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INTERNALITY OF GENERALIZED AVERAGED GAUSS RULES AND THEIR TRUNCATIONS FOR BERNSTEIN-SZEGŐ WEIGHTS*

D. LJ. DJUKIĆ†, L. REICHEL‡, M. M. SPALEVIĆ†, AND J. D. TOMANOVIĆ†

Abstract. Generalized averaged Gauss quadrature formulas may have nodes outside the interval of integration. Quadrature rules with nodes outside the interval of integration cannot be applied to approximate integrals with an integrand that is defined on the interval of integration only. This paper investigates when generalized averaged Gauss quadrature rules for Bernstein-Szegő weight functions have all nodes in the interval of integration. Also truncated variants of these quadrature rules are considered. The relation between generalized averaged Gauss quadrature formulas and Gauss-Kronrod rules is explored.

Key words. Gauss quadrature, averaged Gauss quadrature, truncated generalized averaged Gauss quadrature, internality of quadrature rule

AMS subject classifications. 65D32, 65D30

1. Introduction. Let w be a given weight function on a bounded interval $[a, b]$ with infinitely many points of support. We call an *interpolatory quadrature formula* of the form

$$I[f] = \int_a^b f(t)\,w(t)dt = Q_n[f] + R_n[f], \quad Q_n[f] = \sum_{j=1}^{n} \omega_j f(t_j), \tag{1.1}$$

a $(2n-m-1, n, w)$ quadrature formula (q.f.) if the remainder term satisfies $R_n[f] = 0$ for all $f \in \mathbb{P}_{2n-m-1}$. Here $t_1 < t_2 < \cdots < t_n$ are distinct nodes, $\omega_1, \omega_2, \ldots, \omega_n$ are weights, $\mathbb{P}_k$ denotes the set of all polynomials of degree at most k, and $0 \le m \le n$. A $(2n-m-1, n, w)$ q.f. is said to be *internal* if all nodes are in the closed interval $[a, b]$. A node not belonging to the interval $[a, b]$ is said to be *external*. We say that a polynomial

$$q_n(t) = \prod_{j=1}^{n} (t - t_j)$$

with distinct nodes $t_1 < t_2 < \cdots < t_n$ generates a $(2n-m-1, n, w)$ q.f. if the interpolatory q.f. (1.1) with these nodes is a $(2n-m-1, n, w)$ q.f.

Let $\pi_0, \pi_1, \pi_2, \ldots$ denote the monic orthogonal polynomials with respect to the inner product $(f, g) = I[fg]$. Thus, π_k is of degree k and

$$(t^j, \pi_k) = 0, \quad j = 0, 1, \ldots, k-1.$$

The polynomials π_j satisfy a three-term recurrence relation of the form

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, \ldots, \tag{1.2}$$

where $\pi_{-1}(t) \equiv 0$, $\pi_0(t) \equiv 1$, and the coefficients β_k are positive.

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†Department of Mathematics, University of Beograd, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade 35, Serbia ({ddjukic,mspalevic,jtomanovic}@mas.bg.ac.rs).

‡Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA (reichel@math.kent.edu).

It is known that the unique interpolatory q.f. with l nodes and the highest possible degree of precision $2l-1$ is the Gauss formula with respect to the weight function w,

$$G_l[f] = \sum_{j=1}^{l} \omega_j^G f(t_j^G).$$

The nodes t_j^G are the eigenvalues of the symmetric tridiagonal Jacobi matrix

$$J_l^G(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & 0 \\ \sqrt{\beta_1} & \ddots & \ddots & \\ & \ddots & \ddots & \sqrt{\beta_{l-1}} \\ 0 & & \sqrt{\beta_{l-1}} & \alpha_{l-1} \end{bmatrix} \in \mathbb{R}^{l\times l}, \tag{1.3}$$

and the weights ω_j^G are the squares of the first components of suitably normalized eigenvectors. The latter was probably first observed by Wilf [21]; see Gautschi [4, 5] and Golub and Meurant [7] for more recent discussions. Golub and Welsch [8] developed an efficient algorithm for the computation of the nodes and weights of G_l based on this characterization.

It is often important to know how accurately a Gauss rule $G_l[f]$ approximates the desired integral $I[f]$. The estimation of the error $I[f] - G_l[f]$ therefore has received considerable attention in the literature. A common approach, when applicable, is to estimate the error in $G_l[f]$ by approximating $I[f]$ by a $(2l+1)$-node Gauss-Kronrod q.f., which we denote by H_{2l+1}. This rule uses the l nodes of G_l and can be expressed as

$$H_{2l+1}[f] = \sum_{j=1}^{l} \omega_j^{GK} f(t_j^G) + \sum_{k=1}^{l+1} \widetilde{\omega}_k^{GK} f(\widetilde{t}_k^{GK})$$

with

$$I[f] = H_{2l+1}[f] + R_{2l+1}^{GK}[f],$$

where the remainder term satisfies $R_{2l+1}^{GK}[f] = 0$ for all $f \in \mathbb{P}_{3l+1}$; see Notaris [15] for a nice recent discussion of Gauss-Kronrod rules. The nodes $\widetilde{t}_k^{GK}$, $k = 1, 2, \dots, l+1$, are the zeros of the so-called Stieltjes polynomial. This polynomial is characterized by an orthogonality relation with respect to a sign-changing weight function and therefore might have complex (non-real) zeros, in which case the Gauss-Kronrod rule H_{2l+1} is said not to exist. Moreover, some of the real nodes $\widetilde{t}_k^{GK}$ of H_{2l+1} may lie outside the interval of integration. These difficulties arise for several of the classical weight functions such as the Hermite and Laguerre weight functions as well as the Gegenbauer and Jacobi weight functions in certain situations; see, e.g., [1, 15] for references and some computed examples.

The fact that certain Gauss-Kronrod formulas have complex (non-real) nodes led Laurie [9] to develop anti-Gauss quadrature formulas and averaged Gauss quadrature rules. These quadrature rules can be used to estimate the error in $G_l[f]$. Both the anti-Gauss and averaged Gauss quadrature rules are guaranteed to exist and have real nodes, at most two of which may be outside of the interval of integration. Moreover, all weights are positive and the quadrature rules can be constructed easily.

Based on work by Peherstorfer [16], Spalević [18] proposed a novel method for constructing generalized averaged Gauss quadrature formulas $\widehat{G}_{2l+1}$. The nodes of these quadrature

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rules are the zeros of the polynomial

$$q_n \equiv q_{2l+1} = \pi_l \cdot F_{l+1}, \tag{1.4}$$

where

$$F_{l+1} = \pi_{l+1} - \widehat{\beta}_{l+1} \cdot \pi_{l-1}. \tag{1.5}$$

It is shown in [18] that $\widehat{G}_{2l+1}$ has algebraic degree of precision $2l+2$ when $\widehat{\beta}_{l+1} = \beta_{l+1}$ in (1.5). In this case, we denote $\widehat{G}_{2l+1}$ by $\widehat{G}^S_{2l+1}$. When, instead, q_n in (1.4) is determined by $\widehat{\beta}_{l+1} = \beta_l$ in (1.5), the quadrature formula $\widehat{G}_{2l+1}$ becomes the averaged Gauss quadrature formula introduced by Laurie [9]. We denote this rule by $\widehat{G}^L_{2l+1}$. It has algebraic degree of precision $2l+1$. Both rules $\widehat{G}^S_{2l+1}$ and $\widehat{G}^L_{2l+1}$ exist for all $l \geq 1$, but they might have nodes outside the interval of integration. The symmetric tridiagonal Jacobi matrix $J^S_{2l+1}(w) \in \mathbb{R}^{(2l+1)\times(2l+1)}$ associated with the rule $\widehat{G}^S_{2l+1}$ is given by

$$\begin{bmatrix}
\alpha_0 & \sqrt{\beta_1} & & & & & & & 0\\
\sqrt{\beta_1} & \ddots & \ddots & & & & & & \\
& \ddots & \ddots & \sqrt{\beta_{l-1}} & & & & & \\
& & \sqrt{\beta_{l-1}} & \alpha_{l-1} & \sqrt{\beta_l} & & & & \\
& & & \sqrt{\beta_l} & \alpha_l & \sqrt{\beta_{l+1}} & & & \\
& & & & \sqrt{\beta_{l+1}} & \alpha_{l-1} & \sqrt{\beta_{l-1}} & & \\
& & & & & \sqrt{\beta_{l-1}} & \ddots & \ddots & \\
& & & & & & \ddots & \ddots & \sqrt{\beta_1}\\
0 & & & & & & & \sqrt{\beta_1} & \alpha_0
\end{bmatrix}.$$

Spalević [19] investigated conditions under which the degree of precision of generalized averaged Gauss formulas $\widehat{G}^S_{2l+1}$ can be as high as $3l+1$. In this situation $\widehat{G}^S_{2l+1}$ provides an attractive alternative to Gauss-Kronrod formulas for estimating the quadrature error in Gauss formulas. This is discussed further below.

Truncated versions of the quadrature formulas $\widehat{G}^S_{2l+1}$ of the same algebraic degree of precision were first considered in [17]. The simplest truncated generalized averaged Gauss quadrature formulas are of the form

$$Q^{(1)}_{l+2}[f] = \sum_{j=1}^{l+2} \omega_j f(t^{(1)}_j).$$

They have been analyzed in more detail in [3]. The nodes of the rule $Q^{(1)}_{l+2}$ are the zeros of the polynomial

$$q_{l+2}(t) = (t - \alpha_{l-1})\pi_{l+1}(t) - \beta_{l+1}\pi_l(t); \tag{1.6}$$

see [3, Equation (4.2)]. Our interest in truncated versions of the quadrature rule $\widehat{G}^S_{2l+1}$ stems from the fact that they may be internal when $\widehat{G}^S_{2l+1}$ is not. The nodes and weights of truncated

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generalized averaged Gauss rules can be computed as the eigenvalues and the squares of the first components of suitably normalized eigenvectors of a truncation of the Jacobi matrix $J^S_{2l+1}(w)$; see [17]. The internality of the quadrature formulas $\widehat{G}^L_{2l+1}$, $\widehat{G}^S_{2l+1}$, and $Q^{(1)}_{l+2}$ $(l \geq 2)$ with classical weight functions has been investigated in [9], [18], and [3], respectively.

This paper discusses the internality of the quadrature formulas $\widehat{G}^L_{2l+1}$, $\widehat{G}^S_{2l+1}$ $(l \geq 1)$, and $Q^{(1)}_{l+2}$ $(l \geq 2)$ for the classes of Bernstein-Szegő weight functions considered by Gautschi and Notaris [6]. We note that the internality of Gauss-Kronrod quadrature formulas H_{2l+1} for Bernstein-Szegő weight functions has been studied in [6, 12, 13, 14].

We conclude this section with some comments on the quality of the estimate of the quadrature error determined by generalized averaged Gauss quadrature rules and truncated variants. Let the integrand f in (1.1) have an expansion in terms of orthonormal polynomials $p_0, p_1, p_2, \ldots$ with respect to the weight function w. Thus, we have

$$f(t) = \sum_{j=0}^{\infty} \eta_j p_j(t), \qquad \eta_j = I[f p_j],$$

where p_j is of degree j and the p_i satisfy

$$I[p_j p_k] = \begin{cases} 1 & j = k, \\ 0 & j \neq k. \end{cases}$$

Assume for notational simplicity that the weight function w is scaled so that $I[1] = 1$. Then

$$I[f] = \sum_{j=0}^{\infty} \eta_j I[p_j] = \eta_0,$$

$$G_{l+1}[f] = \sum_{j=0}^{\infty} \eta_j G_{l+1}[p_j] = \eta_0 + \sum_{j=2l+2}^{\infty} \eta_j G_{l+1}[p_j], \tag{1.7}$$

$$\widehat{G}^S_{2l+1}[f] = \eta_0 + \sum_{j=2l+3}^{\infty} \eta_j \widehat{G}^S_{2l+1}[p_j]. \tag{1.8}$$

Numerical results reported in Section 4 as well as in [17] indicate that the magnitude of the quadrature error

$$|I[f] - G_{l+1}[f]| = \left| \sum_{j=2l+2}^{\infty} \eta_j G_{l+1}[p_j] \right| \tag{1.9}$$

for many integrands and various weight functions w is quite well approximated by the difference

$$\left| G_{l+1}[f] - \widehat{G}^S_{2l+1}[f] \right| = \left| \eta_{2l+2} G_{l+1}[p_{2l+2}] + \sum_{j=2l+3}^{\infty} \eta_j (G_{l+1}[p_j] - \widehat{G}^S_{2l+1}[p_j]) \right|. \tag{1.10}$$

This holds, in particular, when the coefficients η_j decrease to zero rapidly with increasing j because then the right-hand sides of both (1.9) and (1.10) are dominated by $|\eta_{2l+2} G_{l+1}[p_{2l+2}]|$. It also holds when $|\eta_j|$ decreases to zero with increasing index j and $\widehat{G}^S_{2l+1}[p_j] \approx 0$ for some $j = 2l+3, 2l+4, \ldots$. The latter property holds for weight functions considered by

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Spalević [19]; see also Corollary 3.2 below. We remark that the symmetric tridiagonal Jacobi matrices $J^G_{l+1}(w)$ and $J^S_{2l+1}(w)$, with which the quadrature rules $G_{l+1}[f]$ and $\widehat{G}^S_{2l+1}[f]$ are computed, are defined by about the same recurrence coefficients α_j and β_j. Therefore, the determination of the entries of the matrix $J^S_{2l+1}(w)$ is inexpensive when the entries of the matrix $J^G_{l+1}(w)$ are available; see [17] for further discussions on the computation of $J^S_{2l+1}(w)$.

Similarly to (1.8), we have

$$Q^{(1)}_{l+2}[f] = \eta_0 + \sum_{j=2l+3}^{\infty} \eta_j Q^{(1)}_{l+2}[p_j],$$

which suggests that $|G_{l+1}[f] - Q^{(1)}_{l+2}[f]|$ may be used as an estimate for the quadrature error (1.9). The feasibility of this approach to estimate the quadrature error is illustrated in Section 4 as well as in [3, 17].

This paper is organized as follows. Section 2 introduces Bernstein-Szegő weight functions. The internality of generalized averaged Gauss quadrature rules and truncated variants for Bernstein-Szegő weight functions is analyzed in Section 3. Computed examples that illustrate the usefulness of applying the expressions (1.10) and $|G_{l+1}[f] - Q^{(1)}_{l+2}[f]|$ to estimate the magnitude of the quadrature error $|I[f] - G_{l+1}[f]|$ are presented in Section 4, and concluding remarks can be found in Section 5.

2. Bernstein-Szegő weight functions. This section reviews some properties of Bernstein-Szegő weight functions with support in the open interval $(a, b) = (-1, 1)$. These weight functions have been considered by Gautschi and Notaris [6]. Introduce the Bernstein-Szegő weight functions

$$w^{(\pm 1/2)}(t) = \frac{(1-t^2)^{\pm 1/2}}{\rho(t)}$$

and

$$w^{(\pm 1/2, \mp 1/2)}(t) = \frac{(1-t)^{\pm 1/2}(1+t)^{\mp 1/2}}{\rho(t)},$$

where

$$\rho(t) = \rho(t; \alpha, \beta, \delta) = \beta(\beta - 2\alpha)t^2 + 2\delta(\beta - \alpha)t + \alpha^2 + \delta^2$$

with the coefficients α, β, and δ chosen such that

$$0 < \alpha < \beta, \quad \beta \neq 2\alpha, \quad |\delta| < \beta - \alpha.$$

Then $\rho(t)$ is positive for $-1 < t < 1$; see [6, Proposition 2.1]. Denote the associated monic orthogonal polynomials of degree l and the coefficients of the three-term recurrence relation (1.2) by

$$\pi_l^{(\pm 1/2)}, \pi_l^{(\pm 1/2, \mp 1/2)} \quad \text{and} \quad \alpha_k^{(\pm 1/2)}, \beta_k^{(\pm 1/2)}, \quad \alpha_k^{(\pm 1/2, \mp 1/2)}, \beta_k^{(\pm 1/2, \mp 1/2)},$$

respectively. Then we have (see [6])

$$\pi_l^{(-1/2)}(t) = \frac{1}{2^{l-1}} \left[T_l(t) + \frac{2\delta}{\beta} T_{l-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right) T_{l-2}(t) \right], \quad l \geq 2,$$

$$\pi_1^{(-1/2)}(t) = t + \frac{\delta}{\beta-\alpha}, \qquad \left[\pi_0^{(-1/2)}(t) \equiv 1,\ \pi_{-1}^{(-1/2)}(t) \equiv 0\right],$$
$$\pi_l^{(1/2)}(t) = \frac{1}{2^l}\left[U_l(t) + \frac{2\delta}{\beta}U_{l-1}(t) + \left(1-\frac{2\alpha}{\beta}\right)U_{l-2}(t)\right], \quad l \geq 1,$$
$$\left[\pi_0^{(1/2)}(t) \equiv 1,\ \pi_{-1}^{(1/2)}(t) \equiv 0\right],$$
$$\pi_l^{(1/2,-1/2)}(t) = \frac{1}{2^l}\left[W_l(t) + \frac{2\delta}{\beta}W_{l-1}(t) + \left(1-\frac{2\alpha}{\beta}\right)W_{l-2}(t)\right], \quad l \geq 2,$$
$$\pi_1^{(1/2,-1/2)}(t) = t + \frac{\alpha+\delta}{\beta}, \qquad \left[\pi_0^{(1/2,-1/2)}(t) \equiv 1,\ \pi_{-1}^{(1/2,-1/2)}(t) \equiv 0\right],$$
$$\pi_l^{(-1/2,1/2)}(t;\alpha,\beta,\delta) = (-1)^l \pi_l^{(1/2,-1/2)}(-t;\alpha,\beta,-\delta),$$

where for $t = \cos\theta$,

$$T_l(\cos\theta) = \cos l\theta, \quad U_l(\cos\theta) = \frac{\sin(l+1)\theta}{\sin\theta}, \quad W_l(\cos\theta) = \frac{\sin(l+1/2)\theta}{\sin(\theta/2)}$$

are the Chebyshev orthogonal polynomials of the first, second, and fourth kind, respectively, with $T_0(t) \equiv U_0(t) \equiv W_0(t) \equiv 1$ and $U_{-1}(t) \equiv 0$. Moreover,

$$\alpha_0^{(-1/2)} = -\frac{\delta}{\beta-\alpha}, \qquad \beta_1^{(-1/2)} = \alpha\frac{(\beta-\alpha)^2-\delta^2}{\beta(\beta-\alpha)^2},$$
$$\alpha_1^{(-1/2)} = \frac{\alpha\,\delta}{\beta(\beta-\alpha)}, \qquad \beta_2^{(-1/2)} = \frac{\beta-\alpha}{2\beta},$$
$$\alpha_k^{(-1/2)} = 0, \quad k \geq 2, \qquad \beta_k^{(-1/2)} = \frac{1}{4}, \quad k \geq 3,$$

$$\alpha_0^{(1/2)} = -\frac{\delta}{\beta}, \qquad \beta_1^{(1/2)} = \frac{\alpha}{2\beta},$$
$$\alpha_k^{(1/2)} = 0, \quad k \geq 1, \qquad \beta_k^{(1/2)} = \frac{1}{4}, \quad k \geq 2,$$

$$\alpha_0^{(1/2,-1/2)} = -\frac{\alpha+\delta}{\beta}, \qquad \beta_1^{(1/2,-1/2)} = \frac{\alpha(\beta-\alpha-\delta)}{\beta^2},$$
$$\alpha_1^{(1/2,-1/2)} = \frac{2\alpha-\beta}{2\beta}, \qquad \beta_k^{(1/2,-1/2)} = \frac{1}{4}, \quad k \geq 2,$$
$$\alpha_k^{(1/2,-1/2)} = 0, \quad k \geq 2,$$

and

$$\alpha_k^{(-1/2,1/2)}(\alpha,\beta,\delta) = -\alpha_k^{(1/2,-1/2)}(\alpha,\beta,-\delta), \quad k \geq 0,$$
$$\beta_k^{(-1/2,1/2)}(\alpha,\beta,\delta) = \beta_k^{(1/2,-1/2)}(\alpha,\beta,-\delta), \qquad k \geq 0.$$

Note that for all Bernstein-Szegő weight functions, all but the first few diagonal entries of the Jacobi matrix (1.3) vanish and all but the first few subdiagonal entries are independent of the row number.

3. Internality of quadrature formulas. The recursion formulas for the monic orthogonal polynomials π_l with respect to Bernstein-Szegő weight functions are of the form

$$\begin{aligned} &\pi_{l+1}(t) = (t-\alpha_l)\pi_l(t) - \beta_l\pi_{l-1}(t), \quad l = 0,1,\ldots, \\ &\alpha_l = \alpha, \quad \beta_l = \beta \quad \text{for} \quad l \geq r, \end{aligned} \tag{3.1}$$

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where $\alpha_l \in \mathbb{R}$, $\beta_l > 0$, r is a non-negative integer, and $\pi_0(t) \equiv 1$ and $\pi_{-1}(t) \equiv 0$. Thus, the coefficients α_l and β_l are equal to some constants $\alpha \in \mathbb{R}$ and $\beta > 0$, respectively, for all $l \geq r$. We note that any weight function w that yields a recursion relation of the form (3.1) is known to be supported on a finite interval $[a, b]$; see [11]. Let $\mathcal{M}_r^{\alpha,\beta}[a, b]$ denote the set of weight functions w that give a recursion relation of the form (3.1). In addition to the Bernstein-Szegő weight functions, also the Chebyshev weight functions $w(t) = (1 - t^2)^{1/2}$ and $w(t) = (1 - t^2)^{-1/2}$ belong to sets $\mathcal{M}_r^{\alpha,\beta}[a, b]$ for $a = -1$, $b = 1$, and suitable values of $r \geq 0$. Polynomials that satisfy a recursion relation of the form (3.1) also are considered in [3, Example 5.3]. The following result is shown in [20].

THEOREM 3.1. *Let w be a weight function in $\mathcal{M}_r^{\alpha,\beta}[a, b]$. Then, for $l \geq 2r - 1$, the generalized averaged Gauss quadrature formula $\widehat{G}^S_{2l+1}$ has algebraic degree of precision at least $3l+1$. Therefore, it coincides with the corresponding Gauss-Kronrod quadrature formula and the monic polynomial F_{l+1} coincides with the corresponding monic Stieltjes polynomial given by*

$$E_{l+1}(t) = \pi_{l+1}(t) - \beta\pi_{l-1}(t) \quad \textit{for } l \geq 2r - 1.$$

The fact that generalized averaged Gauss quadrature formulas agree with Gauss-Kronrod formulas is important because the former quadrature rules are quite easy to compute; see [17]. The computation of Gauss-Kronrod rules is more complicated. Numerical methods for this task are discussed in [1, 2, 5, 10, 15]. The following result is a consequence of the above theorem.

COROLLARY 3.2. *Let the conditions of Theorem 3.1 hold. Then the error estimate (1.10) can be expressed as*

$$\left|G_{l+1}[f] - \widehat{G}^S_{2l+1}[f]\right| = \left|\sum_{j=2l+2}^{3l+1} \eta_j G_{l+1}[p_j] + \sum_{j=3l+2}^{\infty} \eta_j \left(G_{l+1}[p_j] - \widehat{G}^S_{2l+1}[p_j]\right)\right|.$$

Proof. By Theorem 3.1, the analogue of the expression (1.8) is

$$\widehat{G}^S_{2l+1}[f] = \eta_0 + \sum_{j=3l+2}^{\infty} \eta_j \widehat{G}^S_{2l+1}[p_j].$$

Combining this expression with (1.7) shows the desired result. □

It follows from Theorem 3.1 that for Bernstein-Szegő weights, the quadrature rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ coincide with the corresponding Gauss-Kronrod quadrature formula H_{2l+1} if $l \geq 2r - 1$. The rule $\widehat{G}^L_{2l+1}$ coincides with $\widehat{G}^S_{2l+1}$ and differs from the corresponding rule H_{2l+1} if $r \leq l < 2r - 1$. Finally, $\widehat{G}^L_{2l+1}$ differs from $\widehat{G}^S_{2l+1}$, and both these quadrature formulas generally differ from the corresponding Gauss-Kronrod formula H_{2l+1} for $l < r$.

3.1. The Bernstein-Szegő weight function $w^{(-1/2)}$. For this weight function, recursion formulas of the form (3.1) hold with $r = 3$. Therefore, $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ coincide with the corresponding Gauss-Kronrod quadrature formula H_{2l+1} for $l \geq 5$. Moreover, $\widehat{G}^L_{2l+1}$ coincides with $\widehat{G}^S_{2l+1}$, and both $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ differ from the corresponding Gauss-Kronrod rule H_{2l+1} for $l = 3, 4$. Finally, $\widehat{G}^L_{2l+1}$ differs from $\widehat{G}^S_{2l+1}$, and both $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$, in general, differ from the corresponding Gauss-Kronrod rule H_{2l+1} for $l = 1, 2$.

PROPOSITION 3.3. *The quadrature rules* $\widehat{G}^L_{2l+1}$ *and* $\widehat{G}^S_{2l+1}$ *for the Bernstein-Szegő weight function* $w^{(-1/2)}$ *have the following properties: the rules* $\widehat{G}^L_{2l+1}$ *are internal for* $l \geq 2$ *and the rule* $\widehat{G}^L_3$ *is internal if*

$$|\delta| \leq \frac{(\beta-\alpha)(\beta-2\alpha)}{\alpha}. \tag{3.2}$$

The rules $\widehat{G}^S_{2l+1}$ *are internal for* $l \geq 3$*, the rule* $\widehat{G}^S_5$ *is internal if* $\beta > 2\alpha$*, and the rule* $\widehat{G}^S_3$ *is internal if*

$$|\delta| \leq \frac{1}{2}(\beta-\alpha). \tag{3.3}$$

Proof. In addition to verifying the proposition, the proof also provides findings on whether the rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ coincide with the corresponding Gauss-Kronrod quadrature formula H_{2l+1}. This connection and results by Gautschi and Notaris [6] for the rules H_{2l+1} can in some cases be used to determine if the quadrature formulas $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ are internal.

It follows from the properties $T_l(1) = 1$ and $T_l(-1) = (-1)^l$ for $l = 0, 1, 2, \ldots$, and from the relations of Section 2, that

$$\begin{aligned}
&\pi_0^{(-1/2)}(1) = 1,\\
&\pi_1^{(-1/2)}(1) = 1 + \frac{\delta}{\beta-\alpha} = \frac{\delta+\beta-\alpha}{\beta-\alpha},\\
&\pi_l^{(-1/2)}(1) = \frac{1}{2^{l-1}}\left[1 + \frac{2\delta}{\beta}\cdot 1 + \left(1-\frac{2\alpha}{\beta}\right)\cdot 1\right] = \frac{\delta+\beta-\alpha}{2^{l-2}\beta}, \quad l \geq 2,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\pi_0^{(-1/2)}(-1) &= 1,\\
\pi_1^{(-1/2)}(-1) &= -1 + \frac{\delta}{\beta-\alpha} = \frac{\delta-\beta+\alpha}{\beta-\alpha},\\
\pi_l^{(-1/2)}(-1) &= \frac{1}{2^{l-1}}\left[(-1)^l + \frac{2\delta}{\beta}(-1)^{l-1} + \left(1-\frac{2\alpha}{\beta}\right)(-1)^{l-2}\right]\\
&= (-1)^l\frac{\beta-\alpha-\delta}{2^{l-2}\beta}, \quad l \geq 2.
\end{aligned} \tag{3.5}$$

These equations are used to show whether the quadrature rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ are internal.

For $l \geq 5$, the rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ coincide with the corresponding Gauss-Kronrod quadrature formula H_{2l+1}. This formula is internal if the conditions (3.6) below hold. Alternatively, we can use [6, Theorem 5.1 (b)], which shows that H_{2l+1} is internal and therefore so are $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$.

For $l = 3, 4$, the quadrature formulas $\widehat{G}^S_{2l+1}$ and $\widehat{G}^L_{2l+1}$ are internal if $F_{l+1}(1) \geq 0$ and $(-1)^{l+1}F_{l+1}(-1) \geq 0$, i.e., if

$$\frac{4\pi_{l+1}^{(-1/2)}(1)}{\pi_{l-1}^{(-1/2)}(1)} \geq 1 \quad \text{and} \quad \frac{4\pi_{l+1}^{(-1/2)}(-1)}{\pi_{l-1}^{(-1/2)}(-1)} \geq 1. \tag{3.6}$$

Substituting (3.4) and (3.5) into the above expressions shows that $\widehat{G}^S_{2l+1}$ and $\widehat{G}^L_{2l+1}$ are internal.

For $l = 1, 2$, the rule $\widehat{G}^L_{2l+1}$ is internal if $F_{l+1}(1) \geq 0$ and $(-1)^{l+1}F_{l+1}(-1) \geq 0$, i.e., if

$$\frac{\pi_{l+1}^{(-1/2)}(1)}{\beta_l^{(-1/2)}\pi_{l-1}^{(-1/2)}(1)} \geq 1 \quad \text{and} \quad \frac{\pi_{l+1}^{(-1/2)}(-1)}{\beta_l^{(-1/2)}\pi_{l-1}^{(-1/2)}(-1)} \geq 1. \tag{3.7}$$

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If $l = 2$, it follows from the identities (3.4) and (3.5) that $\widehat{G}_5^L$ is internal. In case $l = 1$, the conditions (3.7) are equivalent to (3.2). Thus, $\widehat{G}_3^L$ is internal if (3.2) holds.

For $l = 2$, the quadrature formula $\widehat{G}_5^S$ is internal if $F_3(1) \geq 0$ and $-F_3(-1) \geq 0$, which can be expressed as

$$\frac{4\pi_3^{(-1/2)}(1)}{\pi_1^{(-1/2)}(1)} \geq 1 \quad \text{and} \quad \frac{4\pi_3^{(-1/2)}(-1)}{\pi_1^{(-1/2)}(-1)} \geq 1.$$

Straightforward computations show that $\widehat{G}_5^S$ is internal if $\beta > 2\alpha$.

For $l = 1$, the quadrature formula $\widehat{G}_3^S$ has the same algebraic degree of precision $(2l + 2 = 4)$ as the corresponding Gauss-Kronrod quadrature formula H_3 $(3l + 1 = 4)$. Therefore, these quadrature formulas coincide. Gautschi and Notaris [6, Theorem 5.1(b)] show that H_3 is internal if (3.3) holds. $\square$

We turn to the quadrature rules $Q_{l+2}^{(1)}$.

PROPOSITION 3.4. *The quadrature rules $Q_{l+2}^{(1)}$ for the Bernstein-Szegő weight function $w^{(-1/2)}$ are internal for $l \geq 3$. If $\delta \neq 0$, then the rule $Q_4^{(1)}$ is internal provided that*

$$|\delta| \leq \frac{\beta(\beta - \alpha)}{2\alpha}. \tag{3.8}$$

The rule $Q_4^{(1)}$ is internal if $\delta = 0$.

Proof. It follows from [3, Theorem 4.1] that the rules $Q_{l+2}^{(1)}$ are internal for $l \geq 3$ because $\alpha_{l-1}^{(-1/2)} = \alpha_{l+1}^{(-1/2)} = 0$.

If $l = 2$ and $\delta \neq 0$, then we obtain from (1.6) that $Q_4^{(1)}$ is internal if $q_4(1) \geq 0$ and $q_4(-1) \geq 0$, i.e., if

$$\frac{4\left(1 - \alpha_1^{(-1/2)}\right)\pi_3^{(-1/2)}(1)}{\pi_2^{(-1/2)}(1)} \geq 1 \quad \text{and} \quad -\frac{4\left(1 + \alpha_1^{(-1/2)}\right)\pi_3^{(-1/2)}(-1)}{\pi_2^{(-1/2)}(-1)} \geq 1.$$

These conditions simplify to (3.8).

If $l = 2$ and $\delta = 0$, then since $\alpha_1^{(-1/2)} = 0$, we conclude, using [3, Theorem 4.1], that the quadrature formula $Q_4^{(1)}$ is internal. $\square$

EXAMPLE 3.5. We have shown that for the Bernstein-Szegő weight function $w^{(-1/2)}$ and $l = 3$, the quadrature rule $\widehat{G}_7^S$ and therefore also the rule $\widehat{G}_7^L$ are internal. The corresponding Gauss-Kronrod quadrature formula $H_7 = H_7^{(-1/2)}$ is internal if

$$\delta^2 < \frac{1}{32}\frac{(3\beta - 2\alpha)^2(\beta + 6\alpha)}{\beta + 2\alpha}, \quad \beta > 2\alpha$$

(see [6, Theorem 5.1]) and may have exterior nodes otherwise. Table 3.1 displays the exterior nodes of the Gauss-Kronrod quadrature formula $H_7^{(-1/2)}$ $(l = 3)$ for some α, β, δ.

We showed for $l = 2$ that $\widehat{G}_5^L$ is always internal and that $\widehat{G}_5^S$ is internal if $\beta > 2\alpha$. The corresponding Gauss-Kronrod quadrature formula $H_5 = H_5^{(-1/2)}$ $(l = 2)$ is internal if

$$\beta > 2\alpha, \quad |\delta| \leq \beta - 2\alpha;$$

see [6, Theorem 5.1]. Therefore, if $H_5^{(-1/2)}$ is internal, then $\widehat{G}_5^S$ is internal. The converse is not necessarily true. For example, for $\alpha = 0.05$, $\beta = 0.2$, and $\delta = 0.14$, the rule $\widehat{G}_5^S$ is internal, but $H_5^{(-1/2)}$ has an exterior node near -1.0580.

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TABLE 3.1
Approximate values of exterior nodes in the Gauss-Kronrod quadrature formula $H_7 = H_7^{(-1/2)}$ $(l = 3)$ for some α, β, δ.

α	β	δ	Exterior nodes of $H_7^{(-1/2)}$	α	β	δ	Exterior nodes of $H_7^{(-1/2)}$
0.11	0.2	0.01	$-1.0046; 1.0040$	11.0	12	0.0	∓ 1.0408
0.12	0.2	0.01	$-1.0093; 1.0081$	11.2	12	0.0	∓ 1.0427
0.14	0.2	0.01	$-1.0940; 1.0169$	11.4	12	0.0	∓ 1.0446
0.16	0.2	0.01	$-1.0303; 1.0263$	11.6	12	0.0	∓ 1.0466
0.18	0.2	0.01	$-1.0420; 1.0363$	11.8	12	0.0	∓ 1.0485

TABLE 3.2
Approximate values of exterior nodes in the quadrature rule $\widehat{G}_5$ for some $\alpha > 1$, $\beta = 2$, and $\delta = 0$.

α	Exterior nodes of $\widehat{G}_5^S$
1.1	∓ 1.0124
1.2	∓ 1.0247
1.3	∓ 1.0368
1.4	∓ 1.0488
1.5	∓ 1.0607

EXAMPLE 3.6. This example illustrates that for the Bernstein-Szegő weight function $w^{(-1/2)}$, the quadrature rule $\widehat{G}_{2l+1}^S$ may be external while the corresponding truncated rule $Q_{l+2}^{(1)}$ is internal if $l = 2$. Thus, let $l = 2$ and consider the situation when $\beta < 2\alpha$ and $\delta = 0$. Then the quadrature formulas $\widehat{G}_{2l+1}^S$ and $H_{2l+1} = H_{2l+1}^{(-1/2)}$ coincide. This follows from the fact that $H_{2l+1}^{(-1/2)}$ has algebraic degree of precision $7 = 3l + 1$ and, because of parity, the rule $\widehat{G}_{2l+1}^S$ also has algebraic degree of precision $7 = 2l + 3$; see [18]. Table 3.2 displays the exterior nodes of $\widehat{G}_{2l+1}^S$ for several values of $\alpha > 1$.

Since the condition (3.8) holds for the values of α, β, and δ used in Table 3.2, it follows from Proposition 3.4 that the corresponding truncated quadrature rules $Q_{l+2}^{(1)}$ are internal.

3.2. The Bernstein-Szegő weight function $w^{(1/2)}$. In this case, recursion formulas of the form (3.1) hold with $r = 2$. Therefore, the quadrature rules $\widehat{G}_{2l+1}^L$ and $\widehat{G}_{2l+1}^S$ coincide with the corresponding Gauss-Kronrod quadrature formula H_{2l+1} for $l \geq 3$. If $l = 2$, then the rule $\widehat{G}_{2l+1}^L$ coincides with $\widehat{G}_{2l+1}^S$, and none of these rules agrees with the Gauss-Kronrod rule H_{2l+1}. Moreover, if $l = 1$, then $\widehat{G}_{2l+1}^L$ differs from $\widehat{G}_{2l+1}^S$, and generally both these rules differ from the corresponding Gauss-Kronrod rule H_{2l+1}.

PROPOSITION 3.7. *The quadrature rules $\widehat{G}_{2l+1}^L$ and $\widehat{G}_{2l+1}^S$ for the Bernstein-Szegő weight function $w^{(1/2)}$ have the following properties: the rules $\widehat{G}_{2l+1}^L$ and $\widehat{G}_{2l+1}^S$ for $l \geq 2$ are internal, and so is $\widehat{G}_3^L$. Moreover, $\widehat{G}_3^S$ is internal if $\beta \leq 2\alpha$ or*

$$\beta > 2\alpha \quad \textit{and} \quad |\delta| \leq \frac{1}{4}(3\beta - 2\alpha). \tag{3.9}$$

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Proof. The following equations are used in the proof. We obtain from $U_l(1) = l+1$ and $U_l(-1) = (l+1)(-1)^l$, $l = -1, 0, 1, \ldots$, and from the formulas of Section 2 that

$$\begin{aligned}
\pi_0^{(1/2)}(1) &\equiv 1,\\
\pi_l^{(1/2)}(1) &= \frac{1}{2^l}\left[(l+1) + \frac{2\delta}{\beta}l + \left(1 - \frac{2\alpha}{\beta}\right)(l-1)\right] = \frac{(\delta+\beta-\alpha)l+\alpha}{2^{l-1}\beta}, \quad l \geq 1,\\
\pi_0^{(1/2)}(-1) &\equiv 1,\\
\pi_l^{(1/2)}(-1) &= \frac{1}{2^l}\left[(l+1)(-1)^l + \frac{2\delta}{\beta}l(-1)^{l-1} + \left(1 - \frac{2\alpha}{\beta}\right)(l-1)(-1)^{l-2}\right]\\
&= (-1)^l \frac{(\beta-\alpha-\delta)l+\alpha}{2^{l-1}\beta}, \quad l \geq 1.
\end{aligned}$$

For every $l \geq 3$, the rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ coincide with the corresponding Gauss-Kronrod quadrature formula H_{2l+1}. Gautschi and Notaris [6, Theorem 5.2(b)] have shown that H_{2l+1} is internal.

For $l = 2$, the quadrature formula $\widehat{G}^S_5$ and therefore also $\widehat{G}^L_5$ is internal if $F_3(1) \geq 0$ and $-F_3(-1) \geq 0$, i.e., if

$$\frac{4\pi_3^{(1/2)}(1)}{\pi_1^{(1/2)}(1)} \geq 1 \quad \text{and} \quad \frac{4\pi_3^{(1/2)}(-1)}{\pi_1^{(1/2)}(-1)} \geq 1.$$

From the first condition above, it follows that $\delta \geq -(\beta - \alpha)$. This inequality is true. The second condition above yields $\delta \leq \beta - \alpha$. This inequality also is valid. Therefore, $\widehat{G}^S_5$ and $\widehat{G}^L_5$ are internal.

If $l = 1$, then the rule $\widehat{G}^L_3$ is internal if $F_2(1) \geq 0$ and $F_2(-1) \geq 0$, i.e., if

$$\frac{\pi_2^{(1/2)}(1)}{\beta_1^{(1/2)}\pi_0^{(1/2)}(1)} \geq 1 \quad \text{and} \quad \frac{\pi_2^{(1/2)}(-1)}{\beta_1^{(1/2)}\pi_0^{(1/2)}(-1)} \geq 1.$$

These conditions yield

$$\delta \geq -(\beta - \alpha) \quad \text{and} \quad \delta \leq \beta - \alpha,$$

respectively. Both these inequalities hold. Therefore, $\widehat{G}^L_3$ is internal.

Finally, if $l = 1$, then the rule $\widehat{G}^S_3$ has the same algebraic degree of precision ($2l + 2 = 4$) as the corresponding Gauss-Kronrod quadrature formula H_3 ($3l + 1 = 4$). Therefore, these rules coincide. We can apply the analysis of internality of H_3 provided by Gautschi and Notaris [6, Theorem 5.2(b)], who show that H_3 is internal except if $\beta > 2\alpha$. In the latter situation, the rule is internal if (3.9) holds. ∎

PROPOSITION 3.8. *The quadrature rules $Q^{(1)}_{l+2}$ for the Bernstein-Szegő weight function $w^{(1/2)}$ are internal for $l \geq 2$.*

Proof. The result follows from [3, Theorem 4.1] since $\alpha^{(1/2)}_{l-1} = \alpha^{(1/2)}_{l+1} = 0$. ∎

3.3. The Bernstein-Szegő weight function $w^{(1/2,-1/2)}$. The recursion formulas for the orthogonal polynomials are of the form (3.1) with $r = 2$. Therefore, the rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ coincide with the corresponding Gauss-Kronrod quadrature formula H_{2l+1} if $l \geq 3$.

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For $l = 2$, the rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ are the same and differ from the corresponding Gauss-Kronrod rule H_{2l+1}. Finally, if $l = 1$, the rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ are not the same, and, in general, both of them differ from the corresponding Gauss-Kronrod rule H_{2l+1}.

PROPOSITION 3.9. *The quadrature rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ for the Bernstein-Szegő weight function $w^{(1/2,-1/2)}$ have the following properties: the rules $\widehat{G}^L_{2l+1}$ and $\widehat{G}^S_{2l+1}$ are internal for $l \geq 2$. The quadrature formula $\widehat{G}^L_3$ is internal if*

$$\frac{\beta(3\delta + 3\beta - \alpha)}{2\alpha(\beta - \alpha - \delta)} \geq 1 \quad \textit{and} \quad \beta > 2\alpha \quad (\textit{since } \beta \neq 2\alpha), \tag{3.10}$$

and $\widehat{G}^S_3$ is internal if

$$6\delta + 5\beta - 2\alpha \geq 0 \quad \textit{and} \quad 2\delta + 2\alpha - \beta \leq 0. \tag{3.11}$$

Proof. We obtain from $W_l(1) = 2l + 1$ and $W_l(-1) = (-1)^l$, for $l = 0, 1, 2, \ldots$, and from results of Section 2 that

$$\begin{aligned}
\pi_0^{(1/2,-1/2)}(1) &= 1,\\
\pi_1^{(1/2,-1/2)}(1) &= 1 + \frac{\alpha+\delta}{\beta} = \frac{\alpha+\beta+\delta}{\beta},\\
\pi_l^{(1/2,-1/2)}(1) &= \frac{1}{2^l}\left[(2l+1) + \frac{2\delta}{\beta}(2l-1) + \left(1 - \frac{2\alpha}{\beta}\right)(2l-3)\right]\\
&= \frac{(\delta+\beta-\alpha)(2l-1) + 2\alpha}{2^{l-1}\beta}, \quad l \geq 2,\\
\pi_0^{(1/2,-1/2)}(-1) &= 1,\\
\pi_1^{(1/2,-1/2)}(-1) &= -1 + \frac{\alpha+\delta}{\beta} = \frac{\delta-\beta+\alpha}{\beta},\\
\pi_l^{(1/2,-1/2)}(-1) &= \frac{1}{2^l}\left[(-1)^l + \frac{2\delta}{\beta}(-1)^{l-1} + \left(1 - \frac{2\alpha}{\beta}\right)(-1)^{l-2}\right]\\
&= (-1)^l\frac{\beta-\alpha-\delta}{2^{l-1}\beta}, \quad l \geq 2.
\end{aligned}$$

These equations are used to show the internality of some of the quadrature rules.

For $l \geq 3$, the quadrature rules $\widehat{G}^L_{2l+1}$, $\widehat{G}^S_{2l+1}$, and H_{2l+1} coincide. Gautschi and Notaris [6, Theorem 5.3(b)] have shown that the rules H_{2l+1} are internal.

For $l = 2$, the quadrature formulas $\widehat{G}^S_5$ and $\widehat{G}^L_5$ are internal if $F_3(1) \geq 0$ and $-F_3(-1) \geq 0$, i.e., if

$$\frac{4\pi_3^{(1/2,-1/2)}(1)}{\pi_1^{(1/2,-1/2)}(1)} \geq 1 \quad \text{and} \quad \frac{4\pi_3^{(1/2,-1/2)}(-1)}{\pi_1^{(1/2,-1/2)}(-1)} \geq 1.$$

The first condition above is equivalent to $\beta - \alpha + \delta \geq 0$, which holds true. The second condition also is valid.

For $l = 1$, the quadrature rule $\widehat{G}^L_3$ is internal if $F_2(1) \geq 0$ and $F_2(-1) \geq 0$, i.e., if

$$\frac{\pi_2^{(1/2,-1/2)}(1)}{\beta_1^{(1/2,-1/2)}\pi_0^{(1/2,-1/2)}(1)} \geq 1 \quad \text{and} \quad \frac{\pi_2^{(1/2,-1/2)}(-1)}{\beta_1^{(1/2,-1/2)}\pi_0^{(1/2,-1/2)}(-1)} \geq 1.$$

TABLE 4.1
Estimates of the error in $G_{l+1}[f]$ for approximating $I[f]$ with $f(t) = \exp(-t)$ determined by $\widehat{G}^S_{2l+1}[f]$ and $Q^{(1)}_{l+2}[f]$ for some α, β, δ.

α	β	δ	l	*Error*	$\left\|\widehat{G}^S_{2l+1}[f]-G_{l+1}[f]\right\|$	$\left\|Q^{(1)}_{l+2}[f]-G_{l+1}[f]\right\|$
1	$1+\sqrt{2}$	$-1/\sqrt{2}$	4	$1.1255(-09)$	$1.1255(-09)$	$1.1234(-09)$
			9	$1.6635(-24)$	$1.6635(-24)$	$1.6626(-24)$
			14	$1.4977(-41)$	$1.4977(-41)$	$1.4973(-41)$
			19	$4.7675(-60)$	$4.7675(-60)$	$4.7668(-60)$
$\sqrt{5}$	$2+\sqrt{5}$	1	4	$4.0268(-10)$	$4.0268(-10)$	$4.0193(-10)$
			9	$5.6832(-25)$	$5.6832(-25)$	$5.6801(-25)$
			14	$5.0339(-42)$	$5.0339(-42)$	$5.0326(-42)$
			19	$1.5891(-60)$	$1.5891(-60)$	$1.5888(-60)$

These conditions can be expressed as (3.10). It follows that $\widehat{G}^L_3$ is internal if the inequalities (3.10) hold.

The rule $\widehat{G}^S_3$ has the same algebraic degree of precision ($2l+2=4$) as the corresponding Gauss-Kronrod quadrature formula H_3 ($3l+1=4$). Therefore, these rules coincide. We apply [6, Theorem 5.3 (b)] to conclude that $\widehat{G}^S_3$ is internal if both inequalities (3.11) are valid. ∎

PROPOSITION 3.10. *The quadrature rules $Q^{(1)}_{l+2}$ for the Bernstein-Szegő weight function $w^{(1/2,-1/2)}$ are internal for $l \geq 2$.*

Proof. It follows from [3, Theorem 4.1] that $Q^{(1)}_{l+2}$ is internal for $l \geq 3$ since the weights satisfy $\alpha^{(1/2,-1/2)}_{l-1} = \alpha^{(1/2,-1/2)}_{l+1} = 0$.

If $l = 2$, then we obtain from (1.6) that the rule $Q^{(1)}_4$ is internal if $q_4(1) \geq 0$ and $q_4(-1) \geq 0$, i.e., if the inequalities

$$\frac{4\left(1-\alpha^{(1/2,-1/2)}_1\right)\pi^{(1/2,-1/2)}_3(1)}{\pi^{(1/2,-1/2)}_2(1)} \geq 1, \quad -\frac{4\left(1+\alpha^{(1/2,-1/2)}_1\right)\pi^{(1/2,-1/2)}_3(-1)}{\pi^{(1/2,-1/2)}_2(-1)} \geq 1$$

are satisfied. Straightforward computations show that both inequalities hold. ∎

4. Numerical results. This section illustrates the use of the quadrature rules $\widehat{G}^S_{2l+1}$ and $Q^{(1)}_{l+2}$ to estimate the magnitude of the quadrature error $I[f] - G_{l+1}[f]$. In applications, we use the values $\widehat{G}^S_{2l+1}[f]$ or $Q^{(1)}_{l+2}[f]$ as approximations of $I[f]$ together with the computed error estimates because these quadrature rules typically furnish a more accurate approximation of $I[f]$ than $G_{l+1}[f]$. All computations have been carried out in MATLAB with high precision arithmetic.

EXAMPLE 4.1. Consider the estimation of the magnitude of quadrature errors obtained with the Gauss rules $G_{l+1}[f]$ when applied to the approximation of the integral

$$I[f] = \int_{-1}^{1} f(t)w^{(-1/2)}(t)\,dt.$$

Approximation of these integrals has previously been considered by Notaris [13].

We estimate the magnitude of the quadrature error $I[f] - G_{l+1}[f]$ by the differences $|\widehat{G}^S_{2l+1}[f] - G_{l+1}[f]|$ and $|Q^{(1)}_{l+2}[f] - G_{l+1}[f]|$. As shown above, the quadrature formulas $\widehat{G}^S_{2l+1}$ and $Q^{(1)}_{l+2}$ are internal for the cases reported in Tables 4.1 and 4.2. Table 4.1 displays

TABLE 4.2
Estimates of the error in $G_{l+1}[f]$ for approximating $I[f]$ with $f(t) = \ln(2/(2-t))$ determined by $\widehat{G}^S_{2l+1}[f]$ and $Q^{(1)}_{l+2}[f]$ for some α, β, δ.

α	β	δ	l	*Error*	$\left\|\widehat{G}^S_{2l+1}[f]-G_{l+1}[f]\right\|$	$\left\|Q^{(1)}_{l+2}[f]-G_{l+1}[f]\right\|$
1	$1+\sqrt{2}$	$-1/\sqrt{2}$	4	$1.0935(-06)$	$1.0939(-06)$	$1.0278(-06)$
			9	$1.0569(-12)$	$1.0569(-12)$	$9.8786(-13)$
			14	$1.3506(-18)$	$1.3506(-18)$	$1.2596(-18)$
			19	$1.9372(-24)$	$1.9372(-24)$	$1.8046(-24)$
$\sqrt{5}$	$2+\sqrt{5}$	1	4	$2.1627(-07)$	$2.1636(-07)$	$2.0337(-07)$
			9	$2.0435(-13)$	$2.0435(-13)$	$1.9102(-13)$
			14	$2.5904(-19)$	$2.5904(-19)$	$2.4161(-19)$
			19	$3.7002(-25)$	$3.7002(-25)$	$3.4472(-25)$

the results for $f(t) = \exp(-t)$. The column with the header *Error* shows the magnitude of the error $|I[f] - G_{l+1}[f]|$ for several values of the parameters α, β, and δ that define the weight function $w^{(-1/2)}$. The computed estimates of $|I[f] - G_{l+1}[f]|$ are seen to be very accurate for all values of α, β, and δ.

Table 4.2 differs from Table 4.1 only in that the integrand is

$$f(t) = \ln \frac{2}{2-t} \cdot$$

The computed estimates of the magnitude of quadrature error $|I[f] - G_{l+1}[f]|$ are seen to be accurate for all values of α, β, and δ also for this integrand with the estimates $|\widehat{G}^S_{2l+1}[f] - G_{l+1}[f]|$ being somewhat more accurate than the estimates $|Q^{(1)}_{l+2}[f] - G_{l+1}[f]|$. We conclude that the generalized averaged Gauss quadrature rules $\widehat{G}^S_{2l+1}$ and the truncated version $Q^{(1)}_{l+2}$ provide accurate estimates of the quadrature error of the Gauss rule G_{l+1} for different integrands and several values of l.

5. Conclusion. The present paper investigates whether generalized averaged Gauss quadrature rules associated with Bernstein-Szegő weight functions are internal. This issue is important because internal quadrature rules can be applied to a larger class of integrands than rules with one or several external nodes. Also truncated versions of generalized averaged Gauss quadrature rules are studied. Our investigation complements the recent study [3] of the internality of generalized averaged Gauss quadrature rules and truncated variants for classical weights functions. The analysis of this paper shows that in many situations, generalized averaged Gauss quadrature rules coincide with Gauss-Kronrod rules. This implies that the simple numerical methods for computing generalized averaged Gauss quadrature rules described in [17] can be applied to determine Gauss-Kronrod rules. The averaged rules proposed by Laurie [9] are shown to coincide with the generalized averaged Gauss quadrature rules in certain situations. Computed examples illustrate the high accuracy of quadrature error estimates that can be achieved with generalized averaged Gauss quadrature rules and their associated truncated variants.

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