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MONOTONE-COMONOTONE APPROXIMATION BY FRACTAL CUBIC SPLINES AND POLYNOMIALS*

PUTHAN VEEDU VISWANATHAN† AND ARYA KUMAR BEDABRATA CHAND†

Abstract. We develop cubic fractal interpolation functions H^α as continuously differentiable α-fractal functions corresponding to the traditional piecewise cubic interpolant H. The elements of the iterated function system are identified so that the class of α-fractal functions f^α reflects the monotonicity and $\mathcal{C}^1$-continuity of the source function f. We use this monotonicity preserving fractal perturbation to: (i) prove the existence of piecewise defined fractal polynomials that are comonotone with a continuous function, (ii) obtain some estimates for monotone and comonotone approximation by fractal polynomials. Drawing on the Fritsch-Carlson theory of monotone cubic interpolation and the developed monotonicity preserving fractal perturbation, we describe an algorithm that constructs a class of monotone cubic fractal interpolation functions H^α for a prescribed set of monotone data. This new class of monotone interpolants provides a large flexibility in the choice of a differentiable monotone interpolant. Furthermore, the proposed class outperforms its traditional non-recursive counterpart in approximation of monotone functions whose first derivatives have varying irregularity/fractality (smooth to nowhere differentiable).

Key words. Fractal function, cubic Hermite fractal interpolation function, fractal polynomial, Fritsch-Carlson algorithm, comonotonicity

AMS subject classifications. 65D05, 41A29, 41A30, 28A80

1. Introduction. Fractal interpolation functions (FIFs) defined through an iterated function system (IFS) [2, 3] is an advancement to the classical interpolation techniques in numerical analysis. A traditional interpolating spline can be generalized with a family of differentiable FIFs (fractal splines) [4]. In this way, the fractal methodology provides more flexibility and versatility in the choice of an interpolant. Consequently, this function class can be useful for mathematical and engineering problems wherein the classical spline interpolation approach may not work satisfactorily, for instance, when an interpolation/approximation problem combined with optimization is to be approached. Since the cubic Hermite interpolants and the cubic splines have proved to be authoritative tools in fields such as applied mathematics, computer aided geometric design (CAGD), tomography, reverse engineering, and signal processing, efforts have been taken to study their fractal analogues; see, for instance, [11, 13, 30].

In addition to providing a good approximant to a given function, scientists and engineers usually demand that interpolation/approximation methods should represent the physical reality as far as possible. In practice, it is desirable that the shape of the interpolant/approximant is compatible with the given data or function to be approximated. A typical demand in the interpolation problem is that of producing a monotone function to fit a prescribed set of monotone data. In this regard, the traditional monotone cubic spline interpolation has been extensively researched. Fritsch and Carlson [21] proposed a necessary and sufficient condition for a cubic polynomial to be monotone in an interval and used it to develop a two pass algorithm for constructing a monotone cubic interpolant to a given set of monotone data. The algorithm discussed by Eisenstat et al. [18] provides an improvement to the Fritsch-Carlson (FC) algorithm. Fritsch and Butland [20] recommended a modified technique to simplify the FC algorithm. Subsequently, many variants and improvements to the FC algorithm were proposed; see, for instance, [22, 41].

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†Department of Mathematics, Indian Institute of Technology Madras, Chennai-600036, India (amritaviswa@gmail.com, chand@iitm.ac.in).

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There has always been a need for advancement in the methods developed earlier so that new techniques incorporate some additional features of interest and can be utilized for more accurate results; a fractal interpolant is not an exception. In this respect, as the cubic Hermite FIFs that generalize the traditional piecewise cubic interpolant have been studied earlier, a natural question of interest is whether the requirement of monotonicity preservation can be incorporated into the cubic fractal interpolation scheme. This paper is devoted to answer this question in the affirmative. Consequently, it is a sequel to [13], which treats cubic Hermite FIFs and certain constrained interpolation aspects, and it can be viewed as a contribution to unify two methodologies, fractal interpolation and shape preserving interpolation, that seem to be developing independently and in parallel; see also [36, 37]. In this way, the paper is aimed at one of the trending topics among the fractal community, namely, the demonstration that fractals are everywhere [3], through a basic problem in numerical analysis.

In Section 2, we recall some of the requisite basic tools, and we obtain cubic FIFs as α-fractal functions corresponding to the traditional cubic interpolant. For blending the requirement of monotonicity with the cubic FIF, we shall adapt a slightly more general approach in Section 3. We identify the elements of the IFS so that the fractal functions f^α, which are regarded as the fractal perturbation of a given function f, retain the $\mathcal{C}^1$-continuity and monotonicity of the germ f. In particular, if we start with a monotone cubic interpolant H, then this procedure culminates with the construction of the monotone cubic FIFs H^α. The advantage is that one may adapt the most suitable method to construct monotone cubic splines (though, in this paper, we use the classical method by Fritsch and Carlson) which can be then perturbed to obtain monotone cubic FIFs.

The derivative of the traditional $\mathcal{C}^1$-continuous monotone cubic interpolant H has discontinuities only at interior knots corresponding to the partition of the interpolation interval. In contrast to this, its fractal generalization H^α may have a derivative $(H^\alpha)^{(1)}$ which is nondifferentiable on a finite or dense set of points in the interpolation interval. Here we note that conditions on the scaling factors for which a fractal interpolation function is k-times differentiable and some special conditions under which a fractal function is nondifferentiable in a dense subset of the interpolation interval are known [4, 23, 28]. However, the most general conditions on the parameters of the IFS so that the corresponding FIF is nondifferentiable in a dense subset of the interpolation interval still remains an open question. Further, the irregularity in a FIF can be quantified by using the fractal dimension, a quantifier (index) that provides a geometric characterization of the measured variable. Let us note here that various kinds of fractal dimensions, such as Hausdorff dimension and Minkowski dimension of FIFs, are reported repeatedly in the literature; see, for instance, [2, 3, 15, 19, 39]. On the other hand, Besicovitch and Ursell, in the reference [8], proved that the graph of a smooth function has fractal dimension one. In this case, this parameter cannot be used as an index for the complexity of the signal. As a consequence, nonsmoothness is a required condition in order to obtain an approximation of the geometrical complexity of arbitrary signals. For a particular example, we refer the reader to [32], wherein fractal dimension is used to study the complexity of electroencephalographic signals and to discriminate an attention disorder. For many real-world phenomena, fractal dimension has been estimated from the sampled data using different techniques as described in [3, 19]. The Matlab package "boxcount" may be used to estimate the fractal dimension of 1D, 2D, or 3D sets, using the box counting method [25].

There are many practical situations wherein a prescribed data set is to be modeled with a shape preserving interpolant and, at the same time, a data set representing a certain derivative is to be modeled with an irregular curve. Let us cite a particular example in the following. A sphere falling through a fluid is a classical problem in fluid dynamics, which is used to study

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the viscoelastic properties of the fluid. A sphere falling in a viscous Newtonian fluid reaches a steady terminal velocity; the approach to this terminal velocity can be shown to be monotonic. A falling sphere in a polymeric fluid approaches a terminal velocity, though sometimes with an oscillating transient. On the other hand, a sphere falling in a wormlike micellar solution does not approach a steady terminal velocity, instead, undergoes continual oscillations or even chaotic motion as it falls [33]. Hence, to simulate the displacement and velocity profiles of such motions, monotonicity/positive interpolants with varying irregularity (which can be quantified using a suitable index) in the derivatives may be advantageous. Similarly, monotone data with varying irregularity in the variable representing the derivative arise naturally and abundantly in electromechanical systems, e.g., a pendulum-cart system [34]. Therefore, in addition to be of theoretical interest, the proposed method possesses potential applications in various nonlinear and nonequilibrium phenomena.

As far as the recursive construction of the smooth shape preserving interpolants and the ability to generate fractality in the derivatives of the constructed interpolants are concerned, the fractal interpolation schemes present significant similarities with the subdivision schemes. A brief comparison of the two methodologies, the fractal interpolation schemes and the subdivision schemes, is given in [13].

Our approach to finding suitable elements of the IFS so that the $\boldsymbol{\alpha}$-fractal function $f^{\boldsymbol{\alpha}}$ preserves the monotonicity of f paves the way to establish the existence of piecewise defined fractal polynomials that are comonotone with a given function. We deduce inequalities of Jackson's type for monotone and comonotone fractal polynomial approximations in Section 4.

2. Background and preliminaries.

To make our presentation fairly self-contained, the basic tools needed in the course of the exposition are reviewed here. Our sources for this material are [2, 3, 4, 21, 29].

2.1. Classical piecewise cubic interpolation.

Given a partition $\Delta^* = \{x_1, x_2, \dots, x_N\}$ of an interval $I = [x_1, x_N]$ satisfying $x_1 < x_2 < \dots < x_N$ and a set of values $\{y_i\}_{i=1}^N$, a univariate interpolation problem deals with the construction of a continuous function $H : I \to \mathbb{R}$ fulfilling $H(x_i) = y_i$, $i = 1, 2, \dots, N$. A piecewise cubic function $H \in \mathcal{C}^1(I)$ is uniquely determined by $\{y_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^N$, where $d_i = H^{(1)}(x_i)$, $i = 1, 2, \dots, N$. From the Taylor representation for an interpolation polynomial, the i-th polynomial curve $H_i = H|_{I_i}$, $i \in J = \{1, 2, \dots, N-1\}$, defined over the subinterval $I_i = [x_i, x_{i+1}]$, has the form:

$$H_i(x) = \frac{d_i + d_{i+1} - 2\Delta_i}{h_i^2}(x-x_i)^3 + \frac{-2d_i - d_{i+1} + 3\Delta_i}{h_i}(x-x_i)^2 + d_i(x-x_i) + y_i, \tag{2.1}$$

where Δ_i denotes the secant slope given by $\Delta_i = \frac{y_{i+1}-y_i}{h_i}$ and $h_i = x_{i+1} - x_i$.

2.2. Monotone piecewise cubic interpolation.

For brevity and simplicity, we assume that the data to be interpolated is monotone increasing throughout the remainder of the paper, unless specifically stated otherwise. Given a set of monotone increasing data (i.e., $y_i \le y_{i+1}$ for all $i \in J$), Fritsch and Carlson [21] developed an algorithm which ensures that the corresponding cubic interpolant H is monotone. The basis of this algorithm is to check whether a cubic polynomial H defined on an interval $[u, v]$ is monotone on that interval, and it is given in the following lemma. Thanks to Schmidt-Heß conditions for the positivity (nonnegativity) of a quadratic polynomial [35], a proof of this lemma that is relatively simpler than that appearing in [21] can be obtained, and it is supplied in the Appendix.

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LEMMA 2.1. *Let H be a cubic polynomial on $[u, v]$ given by*

$$H(x) = \frac{H^{(1)}(u) + H^{(1)}(v) - 2\Delta}{(v-u)^2}(x-u)^3 + \frac{-2H^{(1)}(u) - H^{(1)}(v) + 3\Delta}{(v-u)}(x-u)^2 + H^{(1)}(u)(x-u) + H(u),$$

where $\Delta = \frac{H(v)-H(u)}{v-u}$. When $\Delta \neq 0$, let $\beta = \frac{H^{(1)}(u)}{\Delta}$, $\gamma = \frac{H^{(1)}(v)}{\Delta}$. Then H is monotone on $[u, v]$ if and only if: (i) $H^{(1)}(u) = H^{(1)}(v) = 0$ if $\Delta = 0$, or (ii) $(\beta, \gamma) \in \mathcal{M}$ if $\Delta \neq 0$, where $\mathcal{M}$ is the closed region bounded by the axes and the "upper half" of the ellipse $x^2 + y^2 + xy - 6x - 6y + 9 = 0$ shown in Figure 2.1.

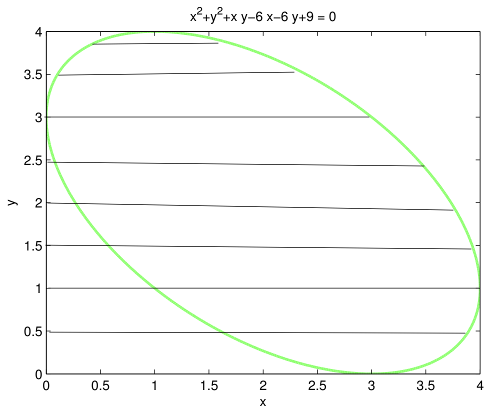

FIG. 2.1. *Fritsch-Carlson monotone region.*

Starting with a given set of data and approximate derivative values $\{d_i\}$ at knots, we construct the cubic Hermite interpolant (cf. (2.1)) for these values, and use the aforementioned region to check whether the interpolant is monotone in each subinterval $I_i = [x_i, x_{i+1}]$, $i \in J$. The cubic polynomial H_i is monotone on $[x_i, x_{i+1}]$ if and only if (d_i, d_{i+1}) lies in the closed region $\mathcal{M}_i$, where $\mathcal{M}_i = \mathcal{M} \cdot \Delta_i = \{(x\Delta_i, y\Delta_i) : (x, y) \in \mathcal{M}\}$, $\Delta_i = \frac{y_{i+1}-y_i}{h_i}$, $i \in J$. If it is not monotone in I_i, then this condition provides a modification rule to make it monotone.

Algorithm (Fritsch-Carlson):

(i) Initialize the derivatives d_i, $i = 1, 2, \ldots, N$, so that $\mathrm{sgn}(d_i) = \mathrm{sgn}(d_{i+1}) = \mathrm{sgn}(\Delta_i)$. If $\Delta_i = 0$, set $d_i = d_{i+1} = 0$.

(ii) For each interval $I_i = [x_i, x_{i+1}]$ in which $(d_i, d_{i+1}) \notin \mathcal{M}_i$, modify d_i and d_{i+1} to d_i^* and d_{i+1}^* such that $(d_i^*, d_{i+1}^*) \in \mathcal{M}_i$.

This kind of algorithm is known as a *fit and modify* type algorithm. Fritsch and Carlson observed that decreasing the magnitude of d_i in moving (d_i, d_{i+1}) into $\mathcal{M}_i$ may force (d_{i-1}, d_i) out of $\mathcal{M}_{i-1}$ and vice versa. Due to this reason, they suggested to work with a subregion $\mathcal{S}$ of $\mathcal{M}$ enjoying the property that if $(x, y) \in \mathcal{S}$ and $0 \leq \tilde{x} \leq x$, $0 \leq \tilde{y} \leq y$, then $(\tilde{x}, \tilde{y}) \in \mathcal{S}$. The recommended regions are (see Figure 2.2):

(i) $\mathcal{S}_1$: region bounded by the lines $x = 0$, $x = 3$, $y = 0$, and $y = 3$.
(ii) $\mathcal{S}_2$: region bounded by $x = 0$, $y = 0$, and the circle $x^2 + y^2 = 3^2$.
(iii) $\mathcal{S}_3$: triangular region determined by the lines $x = 0$, $y = 0$, and $x + y - 3 = 0$.
(iv) $\mathcal{S}_4$: region bounded by $x = 0$, $y = 0$, $2x + y - 3 = 0$, and $x + 2y - 3 = 0$.

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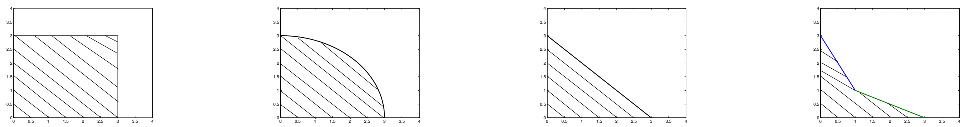

(a): $\mathcal{S}_1$ (b): $\mathcal{S}_2$ (c): $\mathcal{S}_3$ (d): $\mathcal{S}_4$

FIG. 2.2. *Fritsch-Carlson subregions $\mathcal{S}_i$, $i = 1, 2, 3, 4$, for monotone cubic interpolants.*

2.3. Fractal interpolation and α-fractal functions. We begin with the following:

DEFINITION 2.2. *Let (X, d_X) be a complete metric space and $M \in \mathbb{N}$, $M > 1$. If $w_m : X \to X$, $m = 1, 2, \dots, M$, are continuous mappings, then $\mathcal{I} = \{X; w_1, w_2, \dots, w_M\}$ is called an iterated function system (IFS). If, in addition, there exist constants c_m, $0 \le c_m < 1$ such that*

$$d_X\big(w_m(x), w_m(y)\big) \le c_m\, d_X(x, y)$$

for all $x, y \in X$ and $m = 1, 2, \dots, M$, then $\mathcal{I}$ is called a hyperbolic IFS. The constant $c = \max\{c_m : m = 1, 2, \dots, M\}$ is referred to as the contractivity factor of the IFS $\mathcal{I}$.

Associated with the IFS $\mathcal{I}$, there is a set-valued mapping W from the hyperspace $\mathcal{H}(X)$ of nonempty compact subsets of (X, d_X) into itself. More precisely,

$$W : \mathcal{H}(X) \to \mathcal{H}(X), \quad W(E) := \bigcup_{m=1}^{M} w_m(E).$$

The map W is referred to as the collage map to alert that $W(E)$ is a union or collage of sets. The Hausdorff metric $h_{\mathcal{H}(X)}$ completes $\mathcal{H}(X)$. When $\mathcal{I}$ is a hyperbolic IFS with contractivity factor c, it is well-known that W is a contraction on the complete metric space $\big(\mathcal{H}(X), h_{\mathcal{H}(X)}\big)$ with the same contractivity factor c. A basic result in the theory of IFS is the following:

THEOREM 2.3 (Barnsley [2]). *Given a hyperbolic IFS $\mathcal{I}$ on a complete metric space (X, d_X) and any set $A_0 \in \mathcal{H}(X)$, there exists a unique set A, called the attractor of the hyperbolic IFS, such that $A = \lim_{n\to\infty} W^{on}(A_0)$ and $W(A) = A$. Here the limit is taken with respect to the Hausdorff metric and W^{on} denotes the n-fold composition of W with itself.*

Note that the term attractor is chosen to suggest the convergence of A_0 to A under successive applications of W. Next we shall address the question of how to obtain functions whose graphs are attractors of suitable IFSs. For $N \in \mathbb{N}$, $N > 2$, suppose a set of data points $\{(x_i, y_i) \in \mathbb{R}^2 : i = 1, 2, \dots, N\}$ is given, where $x_1 < x_2 < \cdots < x_N$. Let $I = [x_1, x_N]$, and for $i \in J = \{1, 2, \dots, N-1\}$, let $I_i = [x_i, x_{i+1}]$. Suppose $L_i : I \longrightarrow I_i$, $i \in J$ are contraction homeomorphisms satisfying

$$L_i(x_1) = x_i, \qquad L_i(x_N) = x_{i+1}. \tag{2.2}$$

Further, let $F_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions satisfying the conditions

$$|F_i(x, y) - F_i(x, y^*)| \le r_i|y - y^*|, \quad F_i(x_1, y_1) = y_i, \quad F_i(x_N, y_N) = y_{i+1}, \tag{2.3}$$

where $x \in I$, $y, y^* \in \mathbb{R}$, and $0 \le r_i < 1$ for all $i \in J$. Define, $w_i(x, y) = \big(L_i(x), F_i(x, y)\big)$.

PROPOSITION 2.4 (Barnsley [2]). *The IFS $\{I \times \mathbb{R}; w_i, i \in J\}$ admits a unique attractor G, and G is the graph of a continuous function $g : I \to \mathbb{R}$ which obeys $g(x_i) = y_i$ for $i = 1, 2, \dots, N$.*

The function g occurring in Proposition 2.4 is called a fractal interpolation function (FIF) corresponding to the IFS $\{I \times \mathbb{R}; w_i, i \in J\}$. Let $\mathcal{G} := \{g^* \in \mathcal{C}(I) : g^*(x_1) = y_1, g^*(x_N) =$

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$y_N\}$ be endowed with the uniform metric. Define the Read-Bajraktarević operator $T:\mathcal{G}\to\mathcal{G}$ by $Tg^*(x) = F_i\big(L_i^{-1}(x), g^*\circ L_i^{-1}(x)\big)$ for $x\in I_i$, $i\in J$. Then g is the unique function satisfying the functional equation:

$$g(x) = F_i\big(L_i^{-1}(x), g\circ L_i^{-1}(x)\big), \quad x\in I_i, \quad i\in J. \tag{2.4}$$

Though this section is not intended to get into the specifics of particular flavors of fractal interpolation, we shall mention a few for the benefit of the reader. Apart from this recursive functional equation, the FIF g possesses an explicit representation in terms of an infinite series which depends on $(N-1)$-adic expansion of points on $[0,1]$; see [14]. Further, g can be expressed using the technique of operator approximation [12]. As with wavelets and many other new function types, "closed form" expressions for FIFs generally take the form of one of the two types of algorithms, chaos game (a Markov chain Monte Carlo algorithm) and deterministic iteration; both approaches are highly accurate and have been reported in many places in the literature; see, e.g., [2, 5, 10, 24]. In many cases, evaluation of a FIF at a specific point can be achieved by summing a rapidly convergent series. For instance, the attractor of the IFS

$$\left\{I\times\mathbb{R}:\ w_1(x,y) = \left(\frac{x}{2}, \alpha y+\sin(\pi x)\right),\ w_2(x,y) = \left(\frac{x+1}{2}, \alpha y-\sin(\pi x)\right)\right\},$$

where $|\alpha|<1$ and $I=[0,1]$ is the graph of the function $\sum\limits_{n=0}^{\infty}\alpha^n\sin(2^{n+1}\pi x)$ [6]. The main step in the computation of fractal functions relates in a way or other to the evaluation of the Read-Bajraktarević (RB) operator. Reference [7] proposes discretization of the RB operator to deliver values of the full RB operator applied to a function and demonstrates that fractal functions defined by IFSs or local IFSs can be used for easy, cheap, and accurate computations.

Notice that the functional equation for the FIF (2.4) provides a rule to predict the values of the interpolant at refined mesh points, and thus reminds of a subdivision scheme. With a simple example, let us note here that an FIF, in fact, provides a subdivision scheme [7]. However, the fact that it arises from an IFS makes the mathematical treatments such as convergence, smoothness, etc. relatively easier to handle. Let $I=[0,1]$, and $L_1(x)=\frac{x}{2}$, $L_2(x)=\frac{x+1}{2}$. Further, let $F_i(x,y)$ be continuous for $i=1,2$, satisfying conditions prescribed as above. One obtains a subdivision scheme with meshes $N_k = 2^{-k}\mathbb{N}_{2^k}$, where $\mathbb{N}_{2^k}=\{0,1,\dots,2^k\}$, $k=0,1,\dots$ by choosing the refinement rules $R_k:\mathbb{R}^{N_k}\to\mathbb{R}^{N_{k+1}}$ to be

$$(R_k g)(\xi) = \left\{\begin{cases} F_1\big(2\xi, g(2\xi)\big), & \text{if } \xi\in[0,\frac{1}{2})\cap N_{k+1},\\ F_2\big(2\xi-1, g(2\xi-1)\big), & \text{if } \xi\in[\frac{1}{2},1]\cap N_{k+1}.\end{cases}\right.$$

By a similar analysis, we can write a more general FIF defined on $[x_1,x_N]$ (cf. (2.4)) emerging from an IFS with $N-1$ maps L_i and F_i, $i\in J$ as a subdivision scheme.

The most popular structure of IFS for the study of FIFs is:

$$L_i(x) = a_i x + b_i, \quad F_i(x,y) = \alpha_i y + q_i(x), \tag{2.5}$$

where $-1<\alpha_i<1$ and $q_i: I\to\mathbb{R}$ are suitable continuous functions satisfying (2.3). The multiplier α_i is called a scaling factor of the map w_i and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_{N-1})$ is the scale vector in I for the IFS. For a detailed exposition of the smoothness analysis of the corresponding FIF $g\big(L_i(x)\big) = \alpha_i g(x) + q_i(x)$ the reader may consult [12, 40]. However, let us recall here that the Hölder exponent of the FIF g is controlled by the scaling factors. The following result assures the existence of a differentiable FIF (fractal spline) and provides a method for its construction.

PROPOSITION 2.5 (Barnsley and Harrington [4]). *Let* $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$ *be a prescribed set of interpolation data satisfying* $x_1 < x_2 < \cdots < x_N$ *and* $L_i(x) = a_i x + b_i$, $i \in J$, *be affine functions satisfying conditions in* (2.2). *Let* $a_i = L_i'(x) = \frac{x_{i+1}-x_i}{x_N - x_1}$ *and* $F_i(x, y) = \alpha_i y + q_i(x)$, $i \in J$, *satisfy* (2.3). *Suppose that for some integer* $r \geq 0$, $|\alpha_i| < a_i^r$, *and* $q_i \in \mathcal{C}^r(I)$, $i \in J$. *For* $k = 1, 2, \ldots, r$, *let*

$$F_{i,k}(x, y) = \frac{\alpha_i y + q_i^{(k)}(x)}{a_i^k}, \quad y_{1,k} = \frac{q_1^{(k)}(x_1)}{a_1^k - \alpha_1}, \quad y_{N,k} = \frac{q_{N-1}^{(k)}(x_N)}{a_{N-1}^k - \alpha_{N-1}}.$$

If $F_{i-1,k}(x_N, y_{N,k}) = F_{i,k}(x_1, y_{1,k})$ *for* $i = 2, 3, \ldots, N-1$ *and* $k = 1, 2, \ldots, r$, *then the IFS* $\{I \times \mathbb{R}; \big(L_i(x), F_i(x, y)\big), i \in J\}$ *determines a FIF* $g \in \mathcal{C}^r(I)$, *and* $g^{(k)}$ *is the FIF determined by* $\{I \times \mathbb{R};\ \big(L_i(x), F_{i,k}(x, y)\big),\ i \in J\}$ *for* $k = 1, 2, \ldots, r$.

Next we show that any continuous function defined on a compact interval can be regarded as a special case of a class of fractal functions. Let $f \in \mathcal{C}(I)$ and consider the case:

$$q_i(x) = f \circ L_i(x) - \alpha_i b(x), \tag{2.6}$$

where b is a continuous real valued function such that

$$b \neq f, \quad b(x_1) = f(x_1), \quad b(x_N) = f(x_N).$$

Here the interpolation data set is $\{(x_i, f(x_i)) : i = 1, 2, \ldots, N\}$. We define the $\boldsymbol{\alpha}$-fractal function corresponding to f in the following:

DEFINITION 2.6. *The continuous function* $f^{\boldsymbol{\alpha}}$ *defined by the IFS* (2.5)–(2.6) *is the* $\boldsymbol{\alpha}$*-fractal function associated with* f *with respect to the base function* b *and the partition* $\Delta^* = \{x_1, x_2, \ldots, x_N\}$.

According to (2.4), $f^{\boldsymbol{\alpha}}$ satisfies the functional equation:

$$f^{\boldsymbol{\alpha}}(x) = f(x) + \alpha_i[(f^{\boldsymbol{\alpha}} - b) \circ L_i^{-1}(x)], \quad x \in I_i, \quad i \in J. \tag{2.7}$$

Since $\boldsymbol{\alpha} \in (-1, 1)^{N-1}$ is arbitrary and $f^{\mathbf{0}} = f$, the above process endows an entire class of continuous fractal functions $f^{\boldsymbol{\alpha}}$ parameterized by $\boldsymbol{\alpha} \in \mathbb{R}^{N-1}$ with f as its germ. Each function $f^{\boldsymbol{\alpha}}$ interpolates f at data points, and $f^{\boldsymbol{\alpha}}$ may have noninteger Hausdorff-Besicovitch dimension. Therefore, the function $f^{\boldsymbol{\alpha}}$ is also referred to as the fractal perturbation of f, and the following map is called the $\boldsymbol{\alpha}$-fractal operator

$$\mathcal{F}^{\boldsymbol{\alpha}} : \mathcal{C}(I) \to \mathcal{C}(I), \quad \mathcal{F}^{\boldsymbol{\alpha}}(f) = f^{\boldsymbol{\alpha}}.$$

If $p \in \mathcal{C}(I)$ is a polynomial, then $p^{\boldsymbol{\alpha}} = \mathcal{F}^{\boldsymbol{\alpha}}(p)$ is termed a fractal polynomial. For various properties of this fractal operator, we refer the interested reader to [29]. It is worthwhile to note that the α-fractal function $f^{\boldsymbol{\alpha}}$ corresponding to a differentiable function f may not be differentiable, unless the elements of the IFS are appropriately chosen. Conditions for $f^{\boldsymbol{\alpha}}$ to be nondifferentiable in a dense subset of I can be found in [23]. From functional equation (2.7), we can easily infer that by taking corresponding scaling factors equal to zero, $f^{\boldsymbol{\alpha}}$ agrees with f in specified subintervals. Therefore, the irregularity (fractality) can be confined to a small portion of the domain if the corresponding signal shows some complex irregular structure therein.

2.4. Cubic FIFs as α-fractal functions corresponding to a classical piecewise cubic interpolant. First we shall note that the classical $\mathcal{C}^1$-cubic Hermite interpolant is also a fixed point of a suitable IFS, and hence satisfies its own functional equation; for details see [13].

Let a set of data points $\{(x_i, y_i) : i = 1, 2, \dots, N\}$ be given. We let $\alpha_i = 0$ for all $i \in J$, and define the IFS $\{I \times \mathbb{R};\ w_i(x, y),\ i \in J\}$ through the maps $L_i(x) = a_i x + b_i$, $F_i(x, y) = q_i(x)$, where $q_i(x)$ are cubic polynomials.

Assuming the conditions $F_i(x_1, y_1) = y_i$, $F_i(x_N, y_N) = y_{i+1}$, $F_{i,1}(x_1, d_1) = d_i$, and $F_{i,1}(x_1, d_1) = d_{i+1}$ (see Proposition 2.5), the corresponding FIF H obeys:

$$
\begin{aligned}
H\big(L_i(x)\big) = {} & [h_i(d_i + d_{i+1}) - 2(y_{i+1} - y_i)] \left(\frac{x - x_1}{x_N - x_1}\right)^3 \\
& + [-h_i(2d_i + d_{i+1}) + 3(y_{i+1} - y_i)] \left(\frac{x - x_1}{x_N - x_1}\right)^2 \\
& + h_i d_i \left(\frac{x - x_1}{x_N - x_1}\right) + y_i.
\end{aligned}
\tag{2.8}
$$

For $x \in I_i$, using $\frac{L_i^{-1}(x) - x_1}{x_N - x_1} = \frac{x - x_i}{h_i}$, one can see that the above expression coincides with the classical piecewise $\mathcal{C}^1$-cubic Hermite interpolant; cf. (2.1). Following the same procedure but with a scaling vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$, $\mathbf{0} \neq \boldsymbol{\alpha} \in \mathbb{R}^{N-1}$, we obtain cubic Hermite FIFs; see [13]. Here we shall approach the cubic Hermite FIFs through differentiable $\boldsymbol{\alpha}$-fractal function technique.

Given the cubic Hermite interpolant $H \in \mathcal{C}^1(I)$ (cf. (2.8)) corresponding to the data set $\{(x_i, y_i) : i = 1, 2, \dots, N\}$, consider the IFS defined through (2.5)–(2.6). Our goal is to obtain the $\boldsymbol{\alpha}$-fractal function $H^{\boldsymbol{\alpha}} \in \mathcal{C}^1(I)$ corresponding to this cubic Hermite interpolant H via a suitable base function b and a scale vector $\boldsymbol{\alpha}$.

According to Proposition 2.5, we take the scaling factors such that $|\alpha_i| < a_i$ for all $i \in J$. Next we identify appropriate function b so that the functions

$$F_i(x, y) = \alpha_i y + H \circ L_i(x) - \alpha_i b(x), \quad i \in J,$$

satisfy the conditions prescribed in Proposition 2.5 for $k = 1$. Our analysis is patterned after [31]. However, we avoid the following restrictions imposed therein: (i) the partition should be uniformally spaced, and (ii) the scaling factors in each interval should be the same. For the chosen maps F_i, we have

$$F_{i,1}(x, y) = \frac{\alpha_i y + a_i H^{(1)}\big(L_i(x)\big) - \alpha_i b^{(1)}(x)}{a_i}.$$

Therefore, the conditions $F_{i-1,1}(x_N, y_{N,1}) = F_{i,1}(x_1, y_{1,1})$, for $i = 2, 3, \dots, N-1$, become:

$$
\begin{aligned}
& H^{(1)}(x_i) + \frac{\alpha_{i-1}}{a_{i-1}(a_{N-1} - \alpha_{N-1})}[a_{N-1} H^{(1)}(x_N) - \alpha_{N-1} b^{(1)}(x_N)] - \frac{\alpha_{i-1}}{a_{i-1}} b^{(1)}(x_N) \\
& \quad = H^{(1)}(x_i) + \frac{\alpha_i}{a_i(a_1 - \alpha_1)}[a_1 H^{(1)}(x_1) - \alpha_1 b^{(1)}(x_1)] - \frac{\alpha_i}{a_i} b^{(1)}(x_1).
\end{aligned}
\tag{2.9}
$$

It can be readily seen that the following conditions ensure (2.9):

$$b^{(1)}(x_1) = H^{(1)}(x_1), \quad b^{(1)}(x_N) = H^{(1)}(x_N).$$

The preceding analysis demonstrates that we can generate $\boldsymbol{\alpha}$-fractal functions $H^{\boldsymbol{\alpha}} \in \mathcal{C}^1(I)$ (more generally, $f^{\boldsymbol{\alpha}} \in \mathcal{C}^1(I)$) corresponding to the cubic Hermite interpolant H $\big($or, for any $f \in \mathcal{C}^1(I)\big)$ through the IFS (2.5)–(2.6), provided the scaling factors satisfy $|\alpha_i| < a_i$ for all $i \in J$, and $b \in \mathcal{C}^{(1)}(I)$ agrees with H (with f) at the ends of the interval up to the first derivative. An obvious choice for b is the two-point cubic Hermite interpolant for H (for f) with knots at x_1 and x_N. That is,

$$\begin{aligned} b(x) = &[(x_N - x_1)(d_1 + d_N) - 2(y_N - y_1)] \left(\frac{x - x_1}{x_N - x_1}\right)^3 \\ &+ [-(x_N - x_1)(2d_1 + d_N) + 3(y_N - y_1)] \left(\frac{x - x_1}{x_N - x_1}\right)^2 \\ &+ d_1(x - x_1) + y_1. \end{aligned} \tag{2.10}$$

From (2.7), (2.8), and (2.10), we infer that the desired cubic FIFs obey the functional equation:

$$\begin{aligned} H^{\alpha}(L_i(x)) =&\alpha_i H^{\alpha}(x) + H(L_i(x)) - \alpha_i b(x), \\ =&\alpha_i H^{\alpha}(x) + \big\{h_i(d_i + d_{i+1}) - 2(y_{i+1} - y_i) \\ &- \alpha_i[(x_N - x_1)(d_1 + d_N) - 2(y_N - y_1)]\big\} \left(\frac{x - x_1}{x_N - x_1}\right)^3 \\ &+ \big\{ - h_i(2d_i + d_{i+1}) + 3(y_{i+1} - y_i) \\ &- \alpha_i[-(x_N - x_1)(2d_1 + d_N) + 3(y_N - y_1)]\big\} \left(\frac{x - x_1}{x_N - x_1}\right)^2 \\ &+ \{h_i d_i - \alpha_i d_1(x_N - x_1)\} \left(\frac{x - x_1}{x_N - x_1}\right) + y_i - \alpha_i y_1, \end{aligned} \tag{2.11}$$

for $x \in I$ and $i \in J$.

3. Monotone/comonotone α-fractal functions and comonotone cubic FIFs. The cubic FIFs established in the previous section may not preserve the monotonicity property hidden in a given set of data. In this section, we develop sufficient conditions for the cubic FIFs to retain the monotonicity inherent in the prescribed data set. Our approach will be more general in the sense that we find sufficient conditions for f^{α} to be monotone whenever f is so, and particularize this to the traditional monotone cubic interpolant H. We extend this to cover the case of changing monotonicity.

3.1. Monotone α-fractal function. We record the following theorem which identifies a suitable IFS so that the α-fractal function f^{α} preserves smoothness and monotonicity of $f \in \mathcal{C}^1(I)$. The proof can be found in [38].

THEOREM 3.1. *Let $f \in \mathcal{C}^1(I)$ be a monotone increasing function. Let $\Delta^* = \{x_1, x_2, \dots, x_N\}$ be a partition of I such that $x_1 < x_2 < \dots < x_N$, and $b \in \mathcal{C}^1(I)$ be a monotone increasing function satisfying the conditions*

$$b(x_1) = f(x_1), \quad b(x_N) = f(x_N), \quad b^{(1)}(x_1) = f^{(1)}(x_1), \quad b^{(1)}(x_N) = f^{(1)}(x_N).$$

Then, the fractal function f^{α} corresponding to the IFS defined via (2.5)–(2.6) is $\mathcal{C}^1$-continuous and, for M large enough, f^{α} satisfies $0 \le (f^{\alpha})^{(1)}(x) \le M$ (hence, in particular, f^{α} is monotone), provided the scaling factors obey $|\alpha_i| < a_i$ and

$$\max\left\{-\frac{a_i m_i}{M - m_*}, -\frac{a_i(M - M_i)}{M^*}\right\} \le \alpha_i \le \min\left\{\frac{a_i m_i}{M^*}, \frac{a_i(M - M_i)}{M - m_*}\right\}, \quad i \in J,$$

where

$$m_* = \min_{x \in I} b^{(1)}(x), \quad M^* = \max_{x \in I} b^{(1)}(x), \quad m_i = \min_{x \in I_i} f^{(1)}(x), \quad M_i = \max_{x \in I_i} f^{(1)}(x).$$

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REMARK 3.2. Though the monotonicity of $f^{\boldsymbol{\alpha}}$ does not demand an upper bound for $(f^{\boldsymbol{\alpha}})^{(1)}$, the constant M plays a crucial role in admitting negative values for the scaling factors whilst maintaining the positivity of $(f^{\boldsymbol{\alpha}})^{(1)}$. However, if we do not want to bound $(f^{\boldsymbol{\alpha}})^{(1)}$ from above in exchange for the slightly increased generality of negative scaling factors, we can certainly work with $(f^{\boldsymbol{\alpha}})^{(1)}(x) \geq 0$ instead. Conditions on the scaling factors for $(f^{\boldsymbol{\alpha}})^{(1)}(x) \geq 0$ are given by $0 \leq \alpha_i < \min\{a_i, \frac{a_i m_i}{M^*}\}$. We note that the monotonicity condition on b is not claimed to be essential, but it will be desirable later for developing some approximation results.

REMARK 3.3. Suppose the first derivative of f vanishes on I in the preceding theorem. Then f is constant, say $f(x) = \kappa$ for all $x \in I$ and, consequently, the interpolatory condition on the monotonic function b implies $b(x) = \kappa$ for all $x \in I$. Recall that the $\boldsymbol{\alpha}$-fractal function $f^{\boldsymbol{\alpha}}$ corresponding to this f and b satisfies the functional equation (fixed point equation)

$$f^{\boldsymbol{\alpha}}\big(L_i(x)\big) = f\big(L_i(x)\big) + \alpha_i(f^{\boldsymbol{\alpha}} - b)(x), \quad x \in I, \quad i \in J.$$

Substituting $f(x) = b(x) = \kappa$ for all $x \in I$ we obtain

$$f^{\boldsymbol{\alpha}}(L_i(x)) = \kappa + \alpha_i(f^{\boldsymbol{\alpha}}(x) - \kappa), \quad x \in I, \quad i \in J.$$

Since the above equation is satisfied by $f^{\boldsymbol{\alpha}} \equiv \kappa$ and the fixed point is unique, we infer that $f^{\boldsymbol{\alpha}} \equiv f \equiv \kappa$. That is, in this case no fractal perturbation is provided. Further, if $d_i = 0$ for $i \in J$, then the previous theorem prescribes $\alpha_i = 0$ for a monotonic $f^{\boldsymbol{\alpha}}$, and hence $f^{\boldsymbol{\alpha}}$ coincides with f on the subinterval I_i.

REMARK 3.4 (Monotone decreasing $\boldsymbol{\alpha}$-fractal function). On lines similar to that of Theorem 3.1, it can be proved that an $\boldsymbol{\alpha}$-fractal function $f^{\boldsymbol{\alpha}} \in \mathcal{C}^1(I)$ corresponding to f retains the $\mathcal{C}^1$-continuity and monotone decreasing nature of f, if the base function b is a monotone decreasing two-point Hermite interpolant to f, the scaling factors are chosen so that $|\alpha_i| < a_i$, and

$$\max\Big\{-\frac{a_i M_i}{m - M_*}, -\frac{a_i(m - m_i)}{m_*}\Big\} \leq \alpha_i \leq \min\Big\{\frac{a_i M_i}{m_*}, \frac{a_i(m - m_i)}{m - M_*}\Big\}, \quad i \in J.$$

Here m is a real number strictly smaller than M^* and m_i, for any $i \in J$.

REMARK 3.5 (Handling functions of changing monotonicity). The proposed fractal scheme can be modified and extended to produce a piecewise defined $\boldsymbol{\alpha}$-fractal function which is comonotone with a given function $f \in \mathcal{C}^1(I)$, where f changes its monotonicity a finite number of times. For doing this, the interval I has to be partitioned into subintervals, say I_j, $j = 1, 2, \ldots, r$, in such a way that in a typical subinterval I_j the function f is monotone increasing or decreasing throughout. In each of these subintervals I_j, we take a partition Δ_j^*, a base function b_j, and a scaling vector $\boldsymbol{\alpha}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_{m_j}^{(j)})$, where m_j is the number of subintervals in I_j determined by the partition Δ_j^*, so as to meet the specifications in Theorem 3.1 or Remark 3.4. Consequently, we can produce fractal functions $f_j^{\boldsymbol{\alpha}^{(j)}}$ that retain the monotonicity of the functions $f_j = f|_{I_j}$, $j = 1, 2, \ldots, r$. With a slight abuse of notation, let us denote by $\boldsymbol{\alpha}$, the block matrix consisting of the scaling vectors $\boldsymbol{\alpha}^{(j)}$, i.e., $\boldsymbol{\alpha} = [\boldsymbol{\alpha}^{(1)}\ \boldsymbol{\alpha}^{(2)}\ \cdots\ \boldsymbol{\alpha}^{(r)}]$. We define a function denoted by $f[\boldsymbol{\alpha}]$ in a piecewise manner as follows: $f[\boldsymbol{\alpha}]|_{I_j} = f_j^{\boldsymbol{\alpha}^{(j)}}$. Since the fractal function $f_j^{\boldsymbol{\alpha}^{(j)}}$ and its derivative, respectively, interpolate f and its derivative at the knots of the partition Δ_j^*, it is evident that $f[\boldsymbol{\alpha}] \in \mathcal{C}^1(I)$. The piecewise defined fractal function $f[\boldsymbol{\alpha}]$ is comonotone with f (i.e., $f[\boldsymbol{\alpha}](x)f(x) \geq 0$ for

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all $x \in I$). In particular, if f is an algebraic polynomial that is comonotone with an original function Φ, then $f[\boldsymbol{\alpha}]$ gives a piecewise defined fractal polynomial that is comonotone with f and hence with Φ. Thus, we can deduce the existence of comonotone piecewise defined fractal polynomials from the existence of comonotone algebraic polynomials. We may denote $|\boldsymbol{\alpha}^{(j)}|_\infty = \max\{|\alpha_i^{(j)}| : i = 1, 2, \dots, m_j\}$ and $|[\boldsymbol{\alpha}]|_\infty = \max\{|\boldsymbol{\alpha}^{(j)}|_\infty : j = 1, 2, \dots, r\}$.

3.2. Algorithm for monotone cubic FIFs. Combining the Fritsch-Carlson method that produces a monotone cubic interpolant with the monotone $\boldsymbol{\alpha}$-fractal function discussed in the foregoing subsection, we can design an algorithm that produces monotone cubic FIFs for a given set of monotone data. To do this, we shall first calculate the constants m_i, M_i, m_*, and M^*, occurring in the above theorem, where $f = H$ is the cubic interpolant (cf. (2.8)) for the data $\{(x_i, y_i) : i = 1, 2, \dots, N\}$ and b (cf. (2.10)) is the two-point cubic Hermite interpolant corresponding to H.

Considering $L_i : \mathbb{R} \to \mathbb{R}$, it can be verified that $H^{(1)}\big(L_i(x)\big)$ has a unique extremum at

$$x^* = x_1 + \frac{(x_N - x_1)(2d_i + d_{i+1} - 3\Delta_i)}{3(d_i + d_{i+1} - 2\Delta_i)}$$

with the corresponding extremal value

$$H^{(1)}\big(L_i(x^*)\big) = d_i - \frac{(2d_i + d_{i+1} - 3\Delta_i)^2}{3(d_i + d_{i+1} - 2\Delta_i)}.$$

Therefore,

$$M_i = \begin{cases} \max\left\{d_i, d_{i+1}, d_i - \dfrac{(2d_i + d_{i+1} - 3\Delta_i)^2}{3(d_i + d_{i+1} - 2\Delta_i)}\right\}, & \text{if } x^* \in [x_1, x_N], \\ \max\{d_i, d_{i+1}\}, & \text{if } x^* \notin [x_1, x_N], \end{cases}$$

and

$$m_i = \begin{cases} \min\left\{d_i, d_{i+1}, d_i - \dfrac{(2d_i + d_{i+1} - 3\Delta_i)^2}{3(d_i + d_{i+1} - 2\Delta_i)}\right\}, & \text{if } x^* \in [x_1, x_N], \\ \min\{d_i, d_{i+1}\}, & \text{if } x^* \notin [x_1, x_N]. \end{cases}$$

Similarly, $b^{(1)}$ has a unique extremum at

$$x^* = x_1 + \frac{(x_N - x_1)^2(2d_1 + d_N) - 3(y_N - y_1)(x_N - x_1)}{3[(x_N - x_1)(d_1 + d_N) - 2(y_N - y_1)]},$$

and the extremal value is

$$b^{(1)}(x^*) = d_1 - \frac{\left[2d_1 + d_N - 3\frac{y_N - y_1}{x_N - x_1}\right]^2}{3\left[d_1 + d_N - 2\frac{y_N - y_1}{x_N - x_1}\right]}.$$

Hence, we have

$$M^* = \begin{cases} \max\left\{d_1, d_N, d_1 - \dfrac{\left[2d_1 + d_N - 3\frac{y_N - y_1}{x_N - x_1}\right]^2}{3\left[d_1 + d_N - 2\frac{y_N - y_1}{x_N - x_1}\right]}\right\}, & \text{if } x^* \in [x_1, x_N], \\ \max\{d_1, d_N\}, & \text{if } x^* \notin [x_1, x_N], \end{cases}$$

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and

$$m_* = \begin{cases} \min\left\{d_1, d_N, d_1 - \dfrac{\left[2d_1 + d_N - 3\frac{y_N-y_1}{x_N-x_1}\right]^2}{3\left[d_1 + d_N - 2\frac{y_N-y_1}{x_N-x_1}\right]}\right\}, & \text{if } x^* \in [x_1, x_N], \\ \min\{d_1, d_N\}, & \text{if } x^* \notin [x_1, x_N]. \end{cases}$$

Letting $\delta = \frac{d_1(x_N-x_1)}{y_N-y_1}$ and $\rho = \frac{d_N(x_N-x_1)}{y_N-y_1}$, we note that the two-point Hermite interpolant b is monotone if δ and ρ lie in any of the regions given in Section 2.2.

Algorithm: The above discussion suggests the following procedure for constructing monotone increasing cubic FIFs corresponding to a given set of monotone increasing data.

Step 1 *Initialization.* Compute the initial approximate derivative values d_i, $i = 1, 2, \dots, N$. Ensure that each $d_i \geq 0$. If $\Delta_i = 0$, let $d_i = d_{i+1} = 0$.

Step 2 *FC-algorithm for the monotone cubic interpolant H.* For each interval I_i for which $(\beta_i, \gamma_i) = (\frac{d_i}{\Delta_i}, \frac{d_{i+1}}{\Delta_i}) \notin \mathcal{S}$, modify d_i and d_{i+1} to d_i^* and d_{i+1}^* such that $(\beta_i^*, \gamma_i^*) = (\frac{d_i^*}{\Delta_i}, \frac{d_{i+1}^*}{\Delta_i}) \in \mathcal{S}$. Denote by d_i the derivatives at the knots so obtained, satisfying FC conditions; see Section 2.2.

Step 3 *Filtering end derivatives for a monotone b.* For $\delta = \frac{d_1(x_N-x_1)}{y_N-y_1}$ and $\rho = \frac{d_N(x_N-x_1)}{y_N-y_1}$, check whether $(\delta, \rho) \in \mathcal{S}$, where $\mathcal{S}$ is one of the regions suggested by Fritsch and Carlson; see Section 2.2. If not, modify d_1 and d_N.

Step 4 *Scaling parameters for monotonicity preserving fractal perturbation.* Denoting by d_i, $i = 1, 2, \dots, N$, the derivative values obtained at the end of Step 3, calculate the constants M_i, m_i, m_*, and M^*, and select a suitable constant M. Calculate the scaling parameters α_i according to the prescription in Theorem 3.1.

Step 5 *Monotone cubic FIF $H^{\boldsymbol{\alpha}}$.* Use these derivative values and scaling factors as input for the functional equation (2.11) to recursively generate new points obtaining a cubic FIF $H^{\boldsymbol{\alpha}}$. The elements in the IFS generating this cubic FIF $H^{\boldsymbol{\alpha}}$ satisfy the sufficient conditions in Theorem 3.1, and hence $H^{\boldsymbol{\alpha}}$ is monotone.

To get an initial approximation for the derivative values at the knots, one may use various approximation methods, such as the arithmetic mean method, the geometric mean method, etc.; see [16]. For the d_i values obtained in Step 2, the corresponding traditional cubic interpolant H is monotone. Step 3 suggests to modify the end derivatives in the monotone cubic interpolant H obtained from the FC-algorithm, so as to assure the monotonicity of b as required by Theorem 3.1. It is to be noted that, due to the property of the chosen region $\mathcal{S}$, the new end derivatives will not affect the monotonicity of H obtained in Step 2.

3.3. Numerical examples. We consider the following subset of the Akima data (see [1]): $\{(8, 10), (9, 10.5), (11, 15), (12, 50), (14, 60), (15, 85)\}$. The derivative values at the data points are estimated using the following arithmetic mean method [16]. At the interior point x_i; $i = 2, 3, \dots, N-1$, set

$$d_i = \begin{cases} 0, & \text{if } \Delta_i = 0 \text{ or } \Delta_{i-1} = 0, \\ \frac{h_i \Delta_{i-1} + h_{i-1}\Delta_i}{h_i + h_{i-1}}, & \text{otherwise.} \end{cases}$$

At the end points x_1 and x_N a noncentered version of the above formula is used, that is

$$d_1 = \begin{cases} 0, & \text{if } \Delta_1 = 0 \text{ or } \operatorname{sgn}(d_1^*) \neq \operatorname{sgn}(\Delta_1), \\ d_1^* = \Delta_1 + \frac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2}, & \text{otherwise.} \end{cases}$$

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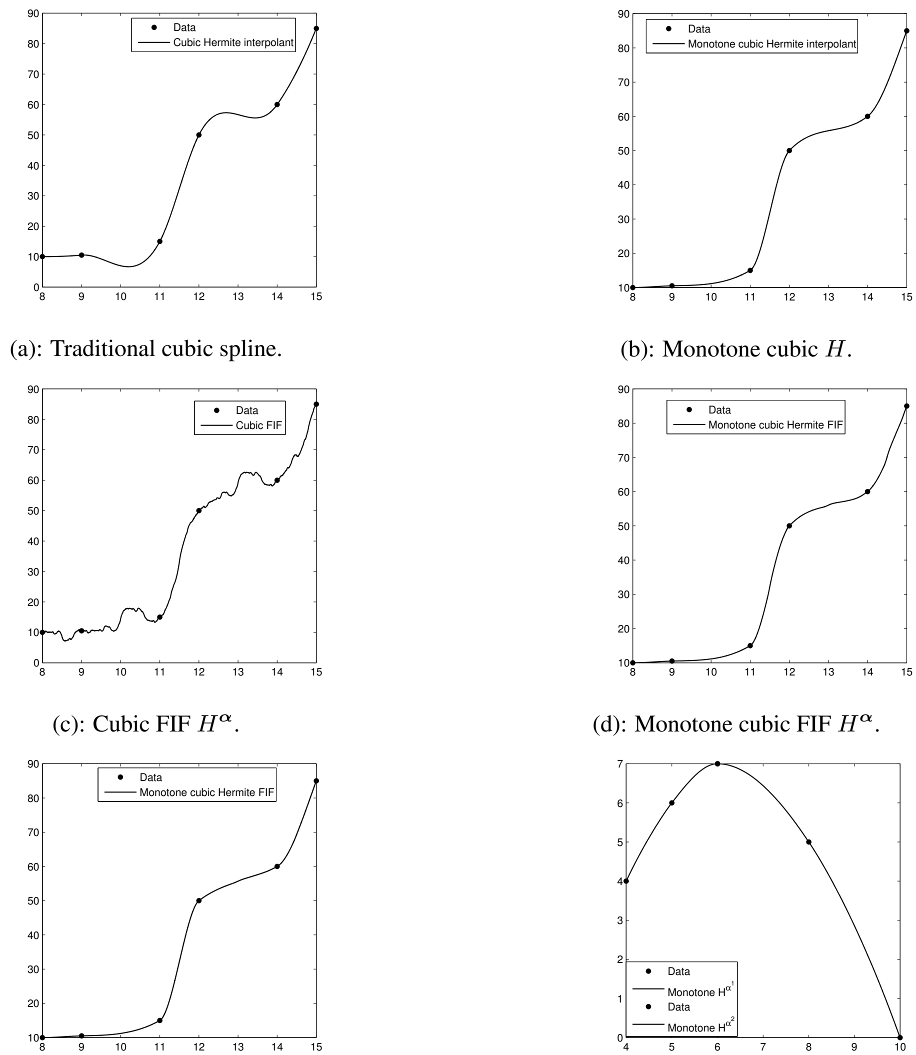
and

$$d_N = \begin{cases} 0, & \text{if } \Delta_{N-1} = 0 \text{ or } \operatorname{sgn}(d_N^*) \neq \operatorname{sgn}(\Delta_{N-1}), \\ d_N^* = \Delta_{N-1} + \frac{(\Delta_{N-1} - \Delta_{N-2}) h_{N-1}}{h_{N-1} + h_{N-2}}, & \text{otherwise.} \end{cases}$$



(a): Traditional cubic spline.

(b): Monotone cubic H.

(c): Cubic FIF $H^{\boldsymbol{\alpha}}$.

(d): Monotone cubic FIF $H^{\boldsymbol{\alpha}}$.

(e): Monotone cubic FIF $H^{\boldsymbol{\alpha}}$.

(f): Comonotone FIF $H^{\boldsymbol{\alpha}}$.

FIG. 3.1. *Traditional piecewise cubic interpolants and cubic FIFs.*

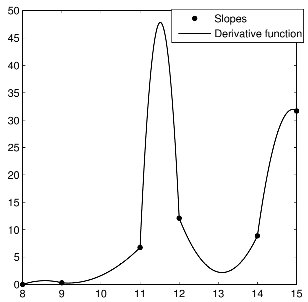
For the present data set, we have $d_1 = 0$, $d_2 = 1.0833$, $d_3 = 24.0833$, $d_4 = 25$, $d_5 = 18.3333$, and $d_6 = 31.6667$. Figure 3.1(a) shows the traditional piecewise cubic interpolant corresponding to this initial choice of derivative values. Note that this cubic interpolant is not monotone. To obtain a monotone cubic interpolant H, we apply the FC-algorithm with monotonicity region $\mathcal{S}_2$, that is, the disc $\beta^2 + \gamma^2 \leq 9$. The mapping $(\beta, \gamma) \to (\beta^*, \gamma^*)$ is the most subtle issue in the FC-algorithm. To 'project' a point $(\beta, \gamma) \notin \mathcal{S}_2$ onto $(\beta^*, \gamma^*) \in \mathcal{S}_2$, we use 'homothetic projection', i.e., (β^*, γ^*) is the point of intersection of the line joining the origin and (β, γ) with the boundary of the disc $\mathcal{S}_2$. This procedure modifies the initial derivative values to $d_1 = 0$, $d_2 = 0.3033$, $d_3 = 6.7432$, $d_4 = 12.0961$, $d_5 = 8.8705$, and $d_6 = 31.6667$. The corresponding monotone cubic spline interpolant H is plotted in Figure 3.1(b).

Recall that for the $\mathcal{C}^1$- continuity of the FIF, we need $|\alpha_i| < a_i$ for all $i = 1, 2, \dots, 5$. The calculated values of a_i for the present data are 0.1428, 0.2857, 0.1428, 0.2857 and 0.1428. We let $\boldsymbol{\alpha} = (-0.12, 0.25, 0.1, 0.25, -0.12)$, whose components α_i are chosen at random, close in magnitude to a_i, while b is the two-point cubic Hermite interpolant corresponding to H; cf. Figure 3.1(b). Iterating the functional equation (2.11), we obtain the $\boldsymbol{\alpha}$-fractal function $H^{\boldsymbol{\alpha}}$ preserving the regularity of H. However, the cubic FIF $H^{\boldsymbol{\alpha}}$ depicted in Figure 3.1(c) does not reflect the monotonicity of H. Next, to obtain a monotone cubic FIF $H^{\boldsymbol{\alpha}}$ we apply Steps 3 and 4 of our monotone cubic FIF algorithm. Observe that the end derivatives $d_1 = 0$ and $d_5 = 8.8705$, obtained through the FC-algorithm, satisfy the condition prescribed in Step 3 and hence there is no need to modify their value, where we take $\mathcal{S}$ to be the disc specified earlier. We calculate the numerical lower and upper bounds for the scaling factors according to the prescription in Theorem 3.1. Taking the scale vector $\boldsymbol{\alpha} = (0, -0.0007, 0.03, 0.0196, 0.04)$, whose random components lie within the calculated bounds, the corresponding fractal perturbation $H^{\boldsymbol{\alpha}}$ that retains the monotonicity of H is plotted in Figure 3.1(d). By applying the FC-algorithm with subregion $\mathcal{S}_3$, where the transformation $(\beta, \gamma) \to (\beta^*, \gamma^*)$ is performed via the projection method indicated earlier, and taking $\boldsymbol{\alpha} = (0, 0.0007, -0.001, 0.009, 0.004)$, we obtain the monotone cubic FIF $H^{\boldsymbol{\alpha}}$ plotted in Figure 3.1(e). Due to the fact that the scaling factors α_i are close to zero, changes in the shape of the monotonic FIFs, given in Figures 3.1(d) and (e), with respect to the traditional monotonic cubic interpolant in Figure 3.1(b), may not be apparently visible. Note that since $\alpha_1 = 0$, the monotonic cubic FIFs $H^{\boldsymbol{\alpha}}$ depicted in Figures 3.1 (d) and (e) exactly coincide with the traditional monotonic cubic interpolant H given in Figure 3.1(b) on the subinterval $I_1 = [8, 9]$.

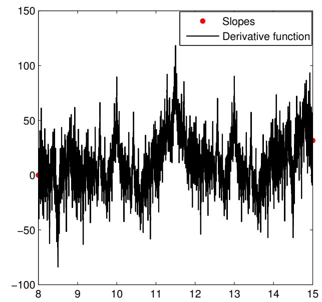
Even though we have chosen the scaling factors arbitrarily, the following points may be noted for "ad-hoc" selection strategies for these multipliers, apart from ensuring the desired shape preservation. As mentioned earlier in the introductory section, the Hölder exponent of $(H^{\boldsymbol{\alpha}})^{(1)}$ is controlled by the scaling factors α_i and hence their selection may be catered so as to have a specified Hölder exponent for $(H^{\boldsymbol{\alpha}})^{(1)}$. Following [23], the conditions under which $(H^{\boldsymbol{\alpha}})^{(1)}$ is nondifferentiable in a dense subset of $I = [8, 15]$ can be obtained, and $\boldsymbol{\alpha}$ may be selected to satisfy this condition along with the desired monotonicity, assuming compatibility of these conditions. Recently, we have proved in [36] that finding a FIF $H^{\boldsymbol{\alpha}}$ close to a function $\Phi \in \mathcal{C}^1(I)$ is a nonlinear convex optimization problem, which we may couple with the proposed monotonicity conditions to obtain a monotonic cubic FIF $H^{\boldsymbol{\alpha}}$ close to a prescribed monotonic function. Overall, various tools available in the literature that provide selection strategies for an "optimal" scale vector may be coupled with the monotonicity conditions derived herein to find an "optimal" monotonic cubic FIF $H^{\boldsymbol{\alpha}}$, which deserves further research.

Next consider the data $\{(4, 4), (5, 6), (6, 7), (8, 5), (10, 0)\}$ which is monotone on $I_1 = [4, 6]$ and $I_2 = [6, 10]$. Let the derivatives at knots be $2.5, 1.5, 0, -1.75, -3.25$. We visit each of these intervals and determine which monotone constraint is to be applied, based on whether the data is increasing or decreasing. The FC-algorithm applied on I_1 with region $\mathcal{S}_1$ does not demand a change in the derivative parameters. Further, the end derivatives are such that the two-point Hermite interpolant b is monotone increasing. Using $\boldsymbol{\alpha}^{(1)} = (0.15, 0)$, we obtain a monotone increasing cubic FIF $H^{\boldsymbol{\alpha}^{(1)}}$ on I_1. Similarly, our monotonic cubic FIF algorithm with $\boldsymbol{\alpha}^{(2)} = (0, 0.13)$ yields a monotone decreasing cubic FIF $H^{\boldsymbol{\alpha}^{(2)}}$ on I_2. For the block matrix $\boldsymbol{\alpha} = [\boldsymbol{\alpha}^{(1)}\ \boldsymbol{\alpha}^{(2)}]$, the FIF $H[\boldsymbol{\alpha}] \in \mathcal{C}^1(I)$ defined in a piecewise manner by

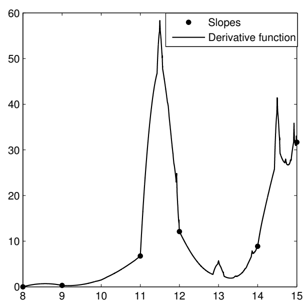
$H[\boldsymbol{\alpha}]|_{I_i} = H^{\boldsymbol{\alpha}^{(i)}}$, $i = 1, 2$, is comonotone with the given data set. If the subinterval I_i in which the given function has a uniform monotonicity property contains only two node points, then we have to introduce an additional node to apply the fractal interpolation scheme. Let



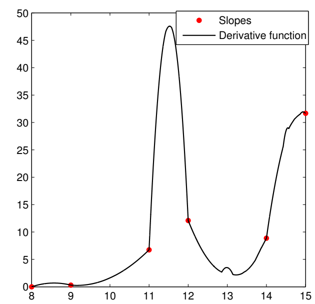
(a): Derivative of monotone cubic H in Figure 3.1(b).



(b): Derivative of cubic FIF $H^{\boldsymbol{\alpha}}$ in Figure 3.1(c).



(c): Derivative of monotone cubic FIF $H^{\boldsymbol{\alpha}}$ in Figure 3.1(d).



(d): Derivative of monotone cubic FIF $H^{\boldsymbol{\alpha}}$ in Figure 3.1(e).

FIG. 3.2. *Derivatives of traditional piecewise cubic interpolant and cubic FIFs.*

us remark here that the strategy of dividing the interval into smaller intervals, applying the FIF scheme in each subinterval separately, and defining the desired interpolant in a piecewise manner can render locality to the FIF scheme. Another problem related to locality is the study of the influence of the scaling factors in the FIF. For the sensitivity analysis of FIFs with respect to perturbations in the scaling factors, the reader may refer [40]. It is to be noted that due to the global nature of FIF, in general, predicting which part of $H^{\boldsymbol{\alpha}}$ is influenced by a perturbation in a particular scaling factor α_i is difficult. On the other hand, the problem is almost trivial once the locality is addressed.

The derivatives of the traditional monotone cubic interpolant H, cubic FIF $H^{\boldsymbol{\alpha}}$, and monotone cubic FIFs $H^{\boldsymbol{\alpha}}$, are given in Figures 3.2(a)-(d). The function $H^{(1)}$ is smooth except possibly at the knots whereas $(H^{\boldsymbol{\alpha}})^{(1)}$ shows irregularity. Further, the irregularity can be quantified using the notion of fractal dimension [2, 39]. It is also known [2] that as $|\alpha_i|$ increases from zero, the dimension of the FIF increases. In geometric modeling and CAGD, in addition to having methods for monotone interpolation, it is desirable to have one or more parameters that can influence the shape of the interpolant and/or its derivative. In this regard, the scaling parameters embedded in the structure of the cubic FIF can be exploited to construct an interpolant satisfying chosen properties such as locality, monotonicity, fractality in the derivative, and convergence order; see the next section.

4. Approximation and convergence results. This section is devoted to shed some light on the approximation properties of $\boldsymbol{\alpha}$-fractal functions, the convergence order of the monotone cubic FIF, and the fractal analogues of Jackson-type estimates for the approximation of functions by monotone/comonotone polynomials. Our results are in fact derived by using the corresponding classical counterparts and the following lemma whose proof follows directly from the functional equation for $f^{\boldsymbol{\alpha}}$; see [27] for details.

LEMMA 4.1. *Let $f^{\boldsymbol{\alpha}}$ be the $\boldsymbol{\alpha}$-fractal function corresponding to the function $f \in \mathcal{C}(I)$ (cf. (2.7)) and $|\boldsymbol{\alpha}|_\infty := \max\{|\alpha_i| : i \in J\}$. Then*

$$\|f^{\boldsymbol{\alpha}} - f\|_\infty \le \frac{|\boldsymbol{\alpha}|_\infty}{1 - |\boldsymbol{\alpha}|_\infty} \|f - b\|_\infty.$$

REMARK 4.2. Let Φ be the original function and f be a traditional non-recursive approximant for Φ. Then, in view of the triangle inequality

$$\|\Phi - f^{\boldsymbol{\alpha}}\|_\infty \le \|\Phi - f\|_\infty + \|f - f^{\boldsymbol{\alpha}}\|_\infty,$$

the previous lemma asserts that for a suitable scale vector $\boldsymbol{\alpha}$ the FIF $f^{\boldsymbol{\alpha}}$ has the same order of convergence as that of its classical counterpart f. To be precise, if f has an order of convergence r, say, then for the scale vector $\boldsymbol{\alpha}$ satisfying $|\alpha_i| < a_i^r = \frac{h_i^r}{(x_N - x_1)^r}$, for all $i \in J$, the fractal function $f^{\boldsymbol{\alpha}}$ also possesses the r-th order convergence.

The next result points to the order of convergence of the monotone cubic FIF scheme.

THEOREM 4.3. *Assume that $\Phi \in \mathcal{C}^3(I)$ is monotone increasing. Let the initial derivative approximations d_i satisfy $|\Phi^{(1)}(x_i) - d_i| \le ch^2$, for all $i \in J$ and some constant c, where $h = \max\{h_i : i \in J\}$. Further, let the closed triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$, be contained in the subregion $\mathcal{S}$, the projection of (β_i, γ_i) onto $\mathcal{S}$ satisfy $\beta_i^* + \gamma_i^* \ge 2$, and the scale vector be such that $|\alpha_i| < a_i^3$ for all $i \in J$. Then the associated monotone cubic FIF $H^{\boldsymbol{\alpha}}$ is a third order approximation to Φ.*

Proof. Under the stated assumptions, we know [18] that the Fritsch-Carlson algorithm is third order accurate, that is, $\|\Phi - H\|_\infty = O(h^3)$. From the triangle inequality, Lemma 4.1, and the monotonicity of b, we have

$$\begin{aligned}
\|\Phi - H^{\boldsymbol{\alpha}}\|_\infty &\le \|\Phi - H\|_\infty + \|H^{\boldsymbol{\alpha}} - H\|_\infty \\
&\le \|\Phi - H\|_\infty + \frac{|\boldsymbol{\alpha}|_\infty}{1 - |\boldsymbol{\alpha}|_\infty} \|H - b\|_\infty \\
&\le \|\Phi - H\|_\infty + \frac{2|\boldsymbol{\alpha}|_\infty}{1 - |\boldsymbol{\alpha}|_\infty} \|H\|_\infty.
\end{aligned}$$

To obtain the last inequality we have also used $\|b\|_\infty = \|H\|_\infty$, which follows from the monotonicity of b and the fact that b coincides with H at the ends of the interval. For the scaling factors satisfying $|\alpha_i| < a_i^3 = \left(\frac{h_i}{x_N - x_1}\right)^3$, we get

$$\frac{|\boldsymbol{\alpha}|_\infty}{1 - |\boldsymbol{\alpha}|_\infty} < \frac{h^3}{(x_N - x_1)^3 - h^3},$$

and hence the result follows. ∎

REMARK 4.4. Assume that $\Phi \in \mathcal{C}^4(I)$ is monotone increasing and the initial derivative approximations are third order accurate, i.e., $|\Phi^{(1)}(x_i) - d_i| < ch^3$ for $i = 1, 2, \dots, N$ and for some constant c. A modification of the FC-algorithm, called extended two-sweep algorithm, that yields fourth-order accuracy is suggested in reference [18]. Let H be the monotone cubic interpolant corresponding to a data generated by Φ obtained by this algorithm. Now we choose scale vectors such that apart from the conditions of monotonicity, $|\alpha_i| < a_i^4$ for $i = 1, 2, \dots, N$ holds. Then following the proof of the preceding theorem, we can infer that the corresponding monotone cubic FIF $H^{\boldsymbol{\alpha}}$ provides fourth order accuracy.

Let us represent the modulus of continuity of f by

$$\omega(f, \epsilon) = \sup_{|h| \le \epsilon} \{|f(x + h) - f(x)|,\ x \in I\}.$$

The following theorem supplements the fractal analogue of the Weierstrass theorem; see [26]. First let us fix some notation. For each $n \in \mathbb{N}$, choose a partition of $I = [-1, 1]$ with $N_n > 2$ points, namely, $\Delta_n^* := \{x_1^{(n)}, x_2^{(n)}, \dots, x_{N_n}^{(n)}\}$, such that $-1 = x_1^{(n)} < x_2^{(n)} < \dots < x_{N_n}^{(n)} = 1$. For $i \in J_n := \{1, 2, \dots, N_n - 1\}$, let $h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)}$. Then $h^{(n)} = \max\{h_i^{(n)} : i \in J_n\}$ is the norm of the partition Δ_n^*. For each $n \in \mathbb{N}$, let $\boldsymbol{\alpha}^{(n)} = (\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_{N_n - 1}^{(n)}) \in \mathbb{R}^{N_n - 1}$ be a scale vector corresponding to the partition Δ_n^* of $I = [-1, 1]$, such that $|\alpha_i^{(n)}| < a_i^{(n)} := \frac{h_i^{(n)}}{2}$. Define $|\boldsymbol{\alpha}^{(n)}|_\infty = \max\{|\alpha_i^{(n)}| : i \in J_n\}$.

THEOREM 4.5. *For each $k \ge 0$ and every monotone increasing $f \in \mathcal{C}^k[-1, 1]$, there are monotone increasing fractal polynomials $p_n^{\alpha^{(n)}}$, where the degree of p_n is less than or equal to n, such that*

$$\|f - p_n^{\boldsymbol{\alpha}^{(n)}}\|_\infty \le c \frac{1 + |\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty} n^{-k} \omega\big(f^{(k)}, \frac{1}{n}\big) + \frac{2|\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty} \|f\|_\infty,$$

where c is an absolute constant independent of f and n.

Proof. Jackson-type estimates for the approximation of monotone functions $f \in \mathcal{C}^k[-1, 1]$ by monotone polynomials are well known in traditional approximation theory. To be specific, we have the following result (see [17]): for each $k \ge 0$ and every monotone increasing $f \in \mathcal{C}^k[-1, 1]$, there are increasing polynomials p_n of degree n such that

$$\|f - p_n\|_\infty \le c\, n^{-k} \omega\big(f^{(k)}, \frac{1}{n}\big), \tag{4.1}$$

where c is an absolute constant independent of f and n. By Theorem 3.1, for each of these polynomials p_n, we can select a scale vector $\boldsymbol{\alpha}^{(n)}$ and a monotone base function b_n so that the fractal polynomial $p_n^{\boldsymbol{\alpha}^{(n)}}$ retains the monotonicity of p_n. We have,

$$\begin{aligned}
\|p_n - p_n^{\boldsymbol{\alpha}^{(n)}}\|_\infty &\le \frac{|\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty} \|p_n - b_n\|_\infty \\
&\le \frac{|\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty} (\|p_n\|_\infty + \|b_n\|_\infty) \\
&\le \frac{2|\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty} \|p_n\|_\infty.
\end{aligned}$$

Lemma 4.1 was utilized in the first step of the preceding analysis, the second step involved the triangle inequality, while the final step can be justified as follows: since b_n is monotone increasing on $I = [-1, 1]$ and matches p_n at the end points of I, $\|b_n\|_\infty = |b_n(1)| =$

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$|p_n(1)| \leq \|p_n\|_\infty$. Therefore,

$$\begin{aligned}\|f - p^{\boldsymbol{\alpha}^{(n)}}\|_\infty &\leq \|f - p_n\|_\infty + \|p_n - p_n^{\boldsymbol{\alpha}^{(n)}}\|_\infty,\\ &\leq \|f - p_n\|_\infty + \frac{2|\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty}\|p_n\|_\infty,\\ &\leq \|f - p_n\|_\infty + \frac{2|\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty}(\|f - p_n\|_\infty + \|f\|_\infty),\\ &= \frac{1 + |\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty}\|f - p_n\|_\infty + \frac{2|\boldsymbol{\alpha}^{(n)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n)}|_\infty}\|f\|_\infty.\end{aligned}$$

Combining this with (4.1), we obtain the desired conclusion. ◻

REMARK 4.6. By noting that $|\boldsymbol{\alpha}^{(n)}|_\infty < \frac{h^{(n)}}{2}$ we obtain

$$\|f - p_n^{\boldsymbol{\alpha}^{(n)}}\|_\infty \leq c\frac{2 + h^{(n)}}{2 - h^{(n)}}n^{-k}\omega\big(f^{(k)}, \frac{1}{n}\big) + \frac{h^{(n)}}{2 - h^{(n)}}\|f\|_\infty,$$

where c is an absolute constant independent of f and n.

We conclude this section with the fractal analogue of a result on comonotone approximation. The reader may recall the notation used in Remark 3.5.

THEOREM 4.7. *Let f be a continuously differentiable function in $[-1, 1]$, which changes monotonicity a finite number of times, say s, in that interval. Then, for each $n \geq 1$ there exists a piecewise defined fractal polynomial denoted by $p_n[\boldsymbol{\alpha}^{(n)}]$, where $[\boldsymbol{\alpha}^{(n)}]$ is a suitable block matrix, the corresponding classical counterpart p_n has degree less than or equal to n, and $p_n[\boldsymbol{\alpha}^{(n)}]$ is comonotone with f on $[-1, 1]$, such that:*

$$\big\|f - p_n[\boldsymbol{\alpha}^{(n)}]\big\|_\infty \leq \frac{1 + |[\boldsymbol{\alpha}^n]|_\infty}{1 - |[\boldsymbol{\alpha}^n]|_\infty}\,\frac{c(s)}{n}\,\omega\big(f^{(1)}, \frac{1}{n}\big) + \frac{2|[\alpha^n]|_\infty}{1 - |[\alpha^n]|_\infty}\|f\|_\infty,$$

where $c(s)$ is a constant depending only on s.

Proof. Assume that f changes monotonicity at $X_s = \{x_1, x_2, \ldots, x_s\}$, and $-1 = x_0 < x_1 < \cdots < x_s < x_{s+1} = 1$. Let I_i, $i = 1, 2, \ldots, s + 1$ be the subintervals of I wherein f has the same monotonicity throughout. With the stated assumptions, we know [9] that for each n there exists a polynomial p_n of degree n which is comonotone with f and which satisfies:

$$\|f - p_n\|_\infty \leq \frac{c(s)}{n}\,\omega\big(f^{(1)}, \frac{1}{n}\big).$$

Following Remark 3.5, in each I_j, $j = 1, 2, \ldots, s + 1$, we select a partition $\Delta^*_{n,j}$, a scale vector $\boldsymbol{\alpha}^{(n,j)}$, and a base function $b_{n,j}$, so that the corresponding fractal function $p_n^{\boldsymbol{\alpha}^{(n,j)}}$ is comonotone with p_n. For a block matrix $\boldsymbol{\alpha}^{(n)} = [\boldsymbol{\alpha}^{(n,1)}\ \boldsymbol{\alpha}^{(n,2)} \cdots \boldsymbol{\alpha}^{(n,s+1)}]$, define the fractal polynomial $p_n[\boldsymbol{\alpha}^{(n)}]$ in a piecewise manner as $p_n[\boldsymbol{\alpha}^{(n)}]|_{I_{n,j}} = p_n^{\boldsymbol{\alpha}^{(n,j)}}$. Using the triangle inequality, Lemma 4.1, and the monotonicity of $b_{n,j}$, we obtain:

$$\begin{aligned}\big\|p_n[\boldsymbol{\alpha}^{(n)}] - p_n\big\|_\infty &= \max\{|p_n[\boldsymbol{\alpha}^{(n)}](x) - p_n(x)| : x \in I\},\\ &= \max_{1\leq j\leq s+1}\max\{|p_n^{\boldsymbol{\alpha}^{(n,j)}}(x) - p_n(x)| : x \in I_{n,j}\},\\ &= \max_{1\leq j\leq s+1}\|p_n^{\boldsymbol{\alpha}^{(n,j)}} - p_n\|_{I_{n,j}},\\ &\leq \max_{1\leq j\leq s+1}\frac{|\boldsymbol{\alpha}^{(n,j)}|_\infty}{1 - |\boldsymbol{\alpha}^{(n,j)}|_\infty}\|p_n - b_{n,j}\|_\infty,\\ &\leq \frac{2|[\boldsymbol{\alpha}^n]|_\infty}{1 - |[\boldsymbol{\alpha}^n]|_\infty}\|p_n\|_\infty,\end{aligned}$$

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where $|[\boldsymbol{\alpha}^{(n)}]|_\infty = \max\{|\boldsymbol{\alpha}^{(n,j)}|_\infty : j = 1, 2, \dots, s+1\}$. The rest of the proof follows exactly as in Theorem 4.5. ∎

5. Conclusions. In this paper, a class of cubic FIFs $H^{\boldsymbol{\alpha}}$ is obtained as differentiable $\boldsymbol{\alpha}$-fractal functions (fractal perturbation) corresponding to the traditional piecewise cubic interpolant H. The advantage of such an approach is that the perturbation may be designed so as to reflect the monotonicity of the cubic interpolant. Suitable algorithms (Fritsch-Carlson, Fritsch-Butland, or any other variant of these) can be applied to obtain a monotone cubic interpolant H, and subsequently a suitable scale vector $\boldsymbol{\alpha}$ can be selected so that $H^{\boldsymbol{\alpha}}$ retains the monotonicity of H. This two-step procedure culminates with the construction of monotone cubic FIFs. In practice, there are many instances where we desire a monotonic approximant with its derivative receiving varying irregularity, and the introduction of monotonicity to cubic FIFs $H^{\boldsymbol{\alpha}}$ accomplishes this. The flexibility offered by the scaling factors, the ability to preserve the approximation order of the traditional monotonic interpolation algorithm, and the strength to provide monotone interpolants with fractality in the derivative, outweigh the cost of a few extra lines of code needed for the monotone fractal perturbation $H^{\boldsymbol{\alpha}}$ of a monotone H. Further, the present approach of obtaining a monotonicity preserving fractal function corresponding to a continuously differentiable monotone function paves the way to fractal analogues of some theorems on monotone and comonotone polynomial approximation. Thus, in conclusion, the fractal methodology can be exploited in the field of shape preserving interpolation/approximation for providing more diverse and flexible shape preserving curves.

Appendix. Here we provide a simple proof for Lemma 2.1. Consider the cubic polynomial $H(x) = \frac{H^{(1)}(u)+H^{(1)}(v)-2\Delta}{(v-u)^2}(x-u)^3 + \frac{-2H^{(1)}(u)-H^{(1)}(v)+3\Delta}{(v-u)}(x-u)^2 + H^{(1)}(u)(x-u) + H(u)$ defined on $I = [u, v]$, where $\Delta = \frac{H(v)-H(u)}{v-u}$. It is clear that the necessary condition for monotonicity is: $\mathrm{sgn}(H^{(1)}(u)) = \mathrm{sgn}(H^{(1)}(v)) = \mathrm{sgn}(\Delta)$.

If $\Delta = 0$ then H is monotone (i.e., constant) on I if and only if $H^{(1)}(u) = H^{(1)}(v) = 0$. Let us assume $\Delta \neq 0$ and set $\beta = \frac{H^{(1)}(u)}{\Delta}, \gamma = \frac{H^{(1)}(v)}{\Delta}$. We have $H^{(1)}(x) = 3(H^{(1)}(u) + H^{(1)}(v) - 2\Delta)(\frac{x-u}{v-u})^2 + 2(-2H^{(1)}(u) - H^{(1)}(v) + 3\Delta)(\frac{x-u}{v-u}) + H^{(1)}(u)$. Letting $\theta = \frac{x-u}{v-u}$ and $H^{(1)}(x) \equiv g(\theta)$, the monotonicity constraint on H reduces to the nonnegativity condition: $g(\theta) \geq 0$ for all $\theta \in [0, 1]$.

We know [35] that the quadratic polynomial $\rho^*(s) = As^2 + Bs + C \geq 0$ for all $s \geq 0$ if and only if one of the following conditions holds: (i) $A \geq 0$, $B \geq 0$, and $C \geq 0$, or (ii) $C \geq 0$ and $4AC \geq B^2$. To put our problem in this framework, we use the substitution $\theta = \frac{s}{s+1}$. Consequently, the desired condition $g(\theta) \geq 0$ for all $\theta \in [0, 1]$ is transformed into $\rho^*(s) = As^2 + Bs + C \geq 0$ for all $s \geq 0$, where $A = H^{(1)}(v)$, $B = -2H^{(1)}(u) - 2H^{(1)}(v) + 6\Delta$, and $C = H^{(1)}(u)$. Applying the Schmidt-Heß conditions for the positivity of a quadratic polynomial, we infer that $\rho^*(s)$ is nonnegative if and only if: (i) $H^{(1)}(u) \geq 0$, $H^{(1)}(v) \geq 0$, and $H^{(1)}(u) + H^{(1)}(v) \leq 3\Delta$, or (ii) $H^{(1)}(u) \geq 0$ and $H^{(1)}(u)^2 + H^{(1)}(v)^2 + 9\Delta^2 + H^{(1)}(u)H^{(1)}(v) - 6H^{(1)}(u)\Delta - 6H^{(1)}(v)\Delta \leq 0$. Therefore, the cubic polynomial H is monotone on $[u, v]$ if and only if $(\beta, \gamma) \in \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ is the region bounded by $x \geq 0, y \geq 0, x + y \leq 3$, and $\mathcal{A}_2$ is the region bounded by $x \geq 0, y \geq 0, (x-3)^2 + (y-3)^2 + xy - 9 \leq 0$. It can be readily seen that $\mathcal{A}_1 \cup \mathcal{A}_2$ coincides with the FC monotone region given in Section 2.2.

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