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OPERATIONAL MÜNTZ-GALERKIN APPROXIMATION FOR ABEL-HAMMERSTEIN INTEGRAL EQUATIONS OF THE SECOND KIND*

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Abstract. Since solutions of Abel integral equations exhibit singularities, existing spectral methods for these equations suffer from instability and low accuracy. Moreover, for nonlinear problems, solving the resulting complex nonlinear algebraic systems numerically requires high computational costs. To overcome these drawbacks, in this paper we propose an operational Galerkin strategy for solving Abel-Hammerstein integral equations of the second kind which applies Müntz-Legendre polynomials as natural basis functions to discretize the problem and to obtain a sparse nonlinear system with upper-triangular structure that can be solved directly. It is shown that our approach yields a well-posed and easy-to-implement approximation technique with a high order of accuracy regardless of the singularities of the exact solution. The numerical results confirm the superiority and effectiveness of the proposed scheme.

Key words. Abel-Hammerstein integral equations, Galerkin method, Müntz-Legendre polynomials, well-posedness

AMS subject classifications. 45E10, 41A25

1. Introduction. Abel's integral equation, one of the very first integral equations seriously studied, and the corresponding integral operator have never ceased to inspire mathematicians to investigate and to generalize them. Abel was led to his equation by a problem of mechanics, the tautochrone problem [16]. However, his equation and some variants of it have found applications in such diverse fields as stereology of spherical particles [3, 18], inversion of seismic travel times [4], cyclic voltametry [5], water wave scattering by two surface-piercing barriers [12], percolation of water [15], astrophysics [19], theory of superfluidity [22], heat transfer between solids and gases under nonlinear boundary conditions [25], propagation of shock-waves in gas fields tubes, subsolutions of a nonlinear diffusion problem [29], etc.

Several analytical and numerical methods have been proposed for solving Abel integral equations such as rational interpolation [2], non-polynomial spline collocation [6], Adomian decomposition [8], product integration [9], homotopy perturbation [20], trapezoidal discretization [13], waveform relaxation [11], Jacobi spectral collocation and Tau methods [14, 21, 27], Taylor expansion [17], Laplace transformation [19], fractional linear multistep methods [23], Runge-Kutta methods [24], modified Tau methods [26], Legendre and Chebyshev wavelets [31, 33], etc. Almost all aforementioned methods have been applied to the linear case, and very few approaches in the literature utilize approximations for a numerical solution of nonlinear Abel integral equations. Thus, providing a suitable numerical strategy which approximates solutions of nonlinear Abel integral equations can be worthwhile and new in the area. In this paper, we consider a Hammerstein-type of nonlinearity in the Abel integral equation.

The Galerkin method is one of the most popular and powerful projection technique for solving integral equations. It belongs to the class of weighted residual methods, in which approximations are sought by forcing the residual to be zero but only in an approximate sense. This method has two main advantages. First, it reduces the given linear (nonlinear) problem to that of solving a linear (nonlinear) system of algebraic equations. Second, it has excellent error estimates with the so-called "exponential-like convergence" being the fastest possible when the solution is infinitely smooth; see [10, 30].

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In this paper, we introduce a numerical scheme based on the Galerkin method to approximate the solutions of the following Abel-Hammerstein integral equation of the second kind

$$u(x) = f(x) + \int_0^x \frac{K(x,t)}{\sqrt{x-t}} G(t,u(t))dt, \qquad x \in \Lambda = [0,1], \tag{1.1}$$

using some simple matrix operations. Throughout the paper, C_i will denote a generic positive constant. The following theorem states the existence and uniqueness result for (1.1).

THEOREM 1.1 ([7]). *Let the following conditions be fulfilled:*

- $f(x)$ *is a continuous function on* Λ,
- $K(x,t)$ *is a continuous function on* $D := \{(x,t) \,|\, 0 \le t \le x \le 1\}$ *with* $K(x,x) \neq 0$, *and*
- $G : \Lambda \times \mathbb{R} \to \mathbb{R}$ *is a Lipschitz-continuous function with respect to the second variable, i.e.,*

$$|G(t,u_1(t)) - G(t,u_2(t))| \le C_1 |u_1(t) - u_2(t)|.$$

Then equation (1.1) *has a unique continuous solution* $u(x)$ *on* Λ. *Here,* $\mathbb{R}$ *denotes the set of all real numbers.*

However, from well-known existence and uniqueness theorems [7, 21, 26, 27], we can conclude that (1.1) typically has a solution whose first derivative is unbounded at the origin and behaves like

$$u'(x) \simeq \frac{1}{\sqrt{x}}. \tag{1.2}$$

After providing criteria for the existence and uniqueness of solutions of (1.1) in the previous theorem, we investigate the well-posedness of the problem. It is well known that linear Abel integral equations of the second kind are well-posed in the sense that solutions depend continuously on the data. In the following theorem we prove that this also holds true for (1.1).

THEOREM 1.2. *Consider the equation* (1.1) *and let* $\tilde{u}(x)$ *be the solution of the perturbed problem*

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x \frac{\tilde{K}(x,t)}{\sqrt{x-t}} G(t,\tilde{u}(t))dt. \tag{1.3}$$

If the assumptions of Theorem 1.1 hold for (1.3), *then we have*

$$\|u(x) - \tilde{u}(x)\|_\infty \le C_3 \|f - \tilde{f}\|_\infty + C_4 \|K - \tilde{K}\|_\infty, \tag{1.4}$$

where $\|.\|_\infty$ *denotes the well-known uniform norm.*

Proof. Subtracting (1.1) from (1.3) yields

$$\begin{aligned} u(x) - \tilde{u}(x) &= f(x) - \tilde{f}(x) + \int_0^x \frac{K(x,t)}{\sqrt{x-t}} G(t,u(t))dt - \int_0^x \frac{\tilde{K}(x,t)}{\sqrt{x-t}} G(t,\tilde{u}(t))dt \\ &= f(x) - \tilde{f}(x) + \int_0^x \frac{K(x,t)}{\sqrt{x-t}} \left(G(t,u(t)) - G(t,\tilde{u}(t))\right) dt \\ &\quad + \int_0^x \frac{\left(K(x,t) - \tilde{K}(x,t)\right)}{\sqrt{x-t}} G(t,\tilde{u}(t))dt. \end{aligned}$$

Since $K(x,t)$ and $G(x,t)$ are continuous functions, we can write

$$|u(x) - \tilde{u}(x)| \leq |f(x) - \tilde{f}(x)| + \|K(x,t)\|_\infty \int_0^x \frac{|G(t,u(t)) - G(t,\tilde{u}(t))|}{\sqrt{x-t}} dt$$
$$+ \|G(t,\tilde{u}(t))\|_\infty \int_0^x \frac{|K(x,t) - \tilde{K}(x,t)|}{\sqrt{x-t}} dt.$$

Since $G(t,u(t))$ satisfies the Lipschitz condition with respect to the second variable, we obtain

$$|u(x) - \tilde{u}(x)| \leq |f(x) - \tilde{f}(x)| + C_2\|K(x,t)\|_\infty \int_0^x \frac{|u(t) - \tilde{u}(t)|}{\sqrt{x-t}} dt$$
$$+ \|G(t,\tilde{u}(t))\|_\infty \int_0^x \frac{|K(x,t) - \tilde{K}(x,t)|}{\sqrt{x-t}} dt.$$

Finally, using the Gronwall inequality (see [26, Lemma 4.2]), we can deduce that

$$\|u(x) - \tilde{u}(x)\|_\infty \leq C_3\|f(x) - \tilde{f}(x)\|_\infty + C_4\|K - \tilde{K}\|_\infty,$$

which gives the desired result (1.4). □

From Theorems 1.1 and 1.2, we can deduce the existence and uniqueness of solutions of (1.1) and also the well-posedness of the problem by assuming Lipschitz-continuity of the nonlinear function $G(t,u(t))$. However, an implementation of spectral methods can lead to some difficulties from the numerical point of view such as a complex nonlinear algebraic system that has to be solved, high computational costs, and thereby possibly low accuracy. In this paper, we show that if $G(t,u(t))$ is a smooth function, then we can obtain its Galerkin solution by solving a well-posed upper-triangular nonlinear algebraic system using forward substitution.

It is well known that the Galerkin method is an efficient tool to solve integral equations with smooth solutions. However, from (1.2) we can conclude that for (1.1) this approach leads to a numerical method with poor convergence behaviour. Moreover, as we mentioned above, for nonlinear problems, its implementation produces a numerical scheme with a high computational cost. Therefore to make it applicable for (1.1), we must prospect for a well-conditioned, easy to use, and highly accurate technique. To this end, in this paper we employ suitable basis functions which have good approximation properties for functions with singularities at the boundaries. Here we consider Müntz-Legendre polynomials [28, 32] as natural basis functions to define the Galerkin solution of (1.1). These functions are of non-polynomial nature and are mutually orthogonal with respect to the weight function $w(x) = 1$. We will show that if we represent the Galerkin solution of (1.1) as a linear combination of these polynomials, we can produce a numerical scheme with a satisfactory order of convergence, and the unknown coefficients of the approximate solution can be characterized by a well-posed upper-triangular system of nonlinear algebraic equations. We will solve it directly by applying the well-known forward substitution method.

The reminder of this paper is organized as follows: in Section 2, we describe the Müntz-Legendre polynomials, which are the basis for our subsequent development. In Section 3, we explain the application of the operational Galerkin method based on the Müntz-Legendre polynomials to obtain an approximate solution of (1.1). Here we also give results on the numerical solvability and a complexity analysis of the resulting system of nonlinear algebraic

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equations. In Section 4, we apply the proposed method from Section 3 to several examples and present some crucial numerical characteristics such as numerical errors, order of convergence, and comparison results with other existing numerical techniques and thereby confirm the efficiency and effectiveness of the proposed method. Section 5, is devoted to our conclusions.

2. Müntz-Legendre polynomials. In this section, we give the definition and the basic properties of the Müntz-Legendre polynomials which are used in the sequel. All the details in this section as well as further ones can be found in [28, 32].

The n-th Müntz-Legendre polynomial is defined as

$$\mathcal{L}_n(x) := \frac{1}{2\pi i}\int_D \prod_{k=0}^{n-1} \frac{t+\xi_k+1}{t-\xi_k}\frac{x^t}{t-\xi_n}dt,$$

where the simple contour D surrounds all the zeros of the denominator in the above integrand and $0=\xi_0<\xi_1<\ldots<\xi_n<\ldots$, with $\xi_n\to\infty$ and $\sum_{k=1}^{\infty}\xi_k^{-1}=\infty$. These polynomials are mutually orthogonal with respect to the weight function $w(x)=1$ such that we have

$$\int_D \mathcal{L}_n(x)\mathcal{L}_m(x)dx = \frac{\delta_{n,m}}{2\xi_n+1}, \qquad n\ge m,$$

where $\delta_{n,m}$ is the Kronecker symbol. It can be shown that the Müntz-Legendre polynomials have the following representation

$$\begin{aligned} \mathcal{L}_n(x) &:= \sum_{k=0}^{n}\rho_{k,n}x^{\xi_k} = \text{Span}\{1, x^{\xi_1}, x^{\xi_2},\ldots,x^{\xi_n}\},\\ \rho_{k,n} &= \frac{\prod_{j=0}^{n-1}(\xi_k+\xi_j+1)}{\prod_{j=0,j\neq k}^{n}(\xi_k-\xi_j)}, \qquad \text{for } n\in\mathbb{N}_0, \end{aligned} \tag{2.1}$$

and they satisfy the recursion formula

$$\mathcal{L}_n(x)=\mathcal{L}_{n-1}(x)-(\xi_n+\xi_{n-1}+1)x^{\xi_n}\int_x^1 t^{-\xi_n-1}\mathcal{L}_{n-1}(t)dt, \qquad x\in(0,1].$$

Here, $\mathbb{N}_0$ is as usual the set of nonnegative integers.

3. The Müntz-Galerkin approach. In this section, we develop an operational Galerkin method which applies the Müntz-Legendre polynomials as basis functions to obtain an approximate solution to (1.1). We present our results in two parts. First, using a sequence of matrix operations, we obtain a suitable nonlinear algebraic form of the Müntz-Galerkin discretization of (1.1). Second, we explain how the unique solution is found. Here we show that the resulting nonlinear system from the first step can be represented by an upper-triangular structure and this enables us to compute the solution directly.

3.1. Outline of the method. In this section, we apply the Müntz-Galerkin approach to convert (1.1) into a suitable nonlinear algebraic form. To this end, we suppose that

$$\xi_i = \frac{i}{2}, \quad i=0,1,\ldots, \tag{3.1}$$

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and define $u_N(x)$ as Müntz-Galerkin approximation of the exact solution $u(x)$ in the following way:

$$u_N(x) := \sum_{i=0}^{\infty} u_i \mathcal{L}_i(x) = \underline{u}\,\underline{\mathcal{L}} = \underline{u}\mathcal{L}\underline{X}, \tag{3.2}$$

where $\underline{u} := [u_0, u_1, \ldots, u_N, 0, \ldots]$ and $\underline{\mathcal{L}} := [\mathcal{L}_0(x), \mathcal{L}_1(x), \ldots, \mathcal{L}_N(x), \ldots]^T$ is a Müntz-Legendre polynomial basis. From the relations (1.2), (2.1), and (3.1), we conclude that our approximate solution (3.2) has the same asymptotic behavior as the exact ones, which is the crucial fact for achieving a satisfactory accuracy in the Galerkin method. Clearly, we can write $\underline{\mathcal{L}} := \mathcal{L}\underline{X}$ where $\mathcal{L} = \{\mathcal{L}_{i,j}\}_{i,j=0}^{\infty}$ is an infinitely non-singular lower-triangular coefficient matrix with degree $\deg(\mathcal{L}_i(x)) \leq \xi_i$ $i = 0, 1, \ldots$, and $\underline{X} := [1, x^{\xi_1}, x^{\xi_2}, \ldots, x^{\xi_N}, \ldots]^T$ are the standard Müntz basis functions. Assume that

$$\begin{aligned} f(x) &\simeq \sum_{i=0}^{N_f} f_i x^{\xi_i} = \underline{f}\,\underline{X} = \underline{f}\mathcal{L}^{-1}\underline{\mathcal{L}}, \quad \underline{f} := [f_0, f_1, \ldots, f_{N_f}, 0, \ldots]^T, \\ G(t, u(t)) &\simeq \sum_{p=0}^{N_G} s_p(t) u^p(t). \end{aligned} \tag{3.3}$$

Substituting (3.2) into (1.1) and using (3.3), our Müntz-Galerkin method is to seek $u_N(x)$ such that we have

$$\begin{aligned} u_N(x) &= \underline{f}\underline{X} + \sum_{p=0}^{N_G} \int_0^x \frac{K(x,t)s_p(t)}{\sqrt{x-t}} u_N^p(t)dt \\ &= \underline{f}\underline{X} + \int_0^x \frac{K(x,t)s_0(t)}{\sqrt{x-t}} dt + \sum_{p=1}^{N_G} \int_0^x \frac{K(x,t)s_p(t)}{\sqrt{x-t}} u_N^p(t)dt. \end{aligned} \tag{3.4}$$

Let us define $K_p(x,t) = K(x,t)s_p(t)$ for $p \geq 1$, and assume that

$$K_p(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k_{i,j}^p x^i t^j, \qquad p \geq 1. \tag{3.5}$$

By inserting (3.5) into (3.4), we obtain

$$u_N(x) = \underline{f}\underline{X} + \int_0^x \frac{K(x,t)s_0(t)}{\sqrt{x-t}} dt + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=1}^{N_G} \int_0^x \frac{k_{i,j}^p x^i t^j}{\sqrt{x-t}} u_N^p(t)dt. \tag{3.6}$$

Now, we intend to derive a matrix representation for the integral terms in (3.6). For this, we first give the following lemma which transform $u_N^p(t)$ into a suitable matrix form. Our strategy is to extend the proof in [14, Theorem 2.1] to the case of Müntz-Legendre basis functions.

LEMMA 3.1. *Suppose that $u_N(x)$ is given by (3.2). Then we have*

$$u_N^p(x) = \underline{u}\mathcal{L}\mathcal{Q}^{p-1}\underline{X},$$

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where $\mathcal{Q}$ is the following infinite upper-triangular matrix

$$\mathcal{Q} := \begin{bmatrix} \underline{u}\mathcal{L}_0 & \underline{u}\mathcal{L}_1 & \underline{u}\mathcal{L}_2 & \cdots \\ 0 & \underline{u}\mathcal{L}_0 & \underline{u}\mathcal{L}_1 & \cdots \\ 0 & 0 & \underline{u}\mathcal{L}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with $\mathcal{L}_j = \{\mathcal{L}_{i,j}\}_{i=0}^{\infty}, \quad j = 0, 1, \ldots$.

Proof. We proceed using mathematical induction on p. For $p = 1$ the lemma is valid. We assume that it holds for p and transit to $p + 1$ as follows:

$$\begin{aligned} u_N^{p+1}(x) &= u_N^p(x) \times u_N(x) \\ &= \left(\underline{u}\mathcal{L}\mathcal{Q}^{p-1}\underline{X}\right) \times (\underline{u}\mathcal{L}\underline{X}) = \underline{u}\mathcal{L}\mathcal{Q}^{p-1}\left(\underline{X} \times (\underline{u}\mathcal{L}\underline{X})\right). \end{aligned} \tag{3.7}$$

Now, it is sufficient to show that

$$\underline{X} \times (\underline{u}\mathcal{L}\underline{X}) = \mathcal{Q}\underline{X}. \tag{3.8}$$

To this end, we can write

$$\begin{aligned} \underline{X} \times (\underline{u}\mathcal{L}\underline{X}) &= \underline{X} \times \left(\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} u_r\mathcal{L}_{r,s}x^{\frac{s}{2}}\right) = \left\{\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} u_r\mathcal{L}_{r,s}x^{\frac{s+i}{2}}\right\}_{i=0}^{\infty} \\ &= \left\{\sum_{j=0}^{\infty}\left(\sum_{r=0}^{\infty} u_r\mathcal{L}_{r,j-i}\right)x^{\frac{j}{2}}\right\}_{i=0}^{\infty} = \left\{\sum_{j=i}^{\infty}\left(\sum_{r=0}^{\infty} u_r\mathcal{L}_{r,j-i}\right)x^{\frac{j}{2}}\right\}_{i=0}^{\infty} \\ &= \mathcal{Q}\underline{X}, \end{aligned}$$

which proves (3.8). Clearly, inserting (3.8) into (3.7) gives the desired result. □

Now applying Lemma 3.1 in the third term on the right-hand side of (3.6) yields

$$\begin{aligned} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=1}^{N_G}\int_0^x \frac{k_{i,j}^p x^i t^j}{\sqrt{x-t}} u_N^p(t)dt &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=1}^{N_G}\int_0^x \frac{k_{i,j}^p x^i t^j}{\sqrt{x-t}} \underline{u}\mathcal{L}\mathcal{Q}^{p-1}\underline{X_t}dt \\ &= \underline{u}\mathcal{L}\sum_{p=1}^{N_G}\mathcal{Q}^{p-1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k_{i,j}^p x^i \left\{\int_0^x \frac{t^{j+\frac{m}{2}}}{\sqrt{x-t}}dt\right\}_{m=0}^{\infty}\right), \end{aligned} \tag{3.9}$$

where $\underline{X_t} := [1, t^{\xi_1}, t^{\xi_2}, \ldots, t^{\xi_N}, \ldots]^T$. Using the relation [27]

$$\left\{\int_0^x \frac{t^{j+\frac{m}{2}}}{\sqrt{x-t}}dt\right\}_{m=0}^{\infty} = \left\{\frac{\sqrt{\pi}\,\Gamma(1+j+\frac{m}{2})}{\Gamma(\frac{3+m}{2}+j)}\,x^{\frac{1+2j+m}{2}}\right\}_{m=0}^{\infty},$$

we can rewrite (3.9) as

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=1}^{N_G}\int_0^x \frac{k_{i,j}^p x^i t^j}{\sqrt{x-t}} u_N^p(t)dt \\
&\quad = \underline{u}\mathcal{L}\sum_{p=1}^{N_G}\mathcal{Q}^{p-1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k_{i,j}^p\left\{\frac{\sqrt{\pi}\,\Gamma(1+j+\frac{m}{2})}{\Gamma(\frac{3+m}{2}+j)}\,x^{\frac{1+2(i+j)+m}{2}}\right\}_{m=0}^{\infty}\right) \\
(3.10)\qquad &\quad = \underline{u}\mathcal{L}\sum_{p=1}^{N_G}\mathcal{Q}^{p-1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k_{i,j}^p\left\{\mathcal{A}_j^m\,x^{\frac{1+2(i+j)+m}{2}}\right\}_{m=0}^{\infty}\right) \\
&\quad = \underline{u}\mathcal{L}\left(\sum_{p=1}^{N_G}\mathcal{Q}^{p-1}\mathcal{A}_p\right)\underline{X} \\
&\quad = \underline{u}\mathcal{L}\mathcal{Q}_p\underline{X},
\end{aligned}
$$

where

$$
\mathcal{A}_j^m = \frac{\sqrt{\pi}\,\Gamma(1+j+\frac{m}{2})}{\Gamma(\frac{3+m}{2}+j)}, \qquad \mathcal{Q}_p = \sum_{p=1}^{N_G}\mathcal{Q}^{p-1}\mathcal{A}_p,
$$

and $\mathcal{A}_p$ is an infinite upper-triangular matrix with nonzero entries

$$
(3.11)\qquad (\mathcal{A}_p)_{s,2l+s+1} := \sum_{j=0}^{l} k_{j,l-j}^p\mathcal{A}_{l-j}^s, \qquad s,l=0,1,2,\dots\,.
$$

It can easily be seen that by proceeding in the same manner as in (3.9)–(3.11), we find that

$$
(3.12)\qquad \int_0^x \frac{K(x,t)s_0(t)}{\sqrt{x-t}}dt = \underline{s_0}\int_0^x \frac{K(x,t)}{\sqrt{x-t}}\underline{X_t}dt = \underline{s_0}\mathcal{A}\underline{X},
$$

where the infinite upper-triangular matrix $\mathcal{A}$ is obtained from (3.11) with $k_{j,l-j}$ in place of $k_{j,l-j}^p$.

Now, to get the algebraic form of the Müntz-Galerkin discretization of (1.1), it is sufficient that we insert the relations (3.2), (3.10), and (3.12) into (3.6). Thus we have

$$
(3.13)\qquad \underline{u}\mathcal{L}\left(\mathrm{id}-\mathcal{Q}_{\mathrm{p}}\right)\underline{X} = \left(\underline{f}+\underline{s_0}\mathcal{A}\right)\underline{X},
$$

where id is the infinite-dimensional identity matrix. Using $\underline{X}=\mathcal{L}^{-1}\underline{\mathcal{L}}$, we can rewrite (3.13) as

$$
(3.14)\qquad \underline{u}\mathcal{L}\left(\mathrm{id}-\mathcal{Q}_{\mathrm{p}}\right)\mathcal{L}^{-1}\underline{\mathcal{L}} = \left(\underline{f}+\underline{s_0}\mathcal{A}\right)\mathcal{L}^{-1}\underline{\mathcal{L}}.
$$

Because of the orthogonality of $\{\mathcal{L}_k(x)\}_{k=0}^{\infty}$, projecting (3.14) onto $\{\mathcal{L}_k(x)\}_{k=0}^{N}$ yields

$$
(3.15)\qquad \underline{u}^N\mathcal{L}^N\left(\mathrm{id}^N-\mathcal{Q}_p^N\right)\left(\mathcal{L}^N\right)^{-1} = \left(\underline{f}^N+\underline{s_0}^N\mathcal{A}^N\right)\left(\mathcal{L}^N\right)^{-1},
$$

where $\underline{u}^N=[u_0,u_1,\dots,u_N]$, $\underline{f}^N=[f_0,f_1,\dots,f_N]$, $\underline{s_0}^N=[s_0^0,s_0^1,\dots,s_0^N]$, and $\mathcal{L}^N$, id^N, $\mathcal{Q}_p^N$, $\left(\mathcal{L}^N\right)^{-1}$, $\mathcal{A}^N$ are the principal submatrices of order $N+1$ of the matrices $\mathcal{L}$, id, $\mathcal{Q}_p$, $\mathcal{L}^{-1}$, and $\mathcal{A}$, respectively. Since the product of upper-triangular matrices gives an upper-triangular matrix, the matrix $\mathcal{Q}_p$ defined in (3.11) has an upper-triangular structure. Consequently, the nonlinear algebraic system (3.15) has an upper-triangular structure, whose solution yields the unknown components of the vector $\underline{u}^N$. In the next section we present more details regarding the complexity analysis and unique solvability of this system.

REMARK 3.2. Clearly, for $G(t, u(t)) = u^p(t)$, the system (3.15) can be represented as

$$\underline{u}^N \mathcal{L}^N \left(\mathrm{id}^N - \left(\mathcal{Q}^{p-1}\right)^N \mathcal{A}^N\right) \left(\mathcal{L}^N\right)^{-1} = \underline{f}^N \left(\mathcal{L}^N\right)^{-1}, \quad p \neq 1, \tag{3.16}$$

$$\underline{u}^N \mathcal{L}^N \left(\mathrm{id}^N - \mathcal{A}^N\right) \left(\mathcal{L}^N\right)^{-1} = \underline{f}^N \left(\mathcal{L}^N\right)^{-1}, \quad p = 1 \text{ (linear case)}, \tag{3.17}$$

where $\left(\mathcal{Q}^{p-1}\right)^N$ is the principal submatrix of order $N+1$ from $\mathcal{Q}^{p-1}$.

3.2. Numerical solvability and complexity analysis. The main object of this section is to provide an existence and uniqueness theorem for the nonlinear algebraic system (3.16) and its complexity analysis. In other words, we explain how we can compute the unknown vector $\underline{u}$ by solving (3.16) by a well-posed technique. Furthermore, we extend the following analysis also to the system (3.15).

Multiplying both sides of (3.16) by $\mathcal{L}^N$ yields

$$\underline{u}^N \mathcal{L}^N \left(\mathrm{id}^N - \left(\mathcal{Q}^{p-1}\right)^N \mathcal{A}^N\right) = \underline{f}^N. \tag{3.18}$$

By defining

$$\underline{\underline{u}} = \underline{u}^N \mathcal{L}^N = [\underline{\underline{u_0}}, \underline{\underline{u_1}}, \ldots, \underline{\underline{u_N}}], \tag{3.19}$$

we can rewrite (3.18) as

$$\underline{\underline{u}} \left(\mathrm{id}^N - \left(\mathcal{Q}^{p-1}\right)^N \mathcal{A}^N\right) = \underline{f}^N.$$

From the structure of the matrix $\mathcal{Q}$ in Lemma 3.1, we have

$$\mathcal{Q} := \begin{bmatrix} \underline{u}\mathcal{L}_0 & \underline{u}\mathcal{L}_1 & \underline{u}\mathcal{L}_2 & \cdots \\ 0 & \underline{u}\mathcal{L}_0 & \underline{u}\mathcal{L}_1 & \cdots \\ 0 & 0 & \underline{u}\mathcal{L}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \underline{\underline{u_0}} & \underline{\underline{u_1}} & \underline{\underline{u_2}} & \cdots \\ 0 & \underline{\underline{u_0}} & \underline{\underline{u_1}} & \cdots \\ 0 & 0 & \underline{\underline{u_0}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

From [14], we observe that $\left(\mathcal{Q}^{p-1}\right)^N$ has the following upper-triangular Toeplitz structure:

$$\left(\mathcal{Q}^{p-1}\right)^N$$

$$= \begin{bmatrix} \left(\underline{\underline{u_0}}\right)^{p-1} & (p-1)\left(\underline{\underline{u_0}}\right)^{p-2}\underline{\underline{u_1}} & \frac{1}{2}(p-1)\left(\underline{\underline{u_0}}\right)^{p-3}\left((p-2)\left(\underline{\underline{u_1}}\right)^2 + 2\underline{\underline{u_0}}\,\underline{\underline{u_2}}\right) & \cdots \\ 0 & \left(\underline{\underline{u_0}}\right)^{p-1} & (p-1)\left(\underline{\underline{u_0}}\right)^{p-2}\underline{\underline{u_1}} & \cdots \\ 0 & 0 & \left(\underline{\underline{u_0}}\right)^{p-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \left(\underline{\underline{u_0}}\right)^{p-1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}}) & \mathcal{Q}^{p-1}_{0,1}(\underline{\underline{u_0}},\underline{\underline{u_1}}) & \mathcal{Q}^{p-1}_{0,2}(\underline{\underline{u_0}},\underline{\underline{u_1}},\underline{\underline{u_2}}) & \cdots & \mathcal{Q}^{p-1}_{0,N}(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_N}}) \\ 0 & \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}}) & \mathcal{Q}^{p-1}_{0,1}(\underline{\underline{u_0}},\underline{\underline{u_1}}) & \cdots & \mathcal{Q}^{p-1}_{0,N-1}(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{N-1}}}) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}}) \end{bmatrix},$$

where $\mathcal{Q}^{p-1}_{i,j}(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_j}})$, $i,j=0,1,\ldots,N$, are nonlinear functions of the elements $\underline{\underline{u_0}}$, $\underline{\underline{u_1}},\ldots,\underline{\underline{u_j}}$. Thus, from the structure of the upper-triangular matrix $\mathcal{A}$ defined by (3.11), we obtain that

$$\left(\mathcal{Q}^{p-1}\right)^N \mathcal{A}^N = \left(\mathcal{Q}^{p-1}\right)^N \begin{bmatrix} 0 & \mathcal{A}_{0,1} & 0 & \mathcal{A}_{0,3} & \cdots & \cdots \\ 0 & 0 & \mathcal{A}_{1,2} & 0 & \mathcal{A}_{1,4} & \cdots \\ 0 & 0 & 0 & \mathcal{A}_{2,3} & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & \mathcal{A}_{N-1,N} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.20}$$

$$= \begin{bmatrix} 0 & \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}})\mathcal{A}_{0,1} & \mathcal{Q}^{p-1}_{0,1}(\underline{\underline{u_0}},\underline{\underline{u_1}})\mathcal{A}_{1,2} & \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}})\mathcal{A}_{0,3}+\mathcal{Q}^{p-1}_{0,2}(\underline{\underline{u_0}},\underline{\underline{u_1}},\underline{\underline{u_2}})\mathcal{A}_{2,3} & \cdots \\ 0 & 0 & \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}})\mathcal{A}_{1,2} & \mathcal{Q}^{p-1}_{0,1}(\underline{\underline{u_0}},\underline{\underline{u_1}})\mathcal{A}_{2,3} & \cdots \\ 0 & 0 & 0 & \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}})\mathcal{A}_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \mathcal{Q}^{p-1}_{0,0}(\underline{\underline{u_0}})\mathcal{A}_{N-1,N} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now with the above relation, we can write

$$\underline{\underline{u}}\mathcal{Q}^{p-1}\mathcal{A} = [0, \mathcal{F}_1(\underline{\underline{u_0}}), \mathcal{F}_2(\underline{\underline{u_0}},\underline{\underline{u_1}}),\ldots,\mathcal{F}_N(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{N-1}}})], \tag{3.21}$$

where $\mathcal{F}_i(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{i-1}}})$, for $i=1,2,\ldots,N$, are nonlinear functions of the elements $\underline{\underline{u_0}}$, $\underline{\underline{u_1}},\ldots,\underline{\underline{u_{i-1}}}$. Substituting (3.21) into (3.16) yields

$$\begin{bmatrix} \underline{\underline{u_0}} \\ \underline{\underline{u_1}} \\ \vdots \\ \underline{\underline{u_N}} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{F}_1(\underline{\underline{u_0}}) \\ \vdots \\ \mathcal{F}_N(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{N-1}}}) \end{bmatrix},$$

and therefore

$$\begin{aligned} \underline{\underline{u_0}} &= f_0, \\ \underline{\underline{u_1}} &= f_1 + \mathcal{F}_1(\underline{\underline{u_0}}), \\ &\vdots \\ \underline{\underline{u_N}} &= f_N + \mathcal{F}_N(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{N-1}}}), \end{aligned} \tag{3.22}$$

gives the unknown components of the vector $\underline{\underline{u}}$. Finally, (3.19) yields the unknown vector $\underline{u}^N$ in the Galerkin representation (3.2). The analysis above is summarized in the next theorem.

THEOREM 3.3. *The nonlinear system* (3.18) *has a unique solution* $\underline{u}^N$ *which is obtained by* (3.22) *and* (3.19).

Obviously, this analysis can be extended similarly to the system (3.15) yielding the following result:

THEOREM 3.4. *The nonlinear system* (3.15) *has a unique solution* $\underline{u}^N$.

Proof. By proceeding in a similar manner as in Section 3.2, we can rewrite (3.15) as

$$\underline{\underline{u}}\left(\mathrm{id}^N - \mathcal{Q}_p^N\right) = \underline{\underline{f}}^N,$$

where $\underline{\underline{f}}^N = \underline{f}^N + \underline{s_0}^N \mathcal{A}^N$. From (3.11), (3.20), the upper-triangular structure of the matrix $\left(\mathcal{Q}^{p-1}\right)^N \mathcal{A}_p^N$, and since a summation of upper-triangular matrices yields an upper-triangular matrix, we similarly obtain that

$$\underline{\underline{u}}\mathcal{Q}_p^N = [0, \tilde{\mathcal{F}}_1(\underline{\underline{u_0}}), \tilde{\mathcal{F}}_2(\underline{\underline{u_0}}, \underline{\underline{u_1}}), \ldots, \tilde{\mathcal{F}}_N(\underline{\underline{u_0}}, \underline{\underline{u_1}}, \ldots, \underline{\underline{u_N}})], \tag{3.23}$$

where $\tilde{\mathcal{F}}_i(\underline{\underline{u_0}}, \underline{\underline{u_1}}, \ldots, \underline{\underline{u_{i-1}}})$, for $i = 1, 2, \ldots, N$, are nonlinear functions of the elements $\underline{\underline{u_0}}$, $\underline{\underline{u_1}}, \ldots, \underline{\underline{u_{i-1}}}$. By adopting the same strategy as in (3.21)–(3.22), we can conclude the result. $\square$

4. Numerical results. In this section, we apply the proposed method of the previous section to approximate the solution of (1.1) and provide some examples. The computations were performed using Mathematica with default precision. In the tables below, the labels "Numerical errors" and "Order" always refer to the L^2-norm of the error function and the order of convergence, respectively. We also compare our results with those obtained by several existing methods for this class of problems and thereby confirm the superiority and effectiveness of our scheme. We also illustrate the well-posedness of the problem and thus numerically verify the theoretical result of Theorem 1.2. In all examples we have $N_G = N_f = N$.

EXAMPLE 4.1. Consider the following integral equations:

$$u(x) = \sqrt{x} - \frac{4\sqrt{x^3}}{3} - \frac{12\sqrt{x^5}}{5} + \int_0^x \frac{1+x+t}{\sqrt{x-t}} u^2(t)dt, \tag{4.1}$$

$$u(x) = \frac{\pi x}{2} + \sqrt{x} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt, \tag{4.2}$$

$$u(x) = \sqrt{x} + \frac{3\pi x^2}{8} - \int_0^x \frac{u^3(t)}{\sqrt{x-t}} dt, \tag{4.3}$$

which all have the same exact solution $u(x) = \sqrt{x}$.

Applying the method described in the previous section with $N = 1$, the Müntz-Galerkin approximation of these problems is given by

$$u_1(x) = \underline{u}^1 \mathcal{L}^1 \underline{X}^1,$$

where

$$\underline{u}^1 = [u_0, u_1], \qquad \mathcal{L}^1 = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \qquad \underline{X}^1 = [1, \sqrt{x}]^T.$$

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TABLE 4.1
The numerical errors of the Legendre-Tau scheme proposed in [27] for (4.1).

N	Errors obtained by the method in [27]
2	3.92×10^{-1}
4	1.53×10^{-2}
6	1.15×10^{-3}
8	2.76×10^{-2}
10	1.58×10^{-4}

Following the technique described in Section 3.2, we find the unknown vector $\underline{u}^1$ by

$$\underline{u}^1 \mathcal{L}^1 = \underline{\underline{u}}, \qquad \underline{\underline{u}} = [\underline{\underline{u_0}}, \underline{\underline{u_1}}], \tag{4.4}$$

and solving the nonlinear system

$$\underline{\underline{u}} \left(\mathrm{id}^1 - \mathcal{Q}^1 \mathcal{A}^1 \right) = \underline{f}^1, \tag{4.5}$$

where

$$\mathrm{id}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathcal{Q}^1 = \begin{bmatrix} \underline{\underline{u_0}} & \underline{\underline{u_1}} \\ 0 & \underline{\underline{u_0}} \end{bmatrix}, \qquad \underline{f}^1 = [0, 1],$$

and

$$\mathcal{A}^1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad \text{for (4.1), and} \qquad \mathcal{A}^1 = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \quad \text{for (4.2) and (4.3).}$$

For all equations in this example, the system (4.5) can be represented as

$$\begin{bmatrix} \underline{\underline{u_0}} \\ \underline{\underline{u_1}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ (-1)^p 2 \underline{\underline{u_0}}^p \end{bmatrix},$$

where

$$p = \begin{cases} 2 & \text{for equation (4.1),} \\ 1 & \text{for equation (4.2),} \\ 3 & \text{for equation (4.3),} \end{cases}$$

which has the exact solution $\underline{\underline{u_0}} = 0$, $\underline{\underline{u_1}} = 1$. Finally, from (4.4) we obtain $u_0 = \frac{2}{3}$, $u_1 = \frac{1}{3}$, and

$$u_1(x) = \underline{u}^1 \mathcal{L}^1 \underline{X}^1 = \frac{2}{3} + \frac{1}{3} \left(-2 + 3\sqrt{x} \right) = \sqrt{x},$$

which is the exact solution of the considered problems. As a consequence, only one term is required to find the solution $u(x) = \sqrt{x}$ exactly. For a comparison, we solve (4.1) by implementing the Legendre-Tau method proposed in [27] and report the results in Table 4.1. Indeed, Table 4.1, shows that by employing Müntz-Legendre polynomials as basis functions in the Galerkin solution of (4.1), we get very accurate and reliable results compared with that of the Legendre polynomials.

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TABLE 4.2
The numerical errors of the proposed scheme in [1] for (4.2).

N	Errors obtained by the method in [1]
2	7.56×10^{-2}
4	4.08×10^{-2}
8	3.07×10^{-2}

TABLE 4.3
The errors of the proposed scheme in [2] with $h = 0.2$ at various values of x for (4.3).

x	Errors obtained by the method in [2]
0.2	9.4×10^{-3}
0.4	1.1×10^{-2}
0.6	1.1×10^{-2}
0.8	1.1×10^{-2}

Furthermore, equation (4.2) has been considered in [1] and solved numerically by employing an appropriately truncated expansion of $\frac{1}{\sqrt{x-t}}$ in terms of orthogonal polynomials. We present the numerical results in Table 4.2, which clearly confirm the superiority and validity of our proposed scheme.

We also provide a comparison of the errors for (4.3). This equation was considered in [2] and solved by applying the rational interpolation method. The results are given in Table 4.3, which show that our strategy with a small degree of approximation performs far better than the rational interpolation method from [2].

EXAMPLE 4.2. Consider the integral equation

$$u(x) = f(x) + x \int_0^x \frac{e^{3t}}{\sqrt{x-t}} u^3(t)dt,$$

where $f(x) = e^{\sqrt{x}-x} - \pi x\sqrt{x}\,(\text{BesselI}[1, 3\sqrt{x}] + \text{StruveL}[-1, 3\sqrt{x}])$ and $u(x) = e^{\sqrt{x}-x}$ is the exact solution. Here, BesselI$[n, z]$ and StruveL$[n, z]$ are the modified Bessel function of the first kind and the modified Struve function, respectively.

We solve this problem using the proposed technique and report the numerical results in Table 4.4 and Figure 4.1. In Table 4.4, the listed errors indicate that our scheme produces powerful approximate solutions for suitable values of N with a high order of accuracy regardless of the singular behavior of the exact solution. In addition, Figure 4.1 shows that the Müntz-Galerkin errors decay in an exponential-like manner for $N \geq 10$ because in this semi-log representation, the logarithm of the error variations is almost a linear function of the degree of approximation N.

EXAMPLE 4.3. Consider the integral equation

$$u(x) = \sin\sqrt{x} - \frac{\pi x}{2} + \int_0^x \frac{\arcsin u(t)}{\sqrt{x-t}} dt,$$

which has the exact solution $u(x) = \sin\sqrt{x}$.

We apply the proposed Müntz-Galerkin method to this example and report the results in Table 4.5 and Figure 4.2. As expected, the errors are decreasing with a high rate of convergence when the approximation degree N is increased. This property confirms the reliability and

TABLE 4.4
The numerical errors for Example 4.2.

N	Numerical errors	Order
5	1.17×10^{-2}	2.57
10	6.92×10^{-4}	4.01
15	3.84×10^{-6}	12.81
20	2.16×10^{-9}	26.02
25	2.35×10^{-11}	20.25
30	1.47×10^{-14}	40.47
35	1.25×10^{-17}	45.84

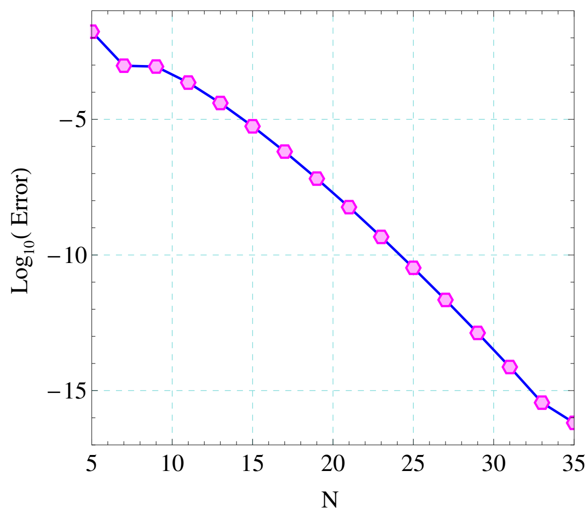

FIG. 4.1. *The log of errors versus N, when the Müntz-Galerkin method is used to obtain an approximate solution for Example 4.2.*

effectiveness of applying the Müntz-Galerkin method to approximate the Abel-Hammerstein integral equations of the second kind.

EXAMPLE 4.4. Consider the integral equation

$$u(x) = 2\sqrt{x} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt,$$

which has the exact solution $u(x) = 1 - e^{\pi x}\mathrm{Erfc}(\sqrt{\pi x})$, where $\mathrm{Erfc}(z)$ is the complimentary error function defined as

$$\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt.$$

The problem in this example occurs in a mathematical model in astrophysics [19]. Since we have a linear Abel integral equation, i.e., $G(t, u(t)) = u(t)$, we can find the Müntz-Galerkin approximations of the problem by solving the linear system given by (3.17). The

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TABLE 4.5
The numerical errors for Example 4.3.

N	Numerical errors	Order
3	2.33×10^{-3}	2.89
5	6.93×10^{-5}	7.58
7	8.64×10^{-7}	13.03
9	7.19×10^{-9}	19.06
11	4.27×10^{-11}	25.54
13	1.91×10^{-13}	32.41
15	6.64×10^{-16}	39.55

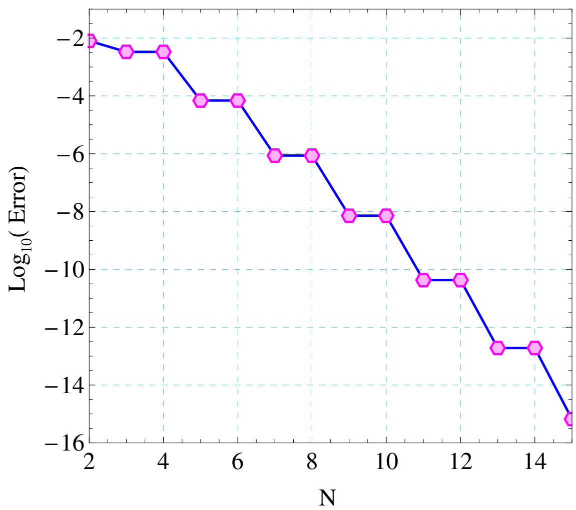

FIG. 4.2. *The log of the errors versus N, when the Müntz-Galerkin method is used to obtain an approximate solution for Example 4.3.*

results are presented in Table 4.6 and Figure 4.3, which approve that the proposed approach is also a well-posed and highly accurate approximate method to obtain the numerical solution of linear Abel integral equations. Again, from Figure 4.3, it is clear that the convergence rate of the Müntz-Galerkin method is exponentially as the semi-log representation of the L^2-errors is linear in N.

For comparison, we also examined the use of the classical Chebyshev polynomials as basis functions in the collocation solution of this problem [10, 30]. We present the numerical errors for various values of N in Table 4.7 to demonstrate the convergence behavior. We observe that the rate of convergence for the Müntz-Galerkin case is much higher in comparison to the classical Chebyshev collocation method. Also, the Chebyshev collocation method gives an ill-conditioned approximate scheme because in this case, for large values of N (e.g., $N \geq 50$), the numerical errors destroy the convergence.

In Theorem 1.2, the behavior of the solution under perturbations of the given data is discussed and the well-posedness of the Abel-Hammerstein integral equations of the second kind is proved in the sense that the solutions depend continuously on the data. In a typical application, an Abel-Hammerstein integral equations of the form (1.3) arises, where the data, namely the right-hand side $f(x)$ and possibly also the kernel function $K(x,t)$, depend on

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TABLE 4.6
The numerical errors for Example 4.4.

N	Numerical errors	Order
10	3.27×10^{-1}	1.69
15	3.59×10^{-2}	5.40
20	1.96×10^{-3}	10.12
25	6.09×10^{-5}	15.55
30	1.21×10^{-6}	21.49
35	1.62×10^{-8}	27.98
40	1.58×10^{-10}	34.74
45	1.13×10^{-12}	41.87
50	6.24×10^{-15}	49.34

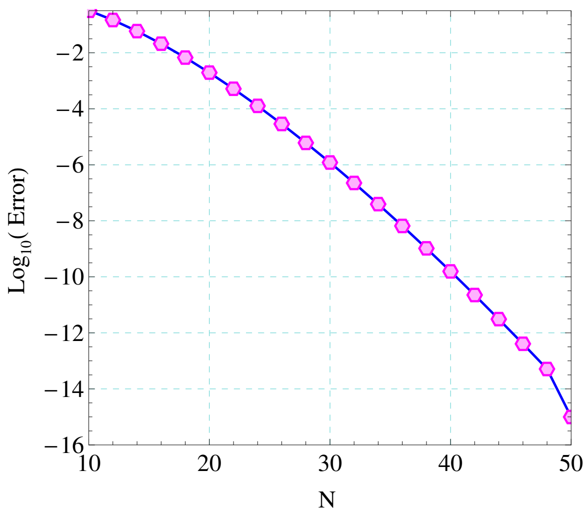

FIG. 4.3. *The log of errors versus N, when the Müntz-Galerkin method is used to obtain an approximate solution for Example 4.4.*

material constants that are only approximately known, usually with a moderate accuracy. For example, in many applications the function $f(x)$ in (1.1) is not given exactly but only noisy measurements of it at certain points. In the next example, we consider a test problem in which $f(x)$ is only approximately available on a set of points, and we illustrate how to find the coefficients f_i in (3.3) then and how the proposed Müntz-Galerkin algorithm works.

EXAMPLE 4.5. Consider the integral equation

$$u(x) = f(x) + \int_0^x \frac{u^2(t)}{\sqrt{x-t}} dt, \tag{4.6}$$

where $f(x)$ is known only at the following points:

$$\begin{array}{c|cccccc} x_i & 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ \hline f_i & 0 & 0.327956 & 0.295145 & 0.154919 & -0.059628 & -0.333333 \end{array}. \tag{4.7}$$

TABLE 4.7
The errors of the Chebyshev collocation method for Example 4.4.

N	Numerical errors
5	1.1×10^{-3}
10	1.17×10^{-4}
15	2.87×10^{-5}
20	1.03×10^{-5}
25	4.57×10^{-6}
30	2.34×10^{-6}
35	1.33×10^{-6}
40	8.09×10^{-7}
45	5.24×10^{-7}
50	2.55×10^{-5}

To make the proposed Müntz-Galerkin approach applicable for this problem, we must determine a suitable approximate function $\tilde{f}(x)$ for $f(x)$ using the data given by (4.7). To this end, we can use various numerical techniques such as least-squares- and interpolation methods. Here we choose an interpolation technique and find $\tilde{f}(x)$ by interpolating the function values gathered from (4.7). From the classical Müntz theorem [32], we have that polynomials based on the standard Müntz basis function $\underline{X}$ are dense in $C(\Lambda)$. Then the continuity of $f(x)$ enables us to represent the interpolation function $\tilde{f}(x)$ by a linear combination of the standard Müntz basis functions $\underline{X}$ as $\tilde{f}(x) = \sum_{i=0}^{5} \tilde{f}_i x^{\xi_i}$, and to find the unknown parameters $\tilde{f}_i$ so that the given data listed in (4.7) satisfy the following linear algebraic system of equations:

$$\sum_{i=0}^{5} \tilde{f}_i \, x_j^{\xi_i} = f_j, \qquad j = 0, 1, \dots, 5. \tag{4.8}$$

Note that this system can be ill-posed when the degree of approximation N is very large, but on a small scale like here ($N = 5$), its condition number can be moderate. Introducing and developing a well-posed interpolation technique for Müntz bases is new and can be considered as future work.

Solving the linear system (4.8) yields

$$\tilde{f}(x) = -4.64 \times 10^{-12} + 1.0x^{\frac{1}{2}} + 1.24 \times 10^{-4} x - 1.33x^{\frac{3}{2}} + 3.98 \times 10^{-4} x^2 - 1.6 \times 10^{-4} x^{\frac{5}{2}}.$$

It can be easily checked that the values listed in (4.7) are the approximate values of the function $f(x) = \sqrt{x}(1 - \frac{4}{3}x)$ at the points x_i, $i = 0, 1, \dots, 5$. Trivially in this case, we have $u(x) = \sqrt{x}$ as the exact solution. But, as we only have the perturbed function $\tilde{f}(x)$ available instead of $f(x)$ on the right-hand side of (4.6), we have a perturbed problem

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x \frac{\tilde{u}(t)}{\sqrt{x-t}} dt, \tag{4.9}$$

with the perturbed exact solution $\tilde{u}(x)$ instead of $u(x)$. The main concern of this example is to solve (4.9) using the Müntz-Galerkin strategy and to monitor the differences of $|f(x) - \tilde{f}(x)|$ and $|u(x) - \tilde{u}_N(x)|$ to confirm the well-posedness of the problem. The numerical results are displayed in Table 4.8. It can be seen that the reported results in Table 4.8 confirm the

TABLE 4.8
The illustration of well-posedness for Example 4.5 with $N = 5$ at various values of x in Λ.

x	$\|u(x) - \tilde{u}_N(x)\|$	$\|f(x) - \tilde{f}(x)\|$
0.1	1.81×10^{-8}	4.59×10^{-7}
0.2	2.01×10^{-8}	6.37×10^{-7}
0.3	4.43×10^{-8}	8.76×10^{-7}
0.4	6.84×10^{-8}	9.15×10^{-7}
0.5	8.17×10^{-8}	7.14×10^{-7}
0.6	1.34×10^{-7}	3.34×10^{-7}
0.7	2.05×10^{-7}	1.12×10^{-7}
0.8	2.94×10^{-7}	4.79×10^{-7}
0.9	4.05×10^{-7}	6.1×10^{-7}
1	5.38×10^{-7}	3.33×10^{-7}

theoretical result of Theorem 1.2 as the difference $|u(x) - \tilde{u}_N(x)|$ is in good agreement and of the same order as $|f(x) - \tilde{f}(x)|$ on Λ:

$$\|u(x) - \tilde{u}_N(x)\|_\infty = 6.1 \times 10^{-7} \leq \|f(x) - \tilde{f}(x)\|_\infty = 9.15 \times 10^{-7} \qquad \text{for } N = 5.$$

5. Conclusion. In this paper, using a Müntz-Galerkin method, we solved Abel-Hammerstein integral equations of the second kind. This technique has two main advantages. First, the approximate solution can be directly obtained by solving an upper-triangular nonlinear algebraic system and second, the Müntz-Galerkin representation of the problem produces an approximate solution with a high order of convergence regardless of a singularity of the exact solution. These advantages are confirmed by some illustrative examples.

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