection of the residual function onto the span of some appropriate set of basis functions, typically arising as eigenfunctions of a singular Sturm-Liouville problem. The auxiliary conditions are imposed as constraints on the expansion coefficients. It is well known that the spectral Tau method based on the classical Jacobi polynomials (Jacobi Tau method) allows the approximation of infinitely smooth solutions of operator equations such that the truncation error approaches zero faster than any negative power of the number of basis functions used in the approximation as that number tends to ∞. This phenomenon is usually referred to as *spectral accuracy*; see [5, 6, 7, 8, 9].

From Lemma 1.1 we can conclude that in (1.1) some derivatives of the exact solution have a discontinuity at the left endpoint of the interval of integration. Thus, the numerical solution of (1.1) using the Jacobi Tau method leads to very poor convergence results. Of the various methods proposed as extensions of the Jacobi Tau method for the numerical solution of (1.1), a regularization approach is usually followed. The main characteristic behind this approach is that the original equation is transformed into a new integral equation that possesses a smooth solution by applying a suitable coordinate transformation. After this process, the Jacobi Tau method can be implemented in a straightforward manner with a satisfactory order of convergence (regularized Jacobi Tau method). However, it is found that this scheme makes the resulting equation and its Jacobi spectral approximation more complicated; see [7].

Recently, Chen et al. [3] introduced generalized Jacobi functions, which are orthogonal with respect to a suitable weight function. These functions are of non-polynomial nature. The attractive fractional calculus properties and remarkable approximability to functions with singular behavior at boundaries are two main advantages of these functions. Hence, we can consider these functions as basis functions for developing modified spectral methods for the numerical solutions of operator equations that have a singular behavior at boundaries. In this work, we shall demonstrate that the modified Tau solution of (1.1) using generalized Jacobi functions as basis functions produces a solution with a high order of accuracy for (1.1) without applying any regularization process.

The paper is organized as follows. In the next section, we first give some preliminaries required for our subsequent development. Afterward, we define generalized Jacobi functions and investigate their basic properties. In Section 3, we outline the modified Tau method for the numerical solution of (1.1) with generalized Jacobi functions as basis functions. Section 4 is devoted to its convergence analysis. In this section, the error estimate of the proposed modified Tau scheme is obtained. In Section 5, we present numerical approximations of selected problems.

2. Preliminaries. In this section we review some basic definitions that will be required in the sequel. In particular, we define the generalized Jacobi functions and give their important properties; see [3].

For $q \in \mathbb{R}^+$($\mathbb{R}^+$ is the set of all positive real numbers), we define the Riemann-Liouville fractional derivative of order q as ([4])

$$\mathcal{D}^q u(t) = \frac{d^{\lceil q \rceil}}{dt^{\lceil q \rceil}} \left(\mathcal{I}^{\lceil q \rceil - q} u \right), \tag{2.1}$$

where the symbol $\lceil q \rceil$ is the smallest integer greater than or equal to q. $\Gamma(.)$ is the gamma function, and $\mathcal{I}^\rho u$ denotes the fractional integral of order ρ and is defined as

$$(\mathcal{I}^\rho u)(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} u(s)\, ds.$$

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We consider the following weighted $L^2_{\alpha,\beta}$-norm of a function $u(t)$ over Λ:

$$\|u\|^2_{\alpha,\beta} = (u,u)_{\alpha,\beta} = \int_\Lambda u^2(t) w^{\alpha,\beta}(t)\,dt < \infty,$$

where $w^{\alpha,\beta}(t) = 2^{\alpha+\beta}(1-t)^\alpha t^\beta$ with parameters α, β is the shifted Jacobi weight function on Λ, and $(.,.)_{\alpha,\beta}$ is the well-known inner product formula; see [9].

We denote generalized Jacobi functions by $\mathcal{G}^{\alpha,\beta}_n(t)$ and define

$$\mathcal{G}^{\alpha,-\beta}_n(t) = (2t)^\beta J^{\alpha,\beta}_n(t), \quad \alpha \in \mathbb{R}, \ \beta > -1, \tag{2.2}$$

where $J^{\alpha,\beta}_n(t)$ are the classical shifted Jacobi polynomials on Λ; see [3]. These functions are of non-polynomial nature, and $\mathcal{G}^{\alpha,-\beta}_n(t) \in \text{span}\{t^\beta, t^{\beta+1}, \ldots, t^{\beta+n}\}$. It follows from [3] that for $\alpha, \beta > -1$, these functions are mutually orthogonal with respect to the weight function $w^{\alpha,-\beta}(t)$, i.e.,

$$\int_\Lambda \mathcal{G}^{\alpha,-\beta}_n(t)\mathcal{G}^{\alpha,-\beta}_m(t) w^{\alpha,-\beta}(t)\,dt = \gamma^{\alpha,\beta}_n \delta_{nm},$$

where

$$\gamma^{\alpha,\beta}_n = \frac{2^{\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}\,. \tag{2.3}$$

Considering the $L^2_{\alpha,-\beta}(\Lambda)$-orthogonal projection $\Pi^{\alpha,-\beta}_N u \in \mathcal{P}^{\alpha,-\beta}_N$ defined by

$$\left(\Pi^{\alpha,-\beta}_N u - u, v_N\right)_{\alpha,-\beta} = 0, \quad \forall v_N \in \mathcal{P}^{\alpha,-\beta}_N, \tag{2.4}$$

with

$$\mathcal{P}^{\alpha,-\beta}_N := \text{span}\{\mathcal{G}^{\alpha,-\beta}_0(t), \mathcal{G}^{\alpha,-\beta}_1(t), \ldots, \mathcal{G}^{\alpha,-\beta}_N(t)\}, \quad \alpha > -1, \ \beta > 0,$$

and by the orthogonality of $\{\mathcal{G}^{\alpha,-\beta}_n\}_{n\geq 0}$, we can expand any $u(t) \in L_{\alpha,-\beta}(\Lambda)$ as

$$u(t) = \sum_{n=0}^{\infty} u_n \mathcal{G}^{\alpha,-\beta}_n(t), \quad \text{with} \quad u_n = \frac{\left(u, \mathcal{G}^{\alpha,-\beta}_n\right)}{\gamma^{\alpha,\beta}_n}\,.$$

Concerning the truncation error of a generalized Jacobi series, the following estimate holds (see [3, Theorem 4.3])

$$\|\Pi^{\alpha,-\beta}_N u - u\|_{\alpha,-\beta} \leq C N^{-\beta} \sqrt{\frac{(N-l+1)!}{(N+l+1)!}} \|\mathcal{D}^{\beta+l} u\|_{\alpha+\beta+l,l}, \tag{2.5}$$

where $0 \leq l \leq N, \alpha > -1, \beta > 0$, and $\|\mathcal{D}^{\beta+l}u\|_{\alpha+\beta+l,l} < \infty$. Throughout the paper, C will denote a generic positive constant that is independent of N.

3. Modified Tau method. In this section, we develop the modified Tau method that combines the spectral Tau method with the generalized Jacobi functions to present a numerical solution for (1.1). Consider (1.1). From Lemma 1.1 we can see that the first derivative of the exact solution $y(t)$ behaves like $\frac{1}{\sqrt{t}}$, thus $y(t) \simeq \sqrt{t}$. Hence, from (2.2) we can consider

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$\{\mathcal{G}_n^{0,-\frac{1}{2}}(t); \quad n \geq 0\}$ as suitable basis functions in the Tau method for (1.1). Thus, we seek an approximated solution $y_N(t)$ of the form

$$y_N(t) = \sum_{i=0}^{\infty} a_i \mathcal{G}_i^{0,-\frac{1}{2}}(t), \quad a_i = 0 \qquad \text{for} \quad i > N. \tag{3.1}$$

The $N+1$ equations for the unknown expansion coefficients $\{a_i\}_{i=0}^N$ are determined from (1.1) by requiring the residual

$$R_N(t) = y_N(t) - f(t) - \lambda \int_0^t \frac{K(t,s)}{\sqrt{t-s}} y_N(s)\, ds$$

to be orthogonal to the basis of $\mathcal{P}_N^{0,-\frac{1}{2}}$ under the weight function $w^{0,-\frac{1}{2}}(t)$. In other words, the modified Tau formulation of (1.1) is defining $y_N \in \mathcal{P}_N^{0,-\frac{1}{2}}$ such that

$$\big(y_N, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}} = \big(f, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}} + \lambda\big(\mathcal{K}y_N, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}}, \quad j = 0,1,\ldots,N, \tag{3.2}$$

where $\mathcal{K}y_N = \int_0^t \frac{K(t,s)}{\sqrt{t-s}} y_N(s)\, ds$. Substituting (3.1) in (3.2) yields

$$\sum_{i=0}^{\infty} a_i \Big\{ \big(\mathcal{G}_i^{0,-\frac{1}{2}}, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}} - \lambda\big(\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}}, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}} \Big\} = \big(f, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}},$$

for $j = 0,1,\ldots,N$.

Using the orthogonality property of $\{\mathcal{G}_i^{0,-\frac{1}{2}}\}_{i\geq 0}$, we can rewrite the equation above as

$$a_j \gamma_j^{0,\frac{1}{2}} - \lambda \sum_{i=0}^{N} a_i \big(\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}}, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}} = \big(f, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}}, \quad j = 0,1,\ldots,N, \tag{3.3}$$

where $\gamma_j^{0,\frac{1}{2}}$ is defined in (2.3). Now, it is sufficient that we calculate $\big(\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}}, \mathcal{G}_j^{0,-\frac{1}{2}}\big)_{0,-\frac{1}{2}}$. To this end, we assume that

$$K(t,s) = \sum_{l=0}^{\infty}\sum_{v=0}^{\infty} k_{lv} \mathcal{G}_l^{0,0}(t)\mathcal{G}_v^{0,0}(s),$$

which can be rearranged as

$$K(t,s) = \sum_{l=0}^{\infty}\sum_{v=0}^{\infty} \tilde{k}_{lv} t^l s^v,$$

and in a similar manner we have $\mathcal{G}_i^{0,-\frac{1}{2}}(t) = \sum_{k=0}^{i} g_k t^{k+\frac{1}{2}}$. Thereby we obtain

$$\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}} = \sum_{k=0}^{i}\sum_{l=0}^{\infty}\sum_{v=0}^{\infty} g_k \tilde{k}_{lv} t^l \int_0^t \frac{s^{\frac{1}{2}+k+v}}{\sqrt{t-s}}\, ds, \quad i \geq 0. \tag{3.4}$$

Using the relation [7]

$$\int_0^t \frac{s^j}{\sqrt{t-s}}\, ds = t^{j+\frac{1}{2}} \mathcal{B}(j+1, \tfrac{1}{2}),$$

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we can rewrite (3.4) as

$$\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}} = \sum_{k=0}^{i}\sum_{l=0}^{\infty}\sum_{v=0}^{\infty} g_k \tilde{k}_{lv} \mathcal{B}_{vk} t^{l+k+v+1}, \quad i \geq 0,$$

with $\mathcal{B}_{vk} = \mathcal{B}(\frac{3}{2}+k+v, \frac{1}{2})$, where $\mathcal{B}(.,.)$ denotes the beta function. From the equation above we have

$$\left(\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}}, \mathcal{G}_j^{0,-\frac{1}{2}}\right)_{0,-\frac{1}{2}} = \sum_{k=0}^{i}\sum_{l=0}^{\infty}\sum_{v=0}^{\infty} g_k \tilde{k}_{lv} \mathcal{B}_{vk} \int_{\Lambda} t^{l+k+v} J_j^{0,\frac{1}{2}}(t)\, dt := \tilde{\mathcal{A}}_{ji}, \tag{3.5}$$

for $i \geq 0$ and $j = 0, 1, \ldots, N$. Now, by substituting (3.5) in (3.3), we can obtain a linear algebraic system $\mathcal{A}\tilde{a} = \tilde{f}$ with

$$\begin{aligned} \mathcal{A} :=& \left(\mathcal{A}_{ji}\right)_{i\geq 0, j=0,1,\ldots,N} := \begin{cases} \gamma_j^{0,\frac{1}{2}} - \lambda\tilde{\mathcal{A}}_{ji}, & j = i, \\ -\lambda\tilde{\mathcal{A}}_{ji}, & j \neq i, \end{cases} \\ \tilde{a} =& [a_0, a_1, \ldots, a_N, 0, 0, \ldots]^T, \\ \tilde{f} =& \left[(f, \mathcal{G}_0^{0,-\frac{1}{2}})_{0,-\frac{1}{2}}, (f, \mathcal{G}_1^{0,-\frac{1}{2}})_{0,-\frac{1}{2}}, \ldots, (f, \mathcal{G}_N^{0,-\frac{1}{2}})_{0,-\frac{1}{2}}\right]^T, \end{aligned} \tag{3.6}$$

which when solved gives us the unknown coefficients $\{a_i\}_{i=0}^N$.

For a special case of (1.1) with $K(t,s) = 1$, the implementation process can be presented in a very simple manner. To this end, from the relation (see [3])

$$\mathcal{I}^{\rho}\left(\mathcal{G}_i^{\alpha,-\beta}\right) = \frac{\Gamma(i+\beta+1)}{\Gamma(i+\beta+\rho+1)} 2^{\beta} t^{\beta+\rho} J_i^{\alpha-\rho,\beta+\rho}(t), \quad \rho \in \mathbb{R}^+,\ i \geq 0,\ \alpha \in \mathbb{R},\ \beta > -1,$$

we can conclude that

$$\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}} = \int_0^t \frac{\mathcal{G}_i^{0,-\frac{1}{2}}(s)}{\sqrt{t-s}}\, ds = \Gamma\left(\frac{1}{2}\right)\mathcal{I}^{\frac{1}{2}}\mathcal{G}_i^{0,-\frac{1}{2}} = \frac{\sqrt{2\pi}\,\Gamma(i+\frac{3}{2})}{\Gamma(i+2)} t J_i^{-\frac{1}{2},1}(t), \quad i \geq 0$$

and thus,

$$\begin{aligned} \left(\mathcal{K}\mathcal{G}_i^{0,-\frac{1}{2}}, \mathcal{G}_j^{0,-\frac{1}{2}}\right)_{0,-\frac{1}{2}} &= \frac{\sqrt{2\pi}\,\Gamma(i+\frac{3}{2})}{\Gamma(i+2)} \int_{\Lambda} t J_i^{-\frac{1}{2},1}(t) \mathcal{G}_j^{0,-\frac{1}{2}}(t)(2t)^{-\frac{1}{2}}\, dt \\ &= \sqrt{\frac{\pi}{2}} \frac{\Gamma(i+\frac{3}{2})}{\Gamma(i+2)} \left(J_i^{-\frac{1}{2},1}, J_j^{0,\frac{1}{2}}\right)_{0,1} := \tilde{\tilde{\mathcal{A}}}_{ji}, \\ & \qquad i \geq 0,\ j = 0, 1, \ldots, N. \end{aligned} \tag{3.7}$$

Finally, substituting (3.7) in (3.3) we obtain a linear algebraic system $\tilde{\mathcal{A}}\tilde{a} = \tilde{f}$ with

$$\tilde{\mathcal{A}} := \left(\tilde{\mathcal{A}}_{ji}\right)_{i\geq 0, j=0,1,\ldots,N} := \begin{cases} \gamma_j^{0,\frac{1}{2}} - \lambda\tilde{\tilde{\mathcal{A}}}_{ji}, & j = i, \\ -\lambda\tilde{\tilde{\mathcal{A}}}_{ji}, & j \neq i, \end{cases}$$

and entries that can be obtained in a very simple manner in comparison with the ones in (3.6).

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4. Convergence analysis. The main topic of this section is to derive an error estimate for the proposed modified Tau scheme which theoretically justifies convergence of this method when approximating non-smooth solutions of (1.1).

In the sequel the symbol $\left(W^k(\Lambda\times\Lambda), \|.\|_{W^k(\Lambda\times\Lambda)}\right)$ will refer to the Sobolev space of order k over $\Lambda\times\Lambda$. Supplementary information of this Hilbert space can be found in [9]. In our analysis we shall apply Hardy's and Gronwall's inequality (see [7]):

LEMMA 4.1 (Generalized Hardy inequality). *For a measurable function $g\geq 0$, the following generalized Hardy inequality*

$$\left(\int_a^b |(\mathcal{N}g)(t)|^q w_1(t)\,dt\right)^{1/q} \leq C\left(\int_a^b |g(t)|^p w_2(t)\,dt\right)^{1/p},$$

holds if and only if

$$\sup_{a<t<b}\left(\int_t^b w_1(t)\,dt\right)^{1/q}\left(\int_a^t w_2^{1-p'}(t)\,dt\right)^{1/p'} < \infty, \quad p'=\frac{p}{p-1},$$

for $1<p\leq q<\infty$. Here, $\mathcal{N}$ is an operator of the form

$$(\mathcal{N}g)(t)=\int_a^t N(t,s)g(s)\,ds,$$

with given kernel $N(t,s)$ and weight functions $w_1(t)$, $w_2(t)$, for $-\infty\leq a<b\leq\infty$.

LEMMA 4.2 (Gronwall's inequality). *Assume that $u(t)$ is a non-negative, locally integrable function defined on Λ that satisfies*

$$u(t)\leq b(t)+B\int_0^t (t-s)^m s^n u(s)\,ds, \quad s\in\Lambda,$$

where $m,n>-1$, $b(t)\geq 0$, and $B\geq 0$. Then there exists a constant C such that

$$u(t)\leq b(t)+C\int_0^t (t-s)^m s^n b(s)\,ds, \quad s\in\Lambda.$$

THEOREM 4.3 (Convergence). *Let $y(t)$ be the exact solution of (1.1), which is assumed to be non-smooth. Let the approximated solution $y_N(t)$ be obtained by using the modified Tau scheme proposed in the previous section. If $\mathcal{D}^{\frac{1}{2}+l}y\in L^2_{\frac{1}{2}+l,l}(\Lambda)$ for $l\geq 0$ and $K(t,s)\in W^{l_1}(\Lambda\times\Lambda)$ for $l_1\geq 1$, then for sufficiently large N, we have the following error estimate*

$$\|e_N\|_{0,-\frac{1}{2}}\leq C\left(N^{-\frac{1}{2}}\sqrt{\frac{(N-l+1)!}{(N+l+1)!}}\|\mathcal{D}^{\frac{1}{2}+l}y\|_{\frac{1}{2}+l,l}+N^{\frac{3}{4}-l_1}\|K(t,s)\|_{W^{l_1}(\Lambda\times\Lambda)}\|y\|_{0,-\frac{1}{2}}\right),$$

where $e_N(t)=y(t)-y_N(t)$ denotes the error function.

Proof. According to the previous section and (2.4), the modified Tau solution $y_N(t)$ for (1.1) satisfies the following operator equation:

$$y_N=\Pi_N^{0,-\frac{1}{2}}f+\lambda\Pi_N^{0,-\frac{1}{2}}\mathcal{K}_N y_N. \tag{4.1}$$

Here $\mathcal{K}_N y_N = \int\limits_0^t \frac{\left(\Pi_N^{0,0} K(t,s)\right) y_N(s)}{\sqrt{t-s}}\, ds$. Subtracting (1.1) from (4.1) yields

$$e_N(t) = e_N^{0,-\frac{1}{2}} f + \lambda\left(\mathcal{K}y - \Pi_N^{0,-\frac{1}{2}} \mathcal{K}_N y_N\right), \tag{4.2}$$

where $e_N^{0,-\frac{1}{2}} f = f - \Pi_N^{0,-\frac{1}{2}} f$ is the truncation error of the generalized Jacobi series. Using some simple manipulations we can obtain

$$\begin{aligned}
\lambda\left(\mathcal{K}y - \Pi_N^{0,-\frac{1}{2}} \mathcal{K}_N y_N\right) &= \lambda\left(e_N^{0,-\frac{1}{2}} \mathcal{K}y + \Pi_N^{0,-\frac{1}{2}}\left(\mathcal{K}y - \mathcal{K}_N y_N\right)\right) \\
&= \lambda\left(e_N^{0,-\frac{1}{2}} \mathcal{K}y + \Pi_N^{0,-\frac{1}{2}}\left(\mathcal{K}e_N - \int_0^t \frac{e_N^{0,0} K(t,s)}{\sqrt{t-s}} y_N(s)\, ds\right)\right) \\
&= \lambda\left(e_N^{0,-\frac{1}{2}} \mathcal{K}y + \mathcal{K}e_N - e_N^{0,-\frac{1}{2}} \mathcal{K}e_N - \Pi_N^{0-\frac{1}{2}} \int_0^t \frac{e_N^{0,0} K(t,s)}{\sqrt{t-s}} y_N(s)\, ds\right) \\
&= e_N^{0,-\frac{1}{2}}\left(y - f\right) + \lambda\left(\mathcal{K}e_N - e_N^{0,-\frac{1}{2}} \mathcal{K}e_N - \Pi_N^{0-\frac{1}{2}} \int_0^t \frac{e_N^{0,0} K(t,s)}{\sqrt{t-s}} y_N(s)\, ds\right).
\end{aligned} \tag{4.3}$$

Substituting (4.3) in (4.2) we get

$$e_N(t) = e_N^{0,-\frac{1}{2}} y + \lambda\left(\mathcal{K}e_N - e_N^{0,-\frac{1}{2}} \mathcal{K}e_N - \Pi_N^{0-\frac{1}{2}} \int_0^t \frac{e_N^{0,0} K(t,s)}{\sqrt{t-s}} y_N(s)\, ds\right),$$

and, hence

$$\begin{aligned}
|e_N| \le |\lambda|\left(\mathcal{K}|e_N|\right) &+ \left(|e_N^{0,-\frac{1}{2}} y|\right. \\
&\left. + |\lambda|\left(|e_N^{0,-\frac{1}{2}} \mathcal{K}e_N| + \left|\Pi_N^{0-\frac{1}{2}} \int_0^t \frac{e_N^{0,0} K(t,s)}{\sqrt{t-s}} y_N(s)\, ds\right|\right)\right).
\end{aligned} \tag{4.4}$$

Using Gronwall's inequality (Lemma 4.2) in (4.4), we have

$$\begin{aligned}
\|e_N\|_{0,-\frac{1}{2}} \le C_1 &\Bigg(\|e_N^{0,-\frac{1}{2}} y\|_{0,-\frac{1}{2}} + \|e_N^{0,-\frac{1}{2}} \mathcal{K}e_N\|_{0,-\frac{1}{2}} \\
&+ \left\|\Pi_N^{0-\frac{1}{2}} \int_0^t \frac{e_N^{0,0} K(t,s)}{\sqrt{t-s}} y_N(s)\, ds\right\|_{0,-\frac{1}{2}}\Bigg).
\end{aligned} \tag{4.5}$$

Since $\Pi_N^{0,-\frac{1}{2}}$ is an orthogonal projection, then $\|\Pi_N^{0,-\frac{1}{2}}\|_{0,-\frac{1}{2}} = 1$; see [1]. Thus, we can rewrite (4.5) as

$$\begin{aligned}
\|e_N\|_{0,-\frac{1}{2}} &\le C_1\Bigg(\|e_N^{0,-\frac{1}{2}} y\|_{0,-\frac{1}{2}} + \|e_N^{0,-\frac{1}{2}} \mathcal{K}e_N\|_{0,-\frac{1}{2}} \\
&\qquad + \left\|\int_0^t \frac{e_N^{0,0} K(t,s)}{\sqrt{t-s}} y_N(s)\, ds\right\|_{0,-\frac{1}{2}}\Bigg) \\
&\le C_1\Bigg(\|e_N^{0,-\frac{1}{2}} y\|_{0,-\frac{1}{2}} + \|e_N^{0,-\frac{1}{2}} \mathcal{K}e_N\|_{0,-\frac{1}{2}} \\
&\qquad + \|e_N^{0,0} K(t,s)\|_\infty \left\|\int_0^t \frac{y_N(s)}{\sqrt{t-s}}\, ds\right\|_{0,-\frac{1}{2}}\Bigg).
\end{aligned} \tag{4.6}$$

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Using the generalized Hardy inequality (Lemma 4.1) in (4.6), we can conclude

$$\begin{aligned}\|e_N\|_{0,-\frac12} &\le C_1\Big(\|e_N^{0,-\frac12}y\|_{0,-\frac12} + \|e_N^{0,-\frac12}\mathcal{K}e_N\|_{0,-\frac12} \\ &\qquad\qquad + \|e_N^{0,0}K(t,s)\|_\infty\|y_N\|_{0,-\frac12}\Big) \\ &\le C_1\Big(\|e_N^{0,-\frac12}y\|_{0,-\frac12} + \|e_N^{0,-\frac12}\mathcal{K}e_N\|_{0,-\frac12} \\ &\qquad\qquad + \|e_N^{0,0}K(t,s)\|_\infty\Big(\|y\|_{0,-\frac12} + \|e_N\|_{0,-\frac12}\Big)\Big).\end{aligned} \tag{4.7}$$

Applying estimate (2.5) in (4.7) yields

$$\begin{aligned}\|e_N\|_{0,-\frac12} \le& C_2N^{-\frac12}\left(\sqrt{\frac{(N-l+1)!}{(N+l+1)!}}\|\mathcal{D}^{\frac12+l}y\|_{\frac12+l,l} + \|\mathcal{D}^{\frac12}\mathcal{K}e_N\|_{\frac12,0}\right) \\ &+ C_1\|e_N^{0,0}K(t,s)\|_\infty\Big(\|y\|_{0,-\frac12} + \|e_N\|_{0,-\frac12}\Big).\end{aligned} \tag{4.8}$$

Now we try to find a suitable upper bound for $\|\mathcal{D}^{\frac12}\mathcal{K}e_N\|_{\frac12,0}$. To this end, using (2.1), we can write

$$\begin{aligned}\|\mathcal{D}^{\frac12}\mathcal{K}e_N\|_{\frac12,0} &= \frac{1}{\sqrt{\pi}}\left\|\frac{d}{dx}\int_0^x\int_0^s (x-s)^{-\frac12}K(s,t)e_N(t)\,dt\,ds\right\|_{\frac12,0} \\ &= \frac{1}{\sqrt{\pi}}\left\|\frac{d}{dx}\int_0^x\int_t^x (x-s)^{-\frac12}K(s,t)e_N(t)\,ds\,dt\right\|_{\frac12,0} \\ &= \frac{1}{\sqrt{\pi}}\left\|\frac{d}{dx}\int_0^x \bar{K}(x,t)e_N(t)\,dt\right\|_{\frac12,0} = \frac{1}{\sqrt{\pi}}\left\|\int_0^x \frac{\partial}{\partial x}\bar{K}(x,t)e_N(t)\,dt\right\|_{\frac12,0},\end{aligned} \tag{4.9}$$

where $\bar{K}(x,t) = \int_t^x (x-s)^{-\frac12}K(s,t)\,ds$. Applying Hardy's inequality (Lemma 4.1) with $w_1(t) = w^{\frac12,0}(t)$ and $w_2(t) = w^{0,-\frac12}(t)$ in (4.9) yields

$$\|\mathcal{D}^{\frac12}\mathcal{K}e_N\|_{\frac12,0} \le C_3\|e_N\|_{0,-\frac12}.$$

Inserting this into (4.8), we obtain

$$\begin{aligned}\|e_N\|_{0,-\frac12} \le& C_4N^{-\frac12}\left(\sqrt{\frac{(N-l+1)!}{(N+l+1)!}}\|\mathcal{D}^{\frac12+l}y\|_{\frac12+l,l} + \|e_N\|_{0,-\frac12}\right) \\ &+ C_1\|e_N^{0,0}K(t,s)\|_\infty\Big(\|y\|_{0,-\frac12} + \|e_N\|_{0,-\frac12}\Big).\end{aligned} \tag{4.10}$$

By applying the following estimate (see [10])

$$\|e_N^{0,0}K(t,s)\|_\infty \le CN^{\frac34-l_1}\|K(t,s)\|_{W^{l_1}(\Lambda\times\Lambda)}$$

in (4.10), the desired result is obtained for sufficiently large N. □

REMARK 4.4. From Lemma 1.1 we conclude that if $f(t)$ and $K(t,s)$ are sufficiently smooth functions, then the exact solution of (1.1) behaves like $y(t) \simeq \sqrt{t}$. It can be easily seen that in this case $\mathcal{D}^{\frac12+l}y$ is a sufficiently smooth function (see [4]), and then l and l_1 are sufficiently large numbers in Theorem 4.3. Thus, Theorem 4.3 implies that the proposed modified Tau method gives a high order of convergence in approximating (1.1) with solutions that may have the given regularity property in Lemma 1.1.

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5. Numerical results. In this section we apply a program written in Mathematica for two numerical examples to demonstrate the accuracy of the method and effectiveness of applying the modified Tau method. The obtained numerical results will be compared with the Jacobi Tau and regularized Jacobi Tau methods, which were proposed in [7] for the numerical solution of (1.1). The numerical error always refers to the weighted $L^2_{0,-\frac{1}{2}}$-norm of the obtained error function.

EXAMPLE 5.1. Consider the following Abel integral equation

$$y(t) = t^{\frac{7}{2}} + \frac{35\pi t^4}{128} - \int_0^t \frac{y(s)}{\sqrt{t-s}}\, ds, \quad t \in \Lambda,$$

with the exact non-smooth solution $y(t) = t^3\sqrt{t}$.

From [3, Remark 4.3] we can deduce that $\mathcal{D}^{\frac{1}{2}+l}(t^3\sqrt{t})$ is analytic for any $l \in \mathbb{N}_0$, so with $l = N$ and by Theorem 4.3 we have spectral accuracy in the numerical solution of this problem using the modified Tau method. We solve this problem by the modified Tau scheme and present the obtained results in Table 5.1. It can be seen from Table 5.1 that the modified Tau method finds the exact solution for $N = 3$ while the Jacobi Tau method gives poor numerical results and the regularized Jacobi Tau method provides a suitable approximate solution for $N \geq 7$. Thus, we can conclude that our algorithm has a significant advantage over the methods in [7]. In particular, we can see that the modified Tau method provides a good approximation of the exact solution with a smaller value of N in comparison with the regularized Jacobi Tau method proposed in [7].

TABLE 5.1
Comparison of errors obtained from our method and the numerical schemes in [7] for Example 5.1.

N	Numerical Errors Jacobi Tau method	 Regularized Jacobi Tau method	 Modified Tau method
1	2.35×10^{-1}	4.24×10^{-1}	8.34×10^{-2}
3	3.81×10^{-3}	7.95×10^{-2}	0
5	3.77×10^{-5}	3.12×10^{-3}	0
7	2.82×10^{-6}	9.78×10^{-15}	0
9	4.46×10^{-7}	1.08×10^{-16}	0

EXAMPLE 5.2 ([7]). Consider the following Abel integral equation

$$y(t) = f(t) - \frac{1}{2}\int_0^t \frac{y(s)}{\sqrt{t-s}}\, ds, \quad t \in I,$$

where $f(t) = \frac{\sin(t)}{\sqrt{t}} + \frac{\pi}{2}\sin\frac{t}{2}J_0(\frac{t}{2})$, with $J_0(t)$ the Bessel function. The exact solution of the problem is $y(t) = \frac{\sin(t)}{\sqrt{t}}$.

The asymptotic behavior of the exact solution is $y(t) \simeq \sqrt{x} - \frac{\sqrt{x^5}}{6} + \frac{\sqrt{x^9}}{120} + O(\sqrt{t^{11}})$. Trivially, $\mathcal{D}^{\frac{1}{2}+l}y$ is analytic for $l \geq 0$. Thus, according to Theorem 4.3, the errors decay exponentially as $N \to \infty$. Numerical errors obtained by solving this problem with the modified Tau method are given in Table 5.2 and Figure 5.1. In Table 5.2, the obtained errors of the modified Tau method are compared with the numerical errors obtained by solving this problem using the methods proposed in [7]. From Table 5.2, we can deduce that our scheme is more powerful and gives more reliable results. In particular, the results for the modified Tau method have clear superiority over those obtained by using the regularized Jacobi Tau

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method. In Figure 5.1, we can deduce an exponential rate of convergence for the modified Tau approximation of this problem since in this semi-log representation, one observes that the error variation is essentially linear versus the degree of approximation.

TABLE 5.2
Comparison of errors obtained from our method and the numerical schemes in [7] for Example 5.2.

	Numerical Errors		
N	Jacobi Tau method	Regularized Jacobi Tau method	Modified Tau method
2	1.44×10^{-3}	1.15×10^{-3}	1.60×10^{-4}
6	1.06×10^{-4}	5.54×10^{-7}	4.94×10^{-10}
10	2.39×10^{-5}	6.09×10^{-11}	1.67×10^{-16}
14	8.58×10^{-6}	5.65×10^{-15}	1.52×10^{-23}

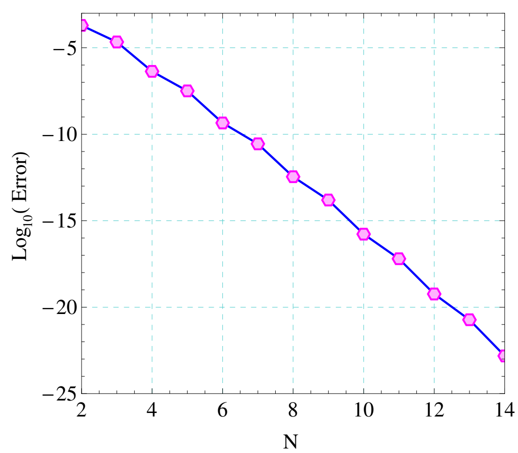

FIG. 5.1. *Errors of Example 5.2 for various values of N.*

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