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ON THE DEVELOPMENT OF PARAMETER-ROBUST PRECONDITIONERS AND COMMUTATOR ARGUMENTS FOR SOLVING STOKES CONTROL PROBLEMS*

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Abstract. The development of preconditioners for PDE-constrained optimization problems is a field of numerical analysis which has recently generated much interest. One class of problems which has been investigated in particular is that of Stokes control problems, that is, the problem of minimizing a functional with the Stokes (or Navier-Stokes) equations as constraints. In this manuscript, we present an approach for preconditioning Stokes control problems using preconditioners for the Poisson control problem and, crucially, the application of a commutator argument. This methodology leads to two block diagonal preconditioners for the problem, one of which was previously derived by W. Zulehner in 2011 [SIAM J. Matrix Anal. Appl., 32 (2011), pp. 536–560] using a nonstandard norm argument for this saddle point problem, and the other of which we believe to be new. We also derive two related block triangular preconditioners using the same methodology and present numerical results to demonstrate the performance of the four preconditioners in practice.

Key words. PDE-constrained optimization, Stokes control, saddle point system, preconditioning, Schur complement, commutator

AMS subject classifications. 65F08, 65F10, 65F50, 76D07, 76D55, 93C20

1. Introduction.

Decades ago, a significant area of research in numerical analysis was the numerical solution of the Stokes and Navier-Stokes equations, two partial differential equations (PDEs) that are crucial to the field of fluid dynamics. Preconditioned Krylov subspace methods for the solutions of the saddle point systems relating to each of these problems are given in [24] and [10], respectively, for instance. Since then, a further area of numerical analysis has become prominent: that of *PDE-constrained optimization*, which involves minimizing a functional with one or more PDEs as constraints. Consequently, the development of solvers for Stokes control problems, one of the most fundamental such problems, has itself become a well researched area.

There has been much success in this field: iterative solvers for a class of these problems that are independent of the mesh-size h have been devised for the time-independent problem in [19] and the time-dependent problem in [23]. Further, a multigrid solver constructed in [8] is shown to be itself independent of h. However, generating Krylov subspace solvers that are robust with respect to the regularization parameter as well as the mesh-size has proved to be a more difficult task—one notable exception is the preconditioned MINRES approach derived in [26] using a nonstandard norm argument, which does exhibit this independence.

In this manuscript, we consider the time-independent Stokes control problem where the velocity and the control variable are included in the cost functional but the pressure is not. We consider these problems using fundamental saddle point theory and explain how it is possible to use this to construct preconditioners for the Stokes control problem using a Poisson control preconditioner along with a commutator argument, the concept of which we shall describe.

There are many reasons why we believe such an investigation is of considerable interest. Firstly, it enables us to re-derive the preconditioner of Zulehner [26] within a pure saddle point framework. We are also able to derive a new block diagonal preconditioner for this problem that is robust with respect to mesh-size and the control regularization coefficient, as well as two block triangular preconditioners which appear to have the same property. Finally, and perhaps most intriguingly, we believe that the theory outlined in this paper can be applied to the much

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harder Navier-Stokes control problem, which we will address in a future manuscript [16]. We are also able to use the methodology presented here to explain why the choice of whether or not to regularize the pressure is crucial from a preconditioning point of view.

This manuscript is structured as follows. In Section 2, we detail two areas of background which we will make use of: those of saddle point theory and preconditioners for Poisson control problems. In Section 3, we combine these with the theory of commutator arguments to derive the four aforementioned preconditioners for Stokes control problems (two block diagonal and two block triangular). We also state the dominant computational operations required to apply our preconditioners and discuss the importance of the inclusion or omission of a pressure term in the cost functional. In Section 4, we provide numerical results to demonstrate how the preconditioners perform in practice, and in Section 5, we make some concluding remarks.

2. Background. In this section, we introduce two fundamental subject areas which we utilize in the remainder of this manuscript. Firstly, in Section 2.1, we outline some basic properties of saddle point systems which we make use of. Secondly, in Section 2.2, we detail the theory of solving Poisson control problems, which we also exploit.

2.1. Saddle point systems. The matrix systems that we consider the iterative solution of in the remainder of this manuscript are all of *saddle point* form, that is, of the form

$$\underbrace{\begin{bmatrix} \Phi & \Psi^T \\ \Psi & -\Theta \end{bmatrix}}_{\Lambda} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \tag{2.1}$$

where $\Phi \in \mathbb{R}^{m\times m}$, $\Psi \in \mathbb{R}^{p\times m}$, $p \leq m$, have full row rank and $\Theta \in \mathbb{R}^{p\times p}$. In the problems that we study, Φ and Θ are also symmetric. A comprehensive review of systems of saddle point type is given in [1].

We are seeking preconditioners for equations of the form (2.1). Therefore we utilize the observations that if we precondition Λ with $\bar{\mathcal{P}}_1$, $\bar{\mathcal{P}}_2$, or $\bar{\mathcal{P}}_3$, where

$$\bar{\mathcal{P}}_1 = \begin{bmatrix} \Phi & 0 \\ 0 & \Theta + \Psi\Phi^{-1}\Psi^T \end{bmatrix}, \quad \bar{\mathcal{P}}_2 = \begin{bmatrix} \Phi & 0 \\ \Psi & \Theta + \Psi\Phi^{-1}\Psi^T \end{bmatrix},$$
$$\bar{\mathcal{P}}_3 = \begin{bmatrix} \Phi & 0 \\ \Psi & -\Theta - \Psi\Phi^{-1}\Psi^T \end{bmatrix},$$

then the non-zero eigenvalues of $\bar{\mathcal{P}}_2^{-1}\Lambda$ and $\bar{\mathcal{P}}_3^{-1}\Lambda$ are given by

$$\lambda(\bar{\mathcal{P}}_2^{-1}\Lambda) = \{\pm 1\}, \quad \lambda(\bar{\mathcal{P}}_3^{-1}\Lambda) = \{1\}$$

for any choice of Θ, and the non-zero eigenvalues of $\bar{\mathcal{P}}_1^{-1}\Lambda$ are given by

$$\lambda(\bar{\mathcal{P}}_1^{-1}\Lambda) = \left\{1, \frac{1}{2}(1 \pm \sqrt{5})\right\},$$

provided $\Theta = 0$. The above results are given in [12, 13] in the case $\Theta = 0$; the eigenvalue results for $\bar{\mathcal{P}}_2^{-1}\Lambda$ and $\bar{\mathcal{P}}_3^{-1}\Lambda$ in the case $\Theta \neq 0$ are shown in [9].

Now, the matrices $\bar{\mathcal{P}}_1^{-1}\Lambda$ and $\bar{\mathcal{P}}_2^{-1}\Lambda$ are diagonalizable but $\bar{\mathcal{P}}_3^{-1}\Lambda$ is not, so consequently a Krylov subspace method for the solution of (2.1) with 'ideal' preconditioners $\bar{\mathcal{P}}_1$, $\bar{\mathcal{P}}_2$, and $\bar{\mathcal{P}}_3$ should converge in 3, 2, and 2 iterations, respectively [13], in the relevant cases provided that Λ is non-singular.[1] Naturally, we are not explicitly constructing $\bar{\mathcal{P}}_1$, $\bar{\mathcal{P}}_2$, and $\bar{\mathcal{P}}_3$ as this would be

[1] In the problem we will consider, the matrix is singular, however, the preconditioners described are often effective for such systems also.

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computationally wasteful but instead construct approximations $\widehat{\mathcal{P}}_1$, $\widehat{\mathcal{P}}_2$, and $\widehat{\mathcal{P}}_3$ such that the actions of the inverses of our preconditioners may be applied efficiently. Having developed these preconditioners, one may consider the MINRES algorithm [14] with preconditioners of the form $\widehat{\mathcal{P}}_1$ and preconditioners of the form $\widehat{\mathcal{P}}_2$ and $\widehat{\mathcal{P}}_3$ used in conjunction with solvers such as GMRES [20] and the Bramble-Pasciak Conjugate Gradient method [3].

We note that the quantity $S := \Theta + \Psi\Phi^{-1}\Psi^T$ is important in all three of the above preconditioners; this term is commonly known as the (negative) *Schur complement* of Λ, and much emphasis will be placed on approximating this quantity of the matrix systems we consider.

2.2. Optimal control of Poisson's equation. In literature including [18, 21, 26], the iterative solution of the matrix system resulting from the distributed Poisson control problem

$$\begin{aligned} \min_{y,u} \ & \frac{1}{2}\left\|y-\widehat{y}\right\|^2_{L_2(\Omega)} + \frac{\beta}{2}\left\|u\right\|^2_{L_2(\Omega)} \\ \text{s.t.} \ & -\triangle y = u, \quad \text{in } \Omega, \\ & \quad\ \ y = g, \quad \text{on } \partial\Omega, \end{aligned}$$

is considered. Here, the domain on which we are working is denoted as $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, with boundary $\partial\Omega$. Moreover, y denotes the *state variable* (with $\widehat{y}$ some *desired state*), u the *control variable*, and β a *regularization parameter* (or *Tikhonov parameter*). The symbol $\triangle$ denotes the Laplacian operator.

Discretizing the above problem using equal-order finite element basis functions for y, u, and p leads to the 2×2 matrix system [18]

$$\begin{bmatrix} M & K \\ K & -\frac{1}{\beta}M \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\widehat{\mathbf{y}} + \mathbf{c} \\ \mathbf{d} \end{bmatrix}, \tag{2.2}$$

where $\mathbf{y}$ and $\widehat{\mathbf{y}}$ are the discretized versions of y and $\widehat{y}$, respectively, $\mathbf{c}$ and $\mathbf{d}$ are vectors corresponding to the boundary conditions, and $\mathbf{p}$ is the discretized version of the *adjoint variable* (or *Lagrange multiplier*) p, which is related to u by $\beta u - p = 0$. Here M denotes a finite element mass matrix and K a finite element stiffness matrix, two frequently used types of matrices, both of which are positive definite.

Two preconditioners which are robust for all values of the mesh-size h and the regularization parameter β, and which we denote $\mathcal{P}_1^P$ and $\mathcal{P}_2^P$ here, have been developed and tested for the matrix system (2.2) in [26] and [18], respectively:

$$\begin{aligned} \mathcal{P}_1^P &= \begin{bmatrix} M + \sqrt{\beta}K & 0 \\ 0 & \frac{1}{\beta}(M + \sqrt{\beta}K) \end{bmatrix}, \\ \mathcal{P}_2^P &= \begin{bmatrix} M & 0 \\ 0 & \left(K + \frac{1}{\sqrt{\beta}}M\right)M^{-1}\left(K + \frac{1}{\sqrt{\beta}}M\right) \end{bmatrix}. \end{aligned}$$

These two preconditioners have been derived in very different ways: $\mathcal{P}_1^P$ was obtained using a nonstandard norm argument in [26] and $\mathcal{P}_2^P$ using the saddle point theory described in Section 2.1.[1] Each of these preconditioners for the Poisson control problem may be extended to an effective preconditioner for Stokes control problems as we demonstrate in Section 3.

[1] The crucial step in constructing $\mathcal{P}_2^P$ is that of approximating the Schur complement of (2.1), $KM^{-1}K + \frac{1}{\beta}M$. In [18], it is shown that if we approximate this by $\left(K + \frac{1}{\sqrt{\beta}}M\right)M^{-1}\left(K + \frac{1}{\sqrt{\beta}}M\right)$, then the eigenvalues of the preconditioned Schur complement are all contained within the interval $\left[\frac{1}{2}, 1\right]$.

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3. Optimal control of the Stokes equations. The problem that we consider for the majority of this section is the following distributed Stokes control problem:

$$\begin{aligned} \min_{\underline{\mathbf{v}},\underline{\mathbf{u}}} \quad & \frac{1}{2}\|\underline{\mathbf{v}}-\widehat{\underline{\mathbf{v}}}\|^2_{L_2(\Omega)}+\frac{\beta}{2}\|\underline{\mathbf{u}}\|^2_{L_2(\Omega)} \\ \text{s.t.} \quad & -\triangle\underline{\mathbf{v}}+\nabla p=\underline{\mathbf{u}}, \quad \text{in } \Omega, \\ & -\nabla\cdot\underline{\mathbf{v}}=0, \quad \text{in } \Omega, \\ & \underline{\mathbf{v}}=\underline{\mathbf{g}}, \quad \text{on } \partial\Omega. \end{aligned}$$

Again we work on a domain $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$, with boundary $\partial\Omega$ and with a regularization parameter β. Here, $\underline{\mathbf{v}}$ denotes the velocity in d dimensions and p the pressure term, both of which are state variables in this problem. $\underline{\mathbf{u}}$ is the control variable in d dimensions. We also introduce at this point the adjoint variables $\underline{\boldsymbol{\lambda}}$ (which is equal to $\beta\underline{\mathbf{u}}$) and μ.

Discretizing this problem results in the matrix system [26]

$$\begin{bmatrix} M & 0 & K & B^T \\ 0 & 0 & B & 0 \\ K & B^T & -\frac{1}{\beta}M & 0 \\ B & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} M\widehat{\mathbf{v}}+\mathbf{c} \\ \mathbf{0} \\ \mathbf{d} \\ \mathbf{f} \end{bmatrix}, \tag{3.1}$$

where M and K here denote $d\times d$ block matrices with mass and stiffness matrices from the velocity space on the block diagonals,[1] and B represents the negative of the divergence operator on the finite element space in matrix form. The vector $\widehat{\mathbf{v}}$ corresponds to the target function $\widehat{\underline{\mathbf{v}}}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are related to the adjoint variables $\underline{\boldsymbol{\lambda}}$ and μ, and the vectors $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{f}$ take account of boundary conditions. We note at this point that this matrix is in general singular, as it is well known that the vector of ones is a member of the nullspace of B^T (see [6, Chapter 5] for instance)—the matrix in (3.1) therefore has two zero eigenvalues (one corresponding to each appearance of B^T).[2] However, this may be avoided by restricting the pressure space to the orthogonal complement of the nullspace as in this case the matrix B^T will clearly no longer have a nullspace.

We also note that in the construction of the functional being minimized in this optimal control problem, we have not regularized the pressure term—the problem where pressure is regularized was considered in [19, 23], for instance. This is extremely important from a preconditioning point of view, and in Section 3.4 we explain why this makes a major difference.

We consider discretizing this problem using the well-studied (inf-sup stable) *Taylor-Hood* finite element basis functions, that is, discretizing the velocity $\underline{\mathbf{v}}$ using $\mathbf{Q2}$-basis functions and the pressure p using $\mathbf{Q1}$-basis functions. We discretize the control $\underline{\mathbf{u}}$ and adjoint variable $\underline{\boldsymbol{\lambda}}$ using $\mathbf{Q2}$-functions and the adjoint variable μ using $\mathbf{Q1}$-functions.

It is not immediately obvious how the preconditioners derived for the Poisson control problem in the previous section can be applied to the more difficult Stokes control problem. In this section, we explain how this may be achieved.

[1] Note that this definition of M and K is slightly different from the definition used in Section 2.2 where these terms simply denoted a single mass or stiffness matrix.

[2] On the continuous level, the zero eigenvalues arise from the fact that an arbitrary constant may be added to the solution of the pressure p or the adjoint variable μ yielding another solution.

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3.1. Derivation of block diagonal preconditioners. To commence our derivation, we reorder the matrix system (3.1) so that we are dealing with the system

$$\underbrace{\begin{bmatrix} M & K & B^T & 0 \\ K & -\frac{1}{\beta}M & 0 & B^T \\ B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\widehat{\mathbf{v}} + \mathbf{c} \\ \mathbf{d} \\ \mathbf{f} \\ \mathbf{0} \end{bmatrix}. \tag{3.2}$$

This is a saddle point system of the form (2.1) with

$$\Phi = \begin{bmatrix} M & K \\ K & -\frac{1}{\beta}M \end{bmatrix}, \quad \Psi = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that the $(1,1)$-block Φ is of the form of the matrix system (2.2) relating to the Poisson control problem. We will use this to motivate two block diagonal preconditioners related to two preconditioners for Poisson control detailed in Section 2.2. These preconditioners will be of the form

$$\mathcal{P} = \begin{bmatrix} \widehat{\Phi} & 0 \\ 0 & (\Psi\Phi^{-1}\Psi^T)_{\text{approx}} \end{bmatrix}. \tag{3.3}$$

Such a strategy also leads to block triangular preconditioners of the form

$$\mathcal{P} = \begin{bmatrix} \widehat{\Phi} & 0 \\ \Psi & (\Psi\Phi^{-1}\Psi^T)_{\text{approx}} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \widehat{\Phi} & 0 \\ \Psi & -(\Psi\Phi^{-1}\Psi^T)_{\text{approx}} \end{bmatrix}. \tag{3.4}$$

We will derive two such block triangular preconditioners in Section 3.2.

3.1.1. First preconditioner. We motivate our first preconditioner for the Stokes control system (3.2) using the preconditioner $\mathcal{P}_1^P$ for the Poisson control problem of Section 2.2. We first note that the $(1,1)$-block of the Stokes control problem (3.2) is of the form of the matrix involved in the Poisson control problem, so we write, in the notation of (3.3),

$$\Phi = \begin{bmatrix} M & K \\ K & -\frac{1}{\beta}M \end{bmatrix} \approx \begin{bmatrix} M + \sqrt{\beta}K & 0 \\ 0 & \frac{1}{\beta}(M + \sqrt{\beta}K) \end{bmatrix} =: \widehat{\Phi}.$$

Here, the notation $\Phi \approx \widehat{\Phi}$ indicates that $\widehat{\Phi}$ has been constructed with the aim that the singular values of $\widehat{\Phi}^{-1}\Phi$ are bounded within a fixed (small) interval.

The next step is to find a good approximation to the Schur complement $\Psi\Phi^{-1}\Psi^T$ of the matrix system (3.3); we justify a potential approximation by writing

$$\begin{aligned} \Psi\Phi^{-1}\Psi^T &= \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M & K \\ K & -\frac{1}{\beta}M \end{bmatrix}^{-1} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \\ &\approx \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M + \sqrt{\beta}K & 0 \\ 0 & \frac{1}{\beta}(M + \sqrt{\beta}K) \end{bmatrix}^{-1} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} =: \Psi\widehat{\Phi}^{-1}\Psi^T \\ &= \begin{bmatrix} B(M + \sqrt{\beta}K)^{-1}B^T & 0 \\ 0 & \beta B(M + \sqrt{\beta}K)^{-1}B^T \end{bmatrix}. \end{aligned}$$

We highlight the fact that, in general, the approximate identity $\widehat{\Phi} \approx \Phi$ does not necessarily tell us that $\Psi\widehat{\Phi}^{-1}\Psi^T \approx \Psi\Phi^{-1}\Psi^T$ (unless Ψ is a square and invertible matrix, which is not

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the case here). However, this seems to be a reasonable motivation for an approximation which is potentially effective, and we indeed find that this strategy does lead to a good approximation of $\Psi\Phi^{-1}\Psi^T$ for this problem. Furthermore, an eigenvalue analysis carried out in [11] for this preconditioner verifies its potency for the Stokes control problem.

At this point, as it is done in [26], we may approximate $B(M+\sqrt{\beta}K)^{-1}B^T$ by $(\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1}$ in the above expression,[1] where M_p and K_p denote finite element mass and stiffness matrices, respectively, of the pressure space. Hence, we may write that

$$\Psi\Phi^{-1}\Psi^T \approx \begin{bmatrix} (\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1} & 0 \\ 0 & \beta(\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1} \end{bmatrix} =: (\Psi\Phi^{-1}\Psi^T)_{\text{approx}}.$$

Therefore, putting all of the above working together, we postulate that

$$\mathcal{P}_1 = \begin{bmatrix} M+\sqrt{\beta}K & 0 & 0 & 0 \\ 0 & \frac{1}{\beta}(M+\sqrt{\beta}K) & 0 & 0 \\ 0 & 0 & (\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1} & 0 \\ 0 & 0 & 0 & \beta(\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1} \end{bmatrix}$$

is an effective preconditioner for $\mathcal{A}$. This is exactly the preconditioner proposed by Zulehner in [26] using a nonstandard norm argument. We will demonstrate the effectiveness of this preconditioner by displaying numerical results in Section 4.

3.1.2. Second preconditioner. We are also able to derive a new block diagonal preconditioner for the Stokes control system (3.2) using the preconditioner $\mathcal{P}_2^P$ for the Poisson control problem. We treat the $(1,1)$-block of the Stokes control system by using the preconditioner for Poisson control, writing (in the notation of (3.3))

$$\begin{aligned}\Phi = \begin{bmatrix} M & K \\ K & -\frac{1}{\beta}M \end{bmatrix} &\approx \begin{bmatrix} M & 0 \\ 0 & KM^{-1}K+\frac{1}{\beta}M \end{bmatrix} \\ &\approx \begin{bmatrix} M & 0 \\ 0 & \left(K+\frac{1}{\sqrt{\beta}}M\right)M^{-1}\left(K+\frac{1}{\sqrt{\beta}}M\right) \end{bmatrix} =: \widehat{\Phi}.\end{aligned}$$

We now again search for a good approximation to the Schur complement—we proceed as follows:

$$\begin{aligned}\Psi\Phi^{-1}\Psi^T &\approx \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & KM^{-1}K+\frac{1}{\beta}M \end{bmatrix}^{-1}\begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \\ &= \begin{bmatrix} BM^{-1}B^T & 0 \\ 0 & B\left(KM^{-1}K+\frac{1}{\beta}M\right)^{-1}B^T \end{bmatrix}.\end{aligned}$$

Once more, we have assumed in the above working that $\widehat{\Phi}$ being a good approximation to Φ leads to $\Psi\widehat{\Phi}^{-1}\Psi^T$ approximating $\Psi\Phi^{-1}\Psi^T$ well; for this problem we find that this heuristic does indeed lead to an effective approximation.

We do not yet have a feasible preconditioner as the matrices $BM^{-1}B^T$ and $B\left(KM^{-1}K+\frac{1}{\beta}M\right)^{-1}B^T$ cannot be inverted without computing the inverses of M or $KM^{-1}K+\frac{1}{\beta}M$. However, it is well known that $BM^{-1}B^T$ may be well approximated

[1] This may be done by applying the commutator argument of Section 3.1.2 with $\mathcal{L} := -\sqrt{\beta}\triangle + I$. This is carried out in a very similar fashion in [23] for matrices of this form for time-dependent Stokes control problems.

by K_p (see [6, Chapter 8]),[1] so we use this for the first block of our Schur complement approximation.

We therefore now seek an idea for approximating $\Sigma := B\left(KM^{-1}K + \frac{1}{\beta}M\right)^{-1}B^T$ so that we obtain a cheap and invertible approximation to the Schur complement. We do this by using a commutator argument, a type of which is described in [6] for the Navier-Stokes equations, for instance. We examine the commutator

$$\mathcal{E} = (\mathcal{L})\nabla - \nabla(\mathcal{L})_p,$$

where $\mathcal{L} = \triangle^2 + \frac{1}{\beta}I$. This is an operator carefully chosen to give us a matrix that we can use to approximate Σ.

Now, discretizing this commutator using finite elements gives

$$\mathcal{E}_h = (M^{-1}L)M^{-1}B^T - M^{-1}B^T(M_p^{-1}L_p),$$

where $L = KM^{-1}K + \frac{1}{\beta}M$. Pre-multiplying by $BL^{-1}M$ and post-multiplying by $L_p^{-1}M_p$, where $L_p = K_pM_p^{-1}K_p + \frac{1}{\beta}M_p$, then gives

$$BM^{-1}B^TL_p^{-1}M_p \approx BL^{-1}B^T,$$

where, crucially, we assume that the commutator $\mathcal{E}_h$ is small.

We may now use the fact that $BM^{-1}B^T \approx K_p$ and substitute in the expression for L to obtain that[2]

$$\Sigma = B\left(KM^{-1}K + \frac{1}{\beta}M\right)^{-1}B^T \approx K_pL_p^{-1}M_p,$$

and therefore that

$$\Sigma^{-1} \approx M_p^{-1}L_pK_p^{-1} = M_p^{-1}\left(K_pM_p^{-1}K_p + \frac{1}{\beta}M_p\right)K_p^{-1} = M_p^{-1}K_pM_p^{-1} + \frac{1}{\beta}K_p^{-1}.$$

We note that such an argument has been used a number of times before—we give a brief summary of some applications in Section 5.

Thus, a second possible preconditioner for $\mathcal{A}$ is

$$\mathcal{P}_2 = \begin{bmatrix} M & 0 & 0 & 0 \\ 0 & \left(K + \frac{1}{\sqrt{\beta}}M\right)M^{-1}\left(K + \frac{1}{\sqrt{\beta}}M\right) & 0 & 0 \\ 0 & 0 & K_p & 0 \\ 0 & 0 & 0 & \left(M_p^{-1}K_pM_p^{-1} + \frac{1}{\beta}K_p^{-1}\right)^{-1} \end{bmatrix},$$

which we postulate being an effective preconditioner. We again verify that this is the case by numerical results presented in Section 4.

We note at this point that this preconditioner is a more "flexible one" as we find that a preconditioner of this form may be applied to the more difficult and general linearizations

[1]The approximation $BM^{-1}B^T \approx K_p$ may be justified by the observations that $-\nabla\cdot\nabla = -\triangle$ on the continuous level, and that the matrices K_p, B, M, and B^T relate to the continuous operators $-\triangle$, $-\nabla\cdot$, I, and ∇, respectively.

[2]An approximation of the form $BL^{-1}B^T \approx K_pL_p^{-1}M_p$ was first introduced by Cahouet and Chabard in [4] for the forward Stokes problem. Such arguments have since been used to develop iterative solvers for a variety of fluid dynamics problems.

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of the Navier-Stokes control problem [16]. In more detail, when a Picard-type iteration is applied to this problem, we may rearrange the matrix system obtained so that we have as the $(1,1)$-block a matrix corresponding to the convection-diffusion control problem as opposed to a Poisson control problem here. Using a preconditioner derived for the convection-diffusion control problem in [17], we may apply a similar commutator argument to approximate the Schur complement of the matrix systems for Navier-Stokes control—for this problem, we find that we need to approximate $BM^{-1}B^T$ and $B\left(FM^{-1}F^T+\frac{1}{\beta}M\right)^{-1}B^T$, where F arises from the differential operator relating to the Navier-Stokes equations. By doing this we arrive at iterative solvers for the Navier- Stokes control problem. It is likely that such strategies could also be applied to the linear systems obtained when Newton iteration is applied to the problem.

3.2. Block triangular preconditioners. A useful aspect of our approach is that we may consider developing robust preconditioners for the Stokes control problem that are not of the block diagonal form of $\mathcal{P}_1$ and $\mathcal{P}_2$. We do this by considering various block triangular preconditioners of the Poisson control matrix system.

Firstly, we may consider a preconditioner of the form $\begin{bmatrix}\widehat{\Phi} & 0\\ \Psi & (\Psi\Phi^{-1}\Psi^T)_{\text{approx}}\end{bmatrix}$ stated in (3.4) that is in some sense analogous to $\mathcal{P}_1$ as derived in Section 3.1.1. We could in fact consider the same approximations $\widehat{\Phi}$ and $(\Psi\Phi^{-1}\Psi^T)_{\text{approx}}$ as we did to construct $\mathcal{P}_1$,

$$\widehat{\Phi}=\begin{bmatrix}M+\sqrt{\beta}K & 0\\ 0 & \frac{1}{\beta}(M+\sqrt{\beta}K)\end{bmatrix},$$

$$(\Psi\Phi^{-1}\Psi^T)_{\text{approx}}=\begin{bmatrix}(\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1} & 0\\ 0 & \beta(\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1}\end{bmatrix},$$

to develop the following block triangular preconditioner for $\mathcal{A}$:

$$\mathcal{P}_3=\begin{bmatrix}M+\sqrt{\beta}K & 0 & 0 & 0\\ 0 & \frac{1}{\beta}(M+\sqrt{\beta}K) & 0 & 0\\ B & 0 & (\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1} & 0\\ 0 & B & 0 & \beta(\sqrt{\beta}M_p^{-1}+K_p^{-1})^{-1}\end{bmatrix},$$

which may be applied within the GMRES algorithm.

In addition to this preconditioner, we may form a block lower triangular preconditioner for the Stokes control problem that is based on the following block triangular preconditioner $\mathcal{P}_3^P$ for the Poisson control problem:

$$\mathcal{P}_3^P=\begin{bmatrix}M & 0\\ K & -\left(K+\frac{1}{\sqrt{\beta}}M\right)M^{-1}\left(K+\frac{1}{\sqrt{\beta}}M\right)\end{bmatrix},$$

which was shown to be effective for that problem in [18]. We may, once again, use this as an approximation to the $(1,1)$-block of the Stokes control matrix $\mathcal{A}$.

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Let us consider how we may precondition the Schur complement of $\mathcal{A}$ while using this approximation of the $(1,1)$-block. We write, in the notation of (3.4),

$$\begin{aligned}
\Psi\Phi^{-1}\Psi^T &= \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M & K \\ K & -\frac{1}{\beta}M \end{bmatrix}^{-1} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \\
&\approx \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M & 0 \\ K & -\widehat{S}_P \end{bmatrix}^{-1} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \\
&= \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ \widehat{S}_P^{-1}KM^{-1} & -\widehat{S}_P^{-1} \end{bmatrix} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} =: \Psi\widehat{\Phi}^{-1}\Psi^T \\
&= \begin{bmatrix} BM^{-1}B^T & 0 \\ B\widehat{S}_P^{-1}KM^{-1}B^T & -B\widehat{S}_P^{-1}B^T \end{bmatrix} \\
&\approx \begin{bmatrix} K_p & 0 \\ B\widehat{S}_P^{-1}KM^{-1}B^T & -\left(M_p^{-1}K_pM_p^{-1} + \frac{1}{\beta}K_p^{-1} + \frac{2}{\sqrt{\beta}}M_p^{-1}\right)^{-1} \end{bmatrix} \\
&=: (\Psi\Phi^{-1}\Psi^T)_{\text{approx}},
\end{aligned}$$

where

$$\widehat{S}_P = \left(K + \frac{1}{\sqrt{\beta}}M\right) M^{-1} \left(K + \frac{1}{\sqrt{\beta}}M\right).$$

In the working above, we have again used the approximation $BM^{-1}B^T \approx K_p$. To approximate the matrix $B\widehat{S}_P^{-1}B^T$, we have used the same commutator argument as in Section 3.1.2 except with $L = \widehat{S}_P = KM^{-1}K + \frac{1}{\beta}M + \frac{2}{\sqrt{\beta}}K$ and $L_p = K_pM_p^{-1}K_p + \frac{1}{\beta}M_p + \frac{2}{\sqrt{\beta}}K_p$.

Therefore, applying the (block triangular) saddle point theory of Section 2.1, we arrive at a block triangular preconditioner for $\mathcal{A}$, namely,

$$\mathcal{P}_4 = \begin{bmatrix} M & 0 & 0 & 0 \\ K & -\widehat{S}_P & 0 & 0 \\ B & 0 & K_p & 0 \\ 0 & B & B\widehat{S}_P^{-1}KM^{-1}B^T & -\left(M_p^{-1}K_pM_p^{-1} + \frac{1}{\beta}K_p^{-1} + \frac{2}{\sqrt{\beta}}M_p^{-1}\right)^{-1} \end{bmatrix}.$$

Of course, we would not be able to apply the MINRES algorithm with the preconditioners $\mathcal{P}_3$ or $\mathcal{P}_4$; instead we would use the GMRES algorithm of [20]. However, numerical tests indicate that $\mathcal{P}_3$ and $\mathcal{P}_4$ are effective preconditioners for $\mathcal{A}$ nevertheless—we refer to Section 4 for a demonstration of this assertion.

3.3. Further comments. We now wish to make some further observations about the preconditioners which we have proposed. Firstly, it is natural to consider the effectiveness of the new commutator arguments we have introduced as such arguments are heuristic in nature. We therefore carry out numerical tests on our approximations; in particular we look for the maximum and minimum (non-zero) eigenvalues of

$$\left[M_p^{-1}K_pM_p^{-1} + \frac{1}{\beta}K_p^{-1}\right] B \left(KM^{-1}K + \frac{1}{\beta}M\right)^{-1} B^T, \tag{3.5}$$

$$\left[M_p^{-1}K_pM_p^{-1} + \frac{1}{\beta}K_p^{-1} + \frac{2}{\sqrt{\beta}}M_p^{-1}\right] B \left(KM^{-1}K + \frac{1}{\beta}M + \frac{2}{\sqrt{\beta}}K\right)^{-1} B^T, \tag{3.6}$$

which relate to the two new commutator arguments introduced in this paper, and which are utilized in the preconditioners $\mathcal{P}_2$ and $\mathcal{P}_4$, respectively. In Table 3.1, we provide eigenvalues

TABLE 3.1
Maximum and minimum (non-zero) eigenvalues for the commutator approximation (3.5) *used in the block diagonal preconditioner for different values of h and β.*

		β							
		10		10^{-2}		10^{-5}		10^{-8}	
		λ_2	$\lambda_{\max}$	λ_2	$\lambda_{\max}$	λ_2	$\lambda_{\max}$	λ_2	$\lambda_{\max}$
h	2^{-2}	0.0584	1.3315	0.1271	1.2617	0.4537	0.9776	0.4975	1.0096
	2^{-3}	0.0400	1.3495	0.0843	1.3245	0.2988	0.9591	0.5000	1.0090
	2^{-4}	0.0295	1.3730	0.0560	1.3560	0.1721	1.1442	0.4876	0.9994
	2^{-5}	0.0227	1.3645	0.0396	1.3624	0.1065	1.2964	0.3872	0.9968

TABLE 3.2
Maximum and minimum (non-zero) eigenvalues for the commutator approximation (3.6) *used in the block triangular preconditioner for different values of h and β.*

		β							
		10		10^{-2}		10^{-5}		10^{-8}	
		λ_2	$\lambda_{\max}$	λ_2	$\lambda_{\max}$	λ_2	$\lambda_{\max}$	λ_2	$\lambda_{\max}$
h	2^{-2}	0.0653	1.3211	0.1541	1.1475	0.3922	0.9171	0.4924	1.0026
	2^{-3}	0.0446	1.3443	0.1048	1.2563	0.2881	0.9550	0.4812	0.9839
	2^{-4}	0.0326	1.3694	0.0699	1.3167	0.1951	1.0707	0.4355	0.9876
	2^{-5}	0.0249	1.3633	0.0487	1.3466	0.1294	1.2051	0.3418	0.9968

of the matrix (3.5) for a range of mesh-sizes and values of β, and in Table 3.2, we present the same results for (3.6). For the results in both tables, an evenly spaced grid with Taylor-Hood elements was used with the values of h stated corresponding to the distance between **Q2**-nodes. We can see that the approximations are effective ones for a range of parameter values, especially for smaller values of β. We note a benign dependence of the effectiveness of the approximations on h, but the results obtained are still very reasonable.

Another pertinent question is how cheap it is to apply our proposed preconditioners. We therefore detail the main computational operations required to approximate $\mathcal{P}_1^{-1}$, $\mathcal{P}_2^{-1}$, $\mathcal{P}_3^{-1}$, and $\mathcal{P}_4^{-1}$ (excluding matrix multiplications, which are comparatively cheap). For the purposes of these descriptions, we view the preconditioners as 4×4 block matrices and refer to each block in this way.

- Operations needed to apply $\mathcal{P}_1^{-1}$:
  - $(1,1)$: 1 multigrid operation for $M + \sqrt{\beta}K$,
  - $(2,2)$: 1 multigrid operation for $M + \sqrt{\beta}K$,
  - $(3,3)$: 1 Chebyshev semi-iteration for M_p and 1 multigrid operation for K_p,
  - $(4,4)$: 1 Chebyshev semi-iteration for M_p and 1 multigrid operation for K_p,
  - total: 2 Chebyshev semi-iterations and 4 multigrids.
- Operations needed to apply $\mathcal{P}_2^{-1}$:
  - $(1,1)$: 1 Chebyshev semi-iteration for M,
  - $(2,2)$: 2 multigrid operations for $K + \frac{1}{\sqrt{\beta}}M$,
  - $(3,3)$: 1 multigrid operation for K_p,
  - $(4,4)$: 2 Chebyshev semi-iterations for M_p and 1 multigrid operation for K_p,
  - total: 3 Chebyshev semi-iterations and 4 multigrids.

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- Operations needed to apply $\mathcal{P}_3^{-1}$: these are the same as for $\mathcal{P}_1^{-1}$ and hence in total:
  - total: 2 Chebyshev semi-iterations and 4 multigrids.
- Operations needed to apply $\mathcal{P}_4^{-1}$:
  - $(1,1)$: 1 Chebyshev semi-iteration for M,
  - $(2,2)$: 2 multigrid operations for $K + \frac{1}{\sqrt{\beta}}M$,
  - $(3,3)$: 1 multigrid operation for K_p,
  - $(4,3)$: 1 Chebyshev semi-iteration for M and 2 multigrid operations for $K + \frac{1}{\sqrt{\beta}}M$,
  - $(4,4)$: 2 Chebyshev semi-iterations for M_p and 1 multigrid operation for K_p,
  - total: 4 Chebyshev semi-iterations and 6 multigrids.

We can see from this list of operations that the application of each preconditioner (especially $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$) is fairly cheap, and therefore that our methods should be computationally effective ones.

Finally, an important question arising from this work relates to whether the methodology can be applied to other problems of Stokes control type. In more detail, rather than considering distributed control problems of the form described in this manuscript, one could examine formulations where the control is only applied on the boundary or within some subdomain. It is likely that much of the methodology within this paper could be applied to these more diverse problems, however, two major issues will inevitably arise:

- A robust approximation $\widehat{\Phi}$ of the $(1,1)$-block Φ will become harder to construct. In particular deriving an approximation to the Schur complement of the Poisson control problem, which is involved in the construction of $\widehat{\Phi}$, will become heuristic in nature when subdomain problems are considered [15, Chapter 4], as opposed to the rigorous nature of the preconditioners for the Poisson control problem on the whole domain [18, 26] used in Sections 3.1.1 and 3.1.2.
- The application of commutator arguments to build Schur complement approximations $\widehat{S}$ becomes more troublesome as such arguments have not been so widely tested on subdomain problems. The Schur complement approximations will also be impacted by the less robust $(1,1)$-block approximation $\widehat{\Phi}$.

In summary, whereas we believe this work has the potential to be extended to more complex problems of Stokes and Navier-Stokes control type, it is clear that significant investigation will need to be carried out in relation to the validity of the $(1,1)$-block and Schur complement approximations before such an approach could be reliably applied.

3.4. Penalty term applied to pressure. In this section we briefly consider the Stokes control problem

$$\begin{aligned}
\min_{\underline{\mathbf{v}},p,\underline{\mathbf{u}}} \quad & \frac{1}{2}\left\|\underline{\mathbf{v}}-\widehat{\underline{\mathbf{v}}}\right\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\left\|p-\widehat{p}\right\|^2_{L_2(\Omega)} + \frac{\beta}{2}\left\|\underline{\mathbf{u}}\right\|^2_{L_2(\Omega)} \\
\text{s.t.} \quad & -\triangle\underline{\mathbf{v}} + \nabla p = \underline{\mathbf{u}}, \quad \text{in } \Omega, \\
& -\nabla\cdot\underline{\mathbf{v}} = 0, \quad \text{in } \Omega, \\
& \underline{\mathbf{v}} = \underline{\mathbf{g}}, \quad \text{on } \partial\Omega,
\end{aligned}$$

which is identical to the problem we have studied in the previous sections, except that we impose an additional term in the cost functional relating to the pressure (with α being the corresponding penalty parameter).

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It is useful to consider preconditioning of the resulting matrix system [19]

$$\underbrace{\begin{bmatrix} M & K & B^T & 0 \\ K & -\frac{1}{\beta}M & 0 & B^T \\ B & 0 & 0 & 0 \\ 0 & B & 0 & \alpha M_p \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\widehat{\mathbf{v}} + \mathbf{c} \\ \mathbf{d} \\ \mathbf{f} \\ \alpha M_p \widehat{\mathbf{p}} \end{bmatrix} \tag{3.7}$$

in light of the framework discussed in this paper, in particular, whether it is possible to precondition the problem arising from the Stokes control problem *with* a pressure penalty term in the same way as it is done to precondition the system arising *without* this pressure term.

For brevity, we simply consider developing a preconditioner of the form $\mathcal{P}_1$ for the matrix system (3.7) (we find that the same issues arise when trying to construct preconditioners of the form $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$). We may construct an approximation of the $(1,1)$-block of $\mathcal{B}$ exactly as we did for the matrix system $\mathcal{A}$ in Section 3.1.1 (as the $(1,1)$-blocks of $\mathcal{A}$ and $\mathcal{B}$ are the same). When we attempt to construct an approximation of the Schur complement of $\mathcal{B}$ in a similar way for $\mathcal{A}$, in the derivation of $\mathcal{P}_1$, we obtain the following:

$$\begin{aligned} S_{\mathcal{B}} &= -\begin{bmatrix} 0 & 0 \\ 0 & \alpha M_p \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M & K \\ K & -\frac{1}{\beta}M \end{bmatrix}^{-1} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \\ &\approx -\begin{bmatrix} 0 & 0 \\ 0 & \alpha M_p \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M + \sqrt{\beta}K & 0 \\ 0 & \frac{1}{\beta}(M + \sqrt{\beta}K) \end{bmatrix}^{-1} \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \\ &= \begin{bmatrix} B(M + \sqrt{\beta}K)^{-1}B^T & 0 \\ 0 & -\alpha M_p + \beta B(M + \sqrt{\beta}K)^{-1}B^T \end{bmatrix}. \end{aligned}$$

At this point we face a problem—the $(2,2)$-block of our proposed Schur complement approximation could be positive definite, negative definite, or indefinite, depending on the values of α, β, and h used, thus creating major issues when attempting to construct a positive definite preconditioner (which we require for use with MINRES). Even if the values of α, β, and h were such that $-\alpha M_p + \beta B(M + \sqrt{\beta}K)^{-1}B^T$ is positive definite, it is far from clear how we may efficiently approximate this matrix in a similar way as $\beta B(M + \sqrt{\beta}K)^{-1}B^T$ was approximated in Section 3.1.1. We are therefore unable to derive a parameter-robust preconditioner using our approach.

We therefore conclude that the Stokes control problem involving a penalty term for the pressure is a harder problem to solve robustly than the problem without, at least if the methodology discussed in this manuscript is considered. We point the reader to [19] for a solver for the time-independent problem with pressure penalty term that is independent of the mesh-size h (but not the penalty parameter β) and to [23] for an extension of this solver to the time-dependent case.

4. Numerical experiments. Having motivated the theoretical potential of our approach, we now seek to demonstrate how our preconditioners perform in practice. To do this, we consider two test problems. The first problem we look at is an optimal control analogue of the

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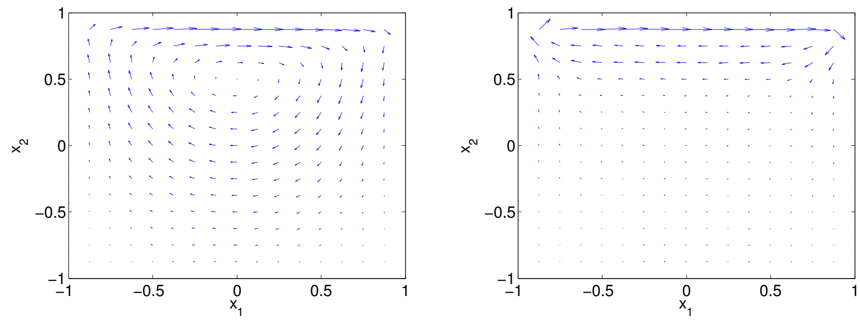
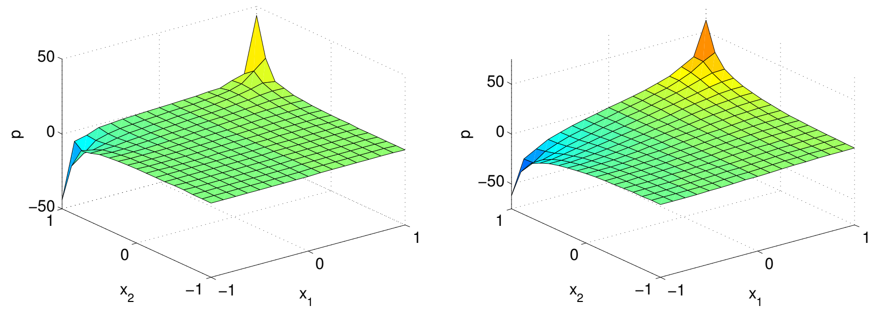


(a) $\beta = 1$ (b) $\beta = 10^{-4}$

FIG. 4.1. *Plots of the computed velocity solution to the first test problem for different* β.



(a) $\beta = 10^{-2}$ (b) $\beta = 10^{-6}$

FIG. 4.2. *Plots of the computed pressure solution to the first test problem for different* β.

lid-driven cavity problem on the domain $\Omega = [-1, 1]^2$:

$$\begin{aligned} \min_{\underline{\mathbf{v}},\underline{\mathbf{u}}} \quad & \frac{1}{2} \|\underline{\mathbf{v}}\|^2_{L_2(\Omega)} + \frac{\beta}{2} \|\underline{\mathbf{u}}\|^2_{L_2(\Omega)} \\ \text{s.t.} \quad & -\triangle \underline{\mathbf{v}} + \nabla p = \underline{\mathbf{u}}, \quad \text{in } \Omega, \\ & -\nabla \cdot \underline{\mathbf{v}} = 0, \quad \text{in } \Omega, \\ & \underline{\mathbf{v}} = \begin{cases} [1,0]^T & \text{on } [-1,1] \times \{1\}, \\ [0,0]^T & \text{on } \partial\Omega \backslash ([-1,1] \times \{1\}) \,. \end{cases} \end{aligned}$$

We wish to observe how well the four preconditioners presented in this paper perform when solving the matrix system relating to this problem. In Table 4.1, we display the number of MINRES iterations and CPU times[1] for solving this problem with preconditioner $\mathcal{P}_1$ to a tolerance of 10^{-6} for a variety of h and β. In Table 4.2, the number of iterations and CPU times for solving the same problem using MINRES with preconditioner $\mathcal{P}_2$ to the same tolerance is given. Finally in Tables 4.3 and 4.4, we report the iteration count and CPU times for solving

[1]The CPU times include the time taken to construct the matrices M_p and K_p involved in the preconditioner. We construct these matrices in the same way as in the Incompressible Flow & Iterative Solver Software (IFISS) package [5, 22]. Where appropriate, we follow the recipe detailed in [6, Chapter 8] of imposing a Dirichlet boundary condition in the matrix K_p at the node on the velocity space corresponding to the *inflow boundary condition*.

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TABLE 4.1
Number of iterations and CPU times (in seconds) when applying MINRES *to the first test problem with preconditioner $\mathcal{P}_1$ for a variety of h and β.*

		SIZE	β: 10^2	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
h	2^{-3}	$1,318$	80 0.281	80 0.283	60 0.216	44 0.189	36 0.156	$(32)^*$ (0.290)	$(26)^*$ (0.232)
	2^{-4}	$4,934$	84 0.755	85 0.766	66 0.601	52 0.488	37 0.475	$(32)^*$ (1.59)	$(26)^*$ (1.38)
	2^{-5}	$19,078$	88 3.03	90 3.08	70 2.41	58 2.04	44 2.05	32 1.54	$(28)^*$ (7.42)
	2^{-6}	$75,014$	86 12.6	90 13.6	74 11.0	62 10.3	50 7.96	33 8.39	$(28)^*$ (40.1)
	2^{-7}	$297,478$	86 62.0	88 58.8	76 53.5	66 46.2	54 39.3	40 29.2	26 28.1

TABLE 4.2
Number of iterations and CPU times (in seconds) when applying MINRES *to the first test problem with preconditioner $\mathcal{P}_2$ for a variety of h and β.*

		SIZE	β: 10^2	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
h	2^{-3}	$1,318$	112 0.502	107 0.480	85 0.388	59 0.317	42 0.222	$(30)^*$ (0.316)	$(25)^*$ (0.249)
	2^{-4}	$4,934$	125 1.51	123 1.49	97 1.18	68 0.847	48 0.768	$(33)^*$ (1.87)	$(25)^*$ (1.52)
	2^{-5}	$19,078$	142 6.68	137 6.42	102 4.79	75 3.59	60 3.54	39 2.37	$(27)^*$ (7.33)
	2^{-6}	$75,014$	156 32.5	148 30.9	106 22.3	80 17.2	67 14.1	48 17.0	$(29)^*$ (46.5)
	2^{-7}	$297,478$	165 141	160 138	106 91.2	84 80.5	72 61.9	54 49.4	34 89.6

the problem to the same tolerance with the GMRES algorithm used in the Incompressible Flow & Iterative Solver Software (IFISS) package[2] [5, 22], preconditioned with the matrices $\mathcal{P}_3$ and $\mathcal{P}_4$. In Figures 4.1 and 4.2, we display solutions to the test problem for velocity and pressure for different values of β. In each of the tables and figures, the value of h indicated corresponds to the spacing between **Q2**-nodes.

When generating these results, we use 20 steps of Chebyshev semi-iteration to approximate the inverses of mass matrices; see [25] for more details. To approximate the inverses of K_p, $M+\sqrt{\beta}K$, and $K+\frac{1}{\sqrt{\beta}}M$ in our preconditioners (note that the last two matrices are the same up to a multiplicative factor), we employ the algebraic multigrid (AMG) routine `HSL_MI20` from the Harwell Subroutine Library (HSL) [2], using 2 V-cycles with 2 pre- and post- (relaxed Jacobi) smoothing steps to approximate each matrix inverse. In all tables in this

[2] All results in Tables 4.1–4.6 are obtained using a tri-core 2.5 GHz workstation.

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TABLE 4.3
Number of iterations and CPU times (in seconds) when applying GMRES *to the first test problem with preconditioner $\mathcal{P}_3$ for a variety of h and β.*

		SIZE	β: 10^2	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
h	2^{-3}	$1,318$	64 0.238	62 0.247	53 0.195	44 0.188	39 0.184	$(33)^*$ (0.287)	$(28)^*$ (0.263)
	2^{-4}	$4,934$	65 0.671	63 0.674	56 0.573	50 0.516	41 0.565	$(38)^*$ (1.98)	$(31)^*$ (1.65)
	2^{-5}	$19,078$	63 2.54	63 2.53	56 2.26	53 2.11	48 2.81	38 2.27	$(35)^*$ (9.42)
	2^{-6}	$75,014$	63 13.7	61 12.5	57 13.1	54 11.5	51 10.9	41 13.8	$(37)^*$ (58.1)
	2^{-7}	$297,478$	63 60.8	62 62.8	56 55.3	52 45.1	52 51.8	48 43.8	39 45.5

TABLE 4.4
Number of iterations and CPU times (in seconds) when applying GMRES *to the first test problem with preconditioner $\mathcal{P}_4$ for a variety of h and β.*

		SIZE	β: 10^2	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
h	2^{-3}	$1,318$	91 0.755	85 0.675	67 0.529	46 0.415	26 0.238	$(19)^*$ (0.376)	$(14)^*$ (0.261)
	2^{-4}	$4,934$	107 2.55	101 2.38	79 1.84	59 1.38	34 1.02	$(24)^*$ (2.47)	$(15)^*$ (1.63)
	2^{-5}	$19,078$	123 11.7	114 10.8	88 8.20	73 6.75	47 5.42	29 3.42	$(21)^*$ (11.7)
	2^{-6}	$75,014$	138 63.5	131 58.5	99 43.7	81 37.0	62 27.6	37 24.1	$(25)^*$ (74.8)
	2^{-7}	$297,478$	156 327	150 287	109 224	89 161	73 130	48 92.3	30 148

section, the symbol * denotes that the coarsening of the AMG routine failed when applied to $M+\sqrt{\beta}K$ or $K+\frac{1}{\sqrt{\beta}}M$—this occurs in the specific and impractical parameter regime where h is large and β is small and is caused by the presence of positive off-diagonal entries. In these cases, we present results obtained using direct solves rather than AMG.

To test our methods further, we also consider the following second test problem on $\Omega=[-1,1]^2$:

$$\begin{aligned}\min_{\underline{\mathbf{v}},\underline{\mathbf{u}}}\quad & \frac{1}{2}\left\|\underline{\mathbf{v}}-\widehat{\underline{\mathbf{v}}}\right\|^2_{L_2(\Omega)}+\frac{\beta}{2}\left\|\underline{\mathbf{u}}\right\|^2_{L_2(\Omega)}\\ \text{s.t.}\quad & -\triangle\underline{\mathbf{v}}+\nabla p=\underline{\mathbf{u}},\quad \text{in }\Omega,\\ & -\nabla\cdot\underline{\mathbf{v}}=0,\quad \text{in }\Omega,\\ & \underline{\mathbf{v}}=\widehat{\underline{\mathbf{v}}},\quad \text{on }\partial\Omega,\end{aligned}$$

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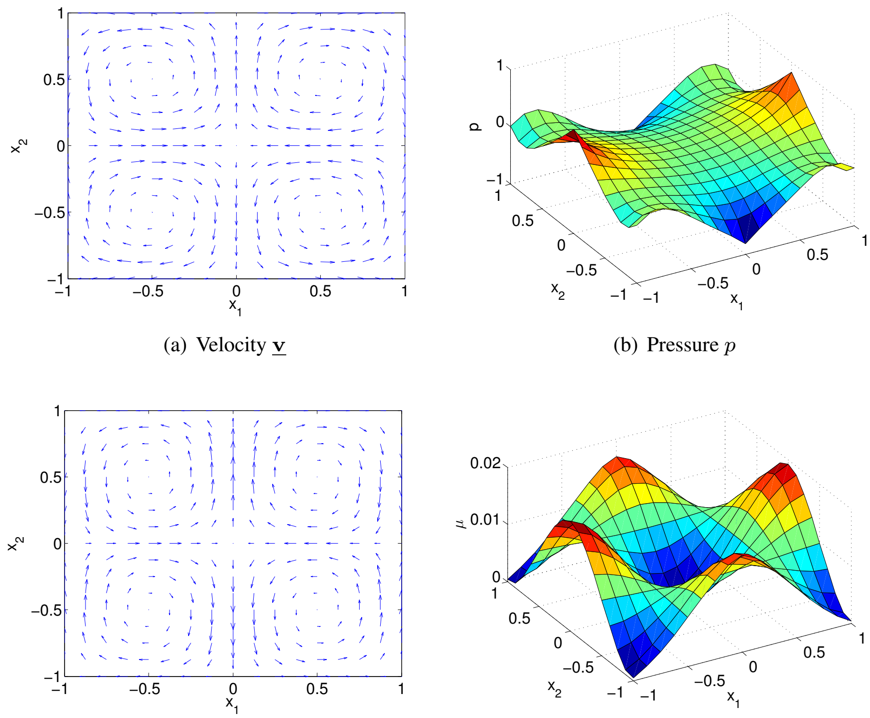

(a) Velocity $\underline{\mathbf{v}}$ (b) Pressure p

(c) Adjoint velocity $\underline{\boldsymbol{\lambda}}$ (d) Adjoint pressure μ

FIG. 4.3. *Plots of the computed solution to the second test problem with* $\beta = 10^{-4}$.

where

$$\widehat{\underline{\mathbf{v}}} = \begin{cases} \left[-\left(\frac{1}{2}-x_2\right)x_1(1+x_1), \left(\frac{1}{2}+x_1\right)x_2(1-x_2)\right]^T & \text{in } [-1,0]\times[0,1], \\ \left[-\left(\frac{1}{2}-x_2\right)x_1(1-x_1), \left(\frac{1}{2}-x_1\right)x_2(1-x_2)\right]^T & \text{in } [0,1]\times[0,1], \\ \left[-\left(\frac{1}{2}+x_2\right)x_1(1+x_1), \left(\frac{1}{2}+x_1\right)x_2(1+x_2)\right]^T & \text{in } [-1,0]\times[-1,0], \\ \left[-\left(\frac{1}{2}+x_2\right)x_1(1-x_1), \left(\frac{1}{2}-x_1\right)x_2(1+x_2)\right]^T & \text{in } [0,1]\times[-1,0], \end{cases}$$

and $\mathbf{x} = [x_1, x_2]^T$ denotes the spatial coordinates. The target state $\widehat{\underline{\mathbf{v}}}$ within this problem setup corresponds to a recirculating flow with symmetry built into the problem. In Figure 4.3, we display solution plots for this problem, and in Tables 4.5 and 4.6, we present numerical results for solving this problem using MINRES preconditioned with $\mathcal{P}_1$ and $\mathcal{P}_2$. Although we do not present results for our GMRES-based solvers for this problem, we note that the numerical features of these solvers are similar to those when tested on the first test problem.

The results shown in Tables 4.1–4.6 indicate that the four preconditioners discussed in this manuscript are robust with respect to mesh-size and regularization parameter.[1] The iteration count is low for all four solvers considering the complexity of the problems. In many practical problems, the value of β is within the range $[10^{-6}, 10^{-1}]$; all methods perform well in this regime. We note that the block diagonal preconditioner $\mathcal{P}_1$ (introduced in [26]) and the block triangular preconditioner $\mathcal{P}_3$ based on it solve the problem in the shortest time in all cases

[1]The only parameter regime where we do not observe complete robustness is that of very small β, when we observe some degradation in the performance of the AMG routine used.

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TABLE 4.5
Number of iterations and CPU times (in seconds) when applying MINRES *to the second test problem with preconditioner $\mathcal{P}_1$ for a variety of h and β.*

		SIZE	$\beta = 10^2$	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
h	2^{-3}	$1,318$	50 0.181	58 0.222	58 0.221	46 0.204	38 0.201	$(32)^*$ (0.337)	$(26)^*$ (0.240)
	2^{-4}	$4,934$	54 0.518	62 0.594	62 0.590	50 0.491	39 0.532	$(32)^*$ (1.72)	$(26)^*$ (1.61)
	2^{-5}	$19,078$	56 2.05	64 2.36	64 2.31	54 1.97	42 2.09	32 1.71	$(24)^*$ (7.15)
	2^{-6}	$75,014$	54 8.96	68 11.3	68 10.4	58 9.69	44 7.07	30 8.63	$(22)^*$ (39.3)
	2^{-7}	$297,478$	52 39.4	68 51.8	70 51.8	60 42.2	46 32.4	28 22.1	21 28.4

TABLE 4.6
Number of iterations and CPU times (in seconds) when applying MINRES *to the second test problem with preconditioner $\mathcal{P}_2$ for a variety of h and β.*

		SIZE	$\beta = 10^2$	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
h	2^{-3}	$1,318$	89 0.423	90 0.440	79 0.376	58 0.315	42 0.235	$(31)^*$ (0.341)	$(25)^*$ (0.276)
	2^{-4}	$4,934$	100 1.26	100 1.26	85 1.07	65 0.835	49 0.813	$(31)^*$ (1.87)	$(25)^*$ (1.54)
	2^{-5}	$19,078$	106 5.14	107 5.16	86 4.17	70 3.43	56 3.42	34 2.17	$(23)^*$ (7.36)
	2^{-6}	$75,014$	116 25.0	116 25.2	89 19.4	74 16.1	58 12.7	35 11.7	$(21)^*$ (39.3)
	2^{-7}	$297,478$	125 120	125 124	95 94.1	78 76.9	59 58.6	33 35.6	23 57.3

considered and with the lowest iteration count in most cases. However, the strategy involved in constructing these preconditioners is highly specific to this problem. We believe that the flexibility in the methodology used to construct $\mathcal{P}_2$ and $\mathcal{P}_4$ would enable us to consider the more general and much harder Navier-Stokes control problem, and therefore it is important to note that these preconditioners also seem to achieve robustness, albeit with larger iteration counts and CPU times than $\mathcal{P}_1$ and $\mathcal{P}_3$.

Of the two preconditioners $\mathcal{P}_2$ and $\mathcal{P}_4$, we note that the preconditioner $\mathcal{P}_4$ solves the problem in fewer iterations than $\mathcal{P}_2$ but greater CPU time due to the added complexity of the GMRES algorithm (though this could partially be offset by using restarts within the GMRES method). We find that in the Navier-Stokes control case, using preconditioners of the form $\mathcal{P}_2$ and $\mathcal{P}_4$ would result in convergence to the solution of the matrix systems involved in similar CPU times [16] because a non-symmetric solver such as GMRES has to be used in both cases as both equivalent preconditioners would be non-symmetric in the Navier-Stokes case. We also note that in the parameter regime of small β, the iteration count when the preconditioner $\mathcal{P}_4$ is

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TABLE 4.7
Comparison of the H^1-norms of the iterative solution $\underline{\mathbf{v}}^{(\ell)}$ and the direct solution $\underline{\mathbf{v}}^{(\ell,dir)}$ for the state $\underline{\mathbf{v}}$ and the L_2-norms of the iterative solution $\underline{\mathbf{u}}^{(\ell)}$ and the direct solution $\underline{\mathbf{u}}^{(\ell,dir)}$ for the control $\underline{\mathbf{u}}$ when applying MINRES *to the first test problem with the preconditioner $\mathcal{P}_2$. Results are given for a variety of mesh levels ℓ (which correspond to $h = 2^{-\ell}$) and values of β.*

	Level ℓ	β: 1	10^{-2}	10^{-4}
$\left\lvert\frac{\lVert\underline{\mathbf{v}}^{(\ell)}\rVert_{H^1(\Omega)}-\lVert\underline{\mathbf{v}}^{(\ell,\mathrm{dir})}\rVert_{H^1(\Omega)}}{\lVert\underline{\mathbf{v}}^{(\ell,\mathrm{dir})}\rVert_{H^1(\Omega)}}\right\rvert$	2	2.024×10^{-7}	6.845×10^{-8}	1.298×10^{-4}
	3	9.952×10^{-8}	5.910×10^{-7}	6.215×10^{-7}
	4	2.029×10^{-6}	3.520×10^{-8}	8.645×10^{-8}
	5	2.443×10^{-6}	2.576×10^{-6}	2.115×10^{-6}
	6	4.912×10^{-6}	7.606×10^{-6}	6.178×10^{-6}
$\left\lvert\frac{\lVert\underline{\mathbf{u}}^{(\ell)}\rVert_{L_2(\Omega)}-\lVert\underline{\mathbf{u}}^{(\ell,\mathrm{dir})}\rVert_{L_2(\Omega)}}{\lVert\underline{\mathbf{u}}^{(\ell,\mathrm{dir})}\rVert_{L_2(\Omega)}}\right\rvert$	2	4.883×10^{-7}	9.289×10^{-7}	1.180×10^{-3}
	3	2.206×10^{-6}	6.080×10^{-7}	1.550×10^{-6}
	4	3.295×10^{-6}	4.870×10^{-7}	1.873×10^{-7}
	5	1.061×10^{-6}	4.897×10^{-7}	1.799×10^{-6}
	6	4.621×10^{-6}	1.618×10^{-6}	4.714×10^{-8}

TABLE 4.8
Comparison of the H^1-norms of the iterative solution $\underline{\mathbf{v}}^{(\ell)}$ and the direct solution $\underline{\mathbf{v}}^{(\ell,dir)}$ for the state $\underline{\mathbf{v}}$ and the L_2-norms of the iterative solution $\underline{\mathbf{u}}^{(\ell)}$ and the direct solution $\underline{\mathbf{u}}^{(\ell,dir)}$ for the control $\underline{\mathbf{u}}$ when applying GMRES *to the first test problem with the preconditioner $\mathcal{P}_4$. Results are given for a variety of mesh levels ℓ (which correspond to $h = 2^{-\ell}$) and values of β.*

	Level ℓ	β: 1	10^{-2}	10^{-4}
$\left\lvert\frac{\lVert\underline{\mathbf{v}}^{(\ell)}\rVert_{H^1(\Omega)}-\lVert\underline{\mathbf{v}}^{(\ell,\mathrm{dir})}\rVert_{H^1(\Omega)}}{\lVert\underline{\mathbf{v}}^{(\ell,\mathrm{dir})}\rVert_{H^1(\Omega)}}\right\rvert$	2	5.318×10^{-7}	2.569×10^{-6}	1.395×10^{-4}
	3	8.897×10^{-8}	4.512×10^{-7}	3.290×10^{-7}
	4	3.892×10^{-7}	2.327×10^{-6}	2.199×10^{-7}
	5	7.023×10^{-7}	6.723×10^{-7}	2.187×10^{-7}
	6	1.038×10^{-6}	9.806×10^{-7}	2.037×10^{-7}
$\left\lvert\frac{\lVert\underline{\mathbf{u}}^{(\ell)}\rVert_{L_2(\Omega)}-\lVert\underline{\mathbf{u}}^{(\ell,\mathrm{dir})}\rVert_{L_2(\Omega)}}{\lVert\underline{\mathbf{u}}^{(\ell,\mathrm{dir})}\rVert_{L_2(\Omega)}}\right\rvert$	2	1.086×10^{-6}	6.335×10^{-7}	7.720×10^{-4}
	3	2.962×10^{-6}	1.121×10^{-6}	1.825×10^{-7}
	4	2.411×10^{-7}	9.819×10^{-7}	4.723×10^{-7}
	5	3.867×10^{-7}	6.506×10^{-6}	6.987×10^{-7}
	6	3.532×10^{-7}	8.998×10^{-9}	1.373×10^{-6}

used is even smaller than that when $\mathcal{P}_1$ (or indeed $\mathcal{P}_2$) is applied. We believe that to extend this methodology to obtain an effective solver for the analogous Navier-Stokes control problem, a preconditioner of the form of either $\mathcal{P}_2$ or $\mathcal{P}_4$ can therefore be considered.

When testing our new methods, it is also desirable to ascertain whether the solutions obtained are accurate reflections of the "true" solutions and are reasonably unaffected by the stopping criteria within MINRES and GMRES (which by definition are related to the preconditioners used). In Tables 4.7 and 4.8 we therefore compare the values of $\lVert\underline{\mathbf{v}}^{(\ell)}\rVert_{H^1(\Omega)}$ and $\lVert\underline{\mathbf{u}}^{(\ell)}\rVert_{L_2(\Omega)}$ for the iterative solution of $\underline{\mathbf{v}}$ and $\underline{\mathbf{u}}$ on each mesh level ℓ with the values $\lVert\underline{\mathbf{v}}^{(\ell,\mathrm{dir})}\rVert_{H^1(\Omega)}$ and $\lVert\underline{\mathbf{u}}^{(\ell,\mathrm{dir})}\rVert_{L_2(\Omega)}$ obtained using a direct method (for the mesh levels where

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we find using a direct method to be feasible and computationally non-prohibitive). We present these results for the first test problem, using the new preconditioners $\mathcal{P}_2$ (with MINRES) and $\mathcal{P}_4$ (with GMRES) for a range of mesh levels and values of β. In these tables we find that the scaled norms are largely around 10^{-6} as expected and depend very little on the mesh level and value of β (and hence the changing preconditioner). This gives a good indication that our iterative schemes are solving the problem well and are presenting accurate solutions to the linear systems tested.

5. Concluding remarks. The use of commutator arguments has been an extremely valuable tool when developing iterative methods for problems in fluid dynamics. In [10] for instance, such an argument was applied in order to develop a solver for the Navier-Stokes equations which performed well for a wide range of values of mesh-size and viscosity. Since then, such arguments have also been applied to good effect when deriving iterative schemes for PDE-constrained optimization problems, for example in [23] to obtain mesh-independent solvers for time-dependent Stokes control and in [26] to arrive at a mesh- and regularization-robust solver for a class of Stokes control problems. Also, in [7], commutator arguments for the Navier-Stokes equations are analyzed for a range of boundary conditions. In this manuscript, we have used new commutator arguments to derive further mesh- and regularization-robust solvers for these problems: block diagonal and block triangular.

We have also explained the role of saddle point theory and that of preconditioners for the Poisson control problem in generating solvers for the more difficult Stokes control problem. We provided numerical results to justify the potency of this approach and explained the importance of the pressure regularization term (or lack of it) from an iterative solver point of view. We believe that the arguments we have introduced in this manuscript may be extended to generate robust solvers for a class of the harder Navier-Stokes control problems—we will discuss this in a future paper [16]. In addition, future research in this area could include the application of these techniques to problems with state or control constraints, boundary control problems, or time-dependent Stokes-type problems, as well as tackling optimal control problems derived specifically from real-world data.

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