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MAPS FOR GLOBAL SEPARATION OF ROOTS*

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Abstract. The global separation of the fixed-points of a real-valued function g on an interval $D = [a, b]$ is considered by introducing the notions of quasi-step maps associated to g and quasi-step maps educated by two predicates. The process of 'education' by the predicates is an a priori global technique which does not require initial guesses. The main properties of these maps are studied and the theoretical results are illustrated by some examples where appropriate quasi-step maps for Newton and Halley methods are applied.

Key words. step function, fixed-point, iteration map, Newton map, Halley map, sieve of Eratosthenes

AMS subject classifications. 65H05, 65H20, 65S05

1. Introduction. Separation of real roots is a classical subject dating back to the seminal work of Lagrange [11] on polynomial equations. In this paper we aim to offer a different computational perspective to the global separation of roots of a general nonlinear equation by constructing certain iteration maps which will be called *quasi-step* maps.

We consider the problem of finding the roots of a given real-valued equation $f(x) = 0$ on a closed interval $D = [a, b]$, which we write as a fixed-point equation $x = g(x)$. Let $\mathcal{Z} = \{z_1, z_2, \dots, z_n\}$ be the non-empty set of (distinct) fixed-points of the map g. If the set $\mathcal{Z}$ was known, then a good model of a map separating the fixed-points of g in the interval D is the following *step map* Ψ:

$$\Psi(x) = \sum_{i=1}^{n} z_i \mathcal{X}_{I_i}(x), \quad x \in D, \tag{1.1}$$

where $\mathcal{X}_{I_i}$ is the *characteristic function* of the subinterval $I_i \subset D$ (i.e., $\mathcal{X}_{I_i}(x) = 1$ for $x \in I_i$ and $\mathcal{X}_{I_i}(x) = 0$ otherwise) and the intervals I_i are pairwise disjoint.

In general, a step map of the form (1.1) is not directly constructable from g since the set of fixed-points $\mathcal{Z}$ is unknown. It is then natural to look for a map $\widetilde{\Psi}$ of the form

$$\widetilde{\Psi}(x) = \sum_{i=1}^{s} \widetilde{g}(x) \mathcal{X}_{J_i}(x), \quad x \in D, \tag{1.2}$$

where:

(a) The union of the intervals J_i is contained in D and each J_i contains at least one fixed-point of g.
(b) The function $\widetilde{g}$ is *continuous* on each subinterval J_i.
(c) The function $\widetilde{g}$ preserves the fixed-points of g on each J_i.

Like the map (1.1), the map (1.2) may be seen as a tool for a separation of points in $\mathcal{Z}$. A map $\widetilde{\Psi}$ of the type (1.2) will be called a *quasi-step map* (see Definition 2.1).

The construction of a map $\widetilde{g}$ as in (1.2) will be done by considering one or more predicates which are based upon the map g and the domain D. The predicates to be used hereafter are denoted by $\mathbb{P}_0$ and $\mathbb{P}_1$. Given a constant $d > 0$, these predicates are:

$$\mathbb{P}_0 : \; x \in D, \;\; y = g(x) \in D \text{ and } |y - x| < d, \tag{1.3}$$

$$\mathbb{P}_1 : \; x \in D, y = g(x) \in D, w = g(y) \in D \text{ and } |w - y| \leq |y - x|. \tag{1.4}$$

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For any point $x \in D$, the predicate $\mathbb{P}_0$ tests the image $y = g(x)$ while $\mathbb{P}_1$ tests two applications of g.

A collection of subintervals $J_i \subset D$ is induced by a sort of *divide and conquer* effect from the action of one or both predicates. Moreover, when we use the predicate $\mathbb{P}_0$ (resp. $\mathbb{P}_1$), the value assigned to $\widetilde{g}(x)$ in (1.2) will be $y = g(x)$ (resp. $w = g(g(x))$) for all points $x \in D$ for which $\mathbb{P}_0$ (resp. $\mathbb{P}_1$) is true. For all the other points $x \in D$ for which the predicate under consideration is false, the value zero will be assigned to $\widetilde{g}(x)$. A map $\widetilde{\Psi}$ as in (1.2) constructed from one or more predicates will be called an 'educated' map in the sense that the construction of this map is based on the action of the predicate(s).

As explained in detail in Section 2, under mild assumptions on g, the above predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ lead to quasi-step maps of type (1.2) separating fixed-points of the initial map g. As we will see, it is convenient to choose an initial map g satisfying the property of attracting points in D which are sufficiently close to the fixed-points. Fortunately, such a choice of maps g does not present any difficulty due to the plethora of iteration maps in the literature enjoying the referred attracting property. Among them we will consider the celebrated Newton-Raphson and Halley maps since, as it is well known, under mild assumptions, both maps have at least linear local convergence and so guarantee the referred attracting property ([5, 6, 9, 14, 17, 22, 23, 27]). The proofs of how the predicates will lead to quasi-step maps separating fixed-points of g, given in Section 2, use mainly the fixed-point theorem for closed and bounded real intervals or the Banach contraction principle (see, for instance, [17, 18, 28]) and properties of nonexpansive iterated contractions [16, 20].

The paper is divided into two parts. In the first part (Section 2) the main theoretical results are established and the second part deals with worked examples (Section 3). In Section 2 we show under which conditions on g or on its fixed-points, the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ will enable the construction of quasi-step maps providing a global separation of the fixed-points of g (Propositions 2.4 and 2.6). A brief reference on how the composition of quasi-step maps may be implemented to achieve accurate approximations of fixed-points of g is also made; see Section 2.1.1.

Section 3 is devoted to examples illustrating the separation of fixed points by constructing quasi-step maps from the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$. We begin with a family of trigonometrical functions f_k presented by Charles Pruitt in [21]. Due to the fact that these functions only admit integer zeros (cf. Proposition 3.3), we are able to numerically construct (Example 3.4) a step map which provides not only all the zeros of f_3 but it also enables to distinguish composite numbers from prime numbers in the interval of definition of the map. Pruitt functions are also used (Examples 3.6 and 3.7) to illustrate the main features of several quasi-step maps derived from Newton and Halley maps educated by the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$.

In Example 3.8, a strongly oscillating transcendental function f is considered. Certain discretized versions of composed Halley educated maps are applied in order to globally separate a large number of zeros of f producing at the same time accurate approximations of them.

2. Separation of fixed-points. In this section we show how the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ given by (1.3) and (1.4) enable the construction of quasi-step maps of type (1.2) from a given function g. Given an interval $D = [a, b]$ and a constant $d > 0$, we note that if x is a fixed-point of g, then the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ hold true for x.

In what follows we assume that a real-valued map g is given, D is the closed interval $D = [a, b] \subset \mathbb{R}$, and $\mathcal{X}_B$ denotes the characteristic function of the set B. We denote by $\mathcal{Z} \subset D$ the (non-empty) finite set of the fixed-points of g, say $z_1, z_2, \ldots, z_n$. Let us define a *quasi-step map*.

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DEFINITION 2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function and $D = [a, b] \subset \mathbb{R}$. A quasi-step map associate to g is a function $\widetilde{\Psi}$ of the form*

$$\widetilde{\Psi}(x) = \sum_{k=1}^{s} \widetilde{g}(x)\, \mathcal{X}_{J_k}(x), \quad \text{for } x \in D, \tag{2.1}$$

where:

(a) Any subinterval J_k contains at least one fixed-point of g, and the union of these intervals is contained in D (i.e., $J = \bigcup_{k=1}^{s} J_k \subseteq D$).
(b) The function $\widetilde{g}$ is continuous *on each subinterval J_k.*
(c) $\widetilde{g}(z) = z$ for any fixed-point z of g belonging to J_k.

We note that if in the above definition all the subintervals J_k are pairwise disjoint and the number s coincides with the number n of the fixed-points of g, then the quasi-step map $\widetilde{\Psi}$ separates all the fixed-points of g in D.

For practical purposes we choose either one or both predicates $\mathbb{P}_0$, $\mathbb{P}_1$ to construct a quasi-step map as in (2.1). Such a map will be called 'educated' by the predicate(s) in the sense that the map is the result of the action of the predicate(s) on the interval D.

DEFINITION 2.2 (Educated map). *Let g be a real-valued map and consider the interval $D \subseteq \mathbb{R}$, $\mathcal{X}_B$ the characteristic function of a set B, d a positive constant, and $\mathbb{P}_0$, $\mathbb{P}_1$ the predicates in* (1.3) *and* (1.4)*, respectively. Let $\{L_k^i\}$ be the collection of subintervals of D where the predicate $\mathbb{P}_i$ ($i = 0, 1$) holds true.*

(a) The quasi-step map

$$\widetilde{\Psi}_i(x) = \sum_{k} \widetilde{g}_i(x)\, \mathcal{X}_{L_k^i}(x), \quad x \in D,$$

is said to be educated by the predicate $\mathbb{P}_i$, if $\widetilde{g}_i(x) = 0$ for all $x \in D \setminus \bigcup_k L_k^i$ and on each L_k^i we have $\widetilde{g}_i = g$ for $i = 0$ and $\widetilde{g}_i = g \circ g$ for $i = 1$.

(b) The quasi-step map

$$\widetilde{\Psi}(x) = \sum_{k} \widetilde{g}(x)\, \mathcal{X}_{J_k}(x), \quad x \in D,$$

is said to be educated by both predicates $\mathbb{P}_0$ and $\mathbb{P}_1$, if we have $\widetilde{g}_i(x) = 0$ for $x \in D \setminus \cup_k (L_k^0 \cap L_k^1)$ and $\widetilde{g} = g \circ g$ in each $J_k = L_k^0 \cap L_k^1$.

Note that a map educated by the predicate $\mathbb{P}_0$ is a map which is necessarily zero at all $x \in D$ where $(x, g(x))$ does not lie in a band of width $2d$ centered at the line $y = x$. This is the reason why we call d the *vertical displacement* parameter. On the other hand, the predicate $\mathbb{P}_1$ tests $y = g(x) \in D$ and $w = g(y) \in D$ satisfying $|w - y| \leq |y - x|$. Since for $y \neq w$ the quantity $(w - y)/(y - x)$ represents the slope of the line through the points (x, y) and (y, w) we call $\mathbb{P}_1$ the *slope predicate*.

We note that the 'education' of a map is an *a priori* global technique (i.e., no initial guesses are required) which may be seen as a counterpart of the classical *a posteriori* stopping criteria used in root solvers algorithms for the local search of roots; see for instance [15] and references therein.

Although in this work we only adopt the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$, other predicates could be considered in order that the respective educated map will satisfy other criteria such as monotony or alternate local convergence. Also, an 'education' of the map f instead of the map g may be of interest, namely in the light of well-known sufficient conditions for local monotone convergence of Newton and Halley methods; see for instance [4, 13].

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2.1. Quasi-step maps from the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$. In this section we address the question what kind of functions g may be chosen in order to construct educated maps from the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ leading to a global separation of fixed-points of g. In particular, with respect to the predicate $\mathbb{P}_0$, we show that the contractivity of g near each of its fixed-points and a conveniently chosen vertical displacement parameter d provide quasi-step maps isolating fixed-points of g in D. In the case of the predicate $\mathbb{P}_1$, under mild hypotheses on g, this predicate implies that g is a strictly non-expansive iterated contraction near fixed-points, which enables the construction of an educated map by $\mathbb{P}_1$ that isolates fixed-points of g as well.

Recall [16, p. 120] that a map g defined on a domain D is called contractive on a set $D_0 \subset D$ if there is an $\alpha < 1$ such that $|g(x) - g(y)| \leq \alpha\, |x - y|$, for all $x, y \in D_0$.

LEMMA 2.3. *Let d be a positive number and z a fixed-point of g belonging to $D = [a, b]$. If g is contractive in the interval*

$$I_z = [z - d, z + d] \subseteq D,$$

and the predicate $\mathbb{P}_0$ holds true for any point of I_z, then there exists a bounded closed interval

$$J_z = [z - \epsilon, z + \epsilon] \subseteq I_z, \quad \epsilon > 0,$$

such that the map

$$\widetilde{\Psi}_z(x) = \widetilde{g}_z(x)\, \mathcal{X}_{J_z}(x) \quad x \in D,$$

with $\widetilde{g}_z(x) = 0$ if $x \in D \setminus J_z$ and $\widetilde{g}_z = g$ in J_z isolates the (unique) fixed-point z of g in D.

Proof. First note that the hypothesis on the predicate $\mathbb{P}_0$ gives

$$|g(x) - x| < d, \quad x \in I_z. \tag{2.2}$$

The points $(x, g(x))$ of the plane satisfying the above inequality (2.2) belong to a closed planar region delimited by the parallel horizontal lines $y = z + d$, $y = z - d$ and the oblique parallel lines $y = x + d$, $y = x - d$. Denote by $\mathcal{P}$ the parallelogram bounding such a region, and note that the diagonals of $\mathcal{P}$ intersect at the point $A = (z, z)$.

Let $\mathcal{D}$ be the closed disk of radius $r = d/\sqrt{2}$ centered at A. This disk is inscribed in the region delimited by $\mathcal{P}$. As by hypothesis g is contractive in I_z, g is continuous in this interval. So, there exists $\delta > 0$ such that

$$x \in [z - \delta, z + \delta] \Longrightarrow |g(x) - z| < r.$$

Also by the contractivity of g, there exists a number K with $0 \leq K < 1$ (K is a contractivity constant) such that the graph of g lies inside a cone section with vertex at A and whose edges form an angle $|\alpha| = \arctan(K) < \pi/4$ at the vertex A. Therefore, there exists a number $0 < \epsilon < r$ such that the square region $\mathcal{S} = [z - \epsilon] \times [z + \epsilon] \subset \mathbb{R}^2$ is contained in the disk $\mathcal{D}$. Moreover, for the closed interval $J_z = [z - \epsilon, z + \epsilon]$ we have $g(J_z) \subseteq J_z$. As J_z is closed and g is contractive in J_z, it follows from the Banach contraction principle that there is a unique fixed-point of g in J_z, which is obviously the point z. It is now immediate that the map $\widetilde{\Psi}_z$ isolates z in D. $\square$

We now assume that the set $\mathcal{Z}$ of the fixed-points of g is ordered and define the *resolution* η of $\mathcal{Z}$ as the minimum of the distances between any pair of consecutive points of $\mathcal{Z}$.

PROPOSITION 2.4. *Let $\mathcal{Z} = \{z_1, \ldots, z_n\}$ be the ordered set of the fixed-points of g and η its resolution. Consider g and $\mathbb{P}_0$ satisfying the conditions of Lemma 2.3 on each interval $I_{z_i} = [z_i - d, z_i + d]$ with*

$$d < \frac{\eta}{2}\cdot$$

Then, there exists a collection of disjoint subintervals $J_{z_i} \subset I_{z_i}$ with $J = \bigcup_{i=1}^n J_{z_i} \subset D$ and maps $\widetilde{g}_{z_i}$ defined as $\widetilde{g}_{z_i}(x) = 0$ for $x \in D \setminus J$ and $\widetilde{g}_{z_i} = g$ in each interval J_{z_i} such that

$$\widetilde{\Psi}(x) = \sum_{i=1}^{n} \widetilde{g}_{z_i}(x)\, \mathcal{X}_{J_{z_i}}(x), \quad x \in D,$$

is an educated map by $\mathbb{P}_0$ separating all the fixed-points of g in D.

Proof. The hypotheses on g and $\mathbb{P}_0$ imply that the conditions in Lemma 2.3 are satisfied for each fixed-point z_i. That is, (i) there exists a closed subinterval $J_{z_i} = [z_i - \epsilon_i, z_i + \epsilon_i]$ of $I_{z_i} = [z_i - d, z_i + d] \subset D$, (ii) the map g is continuous in J_{z_i}, and (iii) the fixed-point z_i is the unique fixed-point of g in J_{z_i}.

Hence, as $d < \eta/2$, the subintervals J_{z_i} are obviously disjoint and, by the definition of an educated map by $\mathbb{P}_0$ (cf. Definition 2.1), the map $\widetilde{\Psi}$ is an educated map separating the fixed-points of g. □

We now explain the role of the slope predicate $\mathbb{P}_1$ in the construction of a quasi-step map. The next lemma shows that if $|g'(x)| \neq 1$ near a given fixed-point z (i.e., a *non-neutral* fixed-point) and the predicate $\mathbb{P}_1$ holds in a certain interval containing z, then it is possible to isolate this fixed-point.

LEMMA 2.5. *Let $z \in D$ be a fixed-point of a map g which is of class C^1 in the interval $X_z = [z - \epsilon, z + \epsilon] \subset D$ ($\epsilon > 0$). Assume that $g(X_z) \subseteq X_z$, $|g'(x)| \neq 1$ for all $x \in X_z$, and $\mathbb{P}_1$ holds true for all non-fixed-points of g belonging to X_z. Then we have:*

(i) There is a closed bounded subinterval $\widetilde{X}_z = [z - \mu, z + \mu]$ of X_z where the map g is a strictly nonexpansive iterated contraction, that is,

$$|g(g(x)) - g(x)| < \alpha\, |g(x) - x|, \quad \text{for } \alpha < 1.$$

(ii) The map

$$\widetilde{\Psi}_z(x) = \widetilde{g}_z(x)\, \mathcal{X}_{\widetilde{X}_z}(x), \quad x \in D, \tag{2.3}$$

with $\widetilde{g}_z(x) = 0$ for $x \in D \setminus \widetilde{X}_z$ and $\widetilde{g}_z = g \circ g$ in $\widetilde{X}_z$ isolates the fixed-point z.

For the notion of an iterated contraction and properties of nonexpansive iterated contractions, we refer to [16, Chapter 12.3] and [20, Section 6].

Proof. (i) Note that the inequality in $\mathbb{P}_1$ means that $|g(g(x)) - g(x)| \leq |g(x) - x|$ for all x in X_z with $x \neq g(x)$. Defining the function

$$q(x) = \frac{g(g(x)) - g(x)}{g(x) - x},$$

we have

$$\left| \frac{g(g(x)) - g(x)}{g(x) - x} \right| \leq 1 \iff |q(x)| \leq 1,$$

which implies that

$$\lim_{x \to z} |q(x)| \leq 1. \tag{2.4}$$

Let $N(x) = g(g(x)) - g(x)$ and $M(x) = g(x) - x$. Since $g \in C^1(X_z)$, both the functions N and M are of class C^1 in X_z. As

$$N'(x) = g'(x)\ (g'(g(x)) - 1) \quad \text{and} \quad M'(x) = g'(x) - 1,$$

L'Hôpital's rule gives

$$\lim_{x \to z} |q(x)| = \lim_{x \to z} \frac{|N'(x)|}{|M'(x)|} = \lim_{x \to z} \frac{|g'(x)\,(g'(g(x)) - 1)|}{|g'(x) - 1|} = |g'(z)| \neq 1,$$

where the last equality follows from the continuity of g and g' and the hypothesis $|g'(x)| \neq 1$. Hence, the inequality (2.4) holds strictly proving that g is a strictly nonexpansive iterated contraction in an interval centered at z, say $\widetilde{X}_z$.

(ii) Since g is strictly nonexpansive in $\widetilde{X}_z$ and $g(\widetilde{X}_z) \subseteq \widetilde{X}_z$, by Edelstein's Theorem ([16, Chapter 12.3]), the sequence $x_{k+1} = g(x_k)$, for any $x_0 \in \widetilde{X}_z$ converges to the unique fixed-point of g in $\widetilde{X}_z$. As $\tilde{g}(\widetilde{X}_z) \subseteq \widetilde{X}_z$, it holds that $g(g(\widetilde{X}_z)) \subseteq \widetilde{X}_z$, and thus the map $\widetilde{\Psi}_z$ in (2.3) isolates the fixed-point z in D. □

We note that if the hypotheses of the above lemma are satisfied for all the fixed-points of g, then we are able to separate all of them. The precise statement is as follows.

PROPOSITION 2.6. *Let $\mathcal{Z} = \{z_1, \dots, z_n\}$ be the fixed-points of a map g. Suppose that for each point z_i in $\mathcal{Z}$ all the assumptions of Lemma 2.5 are satisfied in the subintervals $X_{z_i} = [z_i - \epsilon_i, z_i + \epsilon_i]$. Then, there exists a collection of disjoint subintervals of D, $K_i = [z_i - \delta, z_i + \delta]$, and maps $\widetilde{g}_{z_i}$ with $\widetilde{g}_{z_i}(x) = 0$ for $x \in D \setminus \bigcup_i K_i$ and $\widetilde{g}_{z_i} = g \circ g$ in K_i such that the map*

$$\widetilde{\Psi}(x) = \sum_{i=1}^{n} \widetilde{g}_{z_i}(x)\, \mathcal{X}_{K_i}(x), \quad x \in D, \tag{2.5}$$

is an educated map by the predicate $\mathbb{P}_1$ separating all the fixed-points of g.

Proof. By Lemma 2.5 there are closed subintervals $\widetilde{X}_{z_i} = [z_i - \mu_i, z_i + \mu_i]$ each of them containing a unique fixed-point z_i of g. Taking $\delta = \min\{\mu_1, \dots, \mu_n\}$ and the subintervals $K_i = [z_i - \delta, z_i + \delta] \subset \widetilde{X}_{z_i}$, the collection of the intervals K_i is disjoint and the union of these intervals is contained in D. Also, by the proof of Lemma 2.5, the function g is continuous on each K_i, and so $g \circ g$ is continuous on each K_i. Therefore, the map $\widetilde{\Psi}$ in (2.5) satisfies all the conditions of the definition of an educated map by $\mathbb{P}_1$ and clearly separates all the fixed-points z_i of g. □

2.1.1. Composition of quasi-step maps. It is easy to see that the composition of a quasi-step map with itself is again a quasi-step map. Moreover, if the function $\widetilde{g}$ entering in Definition 2.1 of a quasi-step map $\widetilde{\Psi}$ is contractive in each interval J_i, then it is natural that compositions of $\widetilde{\Psi}$ with itself will provide better approximations of the fixed-points of g than $\widetilde{\Psi}$. Since in Example 3.8 we use compositions of Halley educated maps by $\mathbb{P}_1$, let us briefly explain some of their features.

Let g^r denote the r-fold composition of the map g with itself, that is, $g^r = (g \circ g \circ \cdots \circ g \circ g)$ and $g^1 = g$. Let as before $\mathcal{Z} = \{z_1, \dots, z_n\}$ be the set of fixed-points of a map g and $\widetilde{\Psi}(x)$ a quasi-step map

$$\widetilde{\Psi}(x) = \sum_{k=1}^{n} \widetilde{g}(x)\, \mathcal{X}_{J_k}(x), \quad \text{for } x \in D,$$

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with $\widetilde{g}$ a contractive map on each interval $J_k = [z_k - \delta, z_k + \delta] \subset D$ and $\widetilde{g}(z_i) = z_i$, for all $i = 1, \dots, n$. Due to the contractivity of $\widetilde{g}$ on each J_k, there exists a constant $0 \le L < 1$ such that

$$|\widetilde{g}^r(x) - z_k| \le L^r |x - z_k|, \quad x \in J_k.$$

This implies that the values of $\widetilde{\Psi}^r$ on each subinterval J_k are closer to z_k than the values of $\widetilde{\Psi}$ in the same interval.

We remark that if g is a map satisfying all the conditions of Proposition 2.6, then the respective map $\widetilde{\Psi}$ educated by the predicate $\mathbb{P}_1$ is of the form (1.2). Therefore, for $r > 1$, the map $\widetilde{\Psi}^r$ should be considered as a map in D closer to a step map than the map $\widetilde{\Psi}$.

3. Examples. We present several examples illustrating our procedures for the global separation of roots. The first set of examples deal with a family of functions f_k which we name Pruitt functions (see Definition 3.2) and the second set (Example 3.8) with a strongly oscillating transcendental function.

In Example 3.4 we obtain a step map for the Pruitt function f_3, and in the remaining examples, appropriate quasi-step maps are constructed based on the Newton and Halley maps educated by one or both the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$. For a convergence analysis and historical developments of the Newton and Halley methods, we refer to [1, 4, 9, 13, 19, 20, 27].

As before, the equation $x = g(x)$ is a fixed-point version of $f(x) = 0$. The set of fixed-points of g, $\mathcal{Z}$, is generally unknown, and we aim to find its elements in a given interval D. In all the examples, no initial guesses of the fixed-points are required. We consider for g either the Newton $\mathcal{N}_f$ or the Halley $\mathcal{H}_f$ maps associated to f. These maps are defined as follows.

DEFINITION 3.1 (Newton and Halley maps). *Given a sufficiently differentiable map f, the Newton and Halley maps associate to f are, respectively,*

$$\begin{aligned} \mathcal{N}_f(x) &= x - f(x)/f'(x), \\ \mathcal{H}_f(x) &= x - \frac{2f(x)f'(x)}{2\,f'(x)^2 - f(x)\,f''(x)}. \end{aligned} \tag{3.1}$$

It is well-known (see [13, 17, 19] and references therein) that if f is sufficiently smooth on the interval D, both $\mathcal{N}_f$ and $\mathcal{H}_f$ share the following properties: (a) for simple zeros of f, $\mathcal{N}_f$ and $\mathcal{H}_f$ have a local order of convergence $p > 1$, and (b) for multiple zeros of f, $\mathcal{N}_f$ and $\mathcal{H}_f$ have a local linear convergence, (i.e., $p = 1$). In particular, for simple zeros, the order of convergence of $\mathcal{N}_f$ and of $\mathcal{H}_f$ is, respectively, $p \ge 2$ and $p \ge 3$.

In the light of our purposes, both $\mathcal{N}_f$ and $\mathcal{H}_f$ satisfy a very convenient property concerning the zeros of f: (i) If z is a simple zero of f, then z is a locally super attracting fixed-point of $\mathcal{N}_f$ and $\mathcal{H}_f$. (ii) If z is a zero of f of higher multiplicity, then z is locally an attracting fixed-point of $\mathcal{N}_f$ and $\mathcal{H}_f$. In both cases the assumption $|g'(x)| \ne 1$, for x sufficiently close to z, of Lemma 2.5 is automatically satisfied. The other assumption in this lemma, that is, $g(X_z) \subseteq X_z$ depends on the particular function f. However, in the case that this hypothesis does not hold for f, we can use both predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ in order to separate the fixed-points of g taking an appropriate vertical displacement parameter d in the predicate $\mathbb{P}_0$.

We note that besides $\mathcal{N}_f$ and $\mathcal{H}_f$, one could choose any other iteration map g from the plethora of maps in the literature even with a higher order of convergence, for instance, the family of Halley maps presented in [24], which has maximal order of convergence, or the recursive family of iteration maps in [8] obtained from quadratures which has arbitrary order of convergence.

3.1. Step and quasi-step maps for Pruitt's functions. In this paragraph we consider a family of functions introduced by Pruitt in [21] to illustrate our construction of step and quasi-step maps as tools for global separation of roots. Due to the peculiar fact that these functions only admit integer roots, we begin by showing that in this case we are able to present a step map which provides all the zeros of a Pruitt's function in a given interval.

DEFINITION 3.2. *Let k be a positive integer. The kth Pruitt function f_k is defined by*

$$f_k(x) = \prod_{i=1}^{k} \sin\left(\frac{x\,\pi}{p_i}\right), \quad \text{for } x \in I_k = [3/2, p_k^2 + 1/3], \tag{3.2}$$

where p_i denotes the ith prime number.

Step maps for Pruitt functions. In what follows we denote by $[x]$ the *closest integer* to $x \in \mathbb{R}$. The next proposition gives a step map of a Pruitt function.

PROPOSITION 3.3. *Let f_k be the kth Pruitt function defined on I_k as in (3.2), $[\,\cdot\,] : I_k \to \mathbb{Z}$ the closest integer function, and $\Psi_k : I_k \to I_k$ the map defined by*

$$\Psi_k(x) = \begin{cases} [x], & \text{if } f_k([x]) = 0, \\ 0, & \text{if } f_k([x]) \neq 0. \end{cases} \tag{3.3}$$

Then,

(i) the set $\mathcal{Z}$ of the fixed-points of Ψ_k coincides with the set of zeros of f_k, and it is given by $\mathcal{Z} = \mathcal{P}_0 \cup \mathcal{M}$ with

$$\mathcal{P}_0 = \{p : \ p \text{ is prime and } 2 \le p \le p_k\},$$
$$\mathcal{M} = \{s : \ s \text{ is a multiple of an element of } \mathcal{P}_0, \text{ and } p_k < s \le p_k^2\},$$

(ii) Ψ_k is a step map in I_k separating all the zeros of f_k in the interval I_k,

(iii) the zeros of Ψ_k which are integers are the primes p such that $p_k \le p < p_k^2$.

Proof. (i)–(ii) Noting that the solutions of $\sin(\pi\,x/p) = 0$ are $x = r\,p$ (with $r \in \mathbb{Z}$), it follows from the definition of f_k that a zero of f_k is an integer which belongs either to the set of primes $\mathcal{P}_0$ or to the set $\mathcal{M}$ of the multiples of the elements in $\mathcal{P}_0$. As the fixed-points of Ψ_k are the integers of I_k which are zeros of f_k, the fixed-point set of Ψ_k is precisely $\mathcal{Z} = \mathcal{P}_0 \cup \mathcal{M}$.

It is now obvious that (3.3) can be written as the step map

$$\Psi_k(x) = \sum_{i=1}^{p_k^2-1} c_i\, \mathcal{X}_{I_i}(x), \quad x \in \mathbb{R},$$

where c_i is a non-negative integer and the intervals I_i are defined as

$$I_i = [i + 1/2, i + 3/2], \qquad \text{for } i = 1, \ldots, p_k^2 - 2,$$
$$I_{p_k^2-1} = [p_k^2 - 1/2, p_k^2 + 1/3],$$

with $I = \bigcup_{i=1}^{p_k^2-1} I_i$, which means that Ψ_k separates the fixed-points of f_k.

(iii) From the proof of (i)–(ii) it is immediate that the integers in $I_k \setminus \mathcal{Z}$ are the prime numbers p belonging to $[p_k - 1/2, p_k^2 + 1/3]$, that is, $p_k \le p < p_k^2$. ∎

We remark that from the above proposition, the zeros of f_k in $I_k = [2, p_k^2]$ are the first p_k primes, $2, 3, \ldots, p_k$, and all their multiples which are less or equal to p_k^2. Furthermore, any integer j with $p_k < j \le p_k^2$, is a *composite* number if $f_k(j) = 0$ and is a prime number if $f_k(j) \neq 0$.

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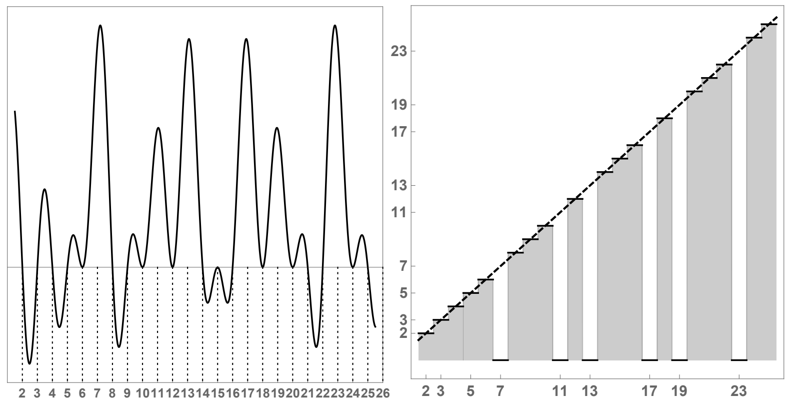

FIG. 3.1. *Left: the graph of the Pruitt function* f_3 *on* $D = [1.5, 25.5]$. *Right: the graph of the step map* Ψ_3 *and the line* $y = x$ *(dashed).*

In the following example the step map given in Proposition 3.3 is constructed for the 3rd Pruitt function.

EXAMPLE 3.4. Let us consider $k = 3$ and the Pruitt function f_3:

$$f_3(x) = \sin(x\,\pi/2)\,\sin(x\,\pi/3)\,\sin(x\,\pi/5), \qquad x \in I_3 = [2 - 1/2, 25 + 1/3].$$

The system *Mathematica* [25] has been utilized to compute the step map Ψ_3 defined in (3.3). For this purpose, the built-in system function $Round[\,]$ is used as the closest integer function. Note that for numbers γ of the form $\gamma = x.5$, $Round[\gamma]$ produces the nearest *even* integer.

The graphs of f_3 and Ψ_3 are displayed in Figure 3.1. Projecting the graph of Ψ_3 onto the y-axis we obtain the zeros of f_3. In fact, as predicted by Proposition 3.3, this set is $\mathcal{Z} = \mathcal{P}_0 \cup \mathcal{M}$, with

$$\mathcal{P}_0 = \{2, 3, 5\}, \quad \mathcal{M} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25\}, \tag{3.4}$$

where $\mathcal{P}_0$ is the set of the first 3 primes and $\mathcal{M}$ the set of composite numbers which are multiples of the primes in $\mathcal{P}_0$. Moreover, by Proposition 3.3 (iii), the step map Ψ_3 may also be used to detect the prime numbers greater than 5. It is clear from the graph of Ψ_3 in Figure 3.1, that these prime numbers are the integers (on the x-axis) which belong to those intervals where Ψ_3 is equal to zero. These primes are in the set $\mathcal{P} = \{7, 11, 13, 17, 19, 23\}$.

REMARK 3.5. The sieve of Eratosthenes is an algorithm [3] providing the prime numbers less than a given integer $n \geq 2$. One of the versions of this algorithm computes precisely the set of primes $\mathcal{P}$ from the set $\mathcal{P}_0 \cup \mathcal{M}$ in Proposition 3.3. Thus, the step function (3.3) may be seen as another computational version of the Eratosthenes algorithm. We refer to [12] for a discussion of efficient practical versions of the sieve of Eratosthenes using appropriate data structures.

Halley and Newton quasi-step maps for Pruitt functions. In the examples below we illustrate the construction of Halley and Newton quasi-step maps educated by the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ for the Pruitt family. Although such construction does not depend on an *a priori* knowledge of the multiplicities of the zeros of the function, we note that for a Pruitt function f_k, we are dealing with both simple and multiple zeros in the interval I_k. In fact, the multiplicities of the zeros of f_k satisfy an interesting property: on I_k there are zeros of multiplicity m, where m takes all the integer values $m = 1, 2, \ldots, k - 1$. This property follows from the particular form of f_k, and although it can be proved in full generality, we briefly explain it through a particular example, namely, with the Pruitt function f_4.

The set of the first four primes is $\mathcal{P}_4 = \{2, 3, 5, 7\}$. Let $\mathcal{M}_1$ be the set of all the multiples of the elements of $\mathcal{P}_4$ which belong to the interval $I_4 = [4 - 1/2, 49 + 1/3]$. That is,

$$\mathcal{M}_1 = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, \\ 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 49\}.$$

The elements of $\mathcal{M}_1$ are zeros of f_4 having multiplicity at least one. Consider now the $6 = \binom{4}{2}$ numbers which are products of two elements of $\mathcal{P}_4$. These products are

$$s_1 = 2 \times 3,\ s_2 = 2 \times 5,\ s_3 = 2 \times 7,\ s_4 = 3 \times 5,\ s_5 = 3 \times 7,\ s_6 = 5 \times 7.$$

Let $\mathcal{M}_2$ be the set of the multiples of $s_1, s_2, \ldots, s_6$ which belong to I_4:

$$\mathcal{M}_2 = \{6, 10, 12, 14, 15, 18, 20, 21, 24, 28, 30, 35, 36, 40, 42, 45, 48\}.$$

The elements of $\mathcal{M}_2$ are zeros of f_4 with multiplicity at least two. Thus, the simple zeros of f_4 belong to the set $\mathcal{Z}_1 = \mathcal{M}_1 \backslash \mathcal{M}_2$, namely to

$$\mathcal{Z}_1 = \{2, 3, 4, 5, 7, 8, 9, 16, 22, 25, 26, 27, 32, 33, 34, 38, 39, 44, 46, 49\}.$$

Taking the products of three elements of $\mathcal{P}_4$, we obtain

$$s_1 = 2 \times 3 \times 5,\ s_2 = 2 \times 3 \times 7,\ s_3 = 2 \times 5 \times 7,\ s_4 = 3 \times 5 \times 7.$$

The set $\mathcal{M}_3$ of all the multiples of s_1 to s_4 belonging to I_4 is

$$\mathcal{M}_3 = \{s_1, s_2\} = \{30, 42\}.$$

As all the elements of $\mathcal{M}_3$ have multiplicity greater or equal to three, the set $\mathcal{Z}_2 = \mathcal{M}_2 \setminus \mathcal{M}_3$ is the set of double zeros of f_4:

$$\mathcal{Z}_2 = \{6, 10, 12, 14, 15, 18, 20, 21, 24, 28, 35, 36, 40, 45, 48\} \quad \text{(double zeros)}.$$

Finally, as the product of all the elements of $\mathcal{P}_4$ is $2 \times 3 \times 5 \times 7 > 49$, there are no zeros of multiplicity greater or equal to four in I_4. This means that

$$\mathcal{Z}_3 = \mathcal{M}_3 = \{30, 42\} \quad \text{(triple zeros)}$$

is the set of the zeros of f_4 with multiplicity $m = 3$ in the interval I_4.

Note that for the function f_3 treated in Example 3.4, either using the previous reasoning or just by an inspection of the graph of f_3 in Figure 3.1, we draw the following conclusions:

- The set of simple zeros of f_3 is

$$\{2, 3, 4, 5, 8, 9, 14, 16, 21, 22, 25\}. \tag{3.5}$$

- The set of double zeros of f_3 is

$$\{6, 10, 12, 15, 18, 20, 24\}. \tag{3.6}$$

The following two examples illustrate several features of the Halley and Newton quasi-step maps associated to Pruitt functions educated by the predicates $\mathbb{P}_0$ and $\mathbb{P}_1$ in (1.3) and (1.4), respectively. A quasi-step map educated by the predicate $\mathbb{P}_0$ will be denoted by $\widetilde{\Psi}_0$ and by $\widetilde{\Psi}_1$ in the case of $\mathbb{P}_1$.

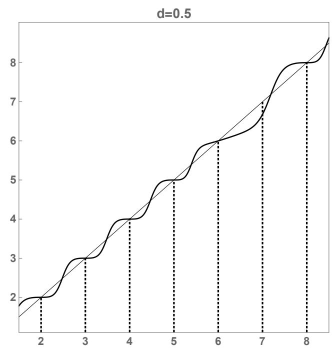

FIG. 3.2. *The Halley educated map* $\widetilde{\Psi}_0$ *for* $d = 0.5$ *and* $D = [1.5, 8.5]$.

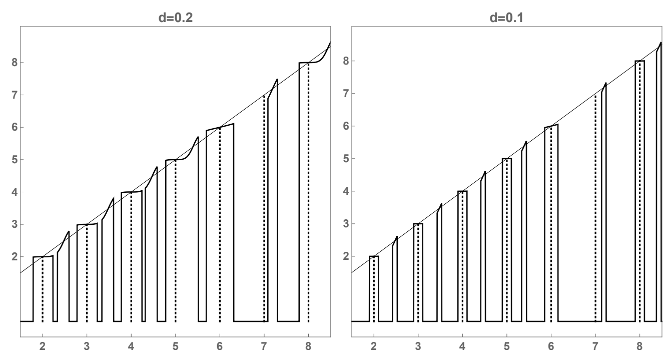

FIG. 3.3. *(A) Left: the Halley educated map* $\widetilde{\Psi}_0$ *for* $d = 0.2$*. (B) Right: the Halley educated map* $\widetilde{\Psi}_0$ *for* $d = 0.1$*.*

EXAMPLE 3.6 (Halley quasi-step maps for f_3). Let f_3 be the 3rd Pruitt function restricted to the interval $D = [1.5, 8.5]$ and $g = \mathcal{H}_{f_3}$ the respective Halley map. Let $\widetilde{\Psi}_0$ be a Halley map educated by the predicate $\mathbb{P}_0$.

Recall from (3.5) and (3.6) that in D all the zeros of f_3 are simple except $z = 6$, which is a double zero. So, $2, 3, 4, 5, 8$ are super attracting fixed-points of $\mathcal{H}_{f_3}$ while $z = 6$ is only attracting. Between two consecutive zeros of f_3 there is exactly one repelling fixed-point of $\mathcal{H}_{f_3}$. This means that $\mathcal{H}_{f_3}$ has fixed-points which are not roots of $f_3(x) = 0$. Moreover, the fixed-points of $\mathcal{H}_{f_3}$ which are zeros of f_3 are precisely the attracting fixed-points of $\mathcal{H}_{f_3}$ as it can be observed in Figure 3.2. The resolution of the set of the attracting fixed-points of $\mathcal{H}_{f_3}$ is $\eta = 1$; see Proposition 2.4. By this proposition, if a vertical displacement d is chosen such that $d < \eta/2 = 0.5$, then $\widetilde{\Psi}_0$ must separate the attracting fixed-points of $\mathcal{H}_{f_3}$ (i.e., the zeros of f_3) in D. This does not mean that $\widetilde{\Psi}_0$ separates all the fixed-points of $\mathcal{H}_{f_3}$ as it is clear from Figure 3.3(A) where the graph of $\widetilde{\Psi}_0$ for $d = 0.2$ shows that in the interval containing 5, there are two fixed-points of $\mathcal{H}_{f_3}$.

In Figure 3.2 and Figure 3.3(B) we display the graphs of the quasi-step map $\widetilde{\Psi}_0$ obtained for the vertical displacement $d = 0.5$ and $d = 0.1$. We observe that for $d = 0.5$, the educated map $\widetilde{\Psi}_0$ coincides with $\mathcal{H}_{f_3}$ while for $d = 0.1$ all the fixed-points of $\mathcal{H}_{f_3}$ (not just the attracting ones) have been separated in D.

We now consider f_3 defined in $D = [1.5, 25.5]$ and the corresponding Halley map $\mathcal{H}_{f_3}$. As before, the graph of $\mathcal{H}_{f_3}$ in Figure 3.4(A) clearly shows the nature of the fixed-points of this map. Namely, those fixed-points of $\mathcal{H}_{f_3}$ that are roots of $f_3(x) = 0$ are either attracting or

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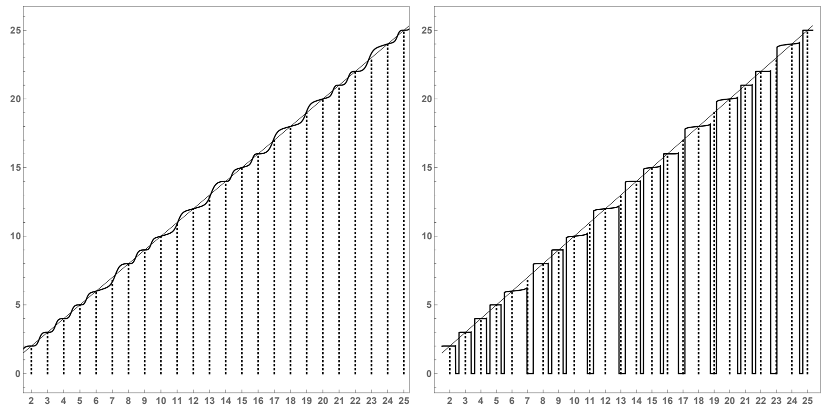

FIG. 3.4. *(A) Left: the Halley's map applied to the Pruitt function f_3, in $D = [1.5, 25.5]$. (B) Right: the graph of $\widetilde{\Psi}_1$ separating all the zeros of f_3.*

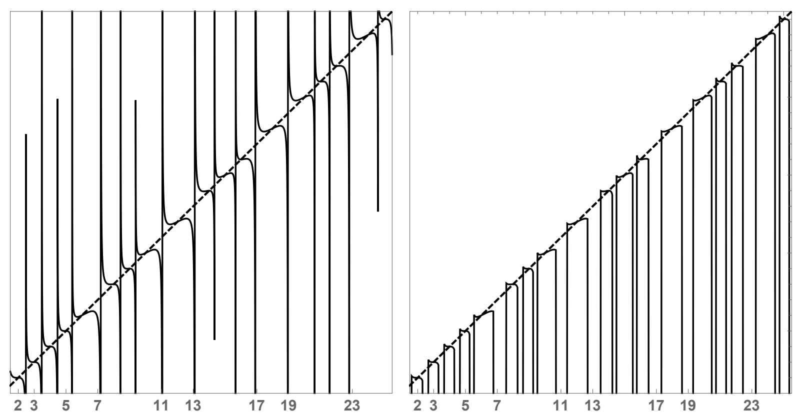

FIG. 3.5. *(A) Left: the Newton map $\mathcal{N}_{f_3}$ in $D = [1.5, 25.5]$. (B) Right: the Newton educated map $\widetilde{\Psi}_0$, for $d = 0.5$.*

super attracting, and the remaining fixed-points are repelling fixed-points. The repelling fixed-points of $\mathcal{H}_{f_3}$ do not satisfy the predicate $\mathbb{P}_1$ since they are in contradiction with Lemma 2.5. Therefore, a map $\widetilde{\Psi}_1$, educated by $\mathbb{P}_1$, separates all the roots of $f_3(x) = 0$ (cf. Proposition 2.6) as it is clear from the graph of $\widetilde{\Psi}_1$ displayed in Figure 3.4(B).

We remark that for certain classes of functions such as the Pruitt family f_k, the corresponding Halley map enjoys the important property of continuity. This property is not satisfied in the case of the Newton map which will be treated in the next example. Moreover, the analysis of $\mathcal{H}_{f_3}$ in Figure 3.2 confirms what is expected concerning the global convergence of the Halley method with respect to the monotone behaviour of these iterative processes as discussed in [4, 10].

EXAMPLE 3.7 (Newton quasi-step maps for f_3). Let us consider the Newton map $\mathcal{N}_{f_3}$ corresponding to the Pruitt function f_3 defined in $D = [1.5, 25.5]$. In contrast with the Halley function of the previous example, $\mathcal{N}_{f_3}$ is not continuous in D as the graph of $\mathcal{N}_{f_3}$ in Figure 3.5(A) shows. This graph presents several vertical asymptotes which are due to the singularities of the function f_3 in the interval. However, this time the fixed-points of $\mathcal{N}_{f_3}$ coincide with the roots of $f_3(x) = 0$, that is, $\mathcal{Z} = \mathcal{P}_0 \cup \mathcal{M}$ with $\mathcal{P}_0$ and $\mathcal{M}$ given by (3.4).

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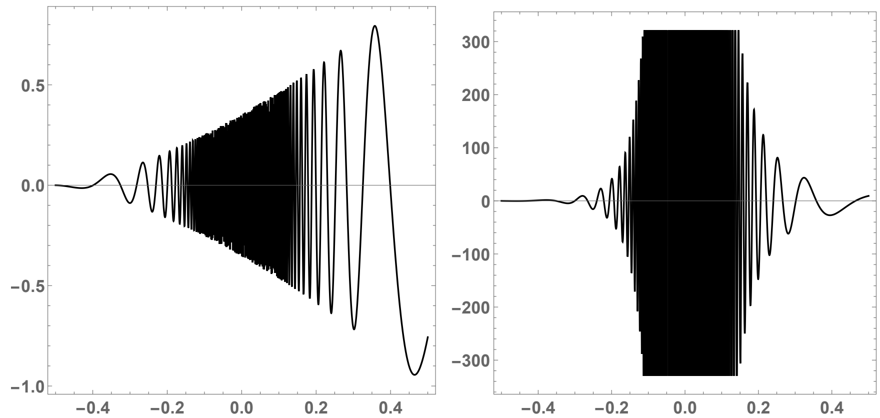

FIG. 3.6. *Left: the strongly oscillating function f. Right: its unbounded derivative.*

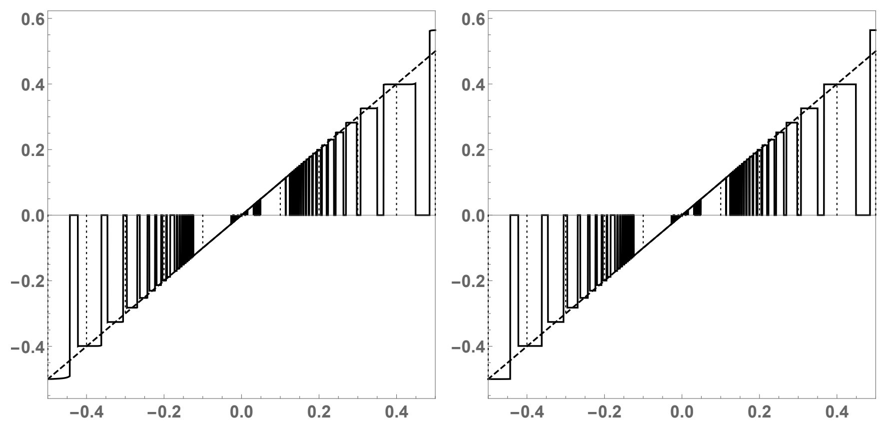

FIG. 3.7. *Halley educated maps $\widetilde{\mathcal{H}}_f$ (left) and $\widetilde{\mathcal{H}}_f^2$ (right).*

Obviously, the resolution of $\mathcal{Z}$ is $\eta = 1$, and all the fixed-points of $\mathcal{N}_{f_3}$ satisfy the attractivity property referred in Proposition 2.4. By this proposition, if one chooses a vertical displacement $d < 0.5$, then all the roots of f_3 will be separated by the quasi-step map $\widetilde{\Psi}_0$ educated by the predicate $\mathbb{P}_0$. Figure 3.5(B) shows that the $\mathcal{N}_{f_3}$-educated map $\widetilde{\Psi}_0$ for $d = 0.5$ globally separates the zeros of f_3.

3.2. Quasi-step maps for a strongly oscillating function. The composition of quasi-step maps can be particularly useful to compute highly accurate roots of strongly oscillating functions as it is illustrated in the following numerical example.

EXAMPLE 3.8 (Strongly oscillating function). We consider the function $f(0) = 0$, $f(x) = (x + 1/2)^{3/2} \sin(1/x)$, for $x \neq 0$, defined in the interval $D = [-1/2, 1/2]$. The function f is differentiable with unbounded derivative and so it is not locally Lipschitzian. Close to the point $x = 0$ the function is strongly oscillating (see Figure 3.6).

As before, let $\mathcal{H}_f$ be the Halley map associated to f as defined in (3.1). The map $\widetilde{\mathcal{H}}_f$ denotes the Halley map educated by the slope predicate $\mathbb{P}_1$ and $\widetilde{\mathcal{H}}_f^r$ denotes its r-fold composition.

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TABLE 3.1
CPU time (in seconds) to run $FindInstance$ to simultaneously compute a sample of s roots in the interval $J = [-0.0097, 0.0097]$.

s	CPU time
2	0.38
100	16.3
250	40.5
500	80.5

In Figure 3.7 the maps $\widetilde{\mathcal{H}}_f$ and $\widetilde{\mathcal{H}}_f^2$ are displayed, which shows the quasi-step nature of these maps near the endpoints of the interval. Moreover, the same figure shows that in the subinterval $I = [-0.2, 0.2]$ there is a large number of fixed-points clustering around the origin.

We used the system *Mathematica* to compute approximations of the fixed-points of $\widetilde{\mathcal{H}}_f$ in a certain interval centered at the origin. All the computations were carried out on a 2GHz Intel Core $i7$ personal computer using standard double-precision floating-point arithmetic. Our computations were compared with those obtained by using some built-in routines of the system *Mathematica*. In particular, the command $FindInstance[expr, vars, Reals, s]$, with $vars$ and $expr$ assumed to be real, was used to compute an (unsorted) sample of s real roots of $f(x) = 0$. Notably, in the interval $J = [x_{min}, x_{max}] = [-0.0097, 0.0097]$, the code line $FindInstance[\{f[u] == 0., xmin \leq u \leq xmax\}, u, s]$ was run for several values of s. The respective CPU time (in seconds) is given in Table 3.1, which shows that for large s the CPU time approximately doubles with s. Unless one uses the system option $WorkingPrecision$, Table 3.1 shows that for $s = 200$ zeros of f in the interval J, the referred time of 80.5 seconds is quite unacceptable. In contrast, using the same default double-precision computer arithmetic, the Halley educated map $\widetilde{\mathcal{H}}_f$ can separate, in less than one second, $s = 200$ fixed-points in the interval J.

We considered discretized versions of the maps $\widetilde{\mathcal{H}}_f$, $\widetilde{\mathcal{H}}_f^2$, and $\widetilde{\mathcal{H}}_f^3$ obtained by dividing the interval $J = [x_{min}, x_{max}] = [-0.0097, 0.0097]$ into $N = 1500$ parts of length $h = (x_{max} - x_{min})/N$. The respective $N+1$ values of the maps at the points $x_i = x_{min} + i\,h$, with $i = 0, \ldots, N$, were tabulated. Due to the large number of points considered, Table 3.2 only presents some of these values, namely those which are near the endpoints of J. For convenience, we call $\mathcal{T}$ the full set of data obtained.

It is clear from Table 3.2 that the points x_i where the value zero occurs for all the maps $\widetilde{\mathcal{H}}_f$, $\widetilde{\mathcal{H}}_f^2$, and $\widetilde{\mathcal{H}}_f^3$ provide a collection of the subintervals where the slope predicate holds true. In the full set $\mathcal{T}$ we found 273 of such subintervals. In each of these subintervals there exists at least one fixed-point of $\widetilde{\mathcal{H}}_f$. The analysis of $\mathcal{T}$ also shows that the maps $\widetilde{\mathcal{H}}_f^3$ and $\widetilde{\mathcal{H}}_f^2$ are numerically invariant (see also Table 3.2), and so all the computed nonzero values $\widetilde{\mathcal{H}}_f^2(x_i)$ are approximations of fixed-points of the Halley map $\mathcal{H}_f$ with eight significant decimal digits.

The previous procedures may be implemented in order to obtain high-precision approximations of the fixed-points of $\mathcal{H}_f$. This can be achieved by considering in the computations not only a convenient machine precision but also an appropriate r-fold composition of $\widetilde{\mathcal{H}}_f$. For instance, taking the same sample of 1501 points x_i in J, an extended precision of 1000 decimal digits, and computing $\widetilde{\mathcal{H}}_f^r(x_i)$, for $r = 2$ to $r = 5$, we obtained a new table of data in about 10 seconds of CPU time. In this case one can verify that $\widetilde{\mathcal{H}}_f^4$ and $\widetilde{\mathcal{H}}_f^5$ are numerically invariant. In particular, for the point $x_{N-1} = 0.0097 - h \simeq 0.0096870667$, we have $z_{N-1} = \widetilde{\mathcal{H}}_f^5(x_{N-1})$

TABLE 3.2
Educated maps $\widetilde{\mathcal{H}}_f^r(x)$, *for* $r = 1, 2, 3$.

x_i	$\widetilde{\mathcal{H}}_f(x_i)$	$\widetilde{\mathcal{H}}_f^2(x_i)$	$\widetilde{\mathcal{H}}_f^3(x_i)$
-0.0097	-0.0097000521	-0.0097000521	-0.0097000521
-0.0096870667	-0.0096871749	-0.0096871749	-0.0096871749
-0.0096741333	-0.0096743489	-0.0096743489	-0.0096743489
-0.0096612	-0.0096615735	-0.0096615738	-0.0096615738
-0.0096482667	0	0	0
-0.0096353333	0	0	0
-0.0096224	-0.0096221501	-0.0096221501	-0.0096221501
-0.0096094667	-0.0096095803	-0.0096095803	-0.0096095803
...	...	...	...
...	...	...	...
0.0096094667	0.0096095803	0.0096095803	0.0096095803
0.0096224	0.0096221501	0.0096221501	0.0096221501
0.0096353333	0	0	0
0.0096482667	0	0	0
0.0096612	0.0096615735	0.0096615738	0.0096615738
0.0096741333	0.0096743489	0.0096743489	0.0096743489
0.0096870667	0.0096871749	0.0096871749	0.0096871749
0.0097	0.0097000521	0.0097000521	0.0097000521

with

$$\begin{aligned} z_{N-1} = &0.00968717492126197578990697768061822598 \\ &\vdots \\ &37857348618418588114840510250806403928\,94, \end{aligned} \tag{3.7}$$

where only a certain number of the initial and the final 1000 decimal machine digits are displayed. Computing the residual $f(z_{N-1})$, we obtain

$$f(z_{N-1}) \simeq 0. * 10^{-997}.$$

In order to check that z_{N-1} is in fact an accurate root of $f(x) = 0$, we use the *Mathematica* function $FindRoot$ as follows:

$$\begin{aligned} &x_{N-1} = 0.0096870667; \\ &\quad z = x\,/.\,FindRoot[f[x] == 0, \{x, x_{N-1}\}, WorkingPrecision -> 1000]; \end{aligned}$$

The respective value of z is such that $z - z_{N-1} \simeq 0. * 10^{-1002}$, which shows that all the 1000 decimal digits of z_{N-1} in (3.7) are correct.

As a final remark let us mention that we have applied with success several Newton- and Halley-educated maps to a set of test functions suggested in [2, 7, 14, 26]. Suitable discretized versions of the respective quasi-step maps allow the computation of high-accurate roots regardless whether these roots are simple or multiple, and so the approach seems to be particularly useful, in particular for the global separation of zeros of strongly oscillating functions.

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