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A BDDC ALGORITHM FOR SECOND-ORDER ELLIPTIC PROBLEMS WITH HYBRIDIZABLE DISCONTINUOUS GALERKIN DISCRETIZATIONS*

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Abstract. A balancing domain decomposition by constraints (BDDC) algorithm is applied to the linear system arising from a hybridizable discontinuous Galerkin (HDG) discretization of the second-order elliptic problems. Edge/face constraints are enforced across the subdomain interface and the similar condition number bound is obtained as those for conforming finite element discretization. Numerical experiments demonstrate the convergence rate of the proposed algorithm.

Key words. discontinuous Galerkin, HDG, domain decomposition, BDDC

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction. In this paper, a Balancing Domain Decomposition by Constraints (BDDC) algorithm is developed for the hybridizable discontinuous Galerkin (HDG) method. General HDG methods were introduced by Cockburn and his collaborators in [14], and the specific HDG method we consider here is often called LDG-H method, which is constructed by using the local discontinuous Galerkin method on each element. One distinct feature of the HDG method is that the only global coupled degrees of freedom are scalar variables, called "numerical traces". Therefore the resulting global system from the HDG is much smaller than other traditional DG methods. The superconvergence of HDG methods has also been studied in [12, 15]. Recently, in [13], the condition number of the linear system arising from the HDG (LDG-H) discretization of a second-order elliptic problem has been shown to grow like $O(h^{-2})$ if $\tau h \leq C$. Here τ is the typical penalty constant, h is the typical mesh size, and C is a constant. For so-called "super-penalized" cases where τ is chosen to be $O(\frac{1}{h^\alpha})$ with $\alpha > 1$, the condition number grows even faster. Therefore efficient fast solvers for the linear system are necessary.

There are many fast solvers for DG methods and their variants such as multigrid and domain decomposition methods. Geometric Multigrid methods for the interior penalty DG were studied in [25] and extended to other DG methods in [24] using the unified analysis of [4]. Algebraic multigrid methods have been studied in [29, 30]. In [21, 22], two-level additive Schwarz methods were developed for second-order elliptic problems and two-level non-overlapping Schwarz methods were studied for fourth-order biharmonic equations, respectively. Overlapping Schwarz preconditioners were developed for advection-diffusion problems in [31]. In [1, 2, 3, 5], a class of Schwarz preconditioners were studied for different problems. Several nonoverlapping domain decomposition methods are developed in [18, 19, 20] for the discretization using a conforming finite element inside each subdomain and a discontinuous Galerkin method across subdomain boundaries. An overlapping Schwarz and a nonoverlapping (BDDC) domain decomposition methods are studied in [6, 10] for a weakly over-penalized symmetric interior penalty method. Similar algorithms have been developed for a class of staggered discontinuous Galerkin methods in [11, 27]. A BDDC algorithm is studied for more general DG methods in [16] based on the unified analysis of [4].

However, there are relatively few fast solvers for the HDG methods. A multigrid V-cycle method was used as a linear solver for the HDG in [13]. Both overlapping and nonoverlapping

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domain decomposition methods are studied for high-order HDG methods in [36], where the domain decomposition algorithms are applied on the element level (namely one element is considered as a subdomain).

The BDDC algorithms, introduced by Dohrmann for second-order elliptic problem in [17], see also [34, 35], are nonoverlapping domain decomposition methods, which are similar to the balancing Neumann-Neumann (BNN) algorithms. In the BDDC algorithm, the coarse problems are given in terms of a set of primal constraints. An important advantage with such a coarse problem is that the Schur complements that arise in the computations will all be invertible. The BDDC algorithms have been extended to second-order elliptic problem with mixed and hybrid formulations in [37, 39] and the Stokes problem [32].

In this paper, we consider the BDDC algorithm for the linear system arising from the HDG method. The close relationship between HDG and the classical hybridized Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) methods was highlighted in [14]. In [12], it has been shown that a specific HDG method has exactly the same stiffness matrix as the hybridized RT and BDM methods. In [13], an important spectral relation between the bilinear form resulting from the HDG and hybridized RT method is established. As a result, the previously developed preconditioners for the hybrid RT methods can be applied to HDG such as the overlapping Schwarz preconditioner in [23], multigrid preconditioner in [26], and the BDDC preconditioner in [39]. Here, we apply the BDDC preconditioner directly to the HDG bilinear form and estimate the condition number bound of the resulting preconditioned operator using its spectral relation with the hybridized RT method. Compared to the multigrid algorithms studied in [13], the BDDC algorithm is applied directly to the system arising from the HDG method. In [39], only the lowest-order Raviart-Thomas finite element method is considered. Here, in our analysis, we also include high-order elements. For the dependence of the condition number bound on the order of the element, we need to examine such dependence in several norms including those derived from the bilinear forms of the HDG and hybridized RT methods. Refined analysis of the condition number bound is needed for this dependence and will be given in future study. For some results related to this issue, see [7, 8, 36].

The rest of the paper is organized as follows. The mixed formulation for the elliptic problems and its HDG discretization are described in Section 2. We reduce our system to an interface problem in Section 3. In Section 4, we introduce the BDDC algorithms for the HDG discretization. We give some auxiliary results in Section 5. In Section 6, we provide an estimate of the condition number for the system with the BDDC preconditioner. Finally, some computational results are given in Section 7.

2. An elliptic problem and HDG discretization. We consider the following elliptic problem on a bounded polygonal/polyhedral domain Ω, in two/three space-dimensions, with a Dirichlet boundary condition:

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where a is a positive definite matrix function with the entries in $L^\infty(\Omega)$ satisfying

$$\xi^T a(\mathbf{x})\xi \geq \alpha \left\|\xi\right\|^2, \quad \text{for a.e.} \quad \mathbf{x}\in\Omega,$$

for some positive constant α, $f \in L^2(\Omega)$, and $g \in H^{1/2}(\partial\Omega)$. Without loss of generality, we assume that $g = 0$. If Ω is convex or has a C^2 boundary, then equation (2.1) has a unique solution $u \in H^2(\Omega)$; see [9].

We then introduce a new variable $\mathbf{q}$:

$$\mathbf{q} = -a\nabla u.$$

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and let $\rho = a^{-1}$. We obtain the following system for $\mathbf{q}$ and u,

$$\begin{aligned} \rho\mathbf{q} &= -\nabla u && \text{in } \Omega, \\ \nabla\cdot\mathbf{q} &= f && \text{in } \Omega, \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{2.2}$$

We will approximate $\mathbf{q}$ and u by introducing discontinuous finite element spaces. Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω with characteristic element size h and the element in $\mathcal{T}_h$ denoted by κ. Define $\mathcal{E}$ to be the union of edges of elements κ. $\mathcal{E}_i$ and $\mathcal{E}_\partial$ are the sets of the domain interior and boundary edges, respectively.

Let $P_k(D)$ be the space of polynomials of order at most k on D. We set $\mathbf{P}_k(D) = [P_k(D)]^n$ ($n = 2$ or 3 for two and three dimensions, respectively) and define the following finite element spaces:

$$\begin{aligned} \mathbf{V}_k &= \{\mathbf{v}_h \in [L^2(\Omega)]^n : \mathbf{v}_h|_\kappa \in \mathbf{P}_k(\kappa) \quad \forall\kappa\in\Omega\}, \\ W_k &= \{w_h \in L^2(\Omega) : w_h|_\kappa \in P_k(\kappa) \quad \forall\kappa\in\Omega\}, \\ M_k &= \{\mu_h \in L^2(\mathcal{E}) : \mu_h|_e \in P_k(e) \quad \forall e\in\mathcal{E}\}. \end{aligned}$$

Let $\Lambda_k = \{\mu \in M_k : \mu|_e = 0 \ \forall e \in \partial\Omega\}$. To make our notation simple, we drop the superscript k from now on.

For each κ, we find $(\mathbf{q}_h, u_h) \in (\mathbf{P}_k(\kappa), P_k(\kappa))$ such that for all $\kappa \in \mathcal{T}_h$,

$$\begin{aligned} (\rho\mathbf{q}_h, \mathbf{v}_h)_\kappa - (u_h, \nabla\cdot\mathbf{v}_h)_\kappa + \langle \hat{u}_h, \mathbf{v}_h\cdot\mathbf{n}\rangle_{\partial\kappa} &= 0 && \forall \mathbf{v}_h \in \mathbf{P}_k(\kappa), \\ (\mathbf{q}_h, \nabla w_h)_\kappa - \langle \hat{\mathbf{q}}_h\cdot\mathbf{n}, w_h\rangle_{\partial\kappa} &= -(f, w_h)_\kappa && \forall w_h \in P_k(\kappa), \end{aligned} \tag{2.3}$$

where $(\cdot,\cdot)_\kappa$ and $\langle\cdot,\cdot\rangle_{\partial\kappa}$ denote the L^2-inner products on functions or vector-valued functions in κ and $\partial\kappa$, respectively. The functions $\hat{u}_h$ and $\hat{\mathbf{q}}_h$ are the numerical traces that approximate u_h and $\mathbf{q}_h$ on $\partial\kappa$, respectively.

Let $\lambda_h \in \Lambda$ and the numerical trace $\hat{u}_h = \lambda_h$. The numerical flux $\hat{\mathbf{q}}_h\cdot\mathbf{n}$ is more complicated and takes the form:

$$\hat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau_\kappa(u_h - \lambda_h), \quad \text{on } \partial\kappa,$$

where τ_κ is a local stabilization parameter; see [12] for details.

With the definitions of numerical trace λ_h and the numerical flux $\hat{\mathbf{q}}_h\cdot\mathbf{n}$, this discrete problem resulting from HDG discretization can be written as: to find $(\mathbf{q}_h, u_h, \lambda_h) \in \mathbf{V}\times W\times\Lambda$ such that for all $(\mathbf{v}_h, w_h, \mu_h) \in \mathbf{V}\times W\times\Lambda$,

$$\begin{aligned} (\rho\mathbf{q}_h, \mathbf{v}_h)_{\mathcal{T}_h} - (u_h, \nabla\cdot\mathbf{v}_h)_{\mathcal{T}_h} + \langle \lambda_h, \mathbf{v}_h\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} &= 0 \\ (\mathbf{q}_h, \nabla w_h)_{\mathcal{T}_h} - \langle \hat{\mathbf{q}}_h\cdot\mathbf{n}, w_h\rangle_{\partial\mathcal{T}_h} &= -(f, w_h)_\Omega \\ \langle \hat{\mathbf{q}}_h\cdot\mathbf{n}, \mu_h\rangle_{\partial\mathcal{T}_h} &= 0, \end{aligned} \tag{2.4}$$

where $(\cdot,\cdot)_{\mathcal{T}_h} = \sum_{\kappa\in\mathcal{T}_h}(\cdot,\cdot)_\kappa$ and $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} = \sum_{\kappa\in\mathcal{T}_h}\langle\cdot,\cdot\rangle_{\partial\kappa}$. Define

$$\begin{aligned} &A_{\mathbf{qq}} : \mathbf{V}\to\mathbf{V}, \quad A_{u\mathbf{q}} : \mathbf{V}\to W, \quad A_{\lambda\mathbf{q}} : \mathbf{V}\to\Lambda, \\ &A_{uu} : W\to W, \quad A_{\lambda u} : W\to\Lambda, \quad A_{\lambda\lambda} : \Lambda\to\Lambda \end{aligned}$$

as

$$\begin{aligned} &(A_{\mathbf{qq}}\mathbf{q}, \mathbf{v}) = (\rho\mathbf{q}, \mathbf{v})_{\mathcal{T}_h}, \qquad (A_{u\mathbf{q}}\mathbf{q}, w) = -(w, \nabla\cdot\mathbf{q})_{\mathcal{T}_h}, \\ &(A_{\lambda\mathbf{q}}\mathbf{q}, \mu) = \langle \mu, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}, \end{aligned}$$

$$(A_{uu}u, w) = -\sum_{\kappa\in\mathcal{T}_h} \langle \tau_\kappa w, u\rangle_{\partial\kappa}\,, \qquad (A_{\lambda u}u, \lambda) = \sum_{\kappa\in\mathcal{T}_h} \langle \tau_\kappa u, \lambda\rangle_{\partial\kappa}\,,$$
$$(A_{\lambda\lambda}\lambda, \mu) = -\sum_{\kappa\in\mathcal{T}_h} \langle \tau_\kappa \lambda, \mu\rangle_{\partial\kappa}\,,$$

for all $\mathbf{q}, \mathbf{v} \in \mathbf{V}$, $u, w \in W$, and $\lambda, \mu \in \Lambda$. Correspondingly, the matrix form of (2.4) is

$$\begin{bmatrix} A_{\mathbf{qq}} & A_{u\mathbf{q}}^T & A_{\lambda\mathbf{q}}^T \\ A_{u\mathbf{q}} & A_{uu} & A_{\lambda u}^T \\ A_{\lambda\mathbf{q}} & A_{\lambda u} & A_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \\ 0 \end{bmatrix}, \tag{2.5}$$

where we use $\mathbf{q}$, u, and λ to denote the unknowns associated with $\mathbf{q}_h$, u_h, and λ_h, respectively.

In each κ, given the value of λ on $\partial\kappa$, $\mathbf{q}$ and u can be uniquely determined; see [14]. Namely, given λ_h, the solution $(\mathbf{q}_h, u_h)$ of (2.3) is uniquely determined. In matrix form, we note that

$$\begin{bmatrix} A_{\mathbf{qq}} & A_{u\mathbf{q}}^T \\ A_{u\mathbf{q}} & A_{uu} \end{bmatrix}$$

is block diagonal, each block is nonsingular and corresponds to one element κ. Therefore, we can easily eliminate $\mathbf{q}$ and u in each element independently from (2.5) and obtain the system determining only λ,

$$A\lambda = b, \tag{2.6}$$

where

$$A = A_{\lambda\lambda} - [A_{\lambda\mathbf{q}}\; A_{\lambda u}] \begin{bmatrix} A_{\mathbf{qq}} & A_{u\mathbf{q}}^T \\ A_{u\mathbf{q}} & A_{uu} \end{bmatrix}^{-1} \begin{bmatrix} A_{\lambda\mathbf{q}}^T \\ A_{\lambda u}^T \end{bmatrix}$$

and

$$b = -[A_{\lambda\mathbf{q}}\; A_{\lambda u}] \begin{bmatrix} A_{\mathbf{qq}} & A_{u\mathbf{q}}^T \\ A_{u\mathbf{q}} & A_{uu} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ F_h \end{bmatrix}.$$

Once the solution λ of (2.6) is obtained, the solution of (2.5) can be completed by computing $\mathbf{q}$ and u in each element with the given λ.

By [14, Theorem 2.1], the system (2.6) can be considered as the matrix form of the following problem: find $\lambda \in \Lambda$ such that

$$a_h(\lambda, \mu) = b_h(\mu), \quad \forall\, \mu \in \Lambda.$$

Here

$$\begin{aligned} a_h(\eta, \mu) &= \sum_{\kappa\in\mathcal{T}_h} a_\kappa(\eta, \mu) = \sum_{\kappa\in\mathcal{T}_h} (\rho\mathcal{Q}\eta, \mathcal{Q}\mu)_\kappa + \langle \tau_\kappa(\mathcal{U}\eta - \eta, (\mathcal{U}\mu - \mu)\rangle_{\partial\kappa}\,, \\ b_h(\mu) &= \sum_{\kappa\in\mathcal{T}_h} b_\kappa(\eta, \mu) = \sum_{\kappa\in\mathcal{T}_h} (f_h, \mathcal{U}\mu)_\kappa, \end{aligned} \tag{2.7}$$

where $\mathcal{Q}\mu$ and $\mathcal{U}\mu$ are the unique solution $(\mathbf{q}_h = \mathcal{Q}\mu,\ u_h = \mathcal{U}\mu)$ of the local element problem (2.3) with $\lambda = \mu$. We note that, in [13, Theorem 3.6], the bilinear form $a_h(\cdot,\cdot)$ has been proved to be positive definite. More properties of $a_h(\cdot,\cdot)$ will be studied in Section 5. In next two sections, we will develop a BDDC algorithm to solve the system in (2.6) for the numerical trace λ.

3. Reduced subdomain interface problem. We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such triangles forming an individual subdomain is uniformly bounded. We also assume $a(\mathbf{x})$, the coefficient of (2.1), is constant in each subdomain. Let Γ be the interface between subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_{i\neq j}(\partial\Omega_{i,h} \cap \partial\Omega_{j,h})) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$, and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$. We reduce the global problem (2.6) to a subdomain interface problem.

We decompose Λ into the subdomain interior and interface parts as

$$\Lambda = \Lambda_I \bigoplus \widehat{\Lambda}_\Gamma,$$

where $\widehat{\Lambda}_\Gamma$ denotes the degrees of freedom associated with Γ and Λ_I is a direct sum of subdomain interior degrees of freedom, i.e,

$$\Lambda_I = \bigoplus_{i=1}^{N} \Lambda_I^{(i)}.$$

The global problem (2.6) can be written as

$$\begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{I\Gamma}^T & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \lambda_I \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} b_I \\ b_\Gamma \end{bmatrix}.$$

We denote the subdomain interface numerical trace space by $\Lambda_\Gamma^{(i)}$, and the associate product space by $\Lambda_\Gamma = \prod_{i=1}^N \Lambda_\Gamma^{(i)}$. $R_\Gamma^{(i)}$ is the operator which maps functions in the continuous interface numerical trace space $\widehat{\Lambda}_\Gamma$ to their subdomain components in the space $\Lambda_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ.

The global problem (2.6) is assembled from subdomain problems

$$A^{(i)}\lambda^{(i)} = b^{(i)},$$

where

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}, \quad \lambda^{(i)} = \begin{bmatrix} \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} \in \Lambda_I^{(i)} \times \Lambda_\Gamma^{(i)}, \quad \text{and} \quad b^{(i)} = \begin{bmatrix} b_I^{(i)} \\ b_\Gamma^{(i)} \end{bmatrix}.$$

We can eliminate the subdomain interior variables $\lambda_I^{(i)}$ in each subdomain independently and define the subdomain Schur complement $S_\Gamma^{(i)}$ by: given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, determine $S_\Gamma^{(i)}\lambda_\Gamma^{(i)}$ such that

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ S_\Gamma^{(i)}\lambda_\Gamma \end{bmatrix}. \tag{3.1}$$

We denote the direct sum of the $S_\Gamma^{(i)}$ by S_Γ, i.e.,

$$S_\Gamma = \begin{bmatrix} S_\Gamma^{(1)} & & \\ & \ddots & \\ & & S_\Gamma^{(N)} \end{bmatrix}.$$

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$\widehat{S}_\Gamma \lambda_\Gamma = b_\Gamma, \tag{3.2}$$

where $b_\Gamma = \sum_{i=1}^{N} R_\Gamma^{(i)^T} b_\Gamma^{(i)}$, and

$$\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^{N} R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}.$$

Thus, $\widehat{S}_\Gamma$ is defined on the interface space $\widehat{\Lambda}_\Gamma$ and is symmetric and positive definite. We will propose a BDDC preconditioner for solving (3.2) with a preconditioned conjugate gradient method.

4. The BDDC preconditioner. The BDDC (Balancing Domain Decomposition by Constraints) methods, which were introduced and analyzed by Dohrmann, Mandel, and Tezaur in [17, 34, 35], are originally designed for standard finite element discretization of elliptic problems. The BDDC algorithms are similar to the balancing Neumann-Neumann algorithms. However, their coarse problems in BDDC, are given in terms of sets of primal constraints. The main advantage of such coarse problems is that the local subdomain problems, arising in the BDDC algorithms are invertible. They are one of the most tested and popular domain decomposition algorithms and suitable for parallel computation.

In order to introduce the BDDC preconditioner, we first introduce a partially assembled interface space $\widetilde{\mathbf{\Lambda}}_\Gamma$ by

$$\widetilde{\mathbf{\Lambda}}_\Gamma = \widehat{\mathbf{\Lambda}}_\Pi \bigoplus \mathbf{\Lambda}_\Delta = \widehat{\mathbf{\Lambda}}_\Pi \bigoplus \left(\prod_{i=1}^{N} \mathbf{\Lambda}_\Delta^{(i)} \right).$$

Here, $\widehat{\mathbf{\Lambda}}_\Pi$ is the coarse level, primal interface space which is spanned by subdomain interface edge/face basis functions with constant values at the nodes of the edge/face for two/three dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit; see [33] and [28]. The new variables are called primal unknowns. The space $\mathbf{\Lambda}_\Delta$ is the direct sum of the $\mathbf{\Lambda}_\Delta^{(i)}$, which are spanned by the remaining interface degrees of freedom with a zero average over each edge/face. In the space $\widetilde{\Lambda}_\Gamma$, we relax most continuity constraints on the numerical trace across the interface but retain the continuity at the primal unknowns, which makes all the local linear systems nonsingular.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\mathbf{\Lambda}}_\Gamma$ to the components $\mathbf{\Lambda}_\Gamma^{(i)}$ related to the subdomain Ω_i. $R_\Delta^{(i)}$ maps functions from $\widehat{\mathbf{\Lambda}}_\Gamma$ to $\mathbf{\Lambda}_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi}$ is a restriction operator from $\widehat{\mathbf{\Lambda}}_\Gamma$ to its subspace $\widehat{\mathbf{\Lambda}}_\Pi$ and $R_\Pi^{(i)}$ is an operator which maps vectors in $\widehat{\mathbf{\Lambda}}_\Pi$ into their components in $\Lambda_\Pi^{(i)}$. $\overline{R}_\Gamma : \widetilde{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma$ is the direct sum of the $\overline{R}_\Gamma^{(i)}$ and $\widetilde{R}_\Gamma : \widehat{\mathbf{\Lambda}}_\Gamma \to \widetilde{\mathbf{\Lambda}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define a positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} \rho_j^\gamma(x)}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We note that $\delta_i^\dagger(x)$ is constant on each edge/face since we assume that $\rho_i(x)$ is constant in each subdomain.

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Multiplying each row of $R_{\Delta}^{(i)}$ with the scaling factor $\delta_i^{\dagger}(x)$ gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$. Furthermore, $\widetilde{R}_{\Delta}^{(i)}$ maps functions from $\widetilde{\mathbf{\Lambda}}_{\Gamma}$ to $\mathbf{\Lambda}_{\Delta}^{(i)}$, its dual subdomain components. $\widetilde{R}_{\Gamma\Pi}$ is a restriction operator from $\widetilde{\mathbf{\Lambda}}_{\Gamma}$ to its subspace $\widehat{\mathbf{\Lambda}}_{\Pi}$.

We also denote by $\widetilde{\mathbf{F}}_{\Gamma}$, the right hand side space corresponding to $\widetilde{\mathbf{\Lambda}}_{\Gamma}$. We will use the same restriction, extension, and scaled restriction operators for the space $\widetilde{\mathbf{F}}_{\Gamma}$ as for $\widetilde{\mathbf{\Lambda}}_{\Gamma}$.

The interface numerical trace Schur complement $\widetilde{S}_{\Gamma}$, on the partially assembled interface numerical trace space $\widetilde{\Lambda}_{\Gamma}$, is obtained from partial assembly of subdomain Schur complements $S_{\Gamma}^{(i)}$, i.e.,

$$\widetilde{S}_{\Gamma} = \overline{R}_{\Gamma}^{T} S_{\Gamma} \overline{R}_{\Gamma}.$$

The BDDC preconditioner for solving the global interface problem (3.2) is

$$M^{-1} = \widetilde{R}_{D,\Gamma}^{T} \widetilde{S}_{\Gamma}^{-1} \widetilde{R}_{D,\Gamma}.$$

The preconditioned BDDC algorithm is then of the form: find $\lambda_{\Gamma} \in \widehat{\Lambda}_{\Gamma}$, such that

$$\widetilde{R}_{D,\Gamma}^{T} \widetilde{S}_{\Gamma}^{-1} \widetilde{R}_{D,\Gamma} \widehat{S}_{\Gamma} \lambda_{\Gamma} = \widetilde{R}_{D,\Gamma}^{T} \widetilde{S}_{\Gamma}^{-1} \widetilde{R}_{D,\Gamma} b_{\Gamma}. \tag{4.1}$$

This preconditioned problem is symmetric positive definite, and we can use the preconditioned conjugate gradient method to solve it.

5. Some auxiliary results.

In this section we collect a number of results that are needed in our condition number estimate of the preconditioned system (4.1). We define

$$\gamma_{h,\tau} = \max_{\kappa \in \mathcal{T}_h} \{1 + \tau_{\kappa} h_{\kappa}\}, \tag{5.1}$$

where τ_{κ} and h_{κ} are the stabilized parameter and diameter of the element κ, respectively. We use c and C to denote constants that are independent of h, H, τ_{κ}, and the coefficients ρ of (2.2).

We first introduce several useful norms, which are defined in [13, 23]. For any domain D, we denote the L^2 norm by $\|\cdot\|_D$. For any $\lambda \in \Lambda(D)$, define

$$|||\lambda|||_D = \left(\frac{1}{h} \sum_{\kappa \in \mathcal{T}_h, \kappa \subseteq \bar{D}} \|\lambda - m_{\kappa}(\lambda)\|_{L^2(\partial\kappa)}^2 \right)^{1/2},$$

where

$$m_{\kappa} = \frac{1}{|\partial\kappa|} \int_{\partial\kappa} \lambda ds,$$

and $|\partial\kappa|$ is the measure (the length for 2D and area for 3D) of the boundary of κ.

We note that when D is strictly contained in Ω, $|||\lambda|||_D$ is a semi-norm. When $D = \Omega$, we use the simple notation $|||\lambda|||$ for $|||\lambda|||_{\Omega}$. $|||\lambda|||$ is an H^1-like norm since the functions in Λ have zero boundary conditions on $\partial\Omega$.

We recall the bilinear norm $a_h(\eta, \mu)$ in (2.7) and define the norm

$$|\lambda|_A^2 = a_h(\lambda, \lambda), \quad \forall \lambda \in \Lambda.$$

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Given a subdomain Ω_i, let $a_h^{(i)}(\cdot,\cdot)$ be the restriction of $a_h(\cdot,\cdot)$ to Ω_i, and we can define similar norms. Let

$$\left|\lambda^{(i)}\right|_{A^{(i)}}^2 = a_h^{(i)}(\lambda^{(i)}, \lambda^{(i)}), \quad \forall \lambda^{(i)} \in \Lambda^{(i)}.$$

The global norm $|\lambda|_A$ can be assembled from the subdomain norms as

$$|\lambda|_A^2 = \sum_{i=1}^N \left|\lambda^{(i)}\right|_{A^{(i)}}^2,$$

where $\lambda^{(i)} = R_\Gamma^{(i)}(\lambda)$, the restriction of λ to the subdomain Ω_i. The following lemma is in [13, Theorem 3.9] applied to each subdomain Ω_i.

LEMMA 5.1. *For any* $\lambda^{(i)} \in \Lambda^{(i)}$,

$$c\rho_i \left|\!\left|\!\left| \lambda^{(i)} \right|\!\right|\!\right|_{\Omega_i}^2 \le \left|\lambda^{(i)}\right|_{A^{(i)}}^2 \le C\rho_i\gamma_{h,\tau} \left|\!\left|\!\left| \lambda^{(i)} \right|\!\right|\!\right|_{\Omega_i}^2,$$

where $\gamma_{h,\tau}$ *is defined in* (5.1).

Given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, we can define a harmonic extension $\mathcal{H}^{(i)}(\lambda_\Gamma^{(i)}) : \Lambda_\Gamma^{(i)} \to \Lambda^{(i)}$ as

$$\left|\mathcal{H}^{(i)}(\lambda_\Gamma^{(i)})\right|_{A^{(i)}}^2 = \min_{\lambda^{(i)} \in \Lambda^{(i)}, \lambda^{(i)} = \lambda_\Gamma^{(i)} \text{ on } \partial\Omega_i} \left|\lambda^{(i)}\right|_{A^{(i)}}^2.$$

By the definition of $\mathcal{H}^{(i)}$ and (3.1), we have

$$\left|\lambda_\Gamma^{(i)}\right|_{S_\Gamma^{(i)}}^2 := \left(\lambda_\Gamma^{(i)}\right)^T S_\Gamma^{(i)} \lambda_\Gamma^{(i)} = \left|\mathcal{H}^{(i)}(\lambda_\Gamma^{(i)})\right|_{A^{(i)}}^2.$$

The bilinear form $a_h(\cdot,\cdot)$ defined in (2.7) is closely related to the bilinear form of the Lagrange multiplier of the hybridized mixed finite element, [13, 23]. Here we denote the corresponding bilinear form and norms with a superscript RT, referring to the Raviart-Thomas finite element of the same order of the HDG method. We list some results which are useful in our analysis. The following lemma is in [23, Theorem 2.2] applied to each subdomain Ω_i:

LEMMA 5.2. *For any* $\lambda^{(i)} \in \Lambda^{(i)}$,

$$c\rho_i |\!|\!| \lambda |\!|\!|_{\Omega_i}^2 \le |\lambda|_{A^{RT^{(i)}}}^2 \le C\rho_i |\!|\!| \lambda |\!|\!|_{\Omega_i}^2.$$

Given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, we can similar define a harmonic extension $\mathcal{H}^{RT^{(i)}}(\lambda_\Gamma^{(i)}) : \Lambda_\Gamma^{(i)} \to \Lambda^{(i)}$ as

$$\left|\mathcal{H}^{RT^{(i)}}(\lambda_\Gamma^{(i)})\right|_{A^{RT^{(i)}}}^2 = \min_{\lambda^{(i)} \in \Lambda^{(i)}, \lambda^{(i)} = \lambda_\Gamma^{(i)} \text{ on } \partial\Omega_i} \left|\lambda^{(i)}\right|_{A^{RT^{(i)}}}^2$$

and have

$$\left|\lambda_\Gamma^{(i)}\right|_{S_\Gamma^{RT^{(i)}}}^2 := \left(\lambda_\Gamma^{(i)}\right)^T S_\Gamma^{RT^{(i)}} \lambda_\Gamma^{(i)} = \left|\mathcal{H}^{RT^{(i)}}(\lambda_\Gamma^{(i)})\right|_{A^{RT^{(i)}}}^2.$$

Let $\Lambda^{0,(i)}$ be the zero-order numerical trace space in Ω_i and Q_0 be the L^2-orthogonal projection from $\Lambda^{(i)}$ into $\Lambda^{0,(i)}$. By a scaling argument, see [23, (4.9) and (4.10)], we have the following lemma:

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LEMMA 5.3. *For any $\lambda^{(i)} \in \Lambda^{(i)}$,*

$$\left\|\left\| Q_0 \lambda^{(i)} \right\|\right\|_{\Omega_i} \leq C \left\|\left\| \lambda^{(i)} \right\|\right\|_{\Omega_i}, \tag{5.2}$$

and

$$\sum_{\kappa \in \mathcal{T}_h, \kappa \subseteq \Omega_i} \left\| \lambda^{(i)} - Q_0 \lambda^{(i)} \right\|^2_{L^2(\partial\kappa)} \leq Ch \left\|\left\| \lambda^{(i)} \right\|\right\|^2_{\Omega_i}. \tag{5.3}$$

Given a subdomain Ω_i, we define partition of unity functions associated with its edges or faces. An edge/face in the interface Γ only belongs to exactly two subdomains. We denote by $\mathcal{F}^{ij}$ the face shared by Ω_i and Ω_j. Let $\zeta_{\mathcal{F}^{ij}}$ be the characteristic function of $\mathcal{F}^{ij}$, i.e., the function that is identically one on $\mathcal{F}^{ij}_h$ and zero on $\partial\Omega_i^h \backslash \mathcal{F}^{ij}_h$, where $\mathcal{F}^{ij}_h$ contains the degrees of freedom of Ω_i^h on $\mathcal{F}^{ij} \subset \partial\Omega_i$. We clearly have

$$\sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} \zeta_{\mathcal{F}^{ij}}(\mathbf{x}) = 1, \qquad \lambda_\Gamma^{(i)} = \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} \zeta_{\mathcal{F}^{ij}}(\mathbf{x}) \lambda_\Gamma^{(i)},$$

for any $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, the numerical trace space on $\partial\Omega_i$.

Let $\overline{\lambda}^{(i)}_{\mathcal{F}^{ij}} = \frac{1}{|\mathcal{F}^{ij}|} \int_{\mathcal{F}^{ij}} \lambda_\Gamma^{(i)} dx$, the average of $\lambda_\Gamma^{(i)}$ over $\mathcal{F}^{ij}$. Particularly, we have the following lemma for the Lagrange multiplier of the zero-order hybridized mixed finite element, which can be proved using [39, Lemmas 5.4 and 5.5].

LEMMA 5.4. *For any $\lambda_\Gamma^{0,(j)} \in \Lambda_\Gamma^{0,(j)}$, we have*

$$\delta_j^{\dagger^2} \left| \zeta_{\mathcal{F}^{ij}} \left(\lambda_\Gamma^{0,(j)} - \overline{\lambda}_\Gamma^{0,(j)} \right) \right|^2_{S_\Gamma^{RT(i)}} \leq C \left(1 + \log \frac{H}{h} \right)^2 \left| \lambda_\Gamma^{0,(j)} \right|^2_{S_\Gamma^{RT(j)}}.$$

We define the interface averaging operator E_D by

$$E_D = \widetilde{R}_\Gamma \widetilde{R}^T_{D,\Gamma},$$

which computes a weighted average across the subdomain interface Γ and then distributes the averages to the degrees of freedom on the boundary of the subdomain. The interface averaging operator E_D satisfies the following bound:

LEMMA 5.5. *For any $\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma$,*

$$|E_D \lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \leq C \gamma_{h,\tau} \left(1 + \log \frac{H}{h} \right)^2 |\lambda_\Gamma|^2_{\widetilde{S}_\Gamma},$$

where $\gamma_{h,\tau}$ is defined in (5.1).

Proof. Given any $\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma$, we have

$$\begin{aligned} |E_D \lambda_\Gamma|^2_{\widetilde{S}_\Gamma} &\leq 2 \left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + |\lambda_\Gamma - E_D \lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \right) \\ &\leq 2 \left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + \left| \overline{R}_\Gamma \left(\lambda_\Gamma - E_D \lambda_\Gamma \right) \right|^2_{S_\Gamma} \right) = 2 \left(|\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} + \sum_{i=1}^N |v_i|^2_{S_\Gamma^{(i)}} \right), \end{aligned}$$

where v_i is the restriction of $\lambda_\Gamma - E_D \lambda_\Gamma$ to the subdomain Ω_i. Also let $\lambda_\Gamma^{(i)}$ be the restriction of λ_Γ to the subdomain Ω_i.

Using the partition of unity function associated with the edges/faces of Ω_i, we have

$$\zeta_{\mathcal{F}^{ij}} v_i|_{\mathcal{F}^{ij}} = (\lambda_\Gamma - E_D\lambda_\Gamma)|_{\mathcal{F}^{ij}} = \zeta_{\mathcal{F}^{ij}} \delta_j^\dagger \left(\lambda_\Gamma^{(i)} - \lambda_\Gamma^{(j)}\right),$$

where $\lambda_\Gamma^{(i)} = R_\Gamma^{(i)}(\lambda_\Gamma)$ and $\lambda_\Gamma^{(j)} = R_\Gamma^{(j)}(\lambda_\Gamma)$, the restrictions of λ_Γ to Ω_i and Ω_j, respectively, and

$$|v_i|^2_{S_\Gamma^{(i)}} = \left| \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} \zeta_{\mathcal{F}^{ij}} v_i \right|^2_{S_\Gamma^{(i)}} \le C \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} |\zeta_{\mathcal{F}^{ij}} v_i|^2_{S_\Gamma^{(i)}}.$$

We only need to show that

$$|\zeta_{\mathcal{F}^{ij}} v_i|^2_{S_\Gamma^{(i)}} \le C\gamma_{h,\tau}\left(1 + \log\frac{H}{h}\right)^2 \left(\left|\lambda_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}} + \left|\lambda_\Gamma^{(j)}\right|^2_{S_\Gamma^{(j)}} \right).$$

Let $\lambda^{(i)} = \mathcal{H}^{(i)}(\lambda_\Gamma^{(i)})$ and $\lambda^{(j)} = \mathcal{H}^{(j)}(\lambda_\Gamma^{(j)})$. We have

$$\left|\lambda_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}} = \left|\lambda^{(i)}\right|^2_{A^{(i)}} \quad \text{and} \quad \left|\lambda_\Gamma^{(j)}\right|^2_{S_\Gamma^{(j)}} = \left|\lambda^{(j)}\right|^2_{A^{(j)}}.$$

We note that the simple inequality,

$$\rho_i {\delta_j^\dagger}^2 \le \min\{\rho_i, \rho_j\}, \tag{5.4}$$

holds for $\gamma \in [1/2, \infty)$. Let $\overline{\lambda}^{(i)}_{\mathcal{F}^{ij}} = \frac{1}{|\mathcal{F}^{ij}|}\int_{\mathcal{F}^{ij}} \lambda_\Gamma^{(i)} dx$, be the average of $\lambda^{(i)}$ over $\mathcal{F}^{ij}$. We know that $\overline{\lambda}^{(i)}_{\mathcal{F}^{ij}} = \overline{\lambda}^{(j)}_{\mathcal{F}^{ij}}$, and we have

$$\begin{aligned} |\zeta_{\mathcal{F}^{ij}} v_i|^2_{S_\Gamma^{(i)}} &= \left|\mathcal{H}^{(i)}(\zeta_{\mathcal{F}^{ij}} v_i)\right|^2_{A^{(i)}} = \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}} \delta_j^\dagger(\lambda_\Gamma^{(i)} - \lambda_\Gamma^{(j)})\right)\right|^2_{A^{(i)}} \\ &\le 2{\delta_j^\dagger}^2 \left(\left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(i)} - \overline{\lambda}^{(i)}_{\mathcal{F}^{ij}})\right)\right|^2_{A^{(i)}} + \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(j)} - \overline{\lambda}^{(j)}_{\mathcal{F}^{ij}})\right)\right|^2_{A^{(i)}} \right). \end{aligned} \tag{5.5}$$

We only need to estimate the second term in (5.5), and the first term can be estimated similarly. Let $\lambda^{0,(j)} = Q_0\lambda^{(j)} \in \Lambda^{0,(j)}$ and $\lambda_\Gamma^{0,(j)}$ is the restriction of $\lambda^{0,(j)}$ to $\partial\Omega_j$. We have

$$\begin{aligned} &{\delta_j^\dagger}^2 \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(j)} - \overline{\lambda}^{(j)}_{\mathcal{F}^{ij}})\right)\right|^2_{A^{(i)}} \\ &= {\delta_j^\dagger}^2 \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)} + \lambda_\Gamma^{0,(j)} - \overline{\lambda}^{(j)}_{\mathcal{F}^{ij}})\right)\right|^2_{A^{(i)}} \\ &\le 2{\delta_j^\dagger}^2 \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)})\right)\right|^2_{A^{(i)}} + 2{\delta_j^\dagger}^2 \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{0,(j)} - \overline{\lambda}^{(j)}_{\mathcal{F}^{ij}})\right)\right|^2_{A^{(i)}}. \end{aligned} \tag{5.6}$$

We estimate the two terms in (5.6) separately. Let $\mathcal{R}^{(i)}(\lambda_\Gamma^{(i)}) : \Lambda_\Gamma^{(i)} \to \Lambda^{(i)}$ be the zero

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extension of $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$ to $\Lambda^{(i)}$. The first term can be estimated as follows:

$$\begin{aligned}
&\delta_j^{\dagger^2} \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)})\right)\right|_{A^{(i)}}^2 \le C\delta_j^{\dagger^2} \left|\mathcal{R}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)})\right)\right|_{A^{(i)}}^2 \\
&\le C\gamma_{h,\tau}\delta_j^{\dagger^2}\rho_i \left\|\left|\mathcal{R}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)})\right)\right\|\right|_{\Omega_i}^2 \\
&\le C\gamma_{h,\tau}\min\{\rho_i,\rho_j\}\frac{1}{h}\left(\sum_{\kappa\in\mathcal{T}_h,\ \kappa\subseteq\Omega_i} \left\|\zeta_{\mathcal{F}^{ij}}(\lambda^{(j)} - \lambda^{0,(j)})\right\|_{L_2(\partial\kappa)}^2\right) \\
&= C\gamma_{h,\tau}\min\{\rho_i,\rho_j\}\frac{1}{h}\left(\sum_{e\subset(\partial\kappa\cap\mathcal{F}^{ij}),\kappa\in\mathcal{T}_h,\ \kappa\subseteq\Omega_i} \left\|\lambda^{(j)} - \lambda^{0,(j)}\right\|_{L_2(e)}^2\right) \\
&= C\gamma_{h,\tau}\min\{\rho_i,\rho_j\}\frac{1}{h}\left(\sum_{e\subset(\partial\kappa\cap\mathcal{F}^{ij}),\kappa\in\mathcal{T}_h,\ \kappa\subseteq\Omega_j} \left\|\lambda^{(j)} - \lambda^{0,(j)}\right\|_{L_2(e)}^2\right) \\
&\le C\gamma_{h,\tau}\rho_j\frac{1}{h}\left(\sum_{\kappa\in\mathcal{T}_h,\ \kappa\subseteq\Omega_j} \left\|\lambda^{(j)} - \lambda^{0,(j)}\right\|_{L_2(\partial\kappa)}^2\right) \\
&\le C\gamma_{h,\tau}\rho_j\frac{1}{h}h\left\|\left|\lambda^{(j)}\right\|\right|_{\Omega_j}^2 \le C\gamma_{h,\tau}|\lambda^{(j)}|_{A^{(j)}}^2 = C\gamma_{h,\tau}|\lambda_\Gamma^{(j)}|_{S_\Gamma^{(j)}}^2.
\end{aligned}$$

Here we use the definition of $\mathcal{H}^{(i)}$ and $\mathcal{R}^{(i)}$ for the first inequality. Lemma 5.1 is used for the second inequality. (5.4) and the definition of $|||\cdot|||$ are used for the third inequality. (5.3) in Lemma 5.3 is used for the last inequality.

For the second term in (5.6), we have

$$\begin{aligned}
&\delta_j^{\dagger^2} \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{0,(j)} - \overline{\lambda}_{\mathcal{F}^{ij}}^{(j)})\right)\right|_{A^{(i)}}^2 \le C\delta_j^{\dagger^2} |\mathcal{H}^{RT^{(i)}}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{0,(j)} - \overline{\lambda}_{\mathcal{F}^{ij}}^{(j)})\right)|_{A^{(i)}}^2 \\
&\quad\le C\gamma_{h,\tau}\delta_j^{\dagger^2} |\mathcal{H}^{RT^{(i)}}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{0,(j)} - \overline{\lambda}_{\mathcal{F}^{ij}}^{(j)})\right)|_{A^{RT^{(i)}}}^2 \\
&\quad= C\gamma_{h,\tau}\delta_j^{\dagger^2} |\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{0,(j)} - \overline{\lambda}_{\mathcal{F}^{ij}}^{(j)})|_{S_\Gamma^{RT^{(i)}}}^2.
\end{aligned}$$

Here the definition of $\mathcal{H}^{(i)}$ and $\mathcal{H}^{RT^{(i)}}$ are used for the first inequality. Lemmas 5.1 and 5.2 are used for the second inequality and the definition of $\mathcal{H}^{RT^{(i)}}$ is used for the last equality. By the equivalence lemmas Lemma 5.1 and 5.2 for the zeroth-order Lagrange multipliers for the hybridized mixed finite element method and Lemma 5.4, and from the observation that $\overline{\lambda}_{\mathcal{F}^{ij}}^{(j)} = \overline{\lambda}_{\mathcal{F}^{ij}}^{0,(j)}$, we have

$$\begin{aligned}
&\delta_j^{\dagger^2} \left|\mathcal{H}^{(i)}\left(\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{0,(j)} - \overline{\lambda}_{\mathcal{F}^{ij}}^{(j)})\right)\right|_{A^{(i)}}^2 \le C\gamma_{h,\tau}\delta_j^{\dagger^2} |\zeta_{\mathcal{F}^{ij}}(\lambda_\Gamma^{0,(j)} - \overline{\lambda}_{\mathcal{F}^{ij}}^{0,(j)})|_{S_\Gamma^{RT^{(i)}}}^2 \\
&\le C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |\lambda_\Gamma^{0,(j)}|_{S_\Gamma^{RT^{(j)}}}^2 = C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |\mathcal{H}^{RT^{(j)}}(\lambda_\Gamma^{0,(j)})|_{A^{RT^{(j)}}}^2 \\
&\le C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |\lambda^{0,(j)}|_{A^{RT^{(j)}}}^2 \\
&\le C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 \rho_j\left\|\left|\lambda^{0,(j)}\right\|\right|_{\Omega_j}^2 = C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 \rho_j\left\|\left|Q_0\lambda^{(j)}\right\|\right|_{\Omega_j}^2 \\
&\le C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 \rho_j\left\|\left|\lambda^{(j)}\right\|\right|_{\Omega_j}^2
\end{aligned}$$

$$\le C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |\lambda^{(j)}|^2_{A^{(j)}} = C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |\mathcal{H}^{(j)}(\lambda_\Gamma^{(j)})|^2_{A^{(j)}}$$
$$= C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |\lambda_\Gamma^{(j)}|^2_{S_\Gamma^{(j)}}.$$

Here we use Lemma 5.4 for the second inequality. The definition $\mathcal{H}^{RT^{(j)}}$ is used for the third inequality. Lemma 5.2 is used for the fourth inequality. Equation (5.2) in Lemma 5.3 is used for the fifth inequality and Lemma 5.1 is used for the sixth inequality. ∎

6. Condition number estimate for the BDDC preconditioner. We are now ready to formulate and prove our main result; it follows as in the proof of [32, Theorem 1] using Lemma 5.5. Also see the proof of [35, Theorem 25], [41, Lemma 4.6], [40, Lemma 4.7], and [38, Theorem 2.8].

THEOREM 6.1. *The condition number of the preconditioned operator $M^{-1}\widehat{S}_\Gamma$ is bounded by $C\gamma_{h,\tau}(1+\log\frac{H}{h})^2$, where $\gamma_{h,\tau}$ is defined in (5.1).*

Proof. It is enough to prove that, for any $\lambda_\Gamma \in \widehat{\mathbf{\Lambda}}_\Gamma$,

$$\lambda_\Gamma^T M\lambda_\Gamma \le \lambda_\Gamma^T\widehat{S}_\Gamma\lambda_\Gamma \le C\gamma_{h,\tau}\left(1+\log(H/h)\right)^2 \lambda_\Gamma^T M\lambda_\Gamma,$$

Lower bound: Let

$$w_\Gamma = M\lambda_\Gamma = \left(\widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}\right)^{-1}\lambda_\Gamma \in \widehat{\mathbf{\Lambda}}_\Gamma. \tag{6.1}$$

Using the properties $\widetilde{R}_\Gamma^T\widetilde{R}_{D,\Gamma} = \widetilde{R}_{D,\Gamma}^T\widetilde{R}_\Gamma = I$ and (6.1), we have,

$$\lambda_\Gamma^T M\lambda_\Gamma = \lambda_\Gamma^T\left(\widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}\right)^{-1}\lambda_\Gamma = \lambda_\Gamma^T w_\Gamma = \lambda_\Gamma^T\widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma$$
$$= \left\langle \widetilde{R}_\Gamma\lambda_\Gamma, \widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma\right\rangle_{\widetilde{S}_\Gamma} \le \left\langle \widetilde{R}_\Gamma\lambda_\Gamma, \widetilde{R}_\Gamma\lambda_\Gamma\right\rangle_{\widetilde{S}_\Gamma}^{1/2}\left\langle \widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma, \widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma\right\rangle_{\widetilde{S}_\Gamma}^{1/2}$$
$$= \left(\lambda_\Gamma^T\widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma\lambda_\Gamma\right)^{1/2}\left(w_\Gamma^T\widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{S}_\Gamma\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma\right)^{1/2}$$
$$= \left(\lambda_\Gamma^T\widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma\lambda_\Gamma\right)^{1/2}\left(\lambda_\Gamma^T M\lambda_\Gamma\right)^{1/2}.$$

We obtain

$$\lambda_\Gamma^T M\lambda_\Gamma \le \lambda_\Gamma^T\widehat{S}_\Gamma\lambda_\Gamma$$

by canceling a common factor and squaring.

Upper bound: Using the definition of w_Γ in (6.1), the Cauchy-Schwarz inequality, and Lemma 5.5, we obtain the upper bound:

$$\lambda_\Gamma^T\widehat{S}_\Gamma\lambda_\Gamma = \lambda_\Gamma^T\widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma\widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma = \left\langle \widetilde{R}_\Gamma\lambda_\Gamma, E_D\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma\right\rangle_{\widetilde{S}_\Gamma}$$
$$\le \left\langle \widetilde{R}_\Gamma\lambda_\Gamma, \widetilde{R}_\Gamma\lambda_\Gamma\right\rangle_{\widetilde{S}_\Gamma}^{1/2}\left\langle E_D\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma, E_D\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma\right\rangle_{\widetilde{S}_\Gamma}^{1/2}$$
$$\le C\left\langle \widetilde{R}_\Gamma\lambda_\Gamma, \widetilde{R}_\Gamma\lambda_\Gamma\right\rangle_{\widetilde{S}_\Gamma}^{1/2}\gamma_{h,\tau}^{1/2}\left(1+\log(H/h)\right)|\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma|_{\widetilde{S}_\Gamma}$$
$$= C\gamma_{h,\tau}^{1/2}\left(1+\log(H/h)\right)\left(\lambda_\Gamma^T\widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma\lambda_\Gamma\right)^{1/2}\left(w_\Gamma^T\widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{S}_\Gamma\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}w_\Gamma\right)^{1/2}$$
$$= C\gamma_{h,\tau}^{1/2}\left(1+\log(H/h)\right)\left(\lambda_\Gamma^T\widehat{S}_\Gamma\lambda_\Gamma\right)^{1/2}\left(\lambda_\Gamma^T M\lambda_\Gamma\right)^{1/2}.$$

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TABLE 7.1
Performance of solving (4.1) *with* $\rho = 1$, $\tau = 1$.

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	2.21	5	3.47	6	4.47	6
	8×8	2.39	9	3.75	10	4.85	12
	16×16	2.33	8	3.70	10	4.78	12
	24×24	2.33	8	3.69	10	4.77	12
	32×32	2.33	8	3.69	10	4.77	12
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.72	7	2.85	10	3.73	11
	8	2.39	9	3.75	10	4.85	12
	16	3.24	10	4.89	12	6.09	13
	24	3.81	11	5.61	13	6.89	14
	32	4.24	11	6.15	13	7.51	15

Thus,

$$\lambda_\Gamma^T \widehat{S}_\Gamma \lambda_\Gamma \le C\gamma_{h,\tau} \left(1+\log(H/h)\right)^2 \lambda_\Gamma^T M \lambda_\Gamma. \qquad \square$$

7. Numerical experiments. We have applied our BDDC algorithms to the model problem (2.1), where $\Omega = [0,1]^2$. We decompose the unit square into $N \times N$ subdomains with side length $H = 1/N$. Equation (2.1) is discretized, in each subdomain, by the pth-order HDG method with element diameter h. The preconditioned conjugate gradient iteration is stopped when the relative l_2-norm of the residual has been reduced by a factor of 10^6.

We consider three different choices of the penalty constant τ, namely $\tau = 1$, $\tau = \frac{1}{h}$, and $\tau = \frac{1}{h^2}$. For each choice of τ, we have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory. We note that, for $\tau = \frac{1}{h^2}$, $\gamma_{h,\tau} \approx \frac{1}{h}$ and the condition number is linearly increasing with the mesh refinement by Theorem 6.1. The algorithm is not scalable. We also found that the condition number for $k = 1$ is larger than those for $k = 2$, which is not consistent with the other choices of τ. The dependence of the condition numbers on the polynomial degrees will be our future study. Nevertheless, in most practical situations, τ is taken as 1 or $\frac{1}{h}$, anyway.

In the first set of experiments, we take the coefficient $\rho \equiv 1$. Tables 7.1, 7.2, and 7.3 display the iteration counts and the estimate of the condition numbers, when increasing the number of subdomains and increasing the size of the subdomain problems.

In the second set of experiments, we take the coefficient $\rho = 1$ in half the subdomains and set $\rho = 1000$ in the neighboring subdomains, in a checkerboard pattern. Tables 7.4, 7.5, and 7.6 display the iteration counts, and condition number estimates when increasing the number of subdomains.

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TABLE 7.2
Performance of solving (4.1) with $\rho = 1$, $\tau = \frac{1}{h}$.

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	2.17	5	3.63	6	4.56	6
	8×8	2.34	9	3.91	10	4.94	12
	16×16	2.28	8	3.87	10	4.88	12
	24×24	2.28	8	3.86	10	4.87	12
	32×32	2.27	8	3.86	10	4.86	12
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.68	7	2.98	10	3.81	11
	8	2.34	9	3.91	10	4.94	12
	16	3.18	10	5.09	12	6.21	13
	24	3.74	11	5.84	13	7.02	14
	32	4.16	11	6.39	13	7.64	15

TABLE 7.3
Performance of solving (4.1) with $\rho = 1$, $\tau = \frac{1}{h^2}$.

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	2.06	5	8.29	6	7.27	7
	8×8	2.19	8	14.23	14	10.84	15
	16×16	2.17	8	24.65	16	16.82	16
	24×24	2.16	8	35.11	17	22.85	17
	32×32	2.16	8	45.61	17	28.88	17
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.58	7	7.04	12	6.29	12
	8	2.19	8	14.23	14	10.84	15
	16	3.04	10	29.98	16	19.88	18
	24	3.57	11	46.94	18	29.19	18
	32	3.98	11	64.81	19	38.78	19

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TABLE 7.4
Performance of solving (4.1) with ρ in a checkerboard pattern, $\tau = 1$.

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	1.79	3	2.36	4	3.08	4
	8×8	2.07	8	2.74	8	3.45	9
	16×16	2.15	10	2.85	10	3.54	11
	24×24	2.17	10	2.87	10	3.56	11
	32×32	2.17	10	2.88	10	3.57	11
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.67	7	2.32	8	3.23	9
	8	2.07	8	2.74	8	3.45	9
	16	2.49	8	3.16	9	3.75	9
	24	2.74	9	3.41	9	3.94	9
	32	2.91	9	3.59	9	4.09	9

TABLE 7.5
Performance of solving (4.1) with ρ in a checkerboard pattern, $\tau = \frac{1}{h}$.

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	1.77	4	2.43	4	3.08	4
	8×8	2.04	8	2.82	8	3.46	9
	16×16	2.12	10	2.94	10	3.56	11
	24×24	2.14	10	2.96	10	3.58	11
	32×32	2.14	10	2.97	10	3.58	12
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.64	7	2.40	8	3.22	9
	8	2.04	8	2.82	8	3.46	9
	16	2.46	8	3.25	9	3.77	9
	24	2.71	9	3.50	9	3.97	9
	32	2.88	9	3.67	9	4.12	9

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TABLE 7.6
Performance of solving (4.1) with ρ in a checkerboard pattern, $\tau = \frac{1}{h^2}$.

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	1.74	4	4.45	4	3.81	4
	8×8	2.04	9	7.53	10	5.40	9
	16×16	2.16	10	12.69	15	7.74	14
	24×24	2.24	10	17.67	17	9.96	15
	32×32	2.30	10	22.62	19	12.16	15
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.59	7	4.72	9	4.02	9
	8	2.04	9	7.53	10	5.40	9
	16	2.58	10	12.68	11	7.84	10
	24	3.00	11	17.65	11	10.14	10
	32	3.36	12	22.55	11	12.38	11

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