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ROBUST A POSTERIORI ERROR BOUNDS FOR SPLINE COLLOCATION APPLIED TO SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS*

TORSTEN LINSS† AND GORAN RADOJEV‡

Abstract. Collocation with arbitrary order C^1-splines for a singularly perturbed reaction-diffusion problem in one dimension is studied. Robust a posteriori error bounds are derived for the collocation method on arbitrary meshes. These bounds are used to drive an adaptive mesh moving algorithm. Numerical results are presented.

Key words. reaction-diffusion, spline collocation, singular perturbations, a posteriori error estimation

AMS subject classifications. 65L10, 65L11, 65L60

1. Introduction. Consider the reaction-diffusion problem of finding a function u with $u \in C^2(0,1) \cap C[0,1]$ such that

$$\mathcal{L}u := -\varepsilon^2 u'' + ru = f \text{ in } (0,1), \qquad u(0) = \gamma_0, \ u(1) = \gamma_1, \tag{1.1}$$

where $\varepsilon \in (0,1]$ and $r \geq \varrho^2$ on $[0,1]$ with some constant $\varrho > 0$. Standard numerical methods fail to capture the boundary layers present in the solution of (1.1). Layers are regions where a function changes rapidly as the perturbation parameter ε tends to zero. In the vicinity of $x = 0$ and $x = 1$ the solution of u exhibits layers of width $\mathcal{O}(\varepsilon)$. Further layers may form at points in the interior of the domain where f or r (or their derivatives) have discontinuities.

One goal for numerical methods applied to (1.1) is *robustness* with regard to the perturbation parameter, i.e., the method should perform equally well no matter how small the perturbation parameter. Special procedures have been devised to achieve this goal. For a survey, we refer the reader to the recent monographs [13] and earlier books [5, 12] and references therein. One possible approach is the use of layer-adapted meshes; see [9]. In the present contribution we focus on mesh adaptivity based on *a posteriori* error estimation.

Most of the literature is devoted to difference schemes and various types of FEMs, while there are only very few publications on collocation methods. This motivates our interest in the subject.

Collocation methods with polynomial trial functions play a very important role in the context of spectral methods. Section 9.7 of Funaro's monograph [6] is dedicated to the application of these methods to problems with boundary layers. For problem (1.1), polynomials of degree $p \approx \varepsilon^{-1/2}$ must be used to resolve the layers satisfactorily. For small values of ε, this leads to dense linear systems of high dimension. In practice, this is not viable.

A general theory for spline-collocation methods applied to classical, not singularly perturbed, boundary-value problems was derived in [4]. An immediate application of those results to (1.1) yields error bounds with "constants" that tend to infinity when $\varepsilon \to 0$, because they involve norms of certain derivatives of the exact solution.

In the present paper we shall present *a posteriori error* bounds for arbitrary order C^1-spline collocation applied to (1.1), thus extending the analysis for quadratic splines in [11]. These error bounds allow us to judge the quality of a numerical approximation after it has been computed. Unlike *a priori* error bounds, this approach does not require knowledge of the exact

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†Fakultät für Mathematik und Informatik, FernUniversität in Hagen, Lützowstr. 125, 58095 Hagen, Germany, (torsten.linss@fernuni-hagen.de).

‡Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia, (goran.radojev@dmi.uns.ac.rs).

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solution and its derivatives. As we deal with singularly perturbed problems, any dependence of error constants on a singular perturbation parameter is shown explicitly.

Notation. Throughout, C will denote a generic positive constant that is independent of the perturbation parameter ε and of the number N of degrees of freedom. For any set $D \subset [0,1]$ and any function v defined on D we set $\|v\|_{\infty,D} := \sup_{x\in D} |v(x)|$. If $D = [0,1]$ then we drop D from the notation.

2. The collocation method. Let $\Delta : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary partition of $[0,1]$. Let $J_i := [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$, $i = 1,\dots,N$. For $\kappa \in [0,1]$, we set $x_{i-\kappa} := x_i - \kappa h_i$. For $m, \ell \in \mathbb{N}_0$, $m < \ell$, we introduce the spline space

$$\mathcal{S}_\ell^m(\Delta) := \Big\{ s \in C^m[0,1] : s|_{J_i} \in \Pi_\ell, \ \text{for} \ i = 1,\dots,N \Big\}.$$

We shall use this notation with $m = -1$ to denote piecewise polynomial splines of degree ℓ that are discontinuous at the nodes of the partition Δ. For any function $g \in C[0,1]$, we set $g_i := g(x_i)$, $i = 0,\dots,N$.

Fix $k \in \{1,2,\dots\}$. We discretise (1.1) by seeking a spline in $\mathcal{S}_{k+1}^1(\Delta)$ that satifies the boundary conditions and the differential equation (1.1) in certain points. For problems that are not singularly perturbed, it is well known that the best choice for collocation are the zeros of the Legendre polynomials; see [4].

Let $\tau_{i,j}$, $j = 1,\dots,k$, $i = 1,\dots,N$, be the zeros of the local Legendre polynomial

$$M_{k,i}(x) := \frac{\mathrm{d}^k}{\mathrm{d}x^k} \Big((x - x_{i-1})^k (x - x_i)^k \Big).$$

Our discretisation is: Find $u_\Delta \in \mathcal{S}_{k+1}^1(\Delta)$ such that

$$u_{\Delta,0} = \gamma_0, \quad \big(\mathcal{L}u_\Delta\big)(\tau_{i,j}) = f(\tau_{i,j}), \ \ i = 1,\dots,N, \ \ j = 1,\dots,k, \quad u_{\Delta,N} = \gamma_1. \tag{2.1}$$

2.1. A priori error estimates. For classical problems, i.e., problems with $\varepsilon = 1$, it is well known that on a uniform mesh of step size h,

$$\|u - u_\Delta\| \le C h^{k^*} \quad \text{with} \begin{cases} k^* = 2 & \text{for} \ \ k = 1, \\ k^* = k+2 & \text{for} \ \ k > 1; \end{cases}$$

see [4]. In view of this result and based on numerical experiments as well as results for other discretisations [9, 13] we expect that for the singularly perturbed problem (1.1)

$$\|u - u_\Delta\| \le \begin{cases} C N^{-k^*} & \text{for Bakhvalov meshes}, \\ C\big(N^{-1}\ln N\big)^{k^*} & \text{for Shishkin meshes}. \end{cases} \tag{2.2}$$

We shall discuss the construction of these layer-adapted meshes in detail later in Section 3.1. Despite disparate attempts, we have been able to prove (2.2) for quadratic splines on a modified Shishkin mesh only; see [11].

2.2. A posteriori error bounds. The case of quadratic splines, i.e., $k = 1$, is somewhat special as is illustrated by the results mentioned in Section 2.1. For them the order of convergence equals the degree of the spline, while for higher order splines the order of convergence exceeds the degree of the spline by one. As a result, the a posteriori error analyses differ in a number of details. Therefore, we consider $k > 1$ in the present publication. Quadratic splines have been analysed in detail in [11].

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Before presenting our results, let us introduce two interpolation operators:
$I_{k+1}^0 : \varphi \mapsto I_{k+1}^0 \varphi \in \mathcal{S}_{k+1}^0$ with

$$\left.\begin{aligned} \varphi(x_{i-1}) &= \big(I_{k+1}^0\varphi\big)(x_{i-1}), \quad \varphi(x_i) = \big(I_{k+1}^0\varphi\big)(x_i) \\ \varphi(\tau_{i,j}) &= \big(I_{k+1}^0\varphi\big)(\tau_{i,j}), \;\; j = 1,\dots,k, \end{aligned}\right\} \; i = 1,\dots,N.$$

$I_{k-1}^{-1} : \varphi \mapsto I_{k-1}^{-1} \varphi \in \mathcal{S}_{k-1}^{-1}$ with

$$\varphi(\tau_{i,j}) = \big(I_{k-1}^{-1}\varphi\big)(\tau_{i,j}), \;\; j = 1,\dots,k, \;\; i = 1,\dots,N. \tag{2.3}$$

THEOREM 2.1. *Let u be the solution of (1.1) and $u_\Delta \in \mathcal{S}_{k+1}^1(\Delta)$, $k > 1$, its approximation by the collocation method (2.1) on an arbitrary mesh Δ. Then*

$$\|u - u_\Delta\|_\infty \le \eta^k(f - r u_\Delta, \Delta),$$

where $\eta^k(q,\Delta) = \eta^{k,I}(q,\Delta) + \eta^{k,D}(q,\Delta)$,

$$\eta^{k,I}(q,\Delta) := \left\| \frac{I_{k+1}^0 q - q}{r} \right\|_\infty ,$$

$$\eta^{k,D}(q,\Delta) := \frac{3}{2\varrho^2} \max_{i=1,\dots,N} \left[Q_{k,i}^{\max} \min\left\{2, \frac{h_i^2\varrho^2}{4\varepsilon^2}\right\} + Q_{k,i}^d \min\left\{1, \frac{h_i\varrho}{2\varepsilon}\right\}\right]$$

with

$$Q_{k,i}^- := q_{i-1} - \big(I_{k-1}^{-1}q\big)(x_{i-1}+0), \quad Q_{k,i}^+ := q_i - \big(I_{k-1}^{-1}q\big)(x_i - 0),$$

and

$$Q_{k,i}^{\max} := \max\left\{\left|Q_{k,i}^-\right|, \left|Q_{k,i}^+\right|\right\}, \quad Q_{k,i}^d := \left|Q_{k,i}^+ - (-1)^k Q_{k,i}^-\right|.$$

REMARK 2.2. The term $\eta^{k,I}$ captures the data oscillations and inevitably requires sampling of r and f. In view of the collocation condition (2.1), we have $q \approx \varepsilon^2 u_\Delta''$. Therefore, $\eta^{k,D}$ involves approximations of derivatives of u of order $k+2$ and $k+3$.

Proof. The proof is based on a representation for arbitrary $v \in W_0^{1,\infty}(0,1)$ involving the Green's function $\mathcal{G}$ associated with the operator $\mathcal{L}$:

$$v(x) = \int_0^1 \mathcal{G}(x,\xi)\,(\mathcal{L}v)\,(\xi)\,\mathrm{d}\xi. \tag{2.4}$$

We shall use this representation with $v := u - u_\Delta$. For $\mathcal{G}$ and its derivatives we have the following (weighted) L_1-norm estimates; see [7] or [9, Th. 3.31]:

$$\|r\mathcal{G}(x,\cdot)\|_1 \le 1, \quad \|\mathcal{G}_\xi(x,\cdot)\|_1 \le (\varrho\varepsilon)^{-1}, \quad \text{and} \quad \|\mathcal{G}_{\xi\xi}(x,\cdot)\|_1 \le 2\varepsilon^{-2}, \tag{2.5}$$

with the L_1-norm $\|v\|_1 := \int_0^1 |v(x)|\,\mathrm{d}x$. Furthermore, note that as a function of the second argument $\mathcal{G}$ satisfies, for any fixed $x \in (0,1)$,

$$-\varepsilon^2\mathcal{G}_{\xi\xi}(x,\xi) + r(\xi)\mathcal{G}(x,\xi) = \delta(\xi - x) \qquad \xi \in (0,1), \;\; \mathcal{G}(x,0) = \mathcal{G}(x,1) = 0, \tag{2.6}$$

where $\delta(\cdot)$ is the Dirac δ-distribution.

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For arbitrary, but fixed $x \in (0,1)$, set $\Gamma := \mathcal{G}(x,\cdot)$. Using (2.4), we write the error of the method as

$$(u - u_\Delta)(x) = \int_0^1 \Gamma(\xi)\big(\mathcal{L}(u-u_\Delta)\big)(\xi)\,\mathrm{d}\xi = \int_0^1 \Gamma(\xi)\big(f - \mathcal{L}u_\Delta\big)(\xi)\,\mathrm{d}\xi. \tag{2.7}$$

In view of (2.1) and (2.3), we have $I_{k-1}^{-1}(f - \mathcal{L}u_\Delta) \equiv 0$. Therefore,

$$\sum_{i=1}^N \int_{J_i} \Gamma(\xi) I_{k-1}^{-1}(f - \mathcal{L}u_\Delta)\,\mathrm{d}\xi = 0.$$

Set $q := f - ru_\Delta$. Subtracting the last equation from (2.7) and employing that $u''_\Delta \equiv I_{k-1}^{-1}u''_\Delta$ on each J_i, we obtain

$$(u - u_\Delta)(x) = \sum_{i=1}^N \int_{J_i} \Gamma(\xi)\big(q - I_{k-1}^{-1}q\big)(\xi)\,\mathrm{d}\xi.$$

Introducing

$$\Psi_{k,i} := \int_{J_i} \Gamma(\xi)\big(I_{k+1}^0 q - I_{k-1}^{-1}q\big)(\xi)\,\mathrm{d}\xi, \tag{2.8}$$

we have

$$(u - u_\Delta)(x) = \int_0^1 \big(q - I_{k+1}^0 q\big)(\xi)\Gamma(\xi)\,\mathrm{d}\xi + \sum_{i=1}^N \Psi_{k,i}.$$

A triangle inequality and (2.5) yield

$$\begin{aligned} |(u - u_\Delta)(x)| &\le \left\| \frac{I_{k+1}^0 q - q}{r} \right\|_\infty \int_0^1 r(\xi)\Gamma(\xi)\,\mathrm{d}\xi + \sum_{i=1}^N |\Psi_{k,i}| \\ &\le \eta^{k,I}(q,\Delta) + \sum_{i=1}^N |\Psi_{k,i}|. \end{aligned} \tag{2.9}$$

Next, we bound the sum of the $\Psi_{k,i}$. Suppose the following three bounds hold:

$$|\Psi_{k,i}| \le Q_{k,i}^{\max} \int_{J_i} \Gamma(\xi)\,\mathrm{d}\xi, \tag{2.10}$$

$$|\Psi_{k,i}| \le \frac{1}{2} Q_{k,i}^d \int_{J_i} \Gamma(\xi)\,\mathrm{d}\xi + \frac{h_i}{2} Q_{k,i}^{\max} \int_{J_i} |\Gamma'(\xi)|\,\mathrm{d}\xi, \tag{2.11}$$

and

$$|\Psi_{k,i}| \le \frac{h_i \Xi_i}{\varepsilon\varrho} \left\{ \frac{1}{4} Q_{k,i}^d + \frac{h_i\varrho}{8\varepsilon} Q_{k,i}^{\max} \right\}, \tag{2.12}$$

with

$$\Xi_i := \varrho\varepsilon \int_{J_i} |\Gamma'(\xi)|\,\mathrm{d}\xi + \int_{J_i} r(\xi)\Gamma(\xi)\,\mathrm{d}\xi + \int_{J_i} \delta(\xi - x)\,\mathrm{d}\xi.$$

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Inequalities (2.10) and (2.11) imply

$$|\Psi_{k,i}| \le \varrho^{-2} Q_{k,i}^{\max} \Xi_i \quad \text{and} \quad |\Psi_{k,i}| \le \left\{ \frac{1}{2\varrho^2} Q_{k,i}^d + \frac{h_i}{2\varepsilon\varrho} Q_{k,i}^{\max} \right\} \Xi_i, \quad \text{respectively.}$$

From these and (2.12) we get

$$|\Psi_{k,i}| \le \frac{\Xi_i}{2\varrho^2} \left[Q_{k,i}^{\max} \min\left\{ 2, \frac{h_i\varrho}{\varepsilon}, \frac{h_i^2\varrho^2}{4\varepsilon^2} \right\} + Q_{k,i}^d \min\left\{ 1, \frac{h_i\varrho}{2\varepsilon} \right\} \right]. \tag{2.13}$$

Furthermore,

$$\sum_{i=1}^N \Xi_i = \varrho\varepsilon \int_0^1 |\Gamma'(\xi)| \, \mathrm{d}\xi + \int_0^1 r(\xi)\Gamma(\xi) \, \mathrm{d}\xi + \int_0^1 \delta(\xi - x) \, \mathrm{d}\xi \le 3,$$

by (2.5). Therefore, (2.13) and a Hölder inequality applied to the $\Psi_{k,i}$-sum in (2.9) would complete the proof.

We are left with proving (2.10), (2.11), and (2.12). Which we will do now. First, note that $\left(I_{k+1}^0 q - I_{k-1}^{-1} q\right)(\tau_{i,j}) = 0$, $i = 1, \dots, N$, $j = 1, \dots, k$. Therefore,

$$\begin{aligned} &\left(I_{k+1}^0 q - I_{k-1}^{-1} q\right)(\xi) \\ &\quad = M_{k,i}(\xi) \left(Q_{k,i}^+ \frac{\xi - x_{i-1}}{h_i M_{k,i}(x_i)} - Q_{k,i}^- \frac{\xi - x_i}{h_i M_{k,i}(x_{i-1})} \right) =: M_{k,i}(\xi) p_i(\xi), \end{aligned} \tag{2.14}$$

with $Q_{k,i}^-$ and $Q_{k,i}^+$, as defined in Theorem 2.1. The function p_i is linear and attains its extrema at the end points of the interval J_i. Furthermore,

$$\|M_{k,i}\|_{\infty,J_i} = M_{k,i}(x_i) = (-1)^k M_{k,i}(x_{i-1}).$$

Thus,

$$\left\| I_{k+1}^0 q - I_{k-1}^{-1} q \right\|_{\infty,J_i} \le \|p_i\|_{\infty,J_i} \|M_{k,i}\|_{\infty,J_i} \le Q_{k,i}^{\max} \tag{2.15}$$

and

$$\|p_i'\|_{\infty,J_i} \|M_{k,i}\|_{\infty,J_i} \le h_i^{-1} Q_{k,i}^d. \tag{2.16}$$

Combining (2.8) and (2.15), we get (2.10).

From (2.8) and (2.14), an alternative estimate is derived as follows:

$$\begin{aligned} \Psi_{k,i} &= \int_{J_i} \left[\big(\Gamma p_i\big)(\xi) - \big(\Gamma p_i\big)(x_{i-1/2}) \right] M_{k,i}(\xi) \, \mathrm{d}\xi \\ &= \int_{J_i} \int_{x_{i-1/2}}^{\xi} \big(\Gamma p_i\big)'(\sigma) \, \mathrm{d}\sigma M_{k,i}(\xi) \, \mathrm{d}\xi, \end{aligned}$$

because $\int_{J_i} M_{k,i}(\xi) \, \mathrm{d}\xi = 0$. The Hölder inequality, (2.15), and (2.16) give (2.11).

The polynomial $M_{k,i}$ is orthogonal to polynomials of highest degree $k-1$ with respect to the standard $L_2(J_i)$ scalar product. Consequently,

$$\begin{aligned} \Psi_{k,i} &= \int_{J_i} \left[\big(\Gamma p_i\big)(\xi) - \big(\Gamma p_i\big)(x_{i-1/2}) \right] M_{k,i}(\xi) \, \mathrm{d}\xi \\ &= \int_{J_i} \left[\big(\Gamma p_i\big)(\xi) - \big(\Gamma p_i\big)(x_{i-1/2}) - \big(\xi - x_{i-1/2}\big) \big(\Gamma p_i\big)'(x_{i-1/2}) \right] M_{k,i}(\xi) \, \mathrm{d}\xi \\ &= \int_{J_i} \int_{x_{i-1/2}}^{\xi} \int_{x_{i-1/2}}^{\tau} \big(\Gamma p_i\big)''(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}\tau M_{k,i}(\xi) \, \mathrm{d}\xi. \end{aligned}$$

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Thus,

$$
\begin{aligned}
|\Psi_{k,i}| &\leq \|M_{k,i}\|_{\infty,J_i} \int_{x_{i-1}}^{x_i} \int_{\xi}^{x_{i-1/2}} \int_{\tau}^{x_{i-1/2}} \left|(\Gamma p_i)''(\sigma)\right| \,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&\leq \|M_{k,i}\|_{\infty,J_i} \left\{ \int_{x_{i-1}}^{x_{i-1/2}} \int_{\xi}^{x_{i-1/2}} \int_{x_{i-1}}^{x_{i-1/2}} \left|(\Gamma p_i)''(\sigma)\right| \,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}\xi \right. \\
&\qquad \left. + \int_{x_{i-1/2}}^{x_i} \int_{x_{i-1/2}}^{\xi} \int_{x_{i-1/2}}^{x_i} \left|(\Gamma p_i)''(\sigma)\right| \,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}\xi \right\} \\
&\leq \|M_{k,i}\|_{\infty,J_i} \left\{ \frac{h_i^2}{8} \int_{x_{i-1}}^{x_{i-1/2}} \left|(\Gamma p_i)''(\sigma)\right| \,\mathrm{d}\sigma + \frac{h_i^2}{8} \int_{x_{i-1/2}}^{x_i} \left|(\Gamma p_i)''(\sigma)\right| \,\mathrm{d}\sigma \right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\Psi_{k,i}| &\leq \|M_{k,i}\|_{\infty,J_i} \frac{h_i^2}{8} \int_{J_i} \left|(\Gamma p_i)''(\xi)\right| \,\mathrm{d}\xi \\
&\leq \frac{h_i}{4} Q_{k,i}^d \int_{J_i} |\Gamma'(\xi)| \,\mathrm{d}\xi + \frac{h_i^2}{8} Q_{k,i}^{\max} \int_{J_i} |\Gamma''(\xi)| \,\mathrm{d}\xi,
\end{aligned}
$$

by the Hölder inequality, (2.15), and (2.16). Next, note that from (2.6) we have

$$
\varepsilon^2 \int_{J_i} |\Gamma''(\xi)| \,\mathrm{d}\xi \leq \int_{J_i} r(\xi)\Gamma(\xi)\,\mathrm{d}\xi + \int_{J_i} \delta(\xi - x)\,\mathrm{d}\xi.
$$

Therefore, (2.12) holds true. The proof of Theorem 2.1 is complete. ∎

3. Numerical experiments.

3.1. Layer-adapted meshes. Two frequently used layer-adapted meshes are those of Bakhvalov [1] and of Shishkin [14]. The former yields more accurate results, while the latter is in general easier to analyse, because it consists of piecewise uniform subpartitions.

Bakhvalov's mesh-generating function [9] can be expressed as

$$
\varphi_B(t) = \begin{cases} \chi(t) := -\dfrac{\sigma\varepsilon}{\varrho} \log\left(1 - \dfrac{t}{q}\right), & t \in [0,\tau], \\ \pi(t) := \chi(\tau) + \chi'(\tau)(t-\tau), & t \in [\tau, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1]. \end{cases}
$$

where the scaling parameter $q \in (0, 1/2)$ and $\sigma \in (0, q\varrho/\varepsilon)$ are user-chosen parameters. The transition point τ satisfies

$$
\chi'(\tau) = \frac{0.5 - \chi(\tau)}{0.5 - \tau}.
$$

Geometrically this means that the point $(\tau, \chi(\tau))$ is the contact point of the tangent π to χ that passes through the point $(0.5, 0.5)$. An algorithm to rapidly computing the transition point τ is given in [1]. The Bakhvalov mesh with $N+1$ mesh nodes is formed as follows:

$$
\Delta_{B,N} = \left\{ x_i = \varphi_B(t_i), t_i = \frac{i}{N}, i = 0, 1, \ldots, N \right\}.
$$

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Shishkin meshes may be constructed as follows. Assume N, the number of mesh intervals, is divisable by 4. Define the mesh transition parameter

$$\lambda := \min\left\{\frac{\sigma\varepsilon}{\varrho}\ln N, \frac{1}{4}\right\}.$$

Then the mesh $\Delta_{S,N}$ is obtained by uniformly dissecting the intervals $[0,\lambda]$ and $[1-\lambda,1]$ into $N/4$ subintervals of equal length and $[\lambda, 1-\lambda]$ into $N/2$ subintervals.

Numerical results. We verify the theoretical results of the preceding section by applying the collocation method to the test problem

$$-\varepsilon^2 u''(x) + (1+x^2+\cos x)u(x) = e^{-x},\ \ x\in(0,1),\ u(0)=u(1)=0. \tag{3.1}$$

The exact solution of this problem is not available. Therefore, we approximate the errors using the so called double-mesh principle, i.e., we comparing the numerical solution to a solution obtained on a mesh that is twice as fine. Given a mesh $\Delta_N = \{x_0, x_1, \dots x_N\}$, we construct the refined mesh by adding the midpoints of the original mesh to that mesh:

$$\Delta^*_{2N} := \Delta_N \cup \Big\{x_{i-1/2} : i = 1,2,\dots,N\Big\}.$$

Then $\|u-u_{\Delta_N}\|_\infty \approx \|u_{\Delta^*_{2N}} - u_{\Delta_N}\|_\infty$. Computing the latter requires us to find the extrema of polynomials of potentially high degree. This too cannot be done exactly. Thus, we approximate the maximum-norm errors by

$$\begin{aligned}\|u-u_{\Delta_N}\|_\infty &\approx \|u_{\Delta^*_{2N}} - u_{\Delta_N}\|_\infty \\ &\approx \chi_N := \max_{\substack{i=1,\dots,N\\ m=0,\dots,M}} \left|\big(u_{\Delta^*_{2N}} - u_{\Delta_N}\big)(x_{i-1}+mM^{-1}h_i)\right|.\end{aligned}$$

In our experiments we have chosen $M = 7$. Larger values will give more accurate approximations of the actual errors. However, the difference is negligible. The rates of convergence are approximated by

$$p_N := \log_2(\chi_N/\chi_{2N}) \quad \text{and} \quad s_N := \frac{\ln\chi_N - \ln\chi_{2N}}{\ln 2 + \ln\ln N - \ln\ln 2N}. \tag{3.2}$$

The latter approximates the "Shishkin rate" s in the error bound $C\left(N^{-1}\ln N\right)^s$.

Table 3.1 displays the results of cubic-spline collocation applied to the test problem (3.1) on a Bakhvalov mesh. The columns of the table contain the discretisation parameter N, the approximate error χ_N, the two components $\eta^{k,I}$ and $\eta^{k,D}$ of the estimator, and the a posteriori error estimator. In the last column we give the ratio of the error estimator and the actual error, i.e., the efficiency of the estimator.

As expected the method converges with fourth order; cf. Section 2.1. Furthermore, we see that the efficiency of the error estimator is independent of of the level of refinement, although the errors are overestimated by a factor of approximately 30.

In Table 3.2 we fix the discretisation parameter $N = 2^{12}$ and vary the perturbation parameter ε. Looking at the errors and their estimates, we see that both are independent of ε. This illustrates the robustness – with respect to the perturbation parameter – of both the discretisation and the error estimator.

Table 3.3 gives results for uniform meshes. No convergence is observed which is to be expected, because the mesh is not adapted to the boundary layers of the problem. More important in the context of this paper, the a posteriori error estimator does not decrease

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TABLE 3.1
Cubic C^1-splines ($k = 2$) on Bakhvalov meshes ($\sigma = 4$ and $q = 1/4$) applied to (3.1), $\varepsilon = 10^{-6}$.

N	χ_N	p_N	$\eta^{k,I}$	$\eta^{k,D}$	η	χ_N/η
2^6	4.873e-06	4.03	4.187e-10	1.358e-04	1.358e-04	3.588e-02
2^7	2.981e-07	4.01	2.617e-11	8.487e-06	8.487e-06	3.513e-02
2^8	1.850e-08	3.99	1.636e-12	5.304e-07	5.304e-07	3.488e-02
2^9	1.165e-09	3.99	1.022e-13	3.314e-08	3.314e-08	3.514e-02
2^{10}	7.305e-11	4.00	6.390e-15	2.071e-09	2.071e-09	3.527e-02
2^{11}	4.574e-12	4.00	3.994e-16	1.294e-10	1.294e-10	3.534e-02
2^{12}	2.861e-13	—	2.496e-17	8.086e-12	8.086e-12	3.538e-02

TABLE 3.2
Cubic C^1-splines ($k = 2$) on Bakhvalov meshes ($\sigma = 4$ and $q = 1/4$) applied to (3.1), $N = 2^{12}$.

ε	χ_N	$\eta^{k,I}$	$\eta^{k,D}$	η	χ_N/η
10^{-2}	2.861e-13	5.039e-16	6.095e-12	6.095e-12	4.694e-02
10^{-3}	2.861e-13	4.976e-17	6.484e-12	6.484e-12	4.412e-02
10^{-4}	2.861e-13	2.449e-17	8.060e-12	8.060e-12	3.550e-02
10^{-5}	2.861e-13	2.491e-17	8.077e-12	8.077e-12	3.542e-02
10^{-6}	2.861e-13	2.496e-17	8.086e-12	8.086e-12	3.538e-02
10^{-7}	2.861e-13	2.497e-17	8.092e-12	8.092e-12	3.536e-02
10^{-8}	2.940e-13	2.497e-17	8.095e-12	8.095e-12	3.632e-02

with increasing N. Thus, the error estimator correctly indicates that uniform refinement is inappropriate for this kind of problem.

Finally, let us consider a higher-order method and a different mesh. Table 3.4 shows results for sixtic C^1-splines on a Shishkin mesh. Note that the third column displays the "Shishkin rates" s_N computed according to (3.2). It is observed that the error bounds strongly correlate with the actual errors and decreases like $\mathcal{O}\left(N^{-7}\ln^7 N\right)$. This time the errors are overestimated by a factor of approximately 250.

3.2. An adaptive algorithm. Using the a posteriori estimates of the preceding section an adaptive algorithm can be devised. It is based on an idea by de Boor [3] and uses an equidistribution principle. Its convergence in connection with an error estimator for a central difference scheme was recently studied by Kopteva and Chadha [2].

The idea is to adaptively design a mesh for which the local contributions to the a posteriori error estimator

$$\mu_i\left(u_\Delta,\Delta\right) := \left\|\frac{I_{k+1}^0 q - q}{r}\right\|_{\infty,J_i} + \frac{3}{2\varrho^2}\left[Q_{k,i}^{\max}\min\left\{2,\frac{h_i^2\varrho^2}{4\varepsilon^2}\right\}\right. \left. + Q_{k,i}^d\min\left\{1,\frac{h_i\varrho}{2\varepsilon}\right\}\right],$$

$q = ru_\Delta - f$, are the same on each mesh interval, i.e.,

$$\mu_{i-1}\left(u_\Delta,\Delta\right) = \mu_i\left(u_\Delta,\Delta\right), \quad \text{for } i = 1,\ldots,N.$$

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TABLE 3.3
Cubic C^1-splines ($k = 2$) on uniform meshes applied to (3.1), $\varepsilon = 10^{-6}$.

N	χ_N	$\eta^{k,I}$	$\eta^{k,D}$	η	χ_N/η
2^6	1.975e-01	1.101e-04	2.196	2.196	8.994e-02
2^7	1.975e-01	5.504e-05	2.196	2.196	8.995e-02
2^8	1.975e-01	2.752e-05	2.196	2.196	8.995e-02
2^9	1.975e-01	1.376e-05	2.196	2.196	8.995e-02
2^{10}	1.975e-01	6.882e-06	2.196	2.196	8.994e-02
2^{11}	1.975e-01	3.441e-06	2.196	2.196	8.993e-02

TABLE 3.4
Sixtic C^1-splines ($k = 5$) on Shishkin meshes ($\sigma = 7$) applied to (3.1) with $\varepsilon = 10^{-6}$.

N	χ_N	s_N	$\eta^{k,I}$	$\eta^{k,D}$	η	χ_N/η
2^6	9.642e-07	6.37	7.722e-13	2.549e-04	2.549e-04	3.782e-03
2^7	3.119e-08	6.64	9.321e-20	7.980e-06	7.980e-06	3.909e-03
2^8	7.610e-10	6.79	2.496e-21	1.922e-07	1.922e-07	3.960e-03
2^9	1.527e-11	6.88	5.288e-23	3.835e-09	3.835e-09	3.981e-03
2^{10}	2.677e-13	6.93	9.538e-25	6.688e-11	6.688e-11	4.003e-03
2^{11}	4.247e-15	6.96	1.535e-26	1.057e-12	1.057e-12	4.019e-03
2^{12}	6.242e-17	—	2.275e-28	1.550e-14	1.550e-14	4.028e-03

This is equivalent to

$$Q_i\left(u_\Delta,\Delta\right) = \frac{1}{N}\sum_{j=1}^{N} Q_j\left(u_\Delta,\Delta\right), \quad Q_i\left(u_\Delta,\Delta\right) := \mu_i\left(u_\Delta,\Delta\right)^{1/k^*}, \tag{3.3}$$

where k^* is the order of convergence. However, de Boor's algorithm, which we are going to describe now, becomes numerically unstable when the equidistribution principle (3.3) is enforced strongly. Instead, we shall stop the algorithm as proposed in [2, 8] when

$$\tilde{Q}_i\left(u_\Delta,\Delta\right) \le \frac{\gamma}{N}\sum_{j=1}^{N} \tilde{Q}_j\left(u_\Delta,\Delta\right),$$

for some user chosen constant $\gamma > 1$ (in our experiments, we have chosen $\gamma = 2$). Here we have also modified Q_i by choosing

$$\tilde{Q}_i\left(u_\Delta,\Delta\right) := \left(h_i^{k^*} + \mu_i\left(u_\Delta,\Delta\right)\right)^{1/k^*},$$

where $k^* = k+2$. Adding this constant floor to μ_i avoids mesh starvation and smoothes the convergence of the adaptive mesh algorithm.

Algorithm (de Boor [3])

1. Fix N, r, and a constant $\gamma > 1$. The initial mesh $\Delta^{[0]}$ is uniform with mesh size $1/N$.
2. For $k = 0, 1, \ldots$, given the mesh $\Delta^{[k]}$, compute the discrete solution $u_{\Delta^{[k]}}^{[k]}$ on this mesh using the $\mathcal{S}_{k+1}^1$-collocation method. Set $h_i^{[k]} = x_i^{[k]} - x_{i-1}^{[k]}$ for each i. Compute

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TABLE 3.5
The adaptive algorithm with cubic C^1-splines ($k = 2$) applied to (3.1), $\varepsilon = 10^{-6}$.

N	χ_N	p_N	$\eta^{2,I}$	$\tilde{\eta}^{2,D}$	η	χ_N/η	iter
2^6	8.943e-07	4.10	5.923e-09	2.084e-05	2.085e-05	4.290e-02	5
2^7	5.221e-08	2.63	3.630e-10	1.189e-06	1.190e-06	4.389e-02	4
2^8	8.434e-09	5.72	2.645e-11	1.896e-07	1.896e-07	4.448e-02	3
2^9	1.600e-10	4.10	1.362e-12	3.899e-09	3.900e-09	4.103e-02	3
2^{10}	9.299e-12	2.75	8.347e-14	2.421e-10	2.422e-10	3.839e-02	3
2^{11}	1.382e-12	5.02	6.144e-15	3.078e-11	3.078e-11	4.488e-02	2
2^{12}	4.269e-14	—	3.364e-16	9.824e-13	9.827e-13	4.344e-02	2
ave. rate		4.05			4.06		

the piecewise constant monitor function $M^{[k]}$ defined by

$$M^{[k]}(x) := \frac{Q_i^{[k]}}{h_i^{[k]}} := \frac{Q_i\left(u_{\Delta^{[k]}}^{[k]}, \Delta^{[k]}\right)}{h_i^{[k]}} \quad \text{for} \quad x \in \left(x_{i-1}^{[k]}, x_i^{[k]}\right).$$

We define

$$I^{[k]} := \sum_{j=1}^{N} \tilde{Q}_j^{[k]}.$$

3. Test mesh: If

$$\tilde{Q}_j^{[k]} \leq \gamma I^{[k]} N^{-1} \quad \text{for all } \ j = 1, \ldots, N$$

then go to Step 5. Otherwise, continue to Step 4.

4. Generate a new mesh by equidistributing the monitor function $M^{[k]}$, i.e., choose the new mesh $\Delta^{[k+1]}$ such that

$$\int_{x_{i-1}^{[k+1]}}^{x_i^{[k+1]}} M^{[k]}(t)\,\mathrm{d}t = \frac{I^{[k]}}{N}, \quad i = 1, \ldots, N.$$

Return to Step 2.

5. Set $\Delta^* = \Delta^{[k]}$ and $u_{\Delta^*}^* = u_{\Delta^{[k]}}^{[k]}$ then stop.

Numerical results. We first apply the adaptive algorithm to our test problem (3.1). The results are presented in Table 3.5. It contains the same quantities and is organised in a similar way as the previous tables. Furthermore, the last column gives the number of iterations required by the de Boor algorithm to meet the stopping criterion. An additional last row gives the averaged rates for the errors and the error estimator. These rates are close to 4 as expected for a fourth-order method.

Finally, we consider a modification of (3.1) which is taken from [10]:

$$-\varepsilon^2 u''(x) + (1 + x^2 + \cos x)u(x) = x^{3/2} + e^{-x}, \ \ x \in (0,1), \ u(0) = u(1) = 0. \tag{3.4}$$

Because of the term $x^{3/2}$ on the right-hand side, the second derivative of the reduced solution $u_0 = f/c$ has a singularity at $x = 0$. Consequently, a mesh that resolves the layers

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TABLE 3.6
Quintic C^1-splines ($k = 4$) on Shishkin meshes ($\sigma = 6$) for (3.4); $\varepsilon = 10^{-6}$.

N	χ_N	s_N	$\eta^{5,I}$	$\tilde{\eta}^{5,D}$	η	χ_N/η
2^6	1.687e-05	6.05	3.342e-06	2.508e-03	2.511e-03	6.718e-03
2^7	9.823e-07	2.14	1.146e-06	1.320e-04	1.332e-04	7.377e-03
2^8	3.289e-07	2.13	3.812e-07	5.504e-06	5.885e-06	5.590e-02
2^9	1.044e-07	2.23	1.195e-07	1.941e-07	3.136e-07	3.329e-01
2^{10}	3.005e-08	2.45	3.373e-08	1.933e-08	5.306e-08	5.663e-01
2^{11}	7.325e-09	2.78	8.003e-09	5.043e-09	1.305e-08	5.614e-01
2^{12}	1.431e-09	—	1.452e-09	1.060e-09	2.512e-09	5.695e-01

TABLE 3.7
Quintic C^1-splines ($k = 4$) with adaptive mesh movement applied to (3.4), $\varepsilon = 10^{-6}$.

N	χ_N	p_N	$\eta^{5,I}$	$\tilde{\eta}^{5,D}$	η	χ_N/η	iter
2^6	2.949e-08	5.55	2.551e-08	8.630e-08	1.118e-07	2.637e-01	5
2^7	6.280e-10	6.21	5.633e-10	1.336e-09	1.900e-09	3.306e-01	4
2^8	8.465e-12	6.19	7.056e-12	4.693e-11	5.399e-11	1.568e-01	3
2^9	1.160e-13	6.26	9.832e-14	2.862e-13	3.845e-13	3.018e-01	3
2^{10}	1.516e-15	5.73	2.156e-15	1.906e-13	1.928e-13	7.864e-03	2
2^{11}	2.848e-17	6.24	3.612e-16	1.584e-16	5.196e-16	5.482e-02	2
2^{12}	3.764e-19	—	3.280e-17	1.516e-18	3.432e-17	1.097e-02	3
ave. rate		6.03	4.92	5.95	5.27		

only, but not the singularity, will give unsatisfactory approximations. This is confirmed by Table 3.6. The error estimator correctly reflects this behaviour.

In contrast, C^1-collocation with adaptive mesh movement using the de Boor algorithm described earlier preserves the high order of the method; see Table 3.7. Both the errors and their a posteriori bounds are converging with order close to $k + 2$.

The results of the experiments are promising. However, more systematic numerical investigations are required, as is a rigorous theoretical justification for the adaptive algorithm.

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REFERENCES

[1] N. S. BAKHVALOV, *Towards optimization of methods for solving boundary value problems in the presence of boundary layers*, Zh. Vychisl. Mat. i Mat. Fiz., 9 (1969), pp. 841–859.

[2] N. M. CHADHA AND N. V. KOPTEVA, *A robust grid equidistribution method for a one-dimensional singularly perturbed semilinear reaction-diffusion problem*, IMA J. Numer. Anal., 31 (2011), pp. 188–211.

[3] C. DE BOOR, *Good approximation by splines with variable knots*, in Spline Functions and Approximation Theory, Proc. Sympos. Univ. Alberta, Edmonton 1972, A. Meir and A. Sharma, eds, Internat. Ser. Numer. Math., vol. 21, Birkhäuser, Basel, 1973, pp. 57–72.

[4] C. DE BOOR AND B. SWARTZ, *Collocation at Gaussian points*, SIAM J. Numer. Anal., 10 (1973), pp. 582–606.

[5] P. A. FARRELL, A. F. HEGARTY, J. J. H. MILLER, E. O'RIORDAN, AND G. I. SHISHKIN, *Robust Computational Techniques for Boundary Layers*, Chapman & Hall, Boca Raton, 2000.

[6] D. FUNARO, *Polynomial Approximation of Differential Equations*, Springer, Berlin, 1992.

[7] N. KOPTEVA, *Maximum norm a posteriori error estimates for a 1D singularly perturbed semilinear reaction-diffusion problem*, IMA J. Numer. Anal., 27 (2007), pp. 576–592.

[8] N. KOPTEVA AND M. STYNES, *A robust adaptive method for a quasi-linear one-dimensional convection-diffusion problem*, SIAM J. Numer. Anal., 39 (2001), pp. 1446–1467.
[9] T. LINSS, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Springer, Berlin, 2010.
[10] ———, *A posteriori error estimation for arbitrary order FEM applied to singularly perturbed one-dimensional reaction-diffusion problem*, Appl. Math., 59 (2014), pp. 241–256.
[11] T. LINSS, G. RADOJEV, AND H. ZARIN, *Approximation of singularly perturbed reaction-diffusion problems by quadratic C^1-splines*, Numer. Algorithms, 61 (2012), pp. 35–55.
[12] J. J. H. MILLER, E. O'RIORDAN, AND G. I. SHISHKIN, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, River Edge, 1996.
[13] H.-G. ROOS, M. STYNES, AND L. TOBISKA, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, 2nd ed., Springer, Berlin, 2008.
[14] G. I. SHISHKIN, *Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations*, Second doctorial thesis, Keldysh Institute, Moscow, 1990.