Electronic Transactions on Numerical Analysis.
Volume 45, pp. 257–282, 2016.

ISSN 1068–9613.

ETNA
Kent State University
http://etna.math.kent.edu

A DECOMPOSITION RESULT FOR BIHARMONIC PROBLEMS AND THE HELLAN-HERRMANN-JOHNSON METHOD*

WOLFGANG KRENDL†, KATHARINA RAFETSEDER‡, AND WALTER ZULEHNER‡

Abstract. For the first biharmonic problem a mixed variational formulation is introduced which is equivalent to a standard primal variational formulation on arbitrary polygonal domains. Based on a Helmholtz decomposition for an involved nonstandard Sobolev space it is shown that the biharmonic problem is equivalent to three (consecutively to solve) second-order elliptic problems. Two of them are Poisson problems, the remaining one is a planar linear elasticity problem with Poisson ratio 0. The Hellan-Herrmann-Johnson mixed method and a modified version are discussed within this framework. The unique feature of the proposed solution algorithms for the Hellan-Herrmann-Johnson method and its modified variant is that they are solely based on standard Lagrangian finite element spaces and standard multigrid methods for second-order elliptic problems and that they are of optimal complexity.

Key words. biharmonic equation, Hellan-Herrmann-Johnson method, mixed methods, Helmholtz decomposition

AMS subject classifications. 65N22, 65F10, 65N55

1. Introduction.

We consider the first biharmonic boundary value problem: for given f, find w such that

$$\Delta^2 w = f \quad \text{in } \Omega, \qquad w = \partial_n w = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where Ω is an open and bounded set in $\mathbb{R}^2$ with a polygonal Lipschitz boundary Γ and Δ and ∂_n denote the Laplace operator and the derivative in the direction normal to the boundary, respectively. Problems of this type occur, for example, in fluid mechanics, where w is the stream function of a two-dimensional Stokes flow (see, e.g., [22]) and in linear elasticity, where w is the vertical deflection of a clamped Kirchhoff plate; see, e.g., [18].

In this paper we focus on finite element methods for discretizing the continuous problem (1.1). The aim is to construct and analyze efficient iterative methods for solving the resulting linear system. In particular, the Hellan-Herrmann-Johnson (HHJ) mixed finite element method is studied (see [26, 27, 30]), which is strongly related to the non-conforming Morley finite element; see [2, 33]. The proposed iterative method consists of the application of the preconditioned conjugate gradient method to three discretized elliptic problems of second order. The implementation requires only manipulations with standard conforming Lagrangian finite elements for second-order problems. The proposed preconditioners are standard multigrid preconditioners for second-order problems that lead to mesh-independent convergence rates.

The results are based on a decomposition of the continuous problem into three second-order elliptic problems that are to be solved consecutively. The first and the last problem are Poisson problems with Dirichlet conditions, the second problem is a pure traction problem in planar linear elasticity with Poisson ratio 0. The HHJ method is a non-conforming method in this setup. A conforming modification will be discussed as well.

There are many alternative approaches for biharmonic problems discussed in literature. Finite element discretizations range from conforming and classical non-conforming finite

*Received July 30, 2014. Accepted April 26, 2016. Published online on July 22, 2016. Recommended by S. Brenner. The research was supported by the Austrian Science Fund (FWF): W1214-N15, project DK12 and S11702-N23.
†Doctoral Program Computational Mathematics, Johannes Kepler University Linz, 4040 Linz, Austria (wolfgang.krendl@dk-compmath.jku.at).
‡Institute of Computational Mathematics, Johannes Kepler University Linz, 4040 Linz, Austria ({rafetseder,zulehner}@numa.uni-linz.ac.at).

ETNA
Kent State University
http://etna.math.kent.edu

element methods for fourth-order problems, discontinuous Galerkin (dG) methods for fourth-order problems to various mixed methods including mixed dG methods; see, e.g., [4, 5, 7, 15, 18, 20, 29] and the references cited therein. Solution techniques proposed for the linear systems that show mesh-independent or nearly mesh-independent convergence rates are typically based on two-level or multilevel additive or multiplicative Schwarz methods (including multigrid methods); see, e.g., [12, 13, 14, 16, 25, 36, 38, 39] and the references cited therein.

We are not aware of any other approach that is based solely on standard components for second-order elliptic problems (regarding both the discretization and the solver for the discretized problem) and for that optimal convergence behavior could be shown. Regarding the discretization, the method introduced in [5, 7] also uses only standard C^0 finite element spaces for second-order problems but for a different formulation in the kinematic variables w and ∇w. The analysis of the method in [5, 7] is based on a mesh-dependent energy norm, which requires some extra nonstandard techniques for constructing and analyzing an efficient iterative solver. The Hellan-Herrmann-Johnson method is strongly related to the mixed dG method introduced in [29] for plate bending problems. This method allows a reduction to a linear system for an approximation of w from a standard C^0 finite element space, where the associated system matrix can be seen as a discretization matrix of a fourth-order operator (in our case of the biharmonic operator). This again requires some extra techniques beyond the case of second-order problems for constructing and analyzing an efficient iterative solver.

An additional feature of the approach in this paper is a new formulation of the underlying continuous mixed variational problem that is fully equivalent to the original primal variational problem without any further assumptions on Ω like convexity. This is achieved by introducing an appropriate nonstandard Sobolev space.

The results of this paper can be extended to boundary conditions of the form $w = \Delta w = 0$ on Γ, which represent simply supported Kirchhoff plates in linear elasticity, as well as to non-homogeneous variants. An extension to plate bending problems with free boundaries and, more generally, to mixed variants involving all three types of boundary conditions is not straightforward and subject of further investigations.

The paper is organized as follows. Section 2 contains a modification of a standard mixed formulation of the biharmonic problem, for which well-posedness will be shown. A Helmholtz decomposition of an involved nonstandard Sobolev space is derived in Section 3 and the resulting decomposition of the biharmonic problem is presented. In Section 4 the Hellan-Herrmann-Johnson method and a modified version are discussed. Section 5 contains the discrete version of the Helmholtz decomposition of Section 3. In Section 6 we briefly discuss error estimates. The paper closes with a few numerical experiments in Section 7 for illustrating the theoretical results.

2. A modified mixed variational formulation.

Here and throughout the paper we use $L^2(\Omega)$, $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$, and $H_0^m(\Omega)$ with its dual space $H^{-m}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces with corresponding norms $\|\cdot\|_0$, $\|\cdot\|_{m,p}$, $\|\cdot\|_m = \|\cdot\|_{m,2}$, $|\cdot|_m$, and $\|\cdot\|_{-m}$ for $p \geq 1$ and positive integers m; see, e.g., [1]. A standard (primal) variational formulation of (1.1) reads as follows: for given $f \in H^{-1}(\Omega)$, find $w \in H_0^2(\Omega)$ such that

$$\int_\Omega \nabla^2 w : \nabla^2 v \, dx = \langle f, v \rangle \quad \text{for all } v \in H_0^2(\Omega), \tag{2.1}$$

where ∇^2 denotes the Hessian, $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^2 \boldsymbol{A}_{ij}\, \boldsymbol{B}_{ij}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{2\times 2}$, and $\langle \cdot, \cdot \rangle$ denotes the duality product in $H^* \times H$ for a Hilbert space H with dual H^*, here for $H = H_0^1(\Omega)$. Existence and uniqueness of a solution to (2.1) is guaranteed even for more general right-hand sides $f \in H^{-2}(\Omega)$ by the theorem of Lax-Milgram; see, e.g., [32, 34].

ETNA
Kent State University
http://etna.math.kent.edu

For the HHJ mixed method the auxiliary variable

$$\boldsymbol{w} = \nabla^2 w \tag{2.2}$$

is introduced, whose elements can be interpreted as bending moments in the context of linear elasticity. This allows to rewrite the biharmonic equation in (1.1) as a system of two second-order equations,

$$\nabla^2 w = \boldsymbol{w}, \quad \operatorname{div} \mathbf{div}\, \boldsymbol{w} = f \quad \text{in } \Omega, \tag{2.3}$$

with the following notation for a matrix-valued function $\boldsymbol{v}$ and a vector-valued function ϕ,

$$\mathbf{div}\, \boldsymbol{v} = \begin{bmatrix} \partial_1 \boldsymbol{v}_{11} + \partial_2 \boldsymbol{v}_{12} \\ \partial_1 \boldsymbol{v}_{21} + \partial_2 \boldsymbol{v}_{22} \end{bmatrix} \quad \text{and} \quad \operatorname{div} \phi = \partial_1 \phi_1 + \partial_2 \phi_2.$$

In the standard approach a mixed variational formulation is directly derived from the system (2.3). We take here a little detour, which better motivates the nonstandard Sobolev space we use in this paper for a modified mixed variational formulation. Starting point is the following unconstrained optimization problem: find $w \in H_0^2(\Omega)$ such that

$$J(w) = \min_{v \in H_0^2(\Omega)} J(v) \quad \text{with} \quad J(v) = \frac{1}{2} \int_\Omega \nabla^2 v : \nabla^2 v \, dx - \langle f, v \rangle. \tag{2.4}$$

It is well-known that (2.4) is equivalent to (2.1). Actually, (2.1) can be seen as the optimality system characterizing the solution of (2.4). By introducing the auxiliary variable $\boldsymbol{w} = \nabla^2 w \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$ with

$$\boldsymbol{L}^2(\Omega)_{\text{sym}} = \{\boldsymbol{v} \colon \boldsymbol{v}_{ji} = \boldsymbol{v}_{ij} \in L^2(\Omega),\ 1 \le i, j \le 2\},$$

equipped with the standard L^2-norm $\|\boldsymbol{v}\|_0$ for matrix-valued functions $\boldsymbol{v}$, the objective functional becomes a functional depending on the original and the auxiliary variable:

$$J(v, \boldsymbol{v}) = \frac{1}{2} \int_\Omega \boldsymbol{v} : \boldsymbol{v} \, dx - \langle f, v \rangle. \tag{2.5}$$

The weak formulation of (2.2) leads to the constraint

$$c((w, \boldsymbol{w}), \boldsymbol{\tau}) = 0 \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{M}, \tag{2.6}$$

where

$$c((v, \boldsymbol{v}), \boldsymbol{\tau}) = -\int_\Omega \boldsymbol{v} : \boldsymbol{\tau} \, dx - \int_\Omega \nabla v \cdot \mathbf{div}\, \boldsymbol{\tau} \, dx,$$

and $\boldsymbol{M}$ is a (not yet specified) space of sufficiently smooth matrix-valued test functions. By this the unconstrained optimization problem (2.4) is transformed to the following constrained optimization problem: find $(w, \boldsymbol{w}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}$ that minimizes the objective functional (2.5) subject to the constraint (2.6). The Lagrangian functional associated with this constrained optimization problem is given by

$$\mathscr{L}((v, \boldsymbol{v}), \boldsymbol{\tau}) = J(v, \boldsymbol{v}) + c((v, \boldsymbol{v}), \boldsymbol{\tau}),$$

which leads to the following first-order optimality system:

$$\begin{aligned} &\int_\Omega \boldsymbol{w} : \boldsymbol{v} \, dx + c((v, \boldsymbol{v}), \boldsymbol{\sigma}) = \langle f, v \rangle && \text{for all } (v, \boldsymbol{v}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}, \\ &c((w, \boldsymbol{w}), \boldsymbol{\tau}) = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{M}. \end{aligned} \tag{2.7}$$

ETNA
Kent State University
http://etna.math.kent.edu

Here $\boldsymbol{\sigma} \in \boldsymbol{M}$ denotes the Lagrangian multiplier associated with the constraint (2.6). The optimality system is a saddle point problem on the space $\boldsymbol{X} = H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}$, equipped with the standard norm

$$\|(v, \boldsymbol{v})\|_{\boldsymbol{X}} = \left(|v|_1^2 + \|\boldsymbol{v}\|_0^2\right)^{1/2}$$

for the primal variable $(v, \boldsymbol{v})$, and the (not yet specified) Hilbert space $\boldsymbol{M}$, equipped with a norm $\|\boldsymbol{\tau}\|_{\boldsymbol{M}}$ for the dual variable $\boldsymbol{\tau}$. An essential condition for the analysis of (2.7) is the inf-sup condition for the bilinear form c, which reads: there is a constant $\beta > 0$ such that

$$\sup_{0 \neq (v,\boldsymbol{v}) \in \boldsymbol{X}} \frac{c((v,\boldsymbol{v}),\boldsymbol{\tau})}{\|(v,\boldsymbol{v})\|_{\boldsymbol{X}}} \geq \beta \, \|\boldsymbol{\tau}\|_{\boldsymbol{M}}.$$

It is easy to see that

$$\sup_{0 \neq (v,\boldsymbol{v}) \in \boldsymbol{X}} \frac{c((v,\boldsymbol{v}),\boldsymbol{\tau})}{\|(v,\boldsymbol{v})\|_{\boldsymbol{X}}} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2\right)^{1/2} \tag{2.8}$$

for sufficiently smooth functions $\boldsymbol{\tau}$. If the right-hand side in (2.8) is chosen as the norm in $\boldsymbol{M}$, then the inf-sup condition is trivially satisfied with constant $\beta = 1$. This motivates to set $\boldsymbol{M} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ with

$$\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} \in H^{-1}(\Omega)\},$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2\right)^{1/2}. \tag{2.9}$$

Here $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$ is meant in the distributional sense. It is easy to see that $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ is a Hilbert space. In order to have a well-defined bilinear form c, the original definition has to be replaced by

$$c((v,\boldsymbol{v}),\boldsymbol{\tau}) = -\int_\Omega \boldsymbol{v} : \boldsymbol{\tau}\, dx + \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle,$$

which coincides with the original definition if $\boldsymbol{\tau}$ is sufficiently smooth, say $\boldsymbol{\tau} \in \boldsymbol{H}^1(\Omega)_{\text{sym}}$ with

$$\boldsymbol{H}^1(\Omega)_{\text{sym}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \boldsymbol{\tau}_{ij} \in H^1(\Omega),\ 1 \leq i,j \leq 2\},$$

equipped with the standard H^1-norm $\|\boldsymbol{\tau}\|_1$ and H^1-semi-norm $|\boldsymbol{\tau}|_1$ for matrix-valued functions $\boldsymbol{\tau}$. Observe that

$$\boldsymbol{H}^1(\Omega)_{\text{sym}} \subset \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}} \subset \boldsymbol{L}^2(\Omega)_{\text{sym}}.$$

From the first row of the optimality system (2.7) for $v = 0$ it easily follows that $\boldsymbol{w} = \boldsymbol{\sigma}$. So the auxiliary variable $\boldsymbol{w}$ can be eliminated and we obtain after reordering the following reduced optimality system: For $f \in H^{-1}(\Omega)$, find $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ and $w \in H_0^1(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}, \\ &-\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\sigma}, v\rangle = -\langle f, v\rangle && \text{for all } v \in H_0^1(\Omega). \end{aligned} \tag{2.10}$$

ETNA
Kent State University
http://etna.math.kent.edu

REMARK 2.1. The presented approach to derive the mixed method via the optimality system of a constrained optimization problem is the same approach as taken in [19] for the Ciarlet-Raviart mixed method. See [40] for a reformulation involving a similar nonstandard Sobolev space $H^{-1}(\Delta,\Omega) = \{v \in H^1(\Omega) : \Delta v \in H^{-1}(\Omega)\}$ as in this paper.

Problem (2.10) has the typical structure of a saddle point problem

$$
\begin{aligned}
a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau}, w) &= 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{V},\\
b(\boldsymbol{\sigma}, v) &= -\langle f, v\rangle && \text{for all } v \in Q.
\end{aligned}
\tag{2.11}
$$

If the linear operator $\mathcal{A}\colon \boldsymbol{V} \times Q \longrightarrow (\boldsymbol{V} \times Q)^*$ is introduced by

$$
\left\langle \mathcal{A}\begin{bmatrix}\boldsymbol{\sigma}\\ w\end{bmatrix}, \begin{bmatrix}\boldsymbol{\tau}\\ v\end{bmatrix}\right\rangle = a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau}, w) + b(\boldsymbol{\sigma}, v),
$$

then the mixed variational problem (2.11) can be rewritten as a linear operator equation

$$
\mathcal{A}\begin{bmatrix}\boldsymbol{\sigma}\\ w\end{bmatrix} = -\begin{bmatrix}0\\ f\end{bmatrix}.
$$

If the bilinear form a is symmetric and non-negative, i.e., $a(\boldsymbol{\sigma},\boldsymbol{\tau}) = a(\boldsymbol{\tau},\boldsymbol{\sigma})$ and $a(\boldsymbol{\tau},\boldsymbol{\tau}) \geq 0$, which is the case for (2.10), it is well-known that $\mathcal{A}$ is an isomorphism from $\boldsymbol{V} \times Q$ onto $(\boldsymbol{V} \times Q)^*$, if and only if the following conditions are satisfied; see, e.g., [10]:

1. a is bounded: there is a constant $\|a\| > 0$ such that
$$
|a(\boldsymbol{\sigma},\boldsymbol{\tau})| \leq \|a\|\, \|\boldsymbol{\sigma}\|_{\boldsymbol{V}}\, \|\boldsymbol{\tau}\|_{\boldsymbol{V}} \quad \text{for all } \boldsymbol{\sigma},\ \boldsymbol{\tau} \in \boldsymbol{V}.
$$
2. b is bounded: there is a constant $\|b\| > 0$ such that
$$
|b(\boldsymbol{\tau}, v)| \leq \|b\|\, \|\boldsymbol{\tau}\|_{\boldsymbol{V}} \|v\|_Q \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{V},\ v \in Q.
$$
3. a is coercive on the kernel of b: there is a constant $\alpha > 0$ such that
$$
a(\boldsymbol{\tau},\boldsymbol{\tau}) \geq \alpha\, \|\boldsymbol{\tau}\|_{\boldsymbol{V}}^2 \quad \text{for all } \boldsymbol{\tau} \in \ker B
$$
with $\ker B = \{\boldsymbol{\tau} \in \boldsymbol{V} \colon b(\boldsymbol{\tau}, v) = 0 \text{ for all } v \in Q\}$.
4. b satisfies the inf-sup condition: there is a constant $\beta > 0$ such that
$$
\inf_{0\neq v\in Q}\ \sup_{0\neq \boldsymbol{\tau}\in \boldsymbol{V}} \frac{b(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{\boldsymbol{V}}\, \|v\|_Q} \geq \beta.
$$

Here $\|\boldsymbol{\tau}\|_{\boldsymbol{V}}$ and $\|v\|_Q$ denote the norms in $\boldsymbol{V}$ and Q, respectively. We will refer to these conditions as Brezzi's conditions with constants $\|a\|$, $\|b\|$, α, and β. (We tacitly assume that $\|a\|$ and $\|b\|$ are the smallest constants for estimating the bilinear forms a and b. Then $\|a\|$ and $\|b\|$ match the standard notation for the norms of the bilinear forms a and b.)

In the next theorem we show that Brezzi's conditions are satisfied for (2.10). For the proof as well as for later use, we first introduce the following simple but useful notation for a function $v \in H_0^1(\Omega)$:

$$
\boldsymbol{\pi}(v) = v\,\boldsymbol{I} \quad \text{with} \quad \boldsymbol{I} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.
\tag{2.12}
$$

THEOREM 2.2. *The bilinear forms*

$$
a(\boldsymbol{\sigma},\boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx \quad \textit{and} \quad b(\boldsymbol{\tau}, v) = -\langle \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}, v\rangle
$$

ETNA
Kent State University
http://etna.math.kent.edu

satisfy Brezzi's conditions on $\boldsymbol{V} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym}$ *and* $Q = H_0^1(\Omega)$*, equipped with the norms* $\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}$ *and* $|v|_1$*, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = 1 \quad \text{and} \quad \beta = (1+2c_F^2)^{-1/2},$$

where c_F *denotes the constant in the Friedrichs' inequality:* $\|v\|_0 \le c_F\, |v|_1$ *for all* $v \in H_0^1(\Omega)$.

Proof.

1. Let $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$. Then

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})| \le \|\boldsymbol{\sigma}\|_0 \|\boldsymbol{\tau}\|_0 \le \|\boldsymbol{\sigma}\|_{-1,\operatorname{div}\mathbf{div}} \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}.$$

2. Let $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ and $v \in H_0^1(\Omega)$. Then

$$|b(\boldsymbol{\tau}, v)| = |\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle| \le \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}\, |v|_1 \le \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}\, |v|_1.$$

3. Observe that $\ker B = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} : \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} = 0\}$. Therefore,

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}^2 \quad \text{for all } \boldsymbol{\tau} \in \ker B.$$

4. Here we follow the proofs in [10, 17]. For $v \in H_0^1(\Omega)$ it is easy to see that

$$b(\boldsymbol{\pi}(v), v) = |v|_1^2 \quad \text{and} \quad \|\boldsymbol{\pi}(v)\|_{-1,\operatorname{div}\mathbf{div}}^2 = \|\boldsymbol{\pi}(v)\|_0^2 + |v|_1^2 \le (1+2c_F^2)\, |v|_1^2.$$

Therefore

$$\sup_{0\neq\boldsymbol{\tau}\in\boldsymbol{V}} \frac{|b(\boldsymbol{\tau},v)|}{\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}} \ge \frac{|b(\boldsymbol{\pi}(v),v)|}{\|\boldsymbol{\pi}(v)\|_{-1,\operatorname{div}\mathbf{div}}} = \frac{|v|_1^2}{\|\boldsymbol{\pi}(v)\|_0^2 + |v|_1^2)^{1/2}} \ge \frac{1}{(1+2c_F^2)^{1/2}} |v|_1. \qquad \square$$

COROLLARY 2.3. *The problems* (2.1) *and* (2.10) *are fully equivalent, i.e., if* $w \in H_0^2(\Omega)$ *solves* (2.1)*, then* $\boldsymbol{\sigma} = \nabla^2 w \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym}$ *and* $(\boldsymbol{\sigma}, w)$ *solves* (2.10)*. And, vice versa, if* $(\boldsymbol{\sigma}, w) \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym} \times H_0^1(\Omega)$ *solves* (2.1)*, then* $w \in H_0^2(\Omega)$ *and* w *solves* (2.1).

Proof. Both problems are uniquely solvable. Therefore, it suffices to show that $(w, \boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = \nabla^2 w$ solves (2.10), if w solves (2.1). So, assume that $w \in H_0^2(\Omega)$ is a solution of (2.1). Then, obviously, $\boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$ and

$$\int_\Omega \boldsymbol{\sigma} : \nabla^2 v \, dx = \langle f, v\rangle \quad \text{for all } v \in H_0^2(\Omega),$$

which implies that $\operatorname{div}\mathbf{div}\,\boldsymbol{\sigma} = f \in H^{-1}(\Omega)$ in the distributional sense. Therefore, $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ and the second row in (2.10) immediately follows.

By the definition of $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$ in the distributional sense we have

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v \, dx \quad \forall v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, it follows for $v = w$ that

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 w \, dx = \int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma} \, dx,$$

which shows the first row in (2.10). $\square$

REMARK 2.4. The space $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ was already introduced in [35, 37] within linear elasticity problems—not for the plate bending problem of linear elasticity considered here, however.

There is a natural trace operator associated with $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$, which was discussed in [35, 37]. We briefly recall here the basic properties for later reference.

Let the boundary Γ of the polygonal domain Ω be written in the form

$$\Gamma = \bigcup_{k=1}^{K} \overline{\Gamma}_k,$$

where Γ_k, $k = 1, 2, \ldots, K$, are the edges of Γ, considered as open line segments. $\overline{\Gamma}_k$ denotes the corresponding closed line segment. For $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ that are additionally twice continuously differentiable and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ we obtain the following identity by integration by parts,

$$\int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\tau})\, v\, dx = \int_\Omega \boldsymbol{\tau} : \nabla^2 v\, dx - \int_\Gamma \boldsymbol{\tau}_{nn}\, \partial_n v\, ds \tag{2.13}$$

with the outer normal unit vector n of Γ and

$$\boldsymbol{\tau}_{nn} = n^T \boldsymbol{\tau} n.$$

Following standard procedures this identity allows to extend the trace $\boldsymbol{\tau}_{nn}$ to all functions $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ as an element of the dual of the image of the Neumann traces of functions from $H^2(\Omega) \cap H_0^1(\Omega)$, i.e.,

$$\boldsymbol{\tau}_{nn} \in H_{pw}^{-1/2}(\Gamma) = \prod_{k=1}^{K} H^{-1/2}(\Gamma_k),$$

where $H^{-1/2}(\Gamma_k)$ is the dual of $\widetilde{H}^{1/2}(\Gamma_k)$; see [23] for details. Another widely used notation for $\widetilde{H}^{1/2}(\Gamma_k)$ is $H_{00}^{1/2}(\Gamma_k)$; see [32].

From (2.13) we obtain the corresponding Green's formula for $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v\, dx - \langle \boldsymbol{\tau}_{nn}, \partial_n v\rangle_\Gamma\,. \tag{2.14}$$

Here $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality product in a Hilbert space of functions on Γ.

3. A Helmholtz decomposition of $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$.

In this section we study some important structural properties of $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ which are helpful for analyzing the HHJ method in the next sections.

THEOREM 3.1. *For each $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ there is a unique decomposition*

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_1,$$

where $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$ for some $p \in H_0^1(\Omega)$ and $\boldsymbol{\tau}_1 \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$ with $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = 0$. Moreover,

$$\underline{c}\,\left(|p|_1^2 + \|\boldsymbol{\tau}_1\|_0^2\right) \le \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}^2 \le \overline{c}\,\left(|p|_1^2 + \|\boldsymbol{\tau}_1\|_0^2\right)$$

for all $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ with positive constants $\underline{c}$ and $\overline{c}$ that depend only on the constant c_F of the Friedrichs' inequality.

Proof. For $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$, let $p \in H^1_0(\Omega)$ be the unique solution to the variational problem

$$\int_\Omega \nabla p \cdot \nabla v \, dx = -\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle \quad \text{for all } v \in H^1_0(\Omega), \tag{3.1}$$

and set $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$. Since

$$-\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_0, v\rangle = \int_\Omega \nabla p \cdot \nabla v \, dx,$$

it follows that $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_0 = \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$, and, therefore, $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = 0$ for $\boldsymbol{\tau}_1 = \boldsymbol{\tau} - \boldsymbol{\tau}_0$ in the distributional sense. On the other hand, if $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_1$ with $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$ and $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = 0$, then $-\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_0 = -\operatorname{div}\mathbf{div}\,\boldsymbol{\tau} + \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = -\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$, which implies (3.1). This shows the uniqueness.

Furthermore, (3.1) implies $|\boldsymbol{\tau}_0|_1^2 = 2\,|p|_1^2 = 2\,\|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2$. Hence

$$\begin{aligned}\|\boldsymbol{\tau}\|^2_{-1,\operatorname{div}\mathbf{div}} &= \|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2 = \|\boldsymbol{\tau}_0 + \boldsymbol{\tau}_1\|_0^2 + |p|_1^2 \\ &\le 2\,\|\boldsymbol{\tau}_0\|_0^2 + 2\,\|\boldsymbol{\tau}_1\|_0^2 + |p|_1^2 \le \left(1 + 4c_F^2\right)\,|p|_1^2 + 2\,\|\boldsymbol{\tau}_1\|_0^2\end{aligned}$$

and

$$\begin{aligned}|p|_1^2 + \|\boldsymbol{\tau}_1\|_0^2 &= |p|_1^2 + \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\|_0^2 \le |p|_1^2 + 2\,\|\boldsymbol{\tau}\|_0^2 + 2\,\|\boldsymbol{\tau}_0\|_0^2 \\ &\le 2\,\|\boldsymbol{\tau}\|_0^2 + (1 + 4c_F^2)\,|p|_1^2 = 2\,\|\boldsymbol{\tau}\|_0^2 + (1 + 4c_F^2)\,\|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2.\end{aligned}$$

Then the estimates immediately follow with the constants $1/\underline{c} = \overline{c} = \max(2, 1 + 4c_F^2)$. ∎

In short, we have algebraically as well as topologically

$$\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}} = \boldsymbol{\pi}(H^1_0(\Omega)) \oplus \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$$

with

$$\mathscr{H}(\operatorname{div}\mathbf{div},\Omega) = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} = 0\}.$$

Here $\oplus$ denotes the direct sum of Hilbert spaces.

REMARK 3.2. The Helmholtz decomposition of $\boldsymbol{L}^2(\Omega)_{\text{sym}}$ in [28], based on previous results in [6], has the same second component. The first component in [6, 28] is different and requires the solution of a biharmonic problem in contrast to Theorem 3.1, where the first component requires to solve only a Poisson problem.

Next an explicit characterization of $\mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$ is given.

THEOREM 3.3. *Let Ω be simply connected. For each $\boldsymbol{\tau} \in \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$, there is a function $\phi \in \left(H^1(\Omega)\right)^2$ such that*

$$\boldsymbol{\tau} = \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \quad \textit{with} \quad \boldsymbol{H} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{\varepsilon}(\phi)_{ij} = \frac{1}{2}\left(\partial_j \phi_i + \partial_i \phi_j\right).$$

On the other hand, each function of the form $\boldsymbol{\tau} = \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi)\boldsymbol{H}$ with $\phi \in \left(H^1(\Omega)\right)^2$ lies in $\mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$. The function ϕ is unique up to an element from

$$RM = \left\{\boldsymbol{\tau}(x) = a \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + b \colon a \in \mathbb{R},\ b \in \mathbb{R}^2\right\},$$

ETNA
Kent State University
http://etna.math.kent.edu

and there is a constant c_K *such that*

$$c_K \|\phi\|_1 \le \|\boldsymbol{\tau}\|_0 = \|\boldsymbol{\varepsilon}(\phi)\|_0 \le \|\phi\|_1 \quad \textit{for all } \phi \in \left(H^1(\Omega)\right)^2/RM.$$

Proof. From [6, Lemma 1] it follows that $\boldsymbol{\tau} \in \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$ can be written in the following way:

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & -\rho \\ \rho & 0 \end{bmatrix} + \mathbf{Curl}\,\phi \quad \text{with} \quad \mathbf{Curl}\,\phi = \begin{bmatrix} -\partial_2\phi_1 & \partial_1\phi_1 \\ -\partial_2\phi_2 & \partial_1\phi_2 \end{bmatrix}$$

for some $\rho \in L^2_0(\Omega)$ and $\phi \in \left(H^1(\Omega)\right)^2$. The proof in [6] relies on the characterization of the kernel of the divergence operator in $\left(L^2(\Omega)\right)^2$, which is well-known (see, e.g., [22, Theorem 3.1]) and in $\left(H^{-1}(\Omega)\right)^2$, for which we refer to Lemma A.1 in the appendix.

To ensure symmetry of $\boldsymbol{\tau}$, it follows (see [28]) that

$$\rho = \frac{1}{2}\operatorname{div}\phi.$$

Replacing $\phi = (\phi_1,\phi_2)^T$ by $(-\phi_2,\phi_1)^T$ yields the representation. The estimates follow from Korn's inequality. ∎

Therefore, we have the following representation of the solution $\boldsymbol{\sigma}$ to (2.10):

$$\boldsymbol{\sigma} = \boldsymbol{\pi}(p) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H}. \tag{3.2}$$

The analogous representation for the test functions $\boldsymbol{\tau} = \boldsymbol{\pi}(q) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\psi)\boldsymbol{H}$ leads to the following equivalent formulation of (2.10). Find $p \in H^1_0(\Omega)$, $\phi \in \left(H^1(\Omega)\right)^2/\text{RM}$, $w \in H^1_0(\Omega)$ such that

$$\begin{aligned}
\int_\Omega \boldsymbol{\pi}(p) : \boldsymbol{\pi}(q)\,dx + \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi)\,dx + \int_\Omega \nabla w\cdot\nabla q\,dx &= 0,\\
\int_\Omega \boldsymbol{\pi}(p) : \boldsymbol{\varepsilon}(\psi)\,dx + \int_\Omega \boldsymbol{\varepsilon}(\phi) : \boldsymbol{\varepsilon}(\psi)\,dx &= 0,\\
\int_\Omega \nabla p\cdot\nabla v\,dx &= -\langle f, v\rangle
\end{aligned} \tag{3.3}$$

for all $q \in H^1_0(\Omega)$, $\psi \in \left(H^1(\Omega)\right)^2/\text{RM}$, $v \in H^1_0(\Omega)$.

Observe that $\boldsymbol{\pi}(p) : \boldsymbol{\pi}(q) = 2\,p\,q$ and $\boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\psi) = q\,\operatorname{div}\psi$, which allows to simplify parts of (3.3).

In summary, the biharmonic problem is equivalent to three (consecutively to solve) elliptic second-order problems. The first problem is a Poisson problem with Dirichlet boundary conditions for p, which reads in the strong form

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma.$$

The second problem is a pure traction problem in linear elasticity with Poisson ratio 0 for ϕ, which reads in the strong form

$$-\mathbf{div}\,\boldsymbol{\varepsilon}(\phi) = \nabla p \quad \text{in } \Omega, \quad \boldsymbol{\varepsilon}(\phi)\,n = 0 \quad \text{on } \Gamma.$$

And, finally, the third problem is a Poisson problem with Dirichlet boundary conditions for the original variable w, which reads in the strong form

$$\Delta w = 2\,p + \operatorname{div}\phi \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma.$$

ETNA
Kent State University
http://etna.math.kent.edu

4. The Hellan-Herrmann-Johnson method. Let $\mathcal{T}_h$ be an admissible triangulation of the polygonal domain Ω. For $k \in \mathbb{N}$ the standard finite element spaces $\mathcal{S}_h$ and $\mathcal{S}_{h,0}$ are given by

$$\mathcal{S}_h = \{v \in C(\overline{\Omega})\colon v|_T \in P_k \text{ for all } T \in \mathcal{T}_h\} \quad \text{and} \quad \mathcal{S}_{h,0} = \mathcal{S}_h \cap H_0^1(\Omega),$$

where P_k denotes the set of bivariate polynomials of total degree less than or equal to k.

A quite natural discretization of (3.3) is the following conforming method. Find $p_h \in \mathcal{S}_{h,0}$, $\phi_h \in (\mathcal{S}_h)^2/\text{RM}$, $w_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\pi}(q) dx + \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi_h)\, dx + \int_\Omega \nabla w_h \cdot \nabla q\, dx = 0,\\
&\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\varepsilon}(\psi)\, dx + \int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi)\, dx = 0,\\
&\int_\Omega \nabla p_h \cdot \nabla v\, dx = -\langle f, v\rangle
\end{aligned} \tag{4.1}$$

for all $q \in \mathcal{S}_{h,0}$, $\psi \in (\mathcal{S}_h)^2/\text{RM}$, $v \in \mathcal{S}_{h,0}$. In this and the subsequent section we will see that this method is strongly related to the HHJ method, which we introduce next.

For the approximation of the Lagrangian multiplier $\boldsymbol{\sigma}$, the HHJ method uses the finite element space

$$\begin{aligned}
\boldsymbol{V}_h = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}\colon \boldsymbol{\tau}|_T \in P_{k-1} \text{ for all } T \in \mathcal{T}_h, \text{ and}\\
\boldsymbol{\tau}_{nn} \text{ is continuous across inter-element boundaries}\}.
\end{aligned}$$

For the approximation of the original variable w the standard finite element space

$$Q_h = \mathcal{S}_{h,0}$$

is used. So, the HHJ method reads as follows: find $\boldsymbol{\sigma}_h \in \boldsymbol{V}_h$ and $w_h \in Q_h$ such that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau}\, dx - \langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}, w_h\rangle = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{V}_h,\\
&- \langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\sigma}_h, v\rangle = -\langle f, v\rangle && \text{for all } v \in Q_h
\end{aligned} \tag{4.2}$$

with

$$\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}, v\rangle = \sum_T \left\{ \int_T \boldsymbol{\tau} : \nabla^2 v\, dx - \int_{\partial T} \boldsymbol{\tau}_{nn}\, \partial_n v\, ds \right\} \quad \text{for } \boldsymbol{\tau} \in \boldsymbol{V}_h,\ v \in Q_h.$$

Similarly to the linear functional $\operatorname{div} \mathbf{div}\, \boldsymbol{\tau} \in H^{-1}(\Omega)$ (for $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\text{sym}}$), we consider $\operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}$ (for $\boldsymbol{\tau} \in \boldsymbol{V}_h$) as a linear functional from the dual of Q_h. And as introduced in Section 2 we use $\langle \cdot, \cdot \rangle$ as the generic symbol for duality products, here for the Hilbert space Q_h.

If compared with (2.14), this definition of $\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}, v\rangle$ for $\boldsymbol{\tau} \in \boldsymbol{V}_h$ and $v \in Q_h$ in the HHJ method is just an element-wise assembled version of corresponding expressions on the continuous level, a standard technique in non-conforming methods.

REMARK 4.1. Using integration by parts we obtain

$$\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}, v\rangle = - \sum_{T \in \mathcal{T}_h} \left\{ \int_T \mathbf{div}\, \boldsymbol{\tau} \cdot \nabla v\, dx - \int_{\partial T} \boldsymbol{\tau}_{ns}\, \partial_s v\, ds \right\}$$

ETNA
Kent State University
http://etna.math.kent.edu

with the normal vector $n = (n_1, n_2)^T$, the vector $s = (-n_2, n_1)^T$, which is tangent to Γ, the tangential derivative ∂_s, and

$$\boldsymbol{\tau}_{ns} = s^T \boldsymbol{\tau} n.$$

The HHJ method is often formulated with this representation, which allows an extension to all functions $\boldsymbol{\tau}$ from the (mesh-dependent) infinite-dimensional function space

$$\widetilde{\boldsymbol{V}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \boldsymbol{\tau}_{ij}|_T \in H^1(T) \text{ for all } T \in \mathcal{T}_h,\ 1 \le i,j \le 2, \text{ and } \boldsymbol{\tau}_{nn} \text{ is continuous across inter-element boundaries}\}. \tag{4.3}$$

This space was used for the analysis of the method in [3, 17, 21] and others. Existence and uniqueness of a solution for the corresponding variational problem on the continuous level could be shown under additional smoothness assumptions. For the approach taken in this paper, this is not required.

Similar to the continuous case, the well-posedness of the discrete problem can be shown. For the proof of the discrete inf-sup condition, the discrete analogue to $\boldsymbol{\pi}(v)$ (see (2.12)) is needed. For $v_h \in \mathcal{S}_{h,0}$, we define

$$\boldsymbol{\pi}_h(v_h) = \boldsymbol{\Pi}_h \boldsymbol{\pi}(v_h)$$

with the linear operator $\boldsymbol{\Pi}_h$, introduced in [17] by the conditions

$$\int_e ((\boldsymbol{\tau}_h)_{nn} - \boldsymbol{\tau}_{nn})\, q\, ds = 0, \quad \text{for all } q \in P_{k-1},\ \text{for all edges } e \text{ of } T,\ T \in \mathcal{T}_h, \tag{4.4}$$

and

$$\int_T ((\boldsymbol{\tau}_h)_{ij} - \boldsymbol{\tau}_{ij})\, q\, dx = 0, \quad \text{for all } q \in P_{k-2},\ T \in \mathcal{T}_h,\ 1 \le i,j \le 2, \tag{4.5}$$

for $\boldsymbol{\tau}_h = \boldsymbol{\Pi}_h \boldsymbol{\tau} \in \boldsymbol{V}_h$ and $\boldsymbol{\tau} \in \boldsymbol{\pi}(Q_h)$. Observe that $\boldsymbol{\Pi}_h$ was originally introduced in [17] as a linear operator on the infinite-dimensional space $\widetilde{\boldsymbol{V}}$ from above.

From the corresponding properties of $\boldsymbol{\Pi}_h$ in [17, Lemma 4], the next result directly follows.

LEMMA 4.2. *Assume that $\mathcal{T}_h$ is a regular family of triangulation. Then there exists a constant $c_B > 0$ that is independent of h such that*

$$\|\boldsymbol{\pi}_h(v)\|_0 \le c_B\, |v|_1 \quad \textit{for all } v \in \mathcal{S}_{h,0}.$$

Moreover, we need the following simple identity.

LEMMA 4.3. *For all $p, v \in \mathcal{S}_{h,0}$, we have*

$$-\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\pi}_h(p), v\rangle = \int_\Omega \nabla p \cdot \nabla v\, dx.$$

Proof. By integration by parts we have

$$\begin{aligned}
\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\pi}_h(p), v\rangle &= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \boldsymbol{\Pi}_h \boldsymbol{\pi}(p) : \nabla^2 v\, dx - \int_{\partial T} (\boldsymbol{\Pi}_h \boldsymbol{\pi}(p))_{nn}\, \partial_n v\, ds \right\} \\
&= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \boldsymbol{\pi}(p) : \nabla^2 v\, dx - \int_{\partial T} (\boldsymbol{\pi}(p))_{nn}\, \partial_n v\, ds \right\} \\
&= \sum_{T\in\mathcal{T}_h} \left\{ \int_T p\, \Delta v\, dx - \int_{\partial T} p\, \partial_n v\, ds \right\} = -\int_\Omega \nabla p \cdot \nabla v\, dx. \qquad \square
\end{aligned}$$

Now the well-posedness of the discrete problem can be shown.

THEOREM 4.4. *The bilinear forms*

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx, \quad b_h(\boldsymbol{\tau}, v) = -\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v\rangle$$

satisfy Brezzi's conditions on $\boldsymbol{V}_h$ and Q_h, equipped with the norms $\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}$ and $|v|_1$, respectively, where

$$\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}_h \boldsymbol{\tau}\|_{-1,h}^2\right)^{1/2} \tag{4.6}$$

and

$$\|\ell\|_{-1,h} = \sup_{v_h \in S_{h,0}} \frac{|\langle \ell, v_h\rangle|}{|v_h|_1} \quad \textit{for } \ell \in (\mathcal{S}_{h,0})^*,$$

with the constants

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = (1 + c_B^2)^{-1/2},$$

where c_B denotes the constant in Lemma 4.2.

Proof.

1. Let $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{V}_h$. Then

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})| \le \|\boldsymbol{\sigma}\|_0\, \|\boldsymbol{\tau}\|_0 \le \|\boldsymbol{\sigma}\|_{-1,\operatorname{div}\mathbf{div},h}\, \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}.$$

2. Let $\boldsymbol{\tau} \in \boldsymbol{V}_h$ and $v \in Q_h$. Then

$$|b(\boldsymbol{\tau}, v)| = |\langle \operatorname{div}\mathbf{div}_h \boldsymbol{\tau}, v\rangle \le \|\operatorname{div}\mathbf{div}_h \boldsymbol{\tau}\|_{-1,h} |v|_1 \le \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h} \|v\|_1.$$

3. Observe that $\ker B_h = \{\boldsymbol{\tau} \in \boldsymbol{V}_h : \operatorname{div}\mathbf{div}\, \boldsymbol{\tau}_h = 0\}$. Therefore,

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}^2 \quad \text{for } \boldsymbol{\tau} \in \ker B_h.$$

4. From Lemma 4.2 and Lemma 4.3 we obtain for $v \in Q_h$,

$$b_h(\boldsymbol{\pi}_h(v), v) = |v|_1^2$$

and

$$\|\boldsymbol{\pi}_h(v)\|_{-1,\operatorname{div}\mathbf{div},h}^2 = \|\boldsymbol{\pi}_h(v)\|_0^2 + |v|_1^2 \le (1 + c_B^2)\, |v|_1^2.$$

Therefore,

$$\sup_{0 \ne \boldsymbol{\tau} \in \boldsymbol{V}_h} \frac{|b_h(\boldsymbol{\tau}, v)|}{\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}} \ge \frac{|b_h(\boldsymbol{\pi}_h(v), v)|}{\|\boldsymbol{\pi}_h(v)\|_{-1,\operatorname{div}\mathbf{div},h}} = \frac{|v|_1^2}{(\|\boldsymbol{\pi}_h(v)\|_0^2 + |v|_1^2)^{1/2}}$$
$$\ge \frac{1}{(1 + c_B^2)^{1/2}} |v|_1. \qquad \square$$

Observe that the norms introduced for the space $\boldsymbol{V} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$ in (2.9) and its discrete counterpart $\boldsymbol{V}_h$ in (4.6) are similar but different. For the discrete problem the norm is mesh-dependent.

ETNA
Kent State University
http://etna.math.kent.edu

5. A discrete Helmholtz decomposition. We have the following discrete version of Theorem 3.1.

THEOREM 5.1. *For each $\boldsymbol{\tau} \in \boldsymbol{V}_h$, there is a unique decomposition*

$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}_0 + \hat{\boldsymbol{\tau}}_1,$$

where $\hat{\boldsymbol{\tau}}_0 = \boldsymbol{\pi}_h(\hat{p})$ for some $\hat{p} \in Q_h$ and $\hat{\boldsymbol{\tau}}_1 \in \boldsymbol{V}_h$ with $\operatorname{div} \mathbf{div}_h \hat{\boldsymbol{\tau}}_1 = 0$. Moreover,

$$\underline{c}\ \left(|\hat{p}|_1^2 + \|\hat{\boldsymbol{\tau}}_1\|_0^2 \right) \leq \|\boldsymbol{\tau}\|^2_{-1,\operatorname{div}\mathbf{div},h} \leq \overline{c}\ \left(|\hat{p}|_1^2 + \|\hat{\boldsymbol{\tau}}_1\|_0^2 \right)$$

for all $\boldsymbol{\tau} \in \boldsymbol{V}_h$, with positive constants $\underline{c}$ and $\overline{c}$, which depend only on the constant c_B of the inequality in Lemma 4.2.

The proof is completely analogous to the proof for the continuous case and is therefore omitted. The only difference is the use of the estimate from Lemma 4.2 instead of the Friedrichs' inequality.

So, in short,

$$\boldsymbol{V}_h = \boldsymbol{\pi}_h(\mathcal{S}_{h,0}) \oplus \mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$$

with

$$\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = \{\boldsymbol{\tau} \in \boldsymbol{V}_h \colon \langle \operatorname{div}\mathbf{div}_h \boldsymbol{\tau}, v_h\rangle = 0 \text{ for all } v_h \in Q_h\}.$$

For describing the space $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$ more explicitly, we consider the subspace of all functions in $\mathscr{H}(\operatorname{div}\mathbf{div}, \Omega)$ that can be represented by a finite element function $\phi \in (\mathcal{S}_h)^2$, for which we show the following result.

THEOREM 5.2. *Let Ω be simply connected. Then*

$$\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = \{\boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\}.$$

Proof. Let $\phi \in (\mathcal{S}_h)^2$. Then $\boldsymbol{\tau} = \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} \in P_{k-1}$ for all triangles $T \in \mathcal{T}_h$. Furthermore, let e be an edge of a triangle T with outer unit normal vector $n = (n_1, n_2)^T$ and unit tangent vector $s = (-n_2, n_1)^T$. By elementary computations we obtain

$$\boldsymbol{\tau}_{nn} = n^T \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} n = s \cdot \partial_s \phi.$$

So, $\boldsymbol{\tau}_{nn}$ depends only on values of ϕ on the edge e, which immediately implies that $\boldsymbol{\tau}_{nn}$ is continuous on inter-element boundaries. This shows that $\boldsymbol{\tau}$ lies in $\boldsymbol{V}_h$, and therefore the inclusion $\{\boldsymbol{\tau} = \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\} \subset \mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$ follows.

The equality follows by comparing the dimensions. We have

$$\dim\{\boldsymbol{\tau} = \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\} = 2 \dim \mathcal{S}_h - \dim \mathrm{RM} = 2 \dim \mathcal{S}_h - 3.$$

On the other hand, by Theorem 5.1, it follows that

$$\dim \mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = \dim \boldsymbol{V}_h - \dim \mathcal{S}_{h,0}.$$

A simple count of the degrees of freedom for $\boldsymbol{V}_h$ yields

$$\dim \boldsymbol{V}_h = \dim \mathcal{S}_{h,0} + 2\ \dim \mathcal{S}_h - 3.$$

Therefore, $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = 2\ \dim \mathcal{S}_h - 3$, which completes the proof. □

REMARK 5.3. A consequence of the last theorem is the important inclusion

$$\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) \subset \mathscr{H}(\operatorname{div}\mathbf{div}, \Omega),$$

that resembles the corresponding result of [17, Lemma 5].

ETNA
Kent State University
http://etna.math.kent.edu

Therefore, we have the following representation of the approximate solution $\boldsymbol{\sigma}_h \in \boldsymbol{V}_h$ of (4.2):

$$\boldsymbol{\sigma}_h = \boldsymbol{\pi}_h(p_h) + \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi_h)\boldsymbol{H}. \tag{5.1}$$

The analogous representation for the test functions $\boldsymbol{\tau} = \boldsymbol{\pi}_h(q) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\psi)\boldsymbol{H}$ leads to the following equivalent formulation of (4.2). Find $p_h \in \mathcal{S}_{h,0}$, $\phi_h \in (\mathcal{S}_h)^2/\text{RM}$, $w_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned}
&\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \hat{\boldsymbol{\pi}}_h(q) dx + \int_\Omega \hat{\boldsymbol{\pi}}_h(q) : \boldsymbol{\varepsilon}(\phi_h)\, dx + \int_\Omega \nabla w_h \cdot \nabla q \, dx = 0, \\
&\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \boldsymbol{\varepsilon}(\psi)\, dx \ + \int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi)\, dx = 0, \\
&\int_\Omega \nabla p_h \cdot \nabla v \, dx = -\langle f, v\rangle
\end{aligned} \tag{5.2}$$

for all $q \in \mathcal{S}_{h,0}$, $\psi \in (\mathcal{S}_h)^2/\text{RM}$, $v \in \mathcal{S}_{h,0}$, and with

$$\hat{\boldsymbol{\pi}}_h(q) = \boldsymbol{H}\boldsymbol{\pi}_h(q)\boldsymbol{H}^T.$$

Observe that the HHJ method in the form of (5.2) is a non-conforming method for (3.3), while (4.1) can be seen as a conforming variant. Compared to (5.2), the conforming variant is slightly less costly since the linear operator $\boldsymbol{\Pi}_h$ is not needed.

6. Error estimates. Since the new mixed variational formulation is equivalent to the original primal variational formulation (without any additional assumption on Ω) and the finite element space $\boldsymbol{V}_h$ of the original Hellan-Herrmann-Johnson method has not changed (but only its representation, which amounts to a change of basis), all known error estimates for the original Hellan-Herrmann-Johnson method are still valid. For completeness we briefly recall some of the most important estimates from [3, 9, 17, 21] and give a sketch of the proofs using the framework of the new variational formulation. Afterwards we will present new error estimates for the conforming variant (4.1).

6.1. The original Hellan-Herrmann-Johnson method. We closely follow the approach in [3], which is based on two essential assumptions, regularity and consistency. Regularity is ensured in [3] by considering only convex domains Ω. Consistency is ensured in [3] by setting up a framework in which the Hellan-Herrmann-Johnson method becomes a conforming method. Then the estimates easily follow from the existence of two interpolation operators $\boldsymbol{\Pi}_h$ and I_h for $\boldsymbol{\sigma}$ and w, respectively, and their approximation properties.

Here we recall a (weaker) regularity result from [8, 9], which is valid without any additional assumption on Ω.

THEOREM 6.1. *If $f \in H^{-1}(\Omega)$, then the solution w of (2.1) lies in $W^{3,p}(\Omega)$ for some $p \in (4/3, 2]$ and there is a constant $c > 0$ such that*

$$\|w\|_{W^{3,p}(\Omega)} \le c\, \|f\|_{-1}. \tag{6.1}$$

Actually, the regularity results in [8, 9] cover a much larger class of boundary conditions. Although the Hellan-Herrman-Johnson method is a non-conforming method with respect to (2.10), we still have consistency. In order to show this, we need to extend the bilinear form b_h, originally defined on $\boldsymbol{V}_h \times Q_h$ by

$$b_h(\boldsymbol{\tau}, v) = -\sum_{T\in\mathcal{T}_h} \left\{ \int_T \boldsymbol{\tau} : \nabla^2 v\, dx - \int_{\partial T} \boldsymbol{\tau}_{nn}\, \partial_n v \, ds \right\},$$

ETNA
Kent State University
http://etna.math.kent.edu

to the larger domain $\widetilde{\boldsymbol{V}} \times \widetilde{Q}$, given by

$$\widetilde{\boldsymbol{V}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \boldsymbol{\tau}_{ij}|_T \in W^{1,p}(T) \text{ for all } T \in \mathcal{T}_h,\ 1 \leq i,j \leq 2, \text{ and } \\ \boldsymbol{\tau}_{nn} \text{ is continuous across inter-element boundaries}\}$$

and

$$\widetilde{Q} = \{v \in H_0^1(\Omega) \colon v|_T \in H^2(T) \text{ for all } T \in \mathcal{T}_h\}.$$

The well-definedness of b_h on $\widetilde{\boldsymbol{V}} \times \widetilde{Q}$ follows from standard embedding theorems for Sobolev spaces; see, e.g., [1]. Observe that $\widetilde{\boldsymbol{V}}$ coincides with the space in (4.3) for $p = 2$. Therefore, we keep the same notation.

Now we can show the following consistency result.

THEOREM 6.2. *For the solution $(\boldsymbol{\sigma}, w)$ of (2.10), we have*

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) \ + b_h(\boldsymbol{\tau}, w) &= 0 && \text{for all } \boldsymbol{\tau} \in \widetilde{\boldsymbol{V}}, \\ b_h(\boldsymbol{\sigma}, v) &= -\langle f, v\rangle && \text{for all } v \in \widetilde{Q}. \end{aligned} \tag{6.2}$$

Proof. From Theorem 6.1 it follows that $b_h(\boldsymbol{\tau}, w)$ and $b_h(\boldsymbol{\sigma}, v)$ are well-defined for all $(\boldsymbol{\tau}, v) \in \widetilde{\boldsymbol{V}} \times \widetilde{Q}$.

We have

$$\begin{aligned} b_h(\boldsymbol{\tau}, w) &= -\sum_{T\in\mathcal{T}_h} \left\{ \int_T \boldsymbol{\tau} : \nabla^2 w \, dx - \int_{\partial T} \boldsymbol{\tau}_{nn}\, \partial_n w \, ds \right\} \\ &= -\sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\tau} : \nabla^2 w\, dx = -\int_\Omega \boldsymbol{\tau} : \nabla^2 w \, dx \end{aligned}$$

since $\boldsymbol{\tau}_{nn}$ is continuous and $\partial_n w$ changes sign across interelement boundaries and $\partial_n w = 0$ on $\partial\Omega$. With $\nabla^2 w = \boldsymbol{\sigma}$, the first line in (6.2) follows.

Moreover, we have

$$\begin{aligned} b_h(\boldsymbol{\sigma}, v) &= -\sum_{T\in\mathcal{T}_h} \left\{ \int_T \boldsymbol{\sigma} : \nabla^2 v \, dx - \int_{\partial T} \boldsymbol{\sigma}_{nn}\, \partial_n v \, ds \right\} \\ &= \sum_{T\in\mathcal{T}_h} \left\{ \int_T (\mathbf{div}\, \boldsymbol{\sigma}) \cdot \nabla v \, dx - \int_{\partial T} (\boldsymbol{\sigma} n) \cdot \nabla v \, ds + \int_{\partial T} \boldsymbol{\sigma}_{nn}\, \partial_n v \, ds \right\} \\ &= \sum_{T\in\mathcal{T}_h} \left\{ \int_T (\mathbf{div}\, \boldsymbol{\sigma}) \cdot \nabla v \, dx - \int_{\partial T} \boldsymbol{\sigma}_{ns}\, \partial_s v \, ds \right\} = \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{div}\, \boldsymbol{\sigma}) \cdot \nabla v \, dx \end{aligned}$$

since $\partial_s v$ is continuous and $\boldsymbol{\sigma}_{ns}$ changes sign across interelement boundaries and $v = 0$ on $\partial\Omega$. Therefore,

$$b_h(\boldsymbol{\sigma}, v) = \int_\Omega (\mathbf{div}\, \boldsymbol{\sigma}) \cdot \nabla v \, dx = -\langle \operatorname{div} \mathbf{div}\, \boldsymbol{\sigma}, v\rangle,$$

where the last identity follows from the corresponding identity

$$\langle \operatorname{div} \mathbf{div}\, \boldsymbol{\sigma}, v\rangle = \int_\Omega \boldsymbol{\sigma} : \nabla^2 v \, dx = -\int_\Omega (\mathbf{div}\, \boldsymbol{\sigma}) \cdot \nabla v \, dx \quad \text{for all } v \in C_0^\infty(\Omega)$$

ETNA
Kent State University
http://etna.math.kent.edu

by a continuity argument in $H_0^1(\Omega)$. With $\operatorname{div} \mathbf{div}\, \boldsymbol{\sigma} = f$ the second line in (6.2) follows, which completes the proof. □

Having again regularity and consistency we proceed as in [3, 17]. The interpolation operator $\boldsymbol{\Pi}_h$ is defined on $\widetilde{\boldsymbol{V}}$ by the conditions (4.4) and (4.5) for $\boldsymbol{\tau}_h = \boldsymbol{\Pi}_h \boldsymbol{\tau} \in \boldsymbol{V}_h$ and $\boldsymbol{\tau} \in \widetilde{\boldsymbol{V}}$. The second interpolation operator I_h is given by the conditions

$$\begin{aligned} v_h(a) &= v(a) && \text{for all vertices } a \text{ of } T,\ T \in \mathcal{T}_h, \\ \int_e v_h\, q\, ds &= \int_e v\, q\, ds, && \text{for all } q \in P_{k-2},\ \text{for all edges } e \text{ of } T,\ T \in \mathcal{T}_h, \\ \int_T v_h\, q\, dx &= \int_T v\, q\, dx, && \text{for all } q \in P_{k-3},\ T \in \mathcal{T}_h, \end{aligned}$$

for $v_h = I_h v \in Q_h$ and $v \in \widetilde{Q}$. It is easy to see that the two interpolation operators satisfy the following properties.

$$\begin{aligned} b_h(\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}, v_h) &= 0 \quad \text{for all } \boldsymbol{\tau} \in \widetilde{\boldsymbol{V}},\ v_h \in Q_h, \\ b_h(\boldsymbol{\tau}_h, v - I_h v) &= 0 \quad \text{for all } \boldsymbol{\tau}_h \in \boldsymbol{V}_h,\ v \in \widetilde{Q}. \end{aligned}$$

The rest of the arguments are identical to the arguments used in [3] with a straightforward adaptation to the weaker regularity condition (6.1), where it is needed. This leads directly to the following known estimate for $\boldsymbol{\sigma}$; see [3].

THEOREM 6.3. *Let* $(\boldsymbol{\sigma}, w)$ *and* $(\boldsymbol{\sigma}_h, w_h)$ *solve* (2.10) *and* (4.2)*, respectively. Then*

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_0 \le \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_0.$$

The error estimates for w read as follows. For completeness we give a sketch of the proof, which closely follows the arguments in [3].

THEOREM 6.4. *Assume that* (6.1) *is satisfied for some* $p \in (4/3, 2]$. *Then there is a constant* $c > 0$ *such that*

$$|w_h - w|_1 \le |w - I_h w|_1 + \begin{cases} c\, h\, |w|_{3,p} & \text{for } k = 1, \\ c\, h^{2-2/p}\, \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{0,h} & \text{for } k \ge 2, \end{cases}$$

with the mesh-dependent norm $\|\boldsymbol{\tau}\|_{0,h}$*, given by*

$$\|\boldsymbol{\tau}\|_{0,h}^2 = \|\boldsymbol{\tau}\|_0^2 + h \sum_{e \in \mathcal{E}_h} \|(\boldsymbol{\tau})_{nn}\|_{L^2(e)}^2.$$

Here $\mathcal{E}_h$ *denotes the set of all edges of triangles from* $\mathcal{T}_h$.

Proof. By the triangle inequality we have

$$|w_h - w|_1 \le |w - I_h w|_1 + |I_h w - w_h|_1.$$

For estimating $|I_h w - w_h|_1$ a duality trick is used. For $d \in H^{-1}(\Omega)$, let $w_d \in H_0^2(\Omega)$ be the unique solution to

$$\int_\Omega \nabla^2 w_d : \nabla^2 v\, dx = \langle d, v \rangle \quad \text{for all } v \in H_0^2(\Omega),$$

and set $\boldsymbol{\sigma}_d = \nabla^2 w_d$. Then it follows from Theorem 6.1 that $w_d \in W^{3,p}(\Omega)$, $\boldsymbol{\sigma}_d \in \boldsymbol{W}^{1,p}(\Omega)$, and

$$\begin{aligned} a(\boldsymbol{\sigma}_d, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, w_d) &= 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}, \\ b(\boldsymbol{\sigma}_d, v) &= -\langle d, v\rangle && \text{for all } v \in H_0^1(\Omega). \end{aligned}$$

From the consistency result in Theorem 6.2 applied to this problem it follows that:

$$\begin{aligned} a(\boldsymbol{\sigma}_d, \widetilde{\boldsymbol{\tau}}) + b_h(\widetilde{\boldsymbol{\tau}}, w_d) &= 0 && \text{for all } \widetilde{\boldsymbol{\tau}} \in \widetilde{\boldsymbol{V}}, \\ b_h(\boldsymbol{\sigma}_d, \widetilde{v}) &= -\langle d, \widetilde{v}\rangle && \text{for all } \widetilde{v} \in \widetilde{Q}. \end{aligned}$$

In particular, for $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $\widetilde{v} = I_h w - w_h$ we obtain

$$\begin{aligned} a(\boldsymbol{\sigma}_d, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, w_d) &= 0, \\ b_h(\boldsymbol{\sigma}_d, I_h w - w_h) &= -\langle d, I_h w - w_h\rangle. \end{aligned}$$

Using Galerkin orthogonality it follows that

$$\begin{aligned} &-\langle d, I_h w - w_h\rangle \\ &\quad = a(\boldsymbol{\sigma}_d, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, w_d) + b_h(\boldsymbol{\sigma}_d, I_h w - w_h) \\ &\quad = a(\boldsymbol{\sigma}_d, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, w_d) + b_h(\boldsymbol{\sigma}_d, w - w_h) + b_h(\boldsymbol{\sigma}_d, I_h w - w) \\ &\quad = a(\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, w_d - I_h w_d) + b_h(\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d, w - w_h) \\ &\qquad + b_h(\boldsymbol{\sigma}_d, I_h w - w). \end{aligned}$$

The properties of the interpolation operators imply

$$b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, w_d - I_h w_d) = b_h(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}, w_d - I_h w_d)$$

and

$$\begin{aligned} b_h(\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d, w - w_h) &= b_h(\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d, w - I_h w), \\ b_h(\boldsymbol{\sigma}_d, I_h w - w) &= b_h(\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d, I_h w - w). \end{aligned}$$

Therefore,

$$-\langle d, I_h w - w_h\rangle = a(\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + b_h(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}, w_d - I_h w_d),$$

which implies

$$\begin{aligned} |\langle d, w_h - I_h w\rangle| &\le \|\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d\|_0 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + |b_h(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}, w_d - I_h w_d)| \\ &\le \|\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_d\|_0 \, \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_0 + |b_h(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}, w_d - I_h w_d)| \end{aligned}$$

using Theorem 6.3. We have

$$b_h(\widetilde{\boldsymbol{\tau}}, \widetilde{v}) \le \|\widetilde{\boldsymbol{\tau}}\|_{0,h} \, \|\widetilde{v}\|_{2,h}$$

with a second mesh-dependent norm $\|v\|_{2,h}$, given by

$$\|v\|_{2,h}^2 = \sum_{T \in \mathcal{T}_h} \|v\|_{H^2(T)}^2 + \frac{1}{h} \sum_{e \in \mathcal{E}_h} \|[\partial_n v]_e\|_{L^2(e)}^2 \,,$$

ETNA
Kent State University
http://etna.math.kent.edu

where $[\partial_n v]_e$ denotes the jump of $\partial_n v$ across an edge; see [3, Equation (4.48)]. Then it follows that

$$|w_h - I_h w|_1 = \sup_{d \in H^{-1}(\Omega)} \frac{|\langle d, w_h - I_h w\rangle|}{\|d\|_{-1}}$$
$$\leq \left[\sup_{d \in H^{-1}(\Omega)} \frac{\|\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h \boldsymbol{\sigma}_d\|_0 + \|w_d - I_h w_d\|_{2,h}}{\|d\|_{-1}} \right] \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{0,h}$$
$$\leq c\, h^{2(1-1/p)} \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{0,h},$$

using the interpolation error estimates

$$\|\boldsymbol{\sigma}_d - \boldsymbol{\Pi}_h \boldsymbol{\sigma}_d\|_0 \leq c\, h^{2-2/p} |\boldsymbol{\sigma}_d|_{1,p},$$
$$\|w_d - I_h w_d\|_{2,h} \leq c\, h^{2-2/p} |w_d|_{3,p} \quad \text{for } k \geq 2,$$

and the regularity estimates

$$|\boldsymbol{\sigma}_d|_{1,p} \leq c\, \|d\|_{-1} \quad \text{and} \quad |w_d|_{3,p} \leq c\, \|d\|_{-1}.$$

For $k = 1$ we have

$$b_h(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}, w_d - I_h w_d) = b_h(\boldsymbol{\sigma}, w_d - I_h w_d) = \int_\Omega (\mathbf{div}\, \boldsymbol{\sigma}) \cdot \nabla (w_d - I_h w_d)\, dx.$$

Therefore,

$$|b_h(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}, w_d - I_h w_d)| \leq \|\,\mathbf{div}\, \boldsymbol{\sigma}\|_{0,p}\, |w_d - I_h w_d|_{1,q} \leq c\, \|\boldsymbol{\sigma}\|_{1,p}\, |w_d - I_h w_d|_{1,q}$$
$$\leq c\, \|w\|_{3,p}\, |w_d - I_h w_d|_{1,q} \quad \text{with } q = p/(p-1).$$

Now

$$|w_d - I_h w_d|_{1,q} \leq c\, h\, \|w_d\|_{2,q} \leq c\, h\, \|w_d\|_{3,p} \leq c\, h\, \|d\|_{-1},$$

which implies

$$|w_h - I_h w|_1 = \sup_{d \in H^{-1}(\Omega)} \frac{|\langle d, w_h - I_h w\rangle|}{\|d\|_{-1}} \leq c\, h\, \|w\|_{3,p}.$$

This completes the proof. □

Using standard approximation properties of the interpolation operators $\boldsymbol{\Pi}_h$ and I_h (see [3, 9, 17]) one immediately obtains the following consequences.

COROLLARY 6.5. *If $w \in H^{k+2}(\Omega)$, then we have estimates of optimal order for $\boldsymbol{\sigma}$.*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leq c\, h^k\, \|\boldsymbol{\sigma}\|_k.$$

COROLLARY 6.6. *Assume that (6.1) is satisfied for some $p \in (4/3, 2]$.*

1. *For $f \in H^{-1}(\Omega)$ we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leq c\, h^{2-2/p} \|f\|_{-1} \quad \textit{and} \quad |w - w_h|_1 \leq c\, h^{\min(k, 4-4/p)}\, \|f\|_{-1}.$$

2. *If $w \in H^{k+1}(\Omega)$, then we have*

$$|w - w_h|_1 \leq c\, h^{k+1-2/p}\, |w|_{k+1} \quad \textit{for } k \geq 2.$$

ETNA
Kent State University
http://etna.math.kent.edu

3. *If Ω is convex and $w \in H^{k+1}(\Omega)$, then we have estimates of optimal order for w.*

$$|w - w_h|_1 \leq c\, h^k\, |w|_{k+1} \quad \textit{for } k \geq 2.$$

The first estimates in Corollary 6.6 are in accordance with [9]. Observe that $2-2/p > 1/2$ and $4-4/p > 1$ since $p > 4/3$. Therefore, we have convergence rates at least of order $O(h^{1/2})$ and $O(h)$ for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$ and $|w-w_h|_1$, respectively. Observe that $k+1-2/p > k-1/2$ since $p > 4/3$. Therefore, we have a convergence rate at least of order $O(h^{k-1/2})$ for $|w - w_h|_1$, if $w \in H^{k+1}(\Omega)$. If Ω is convex, then (6.1) holds for $p = 2$; see [9].

6.2. The conforming variant (4.1). The only difference to the original HHJ method is the use of the finite element space

$$\boldsymbol{V}_h^{\text{conf}} = \{\boldsymbol{\tau} = \boldsymbol{\pi}(q) + \boldsymbol{H}^T \boldsymbol{\varepsilon}(\psi)\boldsymbol{H} \colon q \in \mathcal{S}_{h,0},\ \psi \in (\mathcal{S}_h)^2/\text{RM}\}$$

instead of $\boldsymbol{V}_h$ for approximating the Lagrangian multiplier $\boldsymbol{\sigma}$. As before the error analysis is based on two interpolation operators $\boldsymbol{\Pi}_h^{\text{conf}}$ and I_h^{conf} satisfying

$$\begin{aligned} b(\boldsymbol{\tau} - \boldsymbol{\Pi}_h^{\text{conf}}\boldsymbol{\tau}, v_h) &= 0 \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}},\ v_h \in Q_h,\\ b(\boldsymbol{\tau}_h, v - I_h^{\text{conf}} v) &= 0 \quad \text{for all } \boldsymbol{\tau}_h \in \boldsymbol{V}_h^{\text{conf}},\ v \in H_0^1(\Omega), \end{aligned}$$

with $I_h^{\text{conf}} = R_h$, and $\boldsymbol{\Pi}_h^{\text{conf}}$ is given by

$$\boldsymbol{\Pi}_h^{\text{conf}}\boldsymbol{\tau} = \boldsymbol{\pi}(R_h q) + \boldsymbol{H}^T \boldsymbol{\varepsilon}(\widetilde{I}_h \phi)\boldsymbol{H} \quad \text{for} \quad \boldsymbol{\tau} = \boldsymbol{\pi}(q) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H}.$$

Here $R_h \colon H_0^1(\Omega) \longrightarrow \mathcal{S}_{h,0}$ denotes the Ritz projection, given by

$$\int_\Omega \nabla R_h v \cdot \nabla q_h \, dx = \int_\Omega \nabla v \cdot \nabla q_h \, dx \quad \text{for all } v \in H_0^1(\Omega),\ q_h \in \mathcal{S}_{h,0},$$

and $\widetilde{I}_h \colon H^1(\Omega)^2/\text{RM} \longrightarrow (\mathcal{S}_h)^2/\text{RM}$ can be any reasonable interpolation operator like a Clément-type interpolation operator; see, e.g., [22]. It turns out that the Ritz projection is the only candidate for I_h^{conf}. Thus, note that we have again the same properties of these interpolation operators. However, these operators are no longer local operators as they were in the case of the original HHJ methods. That does not effect the analog of Theorem 6.3 but it leads to a deterioration of the estimates for the analog of Theorem 6.4 and the subsequent corollaries for non-convex domains. For convex domains, the interpolation operators share the same approximation properties as before provided p and ϕ in (3.2) are sufficiently smooth resulting in the following error estimates.

THEOREM 6.7. *Let Ω be convex.*

1. *If $p \in H^k(\Omega)$ and $\phi \in H^{k+1}(\Omega)$, then we have estimates of optimal order for $\boldsymbol{\sigma}$.*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leq c\, h^k\, \left(|p|_k + |\phi|_{k+1}\right).$$

2. *If $p \in H^{k-1}(\Omega)$ and $\phi \in H^k(\Omega)$, then $w \in H^{k+1}(\Omega)$ and we have estimates of optimal order for w.*

$$|w - w_h|_1 \leq c\, h^k\, \left(|p|_{k-1} + |\phi|_k + |w|_{k+1}\right) \quad \textit{for } k \geq 2.$$

The proof of these estimates is a complete copy of the corresponding proofs for the original HHJ method, however, this time based on the interpolation operators introduced above, and is therefore omitted. Using the same techniques for estimating the errors in the non-convex case leads to a deterioration of the estimates since standard L^2-error estimates for the Ritz projection are not of optimal approximation order.

ETNA
Kent State University
http://etna.math.kent.edu

7. Numerical experiments. The obvious procedure for solving (5.2) consists of three consecutive steps.

Step 1. For given $f \in H^{-1}(\Omega)$, solve

$$\int_\Omega \nabla p_h \cdot \nabla v \, dx = -\langle f, v \rangle \tag{7.1}$$

by the preconditioned conjugate gradient (PCG) method with a standard multigrid preconditioner for a Poisson problem.

Step 2. For p_h, computed in Step 1, solve

$$\int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi) \, dx = -\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \boldsymbol{\varepsilon}(\psi) \, dx \tag{7.2}$$

by the PCG method with a standard multigrid preconditioner for a pure traction problem. Observe that the bilinear form in (7.2) has a non-trivial kernel, namely RM. See, e.g., [24] for the treatment of such singular problems. We have chosen the alternative approach of regularizing the problem by replacing the original bilinear form by

$$\int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi) \, dx + \int_\Omega \phi_h \, dx \cdot \int_\Omega \psi \, dx + \int_\Omega \operatorname{curl} \phi_h \, dx \int_\Omega \operatorname{curl} \psi \, dx,$$

which results in a coercive problem in $(\mathcal{S}_h)^2$ with respect to the H^1-norm.

Step 3. For p_h and ϕ_h, computed in Step 1 and 2, respectively, solve

$$\int_\Omega \nabla w_h \cdot \nabla q \, dx = -\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \hat{\boldsymbol{\pi}}_h(q) \, dx - \int_\Omega \hat{\boldsymbol{\pi}}_h(q) : \boldsymbol{\varepsilon}(\phi_h) \, dx \tag{7.3}$$

by the PCG method with a standard multigrid preconditioner for a Poisson problem.

For the conforming variant (4.1), the right-hand sides in (7.2) and (7.3) have to be replaced by the simpler expressions

$$-\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\varepsilon}(\psi) \, dx = -\int_\Omega p_h \operatorname{div} \psi \, dx$$

and

$$-\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\pi}(q) dx - \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi_h) \, dx = -2 \int_\Omega p_h \, q \, dx - \int_\Omega q \operatorname{div} \phi_h \, dx,$$

respectively.

For each of the three multigrid preconditioners we choose one multigrid V-cycle with one forward and one backward Gauss-Seidel sweep for pre- and post-smoothing, respectively. In each of the three steps the initial guess for the PCG method is set to 0.

We will now discuss the accuracy and computational complexity of this procedure in more detail.

It is well-known that the multigrid V-cycle algorithm described above converges with a convergence rate in the energy norm that is bounded by a constant strictly smaller than 1 uniformly with respect to the mesh size h; see [11]. Therefore, the number of PCG iterations that are necessary to reduce an initial error by a prescribed factor, say $\delta > 0$, in the energy norm is uniformly bounded. In particular, the number of PCG iterations necessary to obtain an approximate solution $\widetilde{p}_h$ to (7.1) such that

$$|\widetilde{p}_h - p_h|_1 \le \delta \, |p_h|_1, \tag{7.4}$$

ETNA
Kent State University
http://etna.math.kent.edu

is uniformly bounded. In the second step the PCG method is applied to (7.2) but with p_h (which is not available) replaced by $\widetilde{p}_h$ on the right-hand side. Let $\bar{\phi}_h$ denote the exact solution of this modified problem. Then the number of PCG iterations necessary to obtain an approximate solution $\widetilde{\phi}_h$ to the modified problem in Step 2 such that

$$\|\varepsilon(\widetilde{\phi}_h - \bar{\phi}_h)\|_0 \le \delta\, \|\varepsilon(\bar{\phi}_h)\|_0, \tag{7.5}$$

is uniformly bounded, too. Analogously, in Step 3 the number of PCG iterations necessary to obtain an approximate solution $\widetilde{w}_h$ to (7.3) but with p_h and ϕ_h replaced by $\widetilde{p}_h$ and $\widetilde{\phi}_h$ on the right-hand side, respectively, such that

$$|\widetilde{w}_h - \bar{w}_h|_1 \le \delta\, |\bar{w}_h|_1, \tag{7.6}$$

is uniformly bounded, where $\bar{w}_h$ denotes the exact solution to the modified problem in Step 3. From $\widetilde{p}_h$ and $\widetilde{\phi}_h$ we obtain an approximation $\widetilde{\boldsymbol{\sigma}}_h$ given by

$$\widetilde{\boldsymbol{\sigma}}_h = \boldsymbol{\pi}_h(\widetilde{p}_h) + \boldsymbol{H}^T \varepsilon(\widetilde{\phi}_h)\boldsymbol{H}.$$

Now we have

LEMMA 7.1. *Assume that (7.4), (7.5), and (7.6) hold for some $\delta \le \delta_0$. Then*

$$\|\widetilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0 \le c_1\, \delta\, \|f\|_{-1} \quad \text{and} \quad |\widetilde{w}_h - w_h|_1 \le c_2\, \delta\, \|f\|_{-1} \tag{7.7}$$

with positive constants c_1 and c_2 that depend only on δ_0 and the constant c_B from Lemma 4.2.

Proof. We first estimate the difference between the solutions of the original and the modified problems in Step 2 and 3 in terms of the difference of the data on the right-hand sides.

Subtracting (7.2) from its modified variant for $\psi = \bar{\phi}_h - \phi_h$ yields

$$\|\varepsilon(\bar{\phi}_h - \phi_h)\|_0^2 = -\int_\Omega \hat{\boldsymbol{\pi}}_h(\widetilde{p}_h - p_h) : \varepsilon(\bar{\phi}_h - \phi_h)\, dx \le c_B\, |\widetilde{p}_h - p_h|_1\, \|\varepsilon(\bar{\phi}_h - \phi_h)\|_0,$$

where Lemma 4.2 was applied. This implies

$$\|\varepsilon(\bar{\phi}_h - \phi_h)\|_0 \le c_B\, |\widetilde{p}_h - p_h|_1. \tag{7.8}$$

Analogously, subtracting (7.3) from its modified variant for $q = \bar{w}_h - w_h$ yields

$$|\bar{w}_h - w_h|_1^2 = -\int_\Omega (\widetilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h) : \boldsymbol{\pi}_h(\bar{w}_h - w_h)\, dx \le c_B\, \|\widetilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0\, |\bar{w}_h - w_h|_1,$$

which implies

$$|\bar{w}_h - w_h|_1 \le c_B\, \|\widetilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0. \tag{7.9}$$

Then we have

$$\begin{aligned}
\|\widetilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0 &\le \|\boldsymbol{\pi}_h(\widetilde{p}_h - p_h)\|_0 + \|\varepsilon(\widetilde{\phi}_h - \phi_h)\|_0 \\
&\le c_B\, |\widetilde{p}_h - p_h|_1 + \|\varepsilon(\widetilde{\phi}_h - \bar{\phi}_h)\|_0 + \|\varepsilon(\bar{\phi}_h - \phi_h)\|_0 \\
&\le 2c_B\, |\widetilde{p}_h - p_h|_1 + \|\varepsilon(\widetilde{\phi}_h - \bar{\phi}_h)\|_0 \le 2c_B\, \delta\, |p_h|_1 + \delta\, \|\varepsilon(\bar{\phi}_h)\|_0,
\end{aligned}$$

by (7.4) and using (7.5) for the last estimate. Next we estimate $\|\varepsilon(\bar{\phi}_h)\|_0$. We have

$$\|\varepsilon(\bar{\phi}_h)\|_0 \le \|\varepsilon(\bar{\phi}_h - \phi_h)\|_0 + \|\varepsilon(\phi_h)\|_0 \le c_B\, \delta\, |p_h|_1 + \|\varepsilon(\phi_h)\|_0$$

ETNA
Kent State University
http://etna.math.kent.edu

using (7.8) and (7.4) and

$$\|\boldsymbol{\varepsilon}(\phi_h)\|_0 = \|\boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi_h)\boldsymbol{H}\|_0 = \|\boldsymbol{\sigma}_h - \boldsymbol{\pi}_h(p_h)\|_0 \le c_B\,|p_h|_1 + \|\boldsymbol{\sigma}_h\|_0$$

using (5.1) and Lemma 4.2, which imply

$$\|\boldsymbol{\varepsilon}(\bar{\phi}_h)\|_0 \le c_B\,(1+\delta)\,|p_h|_1 + \|\boldsymbol{\sigma}_h\|_0.$$

Using this estimate we obtain

$$\begin{aligned}\|\widetilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0 &\le c_B\,\delta(3+\delta)\,|p_h|_1 + \delta\,\|\boldsymbol{\sigma}_h\|_0\\ &\le \delta\,\left(1+[(3+\delta)\,c_B]^2\right)^{1/2}\,\left(|p_h|_1^2 + \|\boldsymbol{\sigma}_h\|_0^2\right)^{1/2}\\ &= \delta\,\left(1+[(3+\delta)\,c_B]^2\right)^{1/2}\,\|\boldsymbol{\sigma}_h\|_{-1,\operatorname{div}\mathbf{div},h}.\end{aligned}$$

Furthermore, using (7.6) we obtain

$$\begin{aligned}|\widetilde{w}_h - w_h|_1 &\le |\widetilde{w}_h - \bar{w}_h|_1 + |\bar{w}_h - w_h|_1 \le \delta\,|\bar{w}_h|_1 + |\bar{w}_h - w_h|_1\\ &\le \delta\,|w_h|_1 + (1+\delta)\,|\bar{w}_h - w_h|_1,\end{aligned}$$

and therefore,

$$|\widetilde{w}_h - w_h|_1 \le \delta\,|w_h|_1 + c_B\,(1+\delta)\,\|\widetilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0$$

using (7.9). From the stability estimates for saddle point problems (see, e.g., [31, Theorem 1])

$$\|\boldsymbol{\sigma}_h\|_{-1,\operatorname{div}\mathbf{div},h} \le \frac{1}{\beta}\,\frac{\|a\|}{\alpha}\,\|f\|_{-1} \quad \text{and} \quad |w_h|_1 \le \frac{\|a\|}{\beta^2}\,\frac{\|a\|}{\alpha}\,\|f\|_{-1}$$

with $\|a\| = \alpha = 1$ and $\beta = (1+c_B^2)^{-1/2}$ (see Theorem 4.4), we finally obtain (7.7) with

$$c_1 = \left(1+[(3+\delta_0)\,c_B]^2\right)^{1/2}\left(1+c_B^2\right)^{1/2}, \quad c_2 = 1 + c_B^2 + c_B\,(1+\delta_0)\,c_1. \qquad \square$$

This lemma eventually shows that the number of PCG iterations necessary to obtain approximate solutions for w_h and $\boldsymbol{\sigma}_h$ with a tolerance of order δ relative to the data $\|f\|_{-1}$ of the problem is bounded independently of the mesh size h. With respect to this criterion of accuracy, the proposed procedure is of optimal computational complexity since the number of arithmetic operations for applying one multigrid V-cycle is proportional to the number of involved unknowns and the number of PCG iterations is uniformly bounded. As expected no additional smoothness of the data f is required for these arguments. In case that δ is not considered as a prescribed fixed quantity but is chosen proportional to the order of the discretization error, full multigrid methods have to be considered instead to restore optimal computational complexity.

REMARK 7.2. An estimate of the form (7.7) can also be shown for the conforming variant with a completely analogous proof.

In order to illustrate the theoretical results we consider the following simple biharmonic test problem:

$$\Delta^2 w = f \quad \text{in } \Omega, \qquad w = \partial_n w = 0 \quad \text{on } \Gamma$$

on two domains, the square $\Omega = \Omega_S = (-1,1)^2$ and the L-shaped domain $\Omega = \Omega_L$ depicted in Figures 7.1 and 7.2, where also the initial meshes (level $\ell = 0$) are shown. (The initial

ETNA
Kent State University
http://etna.math.kent.edu

TABLE 7.1
Number of iterations, $\Omega = \Omega_S$ (square).

L	N_1	iter$_1$	N_2	iter$_2$	N_3	iter$_3$
6	65 025	16	132 098	18	65 025	16
7	261 121	17	526 338	19	261 121	17
8	1 046 529	17	2 101 250	20	1 046 529	17
9	4 190 209	18	8 396 802	21	4 190 209	18

TABLE 7.2
Number of iterations, $\Omega = \Omega_L$ (L-shaped domain).

L	N_1	iter$_1$	N_2	iter$_2$	N_3	iter$_3$
6	48 641	17	99 330	19	48 641	17
7	195 585	18	395 266	20	195 585	18
8	784 385	18	1 576 962	21	784 385	18
9	3 141 633	19	6 299 650	22	3 141 633	19

meshes were created by distorting an originally uniform subdivision of Ω_S into 32 and Ω_L into 24 triangles, in order to avoid any artificial super-convergence effects due to uniformity.) The right-hand side $f(x)$ is chosen such that

$$w(x) = \left[1 - \cos(2\pi x_1)\right] \left[1 - \cos(4\pi x_2)\right]$$

is the exact solution to the problem. The initial meshes are uniformly refined until the final level $\ell = L$. In all experiments the polynomial degree k as introduced in the beginning of Section 4 is chosen equal to 1, which represents the lowest order HHJ method. In each of the

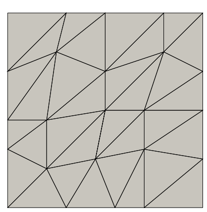

FIG. 7.1. $\Omega = \Omega_S$.

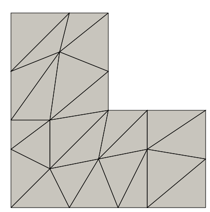

FIG. 7.2. $\Omega = \Omega_L$.

three steps, a reduction of the Euclidean norm of the initial residual by a factor of 10^{-8} was used as stopping criterion for the PCG methods.

Table 7.1 shows the observed number of iterations for the solution procedure as described above for $\Omega = \Omega_S$. The first column contains the level L of refinement. The next three pairs of columns show the total number N_i of degrees of freedom and the number of iterations iter$_i$ of the PCG method for the linear system in Step $i = 1, 2, 3$. Table 7.2 shows the corresponding results for the L-shaped domain $\Omega = \Omega_L$ representing a non-convex case. As expected the number of PCG iterations is bounded uniformly with respect to the mesh size.

Finally, in Tables 7.3 and 7.4 the discretization errors of the original HHJ method (5.2) and its conforming variant (4.1) are shown. For the original HHJ method, the H^1-error of the original variable w and the L^2-error of the auxiliary variable $\boldsymbol{\sigma}$ decrease with the order h, in accordance with known estimates; see Section 6. The last two columns contain the errors of p and ϕ given by (3.2) and measured in the associated norms of the Helmholtz-decomposition

ETNA
Kent State University
http://etna.math.kent.edu

TABLE 7.3
Discretization errors for the HHJ method (5.2), $\Omega = \Omega_L$ *(L-shaped domain).*

L	$\lvert w - w_h\rvert_1$	$\lVert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\rVert_0$	$\lvert p - p_h\rvert_1$	$\lVert \phi - \phi_h\rVert_1$
6	$7.14 * 10^{-1}$	$4.57 * 10^{1}$	$1.24 * 10^{2}$	$9.69 * 10^{0}$
7	$3.56 * 10^{-1}$	$2.30 * 10^{1}$	$6.18 * 10^{1}$	$4.81 * 10^{0}$
8	$1.78 * 10^{-1}$	$1.15 * 10^{1}$	$3.02 * 10^{1}$	$2.35 * 10^{0}$
9	$8.90 * 10^{-2}$	$5.74 * 10^{0}$	$1.35 * 10^{1}$	$1.05 * 10^{0}$

TABLE 7.4
Discretization errors for the conforming variant (4.1), $\Omega = \Omega_L$ *(L-shaped domain).*

L	$\lvert w - w_h\rvert_1$	$\lVert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\rVert_0$	$\lvert p - p_h\rvert_1$	$\lVert \phi - \phi_h\rVert_1$
6	$7.11 * 10^{-1}$	$8.89 * 10^{0}$	$1.24 * 10^{2}$	$9.68 * 10^{0}$
7	$3.56 * 10^{-1}$	$4.45 * 10^{0}$	$6.18 * 10^{1}$	$4.81 * 10^{0}$
8	$1.78 * 10^{-1}$	$2.22 * 10^{0}$	$3.02 * 10^{1}$	$2.35 * 10^{0}$
9	$8.90 * 10^{-2}$	$1.11 * 10^{0}$	$1.35 * 10^{1}$	$1.05 * 10^{0}$

(see Theorems 3.1 and 3.3), where the (analytically not available) exact solutions p and ϕ are replaced by their approximate solutions on level $\ell = 10$. The conforming variant behaves similarly; see Table 7.4.

Acknowledgment. The authors would like to thank the anonymous referees for their valuable comments and suggestions which helped to improve this manuscript considerably.

Appendix A. Divergence-free distributions in $\left(H^{-1}(\Omega)\right)^2$.

The proof of Theorem 3.3 relies on the following result for divergence-free distributions in $\left(H^{-1}(\Omega)\right)^2$, whose proof is given for completeness.

LEMMA A.1. *Let Ω be a simply connected, open and bounded set in $\mathbb{R}^2$ with Lipschitz boundary Γ. For each $f \in \left(H^{-1}(\Omega)\right)^2$ with $\operatorname{div} f = 0$ there exists a function $\rho \in L^2_0(\Omega)$ such that*

$$f = \mathbf{curl}\, \rho.$$

Proof. Let $\langle f, v\rangle = \langle f_1, v_1\rangle + \langle f_2, v_2\rangle \in \left(H^{-1}(\Omega)\right)^2$ with $\operatorname{div} f = 0$, i.e.,

$$\langle f_1, \partial_1 \varphi\rangle + \langle f_2, \partial_2 \varphi\rangle = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Then, by continuity, it follows that

$$\langle f_1, \partial_1 \varphi\rangle + \langle f_2, \partial_2 \varphi\rangle = 0 \quad \text{for all } \varphi \in H_0^2(\Omega).$$

Now, let $v \in \left(H_0^1(\Omega)\right)^2$ with $\operatorname{div} v = 0$. Then from [22, Corollary 3.2] it follows that there is a function $\varphi \in H_0^2(\Omega)$ with

$$v = \mathbf{curl}\, \varphi.$$

Therefore, for $\langle g, v\rangle \equiv \langle f_2, v_1\rangle - \langle f_1, v_2\rangle$, we have

$$\langle g, v\rangle = \langle f_2, v_1\rangle - \langle f_1, v_2\rangle = \langle f_2, \partial_2 \varphi\rangle + \langle f_1, \partial_1 \varphi\rangle = 0.$$

Then [22, Lemma 2.1] implies that there is a function $\rho \in L^2_0(\Omega)$ with

$$g = \nabla \rho, \quad \text{i.e.,} \quad f = \mathbf{curl}\, \rho. \qquad \square$$

REFERENCES

[1] R. A. ADAMS AND J. J. F. FOURNIER, *Sobolev Spaces*, 2nd ed., Academic Press, Amsterdam, 2003.
[2] D. N. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 7–32.
[3] I. BABUŠKA, J. OSBORN, AND J. PITKÄRANTA, *Analysis of mixed methods using mesh dependent norms*, Math. Comp., 35 (1980), pp. 1039–1062.
[4] E. M. BEHRENS AND J. GUZMÁN, *A mixed method for the biharmonic problem based on a system of first-order equations*, SIAM J. Numer. Anal., 49 (2011), pp. 789–817.
[5] L. BEIRÃO DA VEIGA, J. NIIRANEN, AND R. STENBERG, *A family of C^0 finite elements for Kirchhoff plates. I. Error analysis*, SIAM J. Numer. Anal., 45 (2007), pp. 2047–2071.
[6] ———, *A posteriori error estimates for the Morley plate bending element*, Numer. Math., 106 (2007), pp. 165–179.
[7] ———, *A family of C^0 finite elements for Kirchhoff plates. II. Numerical results*, Comput. Methods Appl. Mech. Engrg., 197 (2008), pp. 1850–1864.
[8] H. BLUM AND R. RANNACHER, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Methods Appl. Sci., 2 (1980), pp. 556–581.
[9] ———, *On mixed finite element methods in plate bending analysis I. The first Herrmann scheme.*, Comput. Mech., 6 (1990), pp. 221–236.
[10] D. BOFFI, F. BREZZI, AND M. FORTIN, *Mixed Finite Element Methods and Applications*, Springer, Heidelberg, 2013.
[11] J. H. BRAMBLE AND J. E. PASCIAK, *New estimates for multilevel algorithms including the V-cycle*, Math. Comp., 60 (1993), pp. 447–471.
[12] S. C. BRENNER, *An optimal-order nonconforming multigrid method for the biharmonic equation*, SIAM J. Numer. Anal., 26 (1989), pp. 1124–1138.
[13] ———, *Two-level additive Schwarz preconditioners for nonconforming finite element methods*, Math. Comp., 65 (1996), pp. 897–921.
[14] ———, *Convergence of nonconforming multigrid methods without full elliptic regularity*, Math. Comp., 68 (1999), pp. 25–53.
[15] S. C. BRENNER, T. GUDI, AND L.-Y. SUNG, *A weakly over-penalized symmetric interior penalty method for the biharmonic problem*, Electron. Trans. Numer. Anal., 37 (2010), pp. 214–238.
http://etna.ricam.oeaw.ac.at/vol.37.2010/pp214-238.dir/pp214-238.pdf
[16] S. C. BRENNER AND L.-Y. SUNG, *Multigrid algorithms for C^0 interior penalty methods*, SIAM J. Numer. Anal., 44 (2006), pp. 199–223.
[17] F. BREZZI AND P.-A. RAVIART, *Mixed finite element methods for 4th order elliptic equations*, in Topics in Numerical Analysis III, Proc. Roy. Irish Acad. Conf., Trinity Coll., Dublin, 1976, J. J. H. Miller, ed., Academic Press, London, 1977, pp. 33–56.
[18] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, SIAM, Philadelphia, 2002.
[19] P. G. CIARLET AND P.-A. RAVIART, *A mixed finite element method for the biharmonic equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, Proc. Sympos., Math. Res. Center Univ. Wisconsin 1974, C. de Boor, ed., Publication No. 33. of Math. Res. Center Univ. of Wisconsin-Madison, Academic Press, New York, 1974, pp. 125–145.
[20] G. ENGEL, K. GARIKIPATI, T. J. R. HUGHES, M. G. LARSON, L. MAZZEI, AND R. L. TAYLOR, *Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, Comput. Methods Appl. Mech. Engrg., 191 (2002), pp. 3669–3750.
[21] R. S. FALK AND J. E. OSBORN, *Error estimates for mixed methods*, RAIRO Anal. Numér., 14 (1980), pp. 249–277.
[22] V. GIRAULT AND P.-A. RAVIART, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer, Berlin, 1986.
[23] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, SIAM, Philadelphia, 2011.
[24] W. HACKBUSCH, *Multigrid Methods and Applications*, Springer, Berlin, 1985.
[25] M. R. HANISCH, *Two-level additive Schwarz preconditioners for fourth-order mixed methods*, Electron. Trans. Numer. Anal., 22 (2006), pp. 1–16.
http://etna.ricam.oeaw.ac.at/vol.22.2006/pp1-16.dir/pp1-16.pdf
[26] K. HELLAN, *Analysis of elastic plates in flexure by a simplified finite element method*, Acta Polytech. Scandinav. Civil. Engrg., 46 (1967).
[27] L. HERRMANN, *Finite element bending analysis for plates*, J. Engrg. Mech., 93 (1967), pp. 49–83.
[28] J. HUANG, X. HUANG, AND Y. XU, *Convergence of an adaptive mixed finite element method for Kirchhoff plate bending problems*, SIAM J. Numer. Anal., 49 (2011), pp. 574–607.
[29] X. HUANG AND J. HUANG, *A reduced local C^0 discontinuous Galerkin method for Kirchhoff plates*, Numer. Methods Partial Differential Equations, 30 (2014), pp. 1902–1930.

[30] C. JOHNSON, *On the convergence of a mixed finite-element method for plate bending problems*, Numer. Math., 21 (1973), pp. 43–62.
[31] W. KRENDL, V. SIMONCINI, AND W. ZULEHNER, *Stability estimates and structural spectral properties of saddle point problems*, Numer. Math., 124 (2013), pp. 183–213.
[32] J.-L. LIONS AND E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications. Vol. I*, Springer, New York, 1972.
[33] L. S. D. MORLEY, *The triangular equlibrium element in the solution of plate bending problems*, Aeronaut. Q., 19 (1968), pp. 149–169.
[34] J. NEČAS, *Direct Methods in the Theory of Elliptic Equations*, Springer, Heidelberg, 2012.
[35] A. PECHSTEIN AND J. SCHÖBERL, *Tangential-displacement and normal-normal-stress continuous mixed finite elements for elasticity*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1761–1782.
[36] P. PEISKER, *On the numerical solution of the first biharmonic equation*, RAIRO Modél. Math. Anal. Numér., 22 (1988), pp. 655–676.
[37] A. SINWEL, *A New Family of Mixed Finite Elements for Elasticity*, PhD. Thesis, Insitute of Compuational Mathematics, Johannes Kepler University, Linz, 2009.
[38] X. ZHANG, *Two-level Schwarz methods for the biharmonic problem discretized conforming C^1 elements*, SIAM J. Numer. Anal., 33 (1996), pp. 555–570.
[39] J. ZHAO, *Convergence of V-cycle and F-cycle multigrid methods for the biharmonic problem using the Morley element*, Electron. Trans. Numer. Anal., 17 (2004), pp. 112–132. http://etna.ricam.oeaw.ac.at/vol.17.2004/pp112-132.dir/pp112-132.pdf
[40] W. ZULEHNER, *The Ciarlet-Raviart method for biharmonic problems on general polygonal domains: mapping properties and preconditioning*, SIAM J. Numer. Anal., 53 (2015), pp. 984–1004.