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A COMPARISON OF ADAPTIVE COARSE SPACES FOR ITERATIVE SUBSTRUCTURING IN TWO DIMENSIONS*

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Abstract. The convergence rate of iterative substructuring methods generally deteriorates when large discontinuities occur in the coefficients of the partial differential equations to be solved. In dual-primal Finite Element Tearing and Interconnecting (FETI-DP) and Balancing Domain Decomposition by Constraints (BDDC) methods, sophisticated scalings, e.g., deluxe scaling, can improve the convergence rate when large coefficient jumps occur along or across the interface. For more general cases, additional information has to be added to the coarse space. One possibility is to enhance the coarse space by local eigenvectors associated with subsets of the interface, e.g., edges. At the center of the condition number estimates for FETI-DP and BDDC methods is an estimate related to the P_D-operator which is defined by the product of the transpose of the scaled jump operator B_D^T and the jump operator B of the FETI-DP algorithm. Some enhanced algorithms immediately bring the P_D-operator into focus using related local eigenvalue problems, and some replace a local extension theorem and local Poincaré inequalities by appropriate local eigenvalue problems. Three different strategies, suggested by different authors, are discussed for adapting the coarse space together with suitable scalings. Proofs and numerical results comparing the methods are provided.

Key words. FETI-DP, BDDC, eigenvalue problem, coarse space, domain decomposition, multiscale

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction. Iterative substructuring methods are known to be efficient preconditioners for the large linear systems resulting from the discretization of second-order elliptic partial differential equations, e.g., those of modeling diffusion and linear elasticity. However, it is also known that the convergence rate of domain decomposition methods can deteriorate severely when large coefficient jumps occur. Except for certain special coefficient distributions, e.g., constant coefficients in each subdomain and jumps only across the interface, which can be treated with special scalings, the coarse space has to be enhanced appropriately. One possible approach consists of, given a user-defined tolerance, adaptively solving certain local eigenvalue problems and enhancing the coarse space appropriately using some of the computed eigenvectors; see, e.g., [3, 4, 7, 10, 11, 14, 15, 18, 23, 30, 36, 37].

We compare different adaptive coarse spaces that have been proposed by different authors for the FETI-DP and BDDC domain decomposition methods, in particular, our approach in [23], the classic method in [30], a recent method in [7], and a variant thereof in [21]. Additionally, a proof of the condition number bound for the method in [30] for the two-dimensional case is given. We also introduce cost-efficient variants of the methods in [7, 23] that are based on the ideas of an economic variant of the deluxe scaling given in [9]; deluxe scaling was introduced in [8]; see also [2, 5, 6, 20, 29, 33].

At the center of the condition number estimates for FETI-DP and BDDC methods is an estimate related to the P_D-operator which is defined as the product of the transpose of the scaled jump operator B_D^T and the jump operator B of the FETI-DP algorithm. The algorithms suggested in [30] and [7] involve local eigenvalue problems directly related to the P_D-operator, and the approach in [23] replaces a local extension theorem and local Poincaré inequalities by appropriate local eigenvalue problems. All the adaptive methods have in common that they

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start from an initial coarse space guaranteeing a nonsingular system matrix followed by adding additional constraints that are computed by solving local generalized eigenvalue problems. In this paper, we implement the additional constraints with a method based on projections known as projector preconditioning [17, 26].

The remainder of the paper is organized as follows: in Section 2, we introduce the model problems, their finite element discretization, and the domain decomposition. In Section 3, a short introduction is given to the FETI-DP algorithm and to projector preconditioning and deflation. The latter techniques are used to add the additional local eigenvectors to the coarse problem. A new, more general and direct proof for the condition number estimate of FETI-DP using deflation is given. In Section 4, the first approach considered here, see [7], to adaptively construct a coarse problem is considered. A proof for the condition number estimate is provided, different scalings are considered, and a new economic variant is introduced and analyzed. In Remark 4.9, it is shown that the use of a certain scaling (deluxe scaling) allows us to weaken the requirements on the domain from Jones to John domains in the analysis of the FETI-DP and BDDC methods. In Section 5, as a second approach considered in this paper, the adaptive coarse space construction suggested in [30] is described, and a new condition number estimate for two dimensions is proven. In Section 6, our approach from [23], which concerns the third coarse space analyzed here, is briefly described, and a new variant with a modified deluxe scaling is introduced and analyzed. In Section 7, a brief comparison of the computational cost of the three coarse spaces with different scalings is provided. In Section 8, results of numerical experiments are presented with the three different coarse spaces using different scalings applied to diffusion and almost incompressible linear elasticity. Finally, in Section 9, a conclusion is given.

2. Elliptic model problems, finite elements, and domain decomposition. In this section, we introduce the elliptic model problems and their discretization by finite elements. We consider a scalar diffusion equation discretized by linear finite elements. Additionally, we consider a displacement formulation for almost incompressible linear elasticity which is obtained from a mixed finite element formulation with discontinuous pressure variables by static condensation of the pressure.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, let $\partial\Omega_D \subset \partial\Omega$ be a subset of positive surface measure, and let $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$. We consider the following diffusion problem: find $u \in H_0^1(\Omega, \partial\Omega_D)$ such that

$$a(u,v) = f(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D),$$
$$\text{where} \quad a(u,v) := \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx, \quad f(v) := \int_\Omega f v \, dx + \int_{\partial\Omega_N} g_N v \, ds.$$

Here g_N are the boundary data defined on $\partial\Omega_N$. We assume $\rho(x) > 0$ for $x \in \Omega$ and ρ piecewise constant on Ω; the coefficient $\rho(x)$ is not necessarily constant on the subdomains. This problem is discretized using piecewise linear finite elements. As a second model problem, we consider the mixed displacement-pressure saddle-point system of almost incompressible linear elasticity. With the Lamé parameters λ and μ and the bilinear forms

$$a(u,v) = \int_\Omega 2\mu \, \varepsilon(u) : \varepsilon(v) \, dx, \quad b(v,p) = \int_\Omega \mathrm{div}(v) p \, dx, \quad \text{and} \quad c(p,q) = \int_\Omega \frac{1}{\lambda} pq \, dx \,,$$

the saddle-point variational formulation is of the form: find $(u,p) \in H_0^1(\Omega, \partial\Omega_D)^d \times L^2(\Omega)$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) && \forall \, v \in H_0^1(\Omega, \partial\Omega_D)^d, \\ b(u,q) - c(p,q) &= 0 && \forall \, q \in L_2(\Omega). \end{aligned}$$

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For almost incompressible materials, we use a discretization by mixed finite elements with discontinuous pressures, e.g., $\mathbb{P}2$-$\mathbb{P}0$ elements. In our computations, the pressure variables are statically condensed element-by-element, which again yields a variational formulation in the displacement variables only. In the following, with a slight abuse of notation, we make no distinction between a finite element function and its coordinate vector.

We decompose the domain Ω into N nonoverlapping subdomains $\Omega_i, i = 1,\ldots,N$, where each Ω_i is the union of shape-regular triangular elements of diameter $\mathcal{O}(h)$. We assume that the decomposition is such that the finite element nodes on the boundaries of neighboring subdomains match across the interface $\Gamma := \overline{(\cup_{i=1}^N \partial\Omega_i) \setminus \partial\Omega}$. The interface Γ is the union of edges and vertices where edges are defined as open sets that are shared by two neighboring subdomains and vertices are endpoints of edges. For a more general definition in three dimensions, see [24, 28]. We denote the edge belonging to the subdomains Ω_i and Ω_j by $\mathcal{E}_{ij}$. By $W^h(\Omega_i)$, we denote the standard piecewise linear or quadratic finite element space on Ω_i. We assume that these finite element functions vanish on $\partial\Omega_D$ and that the finite element triangulation is quasi-uniform on each subdomain. By H_i, or generically H, we denote the subdomain diameter of Ω_i. The local stiffness matrices on each subdomain are denoted by $K^{(i)}, i = 1,\ldots,N$.

Let $a_l(u,v)$ be the bilinear form corresponding to the local stiffness matrix on a subdomain Ω_l obtained by a finite element discretization of an elliptic problem. The respective coefficients are denoted by ρ_l in the case of diffusion and by λ_l and μ_l in the case of linear elasticity. For almost incompressible linear elasticity, the subdomain stiffness matrices are defined as $K^{(l)} := A^{(l)} + B^{(l)T} C^{(l)-1} B^{(l)}$, where the matrices

$$u^T A^{(l)} v = \int_{\Omega_l} 2\mu_l\, \varepsilon(u) : \varepsilon(v)\, dx, \quad p^T B^{(l)} u = \int_{\Omega_l} \operatorname{div}(u)\, p\, dx,$$
$$\text{and} \quad p^T C^{(l)} q = \int_{\Omega_l} \frac{1}{\lambda_l} pq\, dx$$

result from a discretization with inf-sup stable $\mathbb{P}2$-$\mathbb{P}0$ finite elements. Other inf-sup stable elements with discontinuous pressures are possible as well. After the elimination of the pressure, we define $a_l(u,v) = u^T K^{(l)} v$.

3. The FETI-DP algorithm and deflation/projector preconditioning.

In this section, we briefly describe the FETI-DP algorithm and a recently introduced method with a second coarse level incorporated by deflation. For more details on the FETI-DP algorithm, see, e.g., [12, 13, 27, 28, 38], and for FETI-DP with deflation and projector preconditioning, see [17, 26].

We start with the local stiffness matrices $K^{(i)}$ associated with the subdomains Ω_i. Let the variables further be partitioned into those in the interior of the subdomain, $u_I^{(i)}$, dual variables on the interface, $u_\Delta^{(i)}$, and primal degrees of freedom on the interface, $u_\Pi^{(i)}$. As primal variables, unknowns associated with subdomain corners can be chosen, but other choices are possible. For the local stiffness matrices, unknowns, and right-hand sides, this yields

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)T} & K_{\Pi I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} & K_{\Pi\Delta}^{(i)T} \\ K_{\Pi I}^{(i)} & K_{\Pi\Delta}^{(i)} & K_{\Pi\Pi}^{(i)} \end{bmatrix}, \quad u^{(i)} = \begin{bmatrix} u_I^{(i)} \\ u_\Delta^{(i)} \\ u_\Pi^{(i} \end{bmatrix}, \quad \text{and} \quad f^{(i)} = \begin{bmatrix} f_I^{(i)} \\ f_\Delta^{(i)} \\ f_\Pi^{(i)} \end{bmatrix}.$$

We obtain the block-diagonal matrices $K_{II} = \operatorname{diag}_{i=1}^N K_{II}^{(i)}$, $K_{\Delta\Delta} = \operatorname{diag}_{i=1}^N K_{\Delta\Delta}^{(i)}$, and $K_{\Pi\Pi} = \operatorname{diag}_{i=1}^N K_{\Pi\Pi}^{(i)}$ from the local blocks. Interior and dual degrees of freedom can be

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combined as the remaining degrees of freedom denoted by the index B. The associated matrices and vectors are then of the form

$$K_{BB}^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} \end{bmatrix}, \quad K_{\Pi B}^{(i)} = \begin{bmatrix} K_{\Pi I}^{(i)} & K_{\Pi\Delta}^{(i)} \end{bmatrix}, \quad \text{and} \quad f_B^{(i)} = \begin{bmatrix} f_I^{(i)T} & f_\Delta^{(i)T} \end{bmatrix}^T.$$

We define a block matrix, a block vector, and a block right-hand side vector

$$K_{BB} = \operatorname{diag}_{i=1}^N K_{BB}^{(i)}, \quad u_B = [u_B^{(1)T}, \ldots, u_B^{(N)T}]^T, \quad f_B = \left[f_B^{(1)T}, \ldots, f_B^{(N)T} \right]^T.$$

We introduce assembly operators $R_\Pi^{(i)T}$ for the primal variables; these matrices consist of zeros and ones only. After assembly, to enforce continuity in the primal variables, we obtain the matrices

$$\widetilde{K}_{\Pi\Pi} = \sum_{i=1}^N R_\Pi^{(i)T} K_{\Pi\Pi}^{(i)} R_\Pi^{(i)}, \quad \widetilde{K}_{\Pi B} = \left[R_\Pi^{(1)T} K_{\Pi B}^{(1)}, \ldots, R_\Pi^{(N)T} K_{\Pi B}^{(N)} \right]$$

and the right-hand side $\widetilde{f} = \left[f_B^T, \left(\sum_{i=1}^N R_\Pi^{(i)T} f_\Pi^{(i)} \right)^T \right]^T$. After elimination of all but the primal degrees of freedom, we obtain the Schur complement $\widetilde{S}_{\Pi\Pi} = \widetilde{K}_{\Pi\Pi} - \widetilde{K}_{\Pi B} K_{BB}^{-1} \widetilde{K}_{\Pi B}^T$. We define a jump matrix $B_B = [B_B^{(1)} \ldots B_B^{(N)}]$ that connects the dual degrees of freedom on the interface such that $B_B u_B = 0$ if u_B is continuous. The FETI-DP system is then given by $F\lambda = d$ with

$$\begin{aligned} F &= B_B K_{BB}^{-1} B_B^T + B_B K_{BB}^{-1} \widetilde{K}_{\Pi B}^T \widetilde{S}_{\Pi\Pi}^{-1} \widetilde{K}_{\Pi B} K_{BB}^{-1} B_B^T, \\ d &= B_B K_{BB}^{-1} f_B + B_B K_{BB}^{-1} \widetilde{K}_{\Pi B}^T \widetilde{S}_{\Pi\Pi}^{-1} \left(\left(\sum_{i=1}^N R_\Pi^{(i)T} f_\Pi^{(i)} \right) - \widetilde{K}_{\Pi B} K_{BB}^{-1} f_B \right). \end{aligned}$$

We have the alternative representation $F = B\widetilde{S}^{-1}B^T$ where $\widetilde{S}$ is obtained by eliminating the interior degrees of freedom from $\begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi B}^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} \end{bmatrix}$ and B is the restriction of B_B to the interface Γ where the primal part is set to zero. The FETI-DP algorithm is the preconditioned conjugate gradients algorithm applied to $F\lambda = d$ with the Dirichlet preconditioner

$$M^{-1} = B_{B,D} \, [0 \quad I_\Delta]^T \left(K_{\Delta\Delta} - K_{\Delta I} K_{II}^{-1} K_{\Delta I}^T \right) [0 \quad I_\Delta] \, B_{B,D}^T = B_D \widetilde{S} B_D^T.$$

Here, $B_{B,D}$ and B_D are scaled variants of B_B and B, respectively; in the simplest case they are scaled by the inverse multiplicity of the nodes, e.g., $1/2$ in two dimensions. Alternatively, we use the approach in, e.g., [25, 34], and introduce scaling weights by

$$\delta_j(x) := \left(\sum_{i \in N_x} \widehat{\rho}_i(x) \right) / \widehat{\rho}_j(x), \qquad \text{where} \quad \widehat{\rho}_j(x) = \max_{x \in \omega(x) \cap \Omega_{j,h}} \rho_j(x)$$

and N_x denotes for each interface node x the set of indices of the subdomains which have x on their boundary. Here, $\omega(x)$ is the support of the finite element basis function associated with the node $x \in \partial\Omega_{j,h} \cap \Gamma_h$, $j = 1, \ldots, N$. The pseudoinverses are defined by

$$\delta_j(x)^\dagger := \widehat{\rho}_j(x) / \sum_{i \in N_x} \widehat{\rho}_i(x)$$

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for $x \in \partial\Omega_{j,h} \cap \Gamma_h$. Each row of $B^{(i)}$ with a nonzero entry connects a point of $\Gamma_h^{(i)}$ with the corresponding point of a neighboring subdomain $x \in \Gamma_h^{(i)} \cap \Gamma_h^{(j)}$. Multiplying each such row with $\delta_j(x)^\dagger$ for each $B^{(i)}$, $i = 1, \dots, N$, results in the scaled operator B_D. We will refer to this scaling as ρ-scaling. For coefficients that are constant on each subdomain but possibly discontinuous across the interface, this approach reduces to the classical ρ-scaling; see, e.g., [38].

Another set of primal constraints can be aggregated as columns of a matrix U; see, e.g., [17, 26]. To enforce $U^T Bu = 0$, i.e., averages of the jump with weights defined by the columns of U vanish, we introduce the F-orthogonal projection $P = U(U^T FU)^{-1} U^T F$. Instead of solving $F\lambda = d$, the deflated and singular but consistent system $(I - P)^T F\lambda = (I - P)^T d$ can be solved. Denoting by λ^* the exact solution of $F\lambda = d$, we define

$$\overline{\lambda} = U(U^T FU)^{-1} U^T d = PF^{-1} d = P\lambda^*.$$

Let λ be the solution of $M^{-1}(I - P)^T F\lambda = M^{-1}(I - P)^T d$ by PCG, where M^{-1} is the classical Dirichlet preconditioner. Then, we can compute

$$\lambda^* = \overline{\lambda} + (I - P)\lambda \in \text{ker}\,(I - P) \oplus \text{range}(I - P).$$

The matrices $P^T F(= FP)$ and $(I - P)^T F(= F(I - P))$ are symmetric. The spectrum is thus not changed by projecting the correction onto range$(I - P)$ in every iteration [26]. Therefore, we obtain the symmetric projector preconditioner

$$M_{PP}^{-1} = (I - P)M^{-1}(I - P)^T.$$

Adding the correction, we compute $\lambda^* = \overline{\lambda} + \lambda$, where λ is the PCG solution of the system $M_{PP}^{-1} F\lambda = M_{PP}^{-1} d$. Alternatively, we can include the computation of $\overline{\lambda}$ into the preconditioner. This results in the balancing preconditioner

$$M_{BP}^{-1} = (I - P)M^{-1}(I - P)^T + PF^{-1}. \tag{3.1}$$

Since $PF^{-1} = U(U^T FU)^{-1} U^T$, this preconditioner is symmetric and can be efficiently computed. Here, $U^T FU$ is usually of much smaller dimension than F.

For each subdomain, we introduce local finite element trace spaces $W_i := W^h(\partial\Omega_i \cap \Gamma)$, $i = 1, \dots, N$. We define the product space $W := \Pi_{i=1}^N W_i$ and denote the subspace of functions $w \in W$ that are continuous in the primal variables by $\widetilde{W}$.

DEFINITION 3.1. *For a symmetric positive semidefinite matrix A and a vector v of appropriate dimension, we denote the induced seminorm by $|v|_A^2 = v^T Av$. If A is positive definite and w is a vector of appropriate dimension, we denote the induced scalar product by $\langle v, w\rangle_A = v^T Aw$.*

The following lemma is an alternative to the proof provided in [26] for projector preconditioning or deflation applied to FETI-DP methods. It directly applies to a larger class of scalings.

LEMMA 3.2. *Let $P_D = B_D^T B$. Assuming that $||P_D w||_{\widetilde{S}}^2 \le C||w||_{\widetilde{S}}^2$ holds for all $w \in \{w \in \widetilde{W} \,|\, U^T Bw = 0\}$ with a constant $C > 0$, we have*

$$\kappa\,(M_{PP}^{-1} F) \le C.$$

Here, the constant C can depend on H/h or η/h (cf. Definition 4.11) and possibly on a prescribed tolerance from local generalized eigenvalue problems.

Proof. Similar to [28, p. 1553], we use $(I-P)^TF = F(I-P)$ with the standard Dirichlet preconditioner M^{-1}. Observing that $w = \widetilde{S}^{-1}B^T(I-P)\lambda \in \widetilde{W}$ and

$$(I-P)U = 0 \Rightarrow U^TB(\widetilde{S}^{-1}B^T(I-P)\lambda) = U^T(I-P)^TB\widetilde{S}^{-1}B^T\lambda = 0,$$

we obtain for the upper bound

$$\begin{aligned}
&\langle M_{PP}^{-1}F\lambda,\lambda\rangle_F = \langle (I-P)M^{-1}(I-P)^TF\lambda, F\lambda\rangle = \langle M^{-1}F(I-P)\lambda, F(I-P)\lambda\rangle\\
=&\langle B_D^TB\widetilde{S}^{-1}B^T(I-P)\lambda, B_D^TB\widetilde{S}^{-1}B^T(I-P)\lambda\rangle_{\widetilde{S}} = |P_D(\widetilde{S}^{-1}B^T(I-P)\lambda)|_{\widetilde{S}}^2\\
=&|P_Dw|_{\widetilde{S}}^2 \le C|w|_{\widetilde{S}}^2 = C|\widetilde{S}^{-1}B^T(I-P)\lambda|_{\widetilde{S}}^2\\
=&C\langle \widetilde{S}^{-1}B^T(I-P)\lambda, \widetilde{S}^{-1}B^T(I-P)\lambda\rangle_{\widetilde{S}} = C\langle (I-P)\lambda,(I-P)\lambda\rangle_F = C\langle\lambda,\lambda\rangle_F.
\end{aligned}$$

Since $\lambda \in \operatorname{range}(I-P)$, we have $(I-P)\lambda = \lambda$. Hence, we have $\lambda_{\max}(M_{PP}^{-1}F) \le C$. We now derive an estimate for the lower bound. With $E_Dw(x) := \sum_{j\in N_x} D^{(j)}w_j(x)$, we see that $P_Dw = B_D^TBw = (I-E_D)w$. Since E_Dw is continuous across the interface, P_D preserves the jump of any function $w \in \widetilde{W}$ in the sense that $Bw = Bw - 0 = B(I-E_D)w = BP_Dw$. Analogously to [28, p. 1552], we obtain for the lower bound

$$\begin{aligned}
\langle\lambda,\lambda\rangle_F^2 &= \langle\lambda, B\widetilde{S}^{-1}B^T\lambda\rangle^2 = \langle\lambda, B\widetilde{S}^{-1}P_D^TB^T\lambda\rangle^2 = \langle\lambda, B\widetilde{S}^{-1}B^TB_DB^T\lambda\rangle^2\\
&= \langle\lambda, B_DB^T\lambda\rangle_F^2 = \langle F\lambda, B_D\widetilde{S}^{1/2}\widetilde{S}^{-1/2}B^T\lambda\rangle^2\\
&\le \langle \widetilde{S}^{1/2}B_D^TF\lambda, \widetilde{S}^{1/2}B_D^TF\lambda\rangle\langle\widetilde{S}^{-1/2}B^T\lambda, \widetilde{S}^{-1/2}B^T\lambda\rangle\\
&= \langle M^{-1}F\lambda, F\lambda\rangle\langle F\lambda,\lambda\rangle = \langle M^{-1}F(I-P)\lambda, F(I-P)\lambda\rangle\langle F\lambda,\lambda\rangle\\
&= \langle M_{PP}^{-1}F\lambda,\lambda\rangle_F\langle F\lambda,\lambda\rangle\,.
\end{aligned}$$

The identity in the penultimate step holds since $\lambda \in \operatorname{range}(I-P)$. Hence, we have $\lambda_{\min}(M_{PP}^{-1}F) \ge 1$. ∎

4. First coarse space.

In this approach, the general eigenvalue problems are based on a localization of the P_D-estimate in contrast to [23], where an edge lemma and a Poincaré-Friedrichs inequality are used; see also Section 6. This section is organized as follows. In Section 4.1, we introduce the relevant notation, and in Section 4.2 we show how the energy of the P_D-operator can be bounded by local estimates. In Section 4.3, we collect some known information on the parallel sum of matrices and show some related spectral estimates. In Sections 4.4 and 4.5, we introduce two approaches to enhance the coarse space with adaptively computed constraints. In both approaches the constraints are computed by solving local generalized eigenvalue problems. This first approach has been proposed in [7] and relies on deluxe scaling. In the second approach, first proposed in [21], any kind of scaling is possible as long it satisfies the partition of unity property (4.2). For the special case of deluxe scaling, the second approach is the same as the first. In Section 4.6, we consider an economic variant solving eigenvalue problems on slabs. Finally, in Section 4.7, we prove a condition number bound for the FETI-DP algorithm with adaptive constraints as described in Sections 4.4.2, 4.5.2, or 4.6.2.

4.1. Notation.

We define the energy-minimal extension of v from the local interface to the interior of the subdomain Ω_l as

$$\mathcal{H}^{(l)}v := \operatorname{argmin}_{u\in V^h(\Omega_l)}\left\{a_l(u,u) : u_{|\partial\Omega_l} = v\right\} \qquad \text{for } l = i,j.$$

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Let $\theta_{E_{ij}}$ be the standard finite element cut-off function, which equals 1 at the nodes on the edge $\mathcal{E}_{ij}$ and is zero on $\partial\Omega_i \setminus \mathcal{E}_{ij}$. With I^h we denote the standard finite element interpolation operator. We make use of the seminorm

$$|v|_{E_l}^2 := a_l(v, v). \tag{4.1}$$

We also make use of an energy-minimal extension from an edge $\mathcal{E}_{ij}$ to the interfaces $\Gamma^{(l)}$, $l = i, j$.

DEFINITION 4.1. *Let $\mathcal{E} \subset \Gamma^{(i)} := \partial\Omega_i \setminus \partial\Omega$ be an edge and $\mathcal{E}^c \subset \Gamma^{(i)}$ be the complement of $\mathcal{E}$ with respect to $\Gamma^{(i)}$, and let $S^{(i)}$ be partitioned as*

$$S^{(i)} = \begin{bmatrix} S^{(i)}_{\mathcal{E}\mathcal{E}} & S^{(i)T}_{\mathcal{E}^c\mathcal{E}} \\ S^{(i)}_{\mathcal{E}^c\mathcal{E}} & S^{(i)}_{\mathcal{E}^c\mathcal{E}^c} \end{bmatrix}.$$

Define the extension operator

$$\mathcal{H}^{(i)}_{\mathcal{E}} v_{|\mathcal{E}} := \begin{bmatrix} v_{|\mathcal{E}} \\ -S^{-1}_{\mathcal{E}^c\mathcal{E}^c} S_{\mathcal{E}^c\mathcal{E}} v_{|\mathcal{E}} \end{bmatrix}$$

and the matrices $S^{(l)}_{E_{ij},0} := S^{(l)}_{\mathcal{E}_{ij}\mathcal{E}_{ij}}$ and $S^{(l)}_{E_{ij}} := S^{(l)}_{\mathcal{E}_{ij}\mathcal{E}_{ij}} - S^{(l)T}_{\mathcal{E}^c_{ij}\mathcal{E}_{ij}} S^{(l)-1}_{\mathcal{E}^c_{ij}\mathcal{E}^c_{ij}} S^{(l)}_{\mathcal{E}^c_{ij}\mathcal{E}_{ij}}$.

The proof of the next lemma follows from a standard variational argument.

LEMMA 4.2. *Using the same notation as in Definition 4.1, for all $w_i \in V^h(\Gamma^{(i)})$, we have $|\mathcal{H}^{(i)}_{\mathcal{E}} w_{i|\mathcal{E}}|^2_{S^{(i)}} \leq |w_i|^2_{S^{(i)}}$.*

With Definition 4.1, we have the following correspondence between (semi)norms and the matrices defined in Definition 4.1:

$$\begin{aligned} |\mathcal{H}^{(l)} I^h(\theta_{E_{ij}} v)|^2_{E_l} &= v^T_{|\mathcal{E}_{ij}} S^{(l)}_{E_{ij},0} v_{|\mathcal{E}_{ij}} \qquad l = i, j \\ |\mathcal{H}^{(l)} \mathcal{H}^{(l)}_{\mathcal{E}_{ij}} v|^2_{E_l} &= v^T_{|\mathcal{E}_{ij}} S^{(l)}_{E_{ij}} v_{|\mathcal{E}_{ij}} \qquad l = i, j. \end{aligned}$$

Let $D^{(l)}_{E_{ij}}$, $l = i, j$, be scaling matrices such that

$$D^{(i)}_{E_{ij}} + D^{(j)}_{E_{ij}} = I, \tag{4.2}$$

where I is the identity matrix. This is a partition of unity.

4.2. Bounding the energy of the jump operator by local contributions. As a classical result in the analysis of iterative substructuring, see, e.g., [28, 38], we have

$$|P_D w|^2_{\widetilde{S}} = |R P_D w|^2_S = \sum_{i=1}^N |R^{(i)} P_D w|^2_{S^{(i)}}.$$

Here, $R^{(i)T}$, $i = 1, \ldots, N$, are the local assembly operators that partially assemble in the primal variables, and $R^T = [R^{(1)T}, \ldots, R^{(N)T}]$; see, e.g., [28]. Let $N_{\mathcal{E}}$ denote the maximum number of edges of a subdomain. For $w \in \widetilde{W}$, we define $w_i = R^{(i)} w$ and $w_j = R^{(j)} w$. In the following, in order to avoid the introduction of additional extension and restriction operators, whenever the difference $w_i - w_j$ is used, we assume that w_i and w_j are first restricted to the edge $\mathcal{E}_{ij}$ and that the difference is then extended by zero to the rest of the interface Γ. Under the assumption that all vertices are primal, we obtain

$$|R^{(i)} P_D w|^2_{S^{(i)}} \leq N_{\mathcal{E}} \sum_{j \in N_i} |\mathcal{H}^{(i)} I^h(\theta_{E_{ij}} D^{(i)}(w_i - w_j))|^2_{E_i},$$

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where N_i denotes the set of indices of the subdomains that share an edge with Ω_i. Hence, we are interested in obtaining bounds for the local contributions on the edges $\mathcal{E}_{ij}$ of the form:

$$
\begin{aligned}
&|\mathcal{H}^{(i)} I^h(\theta_{E_{ij}} D^{(i)}(w_i - w_j))|^2_{E_i} + |\mathcal{H}^{(j)} I^h(\theta_{E_{ij}} D^{(j)}(w_j - w_i))|^2_{E_j} \\
&\qquad \leq C \left(|\mathcal{H}^{(i)} \mathcal{H}^{(i)}_{\mathcal{E}_{ij}} w_{i|\mathcal{E}_{ij}}|^2_{E_i} + |\mathcal{H}^{(j)} \mathcal{H}^{(j)}_{\mathcal{E}_{ij}} w_{j|\mathcal{E}_{ij}}|^2_{E_j} \right) \leq C \left(|w_i|^2_{E_i} + |w_j|^2_{E_j} \right).
\end{aligned}
$$

Using Definition 4.1, this is equivalent to

$$
\begin{aligned}
&(w_i - w_j)^T_{|\mathcal{E}_{ij}} D^{(j)T}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}} (w_i - w_j)_{|\mathcal{E}_{ij}} + (w_j - w_i)^T_{|\mathcal{E}_{ij}} D^{(i)T}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}} (w_j - w_i)_{|\mathcal{E}_{ij}} \\
&\qquad \leq C \left(w_i^T S^{(i)}_{E_{ij}} w_i + w_j^T S^{(j)}_{E_{ij}} w_j \right).
\end{aligned}
$$

Note that C depends on the chosen primal space.

4.3. Parallel sum of matrices and spectral estimates. The next lemma introduces the notion of parallel sum of matrices of two symmetric positive semidefinite matrices and properties of that operation. The definition of a parallel sum of matrices was first given in [1] and for the first time used in our context in [7]. The first two properties of the next lemma are given and proven in [1]. The third property is given, without a proof, in [7].

REMARK 4.3. Using that Ker$(A + B) \subset$ Ker(A) and Ker$(A + B) \subset$ Ker(B) for symmetric positive semidefinite A and B and that $U \subset V$ implies $V^\perp \subset U^\perp$, we obtain Range$(A) \subset$ Range$(A + B)$ and Range$(B) \subset$ Range$(A + B)$. With [32, Theorem 2.1], we conclude that $A : B := A(A + B)^+ B$ is invariant under the choice of the pseudoinverse $(A + B)^+$.

LEMMA 4.4 (Parallel sum of matrices). *Let A, B be symmetric positive semidefinite, and define $A : B = A(A + B)^+ B$ as in Remark 4.3, where $(A + B)^+$ denotes a pseudoinverse with $(A+B)(A+B)^+(A+B) = (A+B)$ and $(A+B)^+(A+B)(A+B)^+ = (A+B)^+$. Then we have*

1. *$A : B \leq A$ and $A : B \leq B$ (spectral estimate).*
2. *$A : B$ is symmetric positive semidefinite.*
3. *Defining $D_A := (A + B)^+ A$ and $D_B := (A + B)^+ B$, we additionally have*

$$
D_A^T B D_A \leq A : B \qquad \textit{and} \qquad D_B^T A D_B \leq A : B. \tag{4.3}
$$

Proof. For the proof of 1. and 2., see [1]. Next, we provide a proof of 3. Since A and B are s.p.s.d., $D_B^T A D_B$ and $D_A^T B D_A$ are also s.p.s.d., and we obtain

$$
D_A^T B D_A + D_B^T A D_B = (A : B) D_A + (A : B) D_B = (A : B)(A + B)^+(A + B).
$$

Since A and B are s.p.s.d., $x^T(A + B)x = 0$ implies $x^T A x = -x^T B x = 0$. Thus, we have Ker$(A + B) =$ Ker$(A) \cap$ Ker(B). For any x we can write $x = x_R + x_K$ with $x_R \in$ Range$(A + B)^+$ and $x_K \in$ Ker$(A + B) =$ Ker$(A) \cap$ Ker(B). Using the fact that $(A + B)^+(A + B)$ is a projection onto Range$(A + B)^+$, we obtain

$$
\begin{aligned}
x^T D_A^T B D_A x + x^T D_B^T A D_B x &= x^T (A : B)(A + B)^+(A + B) x \\
&= x^T (A : B) x_R = x^T (A : B) x. \qquad \square
\end{aligned}
$$

Furthermore, we need some properties of projections on eigenspaces of generalized eigenvalue problems. The next lemma is a well-known result from linear algebra.

LEMMA 4.5. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive semidefinite and $B \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Consider the generalized eigenvalue problem*

$$Ax_k = \lambda_k B x_k \qquad \text{for } k = 1, \ldots, n.$$

Then the eigenvectors can be chosen to be B-orthogonal and such that $x_k^T B x_k = 1$. All eigenvalues are positive or zero.

The proof of the next lemma is based on arguments from classical spectral theory, thus it is omitted here. A related abstract lemma, also based on classical spectral theory, can be found in [36, Lemma 2.11].

LEMMA 4.6. *Let A, B be as in Lemma 4.5 and define $\Pi_m^B := \sum_{i=1}^m x_i x_i^T B$. Let the eigenvalues be sorted in an increasing order $0 = \lambda_1 \leq \ldots \leq \lambda_m < \lambda_{m+1} \leq \ldots \leq \lambda_n$. Then $x = \Pi_n^B x$ and*

$$|x - \Pi_m^B x|_B^2 = (x - \Pi_m^B x)^T B (x - \Pi_m^B x) \leq \lambda_{m+1}^{-1} x^T A x = \lambda_{m+1}^{-1} |x|_A^2.$$

Additionally, we have the stability of Π_m^B in the B-norm

$$|x - \Pi_m^B x|_B^2 \leq |x|_B^2\,.$$

4.4. First approach [7]. The first approach that we discuss was proposed in [7].

4.4.1. Notation. In the following, we define a scaling for the FETI-DP and BDDC method denoted as deluxe scaling, which was first introduced in [8]; for further applications, see [2, 5, 6, 20, 29, 33]. Note that this is not a scaling in the common sense since more than just a multiplication with a diagonal matrix is involved.

DEFINITION 4.7 (Deluxe scaling). *Let $\mathcal{E}_{ij} \subset \Gamma^{(i)}$ be an edge, and let the Schur complements $S_{E_{ij},0}^{(i)}$, $S_{E_{ij},0}^{(j)}$ be as in Definition 4.1. We define the following scaling matrices*

$$D_{E_{ij}}^{(l)} = \left(S_{E_{ij},0}^{(i)} + S_{E_{ij},0}^{(j)}\right)^{-1} S_{E_{ij},0}^{(l)}, \qquad l = i, j.$$

Let $R_{E_{ij}}^{(l)}$ be the restriction operator restricting the degrees of freedom of Lagrange multipliers on Γ to the degrees of freedom of Lagrange multipliers on the open edge $\mathcal{E}_{ij}$. Then, we define the subdomain (deluxe) scaling matrices by

$$D^{(i)} = \sum_{\mathcal{E}_{ij} \subset \Gamma^{(i)}} R_{E_{ij}}^{(i)T} D_{E_{ij}}^{(j)} R_{E_{ij}}^{(i)}.$$

Each pair of the scaling matrices $D_{E_{ij}}^{(i)}$, $D_{E_{ij}}^{(j)}$ satisfies property (4.2). The scaled jump operator B_D in the FETI-DP algorithm is then given by $B_D := [D^{(1)T} B^{(1)}, \ldots, D^{(N)T} B^{(N)}]$, where the transpose is necessary since the $D^{(i)}$ are not symmetric. Using Lemma 4.4, we obtain

$$D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} \leq S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)} \qquad \text{and} \qquad D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \leq S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}.$$

4.4.2. Generalized eigenvalue problem (first approach). We solve the eigenvalue problem

$$S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)} x_m = \mu_m S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)} x_m, \tag{4.4}$$

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where $\mu_m \leq \text{TOL}$, $m = 1, \ldots, k$, and enforce the constraints

$$x_m^T (S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)})(w_i - w_j)_{|\mathcal{E}_{ij}} = 0,$$

e.g., as described in Section 3.

LEMMA 4.8. *We define* $\Pi_k := \sum_{m=1}^k x_m x_m^T S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}$ *using the eigenvectors* x_m *of the generalized eigenvalue problem* (4.4)*. Then we have* $\Pi_k (w_i - w_j)_{|\mathcal{E}_{ij}} = 0$*, and the following inequality holds:*

$$\begin{aligned}(w_i - w_j)_{|\mathcal{E}_{ij}}^T &\left(D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}} \\ &\leq 2(\mu_{k+1})^{-1} (w_{i|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}} + w_{j|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}}).\end{aligned}$$

Proof. The property $\Pi_k (w_i - w_j)_{|\mathcal{E}_{ij}} = 0$ follows directly. We have

$$\begin{aligned}
(w_i - w_j)_{|\mathcal{E}_{ij}}^T & D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} (w_i - w_j)_{|\mathcal{E}_{ij}} + (w_i - w_j)_{|\mathcal{E}_{ij}}^T D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&= (w_i - w_j)_{|\mathcal{E}_{ij}}^T S_{E_{ij},0}^{(j)} (S_{E_{ij},0}^{(i)} + S_{E_{ij},0}^{(j)})^{-1} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&\quad + (w_i - w_j)_{|\mathcal{E}_{ij}}^T S_{E_{ij},0}^{(i)} (S_{E_{ij},0}^{(i)} + S_{E_{ij},0}^{(j)})^{-1} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&= (w_i - w_j)_{|\mathcal{E}_{ij}}^T ((S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}) D_{E_{ij}}^{(j)} + (S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}) D_{E_{ij}}^{(i)}) (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&= (w_i - w_j)_{|\mathcal{E}_{ij}}^T (S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}) (w_i - w_j)_{|\mathcal{E}_{ij}} \qquad (4.5) \\
&\leq 2(\mu_{k+1})^{-1} \left(w_{i|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)} w_{i|\mathcal{E}_{ij}} + w_{j|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}} \right) \\
&\leq 2(\mu_{k+1})^{-1} \left(w_{i|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}} + w_{j|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}} \right).
\end{aligned}$$

For the last two estimates notice that $w_{i|\mathcal{E}_{ij}} - w_{j|\mathcal{E}_{ij}} = w_{i|\mathcal{E}_{ij}} - \Pi_k w_{i|\mathcal{E}_{ij}} - (w_{j|\mathcal{E}_{ij}} - \Pi_k w_{j|\mathcal{E}_{ij}})$, and apply Lemma 4.6 with $A = S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)}$ and $B = S_{E_{ij},0}^{(i)} : S_{E_{ij},0}^{(j)}$. Using the first property of Lemma 4.4, we obtain the desired bound. ∎

REMARK 4.9. Up to equation (4.5), no generalized eigenvalue problem is used but only deluxe scaling. Since the term in (4.5) is bounded by

$$2 \left(w_{i|\mathcal{E}_{ij}}^T S_{E_{ij},0}^{(i)} w_{i|\mathcal{E}_{ij}} + w_{j|\mathcal{E}_{ij}}^T S_{E_{ij},0}^{(j)} w_{j|\mathcal{E}_{ij}} \right),$$

it replaces a classical extension theorem. In [27], the analysis of FETI-DP methods in two dimensions has been extended to uniform domains, which are a subset of John domains. Since all tools were provided for John domains with the exception of the extension theorem, which requires uniform domains, by using deluxe scaling, the analysis carries over to the broader class of John domains.

4.5. Second approach [21]. In this section, we describe a variant of the first approach that allows different kinds of scalings. In the case of standard deluxe scaling, this algorithm is the same as the algorithm in [7]; cf. Section 4.4. A short description of this variant has already been presented in the proceedings article [21].

4.5.1. Notation. We use the notation from Section 4.1.

4.5.2. Generalized eigenvalue problem (second approach). We solve the eigenvalue problem

$$S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)} x_m = \mu_m \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) x_m. \qquad (4.6)$$

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We select the x_m, $m = 1, \dots, k$, for which $\mu_m \leq$ TOL and enforce the constraints

$$x_m^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}} = 0,$$

e.g., as described in Section 3. Note that (4.4) and (4.6) are the same in the case of deluxe scaling. Analogously to Lemma 4.8, we obtain the following bound.

LEMMA 4.10. *Let $\Pi_k := \sum_{m=1}^k x_m x_m^T (D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)})$ using the eigenvectors x_m of the generalized eigenvalue problem (4.6). Then we have $\Pi_k (w_i - w_j)_{|\mathcal{E}_{ij}} = 0$, and the following inequality holds:*

$$\begin{aligned} &(w_i - w_j)_{|\mathcal{E}_{ij}}^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}} \\ &\quad \leq 2(\mu_{k+1})^{-1} (w_{i|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}} + w_{j|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}}), \end{aligned}$$

where $D_{E_{ij}}^{(l)}$, $l = i, j$, are arbitrary scaling matrices that provide a partition of unity, i.e., satisfy (4.2).

Proof. Notice that $w_{i|\mathcal{E}_{ij}} - w_{j|\mathcal{E}_{ij}} = w_{i|\mathcal{E}_{ij}} - \Pi_k w_{i|\mathcal{E}_{ij}} - (w_{j|\mathcal{E}_{ij}} - \Pi_k w_{j|\mathcal{E}_{ij}})$, and apply Lemma 4.6 with $A = S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)}$ and $B = D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)}$. With (4.3), we obtain the desired bound. ∎

4.6. Economic variant of the algorithm. In this section, we introduce a new, more economic variant, solving eigenvalue problems on slabs. Using such a variant for deluxe scaling but without choosing the coarse space adaptively was first introduced and numerically tested in [9]; see Remark 4.14. Let us note that with respect to the eigenvalue problems on slabs, this variant is new. Let us first give the definition of an η-patch; see, e.g., also [38, Lemma 3.10], [23, Definition 6.1], and [34, Definition 2.5 and 2.6].

DEFINITION 4.11. *An η-patch $\omega \subset \Omega$ denotes an open set which can be represented as a union of shape-regular finite elements of diameter $\mathcal{O}(h)$ and which has diam$(\omega) = \mathcal{O}(\eta)$ and a measure of $\mathcal{O}(\eta^2)$.*

The next definition was introduced in 3D in [16]; see also [19, 23].

DEFINITION 4.12. *Let $\mathcal{E}_{ij} \subset \partial\Omega_i$ be an edge. Then a slab $\widetilde{\Omega}_{i\eta}$ is a subset of Ω_i of width η with $\mathcal{E}_{ij} \subset \partial\widetilde{\Omega}_{i\eta}$ which can be represented as the union of η-patches ω_{ik}, $k = 1, \dots, n$, such that $(\partial\omega_{ik} \cap \mathcal{E}_{ij})^\circ \neq \emptyset$, $k = 1, \dots, n$.*

4.6.1. Notation. In addition to $|v|_{E_l}$, cf. (4.1), we define $|v|_{E_l,\eta}^2 := a_{l,\eta}(v, v)$, where $a_{l,\eta}(v, v)$ is the same bilinear form as $a_l(v, v)$ but with an integral over the slab $\widetilde{\Omega}_{l\eta}$. Let $K_\eta^{\mathcal{E},(l)}$ be the locally assembled stiffness matrix of the slab of width η corresponding to an edge $\mathcal{E}$ in the subdomain Ω_l. Here, we use homogeneous Neumann boundary conditions on the part of the boundary of the slab which intersects the interior of Ω_l.

DEFINITION 4.13. *Let $\mathcal{E} \subset \Gamma^{(l)} \cap \partial\widetilde{\Omega}_{l,\eta}$ be an edge and $\mathcal{E}^c \subset \Gamma^{(l)} \cap \partial\widetilde{\Omega}_{l,\eta}$ be the complement of $\mathcal{E}$ with respect to $\Gamma^{(l)} \cap \partial\widetilde{\Omega}_{l,\eta}$. Let $K_\eta^{\mathcal{E},(l)}$ be partitioned as follows:*

$$K_\eta^{\mathcal{E},(l)} = \begin{bmatrix} K_{\eta,II}^{\mathcal{E},(l)} & K_{\eta,\Gamma I}^{\mathcal{E},(l)T} \\ K_{\eta,\Gamma I}^{\mathcal{E},(l)} & K_{\eta,\Gamma\Gamma}^{\mathcal{E},(l)} \end{bmatrix},$$

where the index Γ corresponds to the degrees of freedom on $\Gamma^{(l)} \cap \partial\widetilde{\Omega}_{l,\eta}$ and the index I corresponds to the remaining degrees of freedom in $\overline{\widetilde{\Omega}}_{l,\eta}$. Define the extension operator

$$\mathcal{H}_\eta^{(l)} v = \begin{bmatrix} v_{|\Gamma^{(l)} \cap \partial\widetilde{\Omega}_{l,\eta}} \\ -K_{\eta,II}^{\mathcal{E},(l)-1} K_{\eta,\Gamma I}^{\mathcal{E},(l)T} v_{|\Gamma^{(l)} \cap \partial\widetilde{\Omega}_{l,\eta}} \end{bmatrix}.$$

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Let

$$S_{E,\eta}^{(l)} = S_{\mathcal{E}\mathcal{E},\eta}^{\mathcal{E},(l)} - S_{\mathcal{E}^c\mathcal{E},\eta}^{\mathcal{E},(l)T} S_{\mathcal{E}^c\mathcal{E}^c,\eta}^{\mathcal{E},(l)-1} S_{\mathcal{E}^c\mathcal{E},\eta}^{\mathcal{E},(l)}$$

be the Schur complement of $K_\eta^{\mathcal{E},(l)}$ after elimination of all degrees of freedom except those on the edge. With the discrete energy-minimal extension operator $\mathcal{H}_\eta^{(l)}$ from $\Gamma^{(l)} \cap \partial\widetilde{\Omega}_{l\eta}$ to the interior, we have $|\mathcal{H}_\eta^{(l)}\mathcal{H}_\mathcal{E}^{(l)} v|^2_{E_{l,\eta}} \geq v_E^T S_{E,\eta}^{(l)} v_E$. Let the local finite element space be partitioned into variables on the edge $\mathcal{E}$ and the remaining variables denoted by $\mathcal{E}^*$. Then the local stiffness matrices $K^{(l)}$ can be partitioned accordingly, and we obtain

$$K^{(l)} = \begin{bmatrix} K_{\mathcal{E}\mathcal{E}}^{(l)} & K_{\mathcal{E}^*\mathcal{E}}^{(l)T} \\ K_{\mathcal{E}^*\mathcal{E}}^{(l)} & K_{\mathcal{E}^*\mathcal{E}^*}^{(l)} \end{bmatrix}.$$

Thus, by removing all columns and rows related to the degrees of freedom outside the closure of the slab and those on $(\partial\widetilde{\Omega}_{l\eta} \cap \Gamma^{(l)}) \setminus \mathcal{E}$, we obtain a matrix of the form

$$\begin{bmatrix} K_{\mathcal{E}\mathcal{E}}^{(l)} & K_{I_\eta\mathcal{E}}^{(l)T} \\ K_{I_\eta\mathcal{E}}^{(l)} & K_{I_\eta I_\eta}^{(l)} \end{bmatrix}.$$

Here, the index I_η relates to the degrees of freedom on the closure of the slab except those on $\partial\widetilde{\Omega}_{l\eta} \cap \Gamma^{(l)}$. We define another Schur complement by

$$S_{E,0,\eta}^{(l)} = K_{\mathcal{E}\mathcal{E}}^{(l)} - K_{I_\eta\mathcal{E}}^{(l)T} K_{I_\eta I_\eta}^{(l)-1} K_{I_\eta\mathcal{E}}^{(l)}.$$

We define an extension operator $\mathcal{H}_{\eta,0}^{(l)}$ from the local interface $\partial\widetilde{\Omega}_{l\eta} \cap \Gamma^{(l)}$ of a subdomain Ω_l to the interior by

$$\mathcal{H}_{\eta,0}^{(l)} v = \begin{cases} v, & \text{on } \partial\Omega_l \cap \partial\widetilde{\Omega}_{l\eta}, \\ \text{minimal energy extension}, & \text{in } \overline{\widetilde{\Omega}}_{l,\eta} \cap \Omega_l, \\ 0, & \text{elsewhere.} \end{cases}$$

Then we have $v^T S_{E,0,\eta}^{(l)} v = |\mathcal{H}_{\eta,0}^{(l)} I^h(\theta_\mathcal{E} v)|^2_{E_l}$.

REMARK 4.14 (economic deluxe scaling). In [9], the authors proposed an economic variant of deluxe scaling by replacing the Schur complements $S_{E,0}^{(l)}$, $l = i, j$, by $S_{E,0,\eta}^{(l)}$ with $\eta = h$. As in [9], we will denote this variant by e-deluxe scaling.

4.6.2. Generalized eigenvalue problem (economic version). We solve the eigenvalue problem

$$S_{E_{ij},\eta}^{(i)} : S_{E_{ij},\eta}^{(j)} x_m = \mu_m \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0,\eta}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0,\eta}^{(j)} D_{E_{ij}}^{(i)} \right) x_m, \tag{4.7}$$

where $\mu_m \leq \text{TOL}$, $m = 1, \ldots, k$, and where $D_{E_{ij}}^{(l)} = \left(S_{E_{ij},0,\eta}^{(i)} + S_{E_{ij},0,\eta}^{(j)} \right)^{-1} S_{E_{ij},0,\eta}^{(l)}$ for $l = i, j$. We then enforce the constraints

$$x_m^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0,\eta}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0,\eta}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}} = 0,$$

as in Section 3.

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LEMMA 4.15. *We define*

$$\Pi_k := \sum_{m=1}^{k} x_k x_k^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0,\eta}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0,\eta}^{(j)} D_{E_{ij}}^{(i)} \right)$$

using the eigenvectors x_m of the generalized eigenvalue problem (4.7). Then we have that $\Pi_k(w_i - w_j)_{|\mathcal{E}_{ij}} = 0$, and the following inequality holds:

$$\begin{aligned}
&(w_i - w_j)_{|\mathcal{E}_{ij}}^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&\quad \leq 2(\mu_{k+1})^{-1} \left(w_{i|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}} + w_{j|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}} \right).
\end{aligned}$$

Proof. Since the discrete harmonic extension $|\mathcal{H}^{(l)} I^h(\theta_{E_{ij}} v)|_{E_l}^2 = v^T S_{E_{ij},0}^{(l)} v$ has the smallest energy, we obtain

$$\begin{aligned}
&(w_i - w_j)_{|\mathcal{E}_{ij}}^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&\quad \leq (w_i - w_j)_{|\mathcal{E}_{ij}}^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0,\eta}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0,\eta}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&\quad \leq (\mu_{k+1})^{-1} (w_i - w_j)_{|\mathcal{E}_{ij}}^T S_{E_{ij},\eta}^{(i)} : S_{E_{ij},\eta}^{(j)} (w_i - w_j)_{|\mathcal{E}_{ij}} \\
&\quad \leq 2(\mu_{k+1})^{-1} \left(|w_{i|\mathcal{E}_{ij}}|_{S_{E_{ij},\eta}^{(i)}}^2 + |w_{j|\mathcal{E}_{ij}}|_{S_{E_{ij},\eta}^{(j)}}^2 \right) \\
&\quad \leq 2\,(\mu_{k+1})^{-1} \left(|\mathcal{H}_\eta^{(i)} \mathcal{H}_{\mathcal{E}_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}}|_{E_{i,\eta}}^2 + |\mathcal{H}_\eta^{(j)} \mathcal{H}_{\mathcal{E}_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}}|_{E_{j,\eta}}^2 \right) \\
&\quad \leq 2\,(\mu_{k+1})^{-1} \left(|\mathcal{H}^{(i)} \mathcal{H}_{\mathcal{E}_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}}|_{E_{i,\eta}}^2 + |\mathcal{H}^{(j)} \mathcal{H}_{\mathcal{E}_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}}|_{E_{j,\eta}}^2 \right) \\
&\quad \leq 2\,(\mu_{k+1})^{-1} \left(|\mathcal{H}^{(i)} \mathcal{H}_{\mathcal{E}_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}}|_{E_i}^2 + |\mathcal{H}^{(j)} \mathcal{H}_{\mathcal{E}_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}}|_{E_j}^2 \right) \\
&\quad = 2\,(\mu_{k+1})^{-1} \left(w_{i|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(i)} w_{i|\mathcal{E}_{ij}} + w_{j|\mathcal{E}_{ij}}^T S_{E_{ij}}^{(j)} w_{j|\mathcal{E}_{ij}} \right). \qquad \square
\end{aligned}$$

4.7. Condition number estimate for the first coarse space. Based on the estimates for P_D for the first coarse space given in the previous sections, we now present our condition number estimate.

THEOREM 4.16. *Let $N_\mathcal{E}$ be the maximum number of edges of a subdomain. The condition number $\kappa(\widehat{M}^{-1}F)$ of the FETI-DP algorithm with adaptive constraints defined as in Sections 4.4.2, 4.5.2, or 4.6.2 either enforced by the projector preconditioner $\widehat{M}^{-1} = M_{PP}^{-1}$ or the balancing preconditioner $\widehat{M}^{-1} = M_{BP}^{-1}$ satisfies*

$$\kappa(\widehat{M}^{-1}F) \leq 2N_\mathcal{E}^2\, \mathrm{TOL}^{-1}.$$

Proof. For $w \in \widetilde{W}$, we have the estimate

$$\begin{aligned}
|P_D w|_{\widetilde{S}}^2 &= \sum_{i=1}^{N} |R^{(i)} P_D w|_{S^{(i)}}^2 \leq N_\mathcal{E} \sum_{i=1}^{N} \sum_{j \in N_i} |I^h(\theta_{\mathcal{E}_{ij}} D^{(i)}(w_i - w_j))|_{S^{(i)}}^2 \\
&\leq N_\mathcal{E} \sum_{\mathcal{E}_{ij} \subset \Gamma} (w_i - w_j)_{|\mathcal{E}_{ij}}^T \left(D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(i)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(i)} \right) (w_i - w_j)_{|\mathcal{E}_{ij}}.
\end{aligned}$$

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Using Lemma 4.8 for the coarse space in Section 4.4.2, Lemma 4.10 for the coarse space in Section 4.5.2, and Lemma 4.15 for the coarse space in Section 4.6.2, and using that $\mu_{k+1} \geq \text{TOL}$, we obtain the estimate

$$\begin{aligned}
|P_D w|^2_{\widetilde{S}} &\leq 2N_{\mathcal{E}} \sum_{\mathcal{E}_{ij} \subset \Gamma} \text{TOL}^{-1} \left(w^T_{i|\mathcal{E}_{ij}} S^{(i)}_{E_{ij}} w_{i|\mathcal{E}_{ij}} + w^T_{j|\mathcal{E}_{ij}} S^{(j)}_{E_{ij}} w_{j|\mathcal{E}_{ij}} \right) \\
&\leq 2N_{\mathcal{E}} \sum_{\mathcal{E}_{ij} \subset \Gamma} \text{TOL}^{-1} \left(w_i^T S^{(i)} w_i + w_j^T S^{(j)} w_j \right) \\
&\leq 2N^2_{\mathcal{E}} \, \text{TOL}^{-1} \sum_{i=1}^{N} |R^{(i)} w|^2_{S^{(i)}} \leq 2N^2_{\mathcal{E}} \, \text{TOL}^{-1} \, |w|^2_{\widetilde{S}}. \qquad \square
\end{aligned}$$

5. Second coarse space. We now discuss an approach which has been successfully used in FETI-DP and BDDC for some time [22, 30, 31, 35]. Let us note that this approach is also based on eigenvalue estimates related to the P_D-operator. In the following, we give a brief description of the algorithm in [30] for the convenience of the reader. In Section 5.1, we introduce the relevant notation and in Section 5.2 the specific eigenvalue problem. In Section 5.3, we also give an estimate of the condition number in the case of a two-dimensional problem where all the vertex variables are primal in the initial coarse space.

5.1. Notation. Let $\mathcal{E}_{ij}$ denote the edge between the subdomains Ω_i and Ω_j, and let $B_{E_{ij}} = [B^{(i)}_{E_{ij}} \, B^{(j)}_{E_{ij}}]$ be the submatrix of $[B^{(i)} B^{(j)}]$ with rows consisting of exactly one 1 and one -1 and zeros otherwise. Let $B_{D,E_{ij}} = [B^{(i)}_{D,E_{ij}} \, B^{(j)}_{D,E_{ij}}]$ be obtained by taking the same rows of $[B^{(i)}_D \, B^{(j)}_D] := \begin{bmatrix} D^{(i)T} B^{(i)} & D^{(j)T} B^{(j)} \end{bmatrix}$. Let $S_{ij} = \begin{bmatrix} S^{(i)} & \\ & S^{(j)} \end{bmatrix}$ and $P_{D_{ij}} = B^T_{D,E_{ij}} B_{E_{ij}}$.

The null space ker (ε) is the space of rigid body motions. In the case of two-dimensional linear elasticity, the rigid body modes are given by

$$r^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad r^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad r^{(3)} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \qquad \text{for } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Omega.$$

Here, $r^{(1)}$ and $r^{(2)}$ are translations, and $r^{(3)}$ is a rotation.

By $\widetilde{W}_{ij}$, we denote the space of functions in $W_i \times W_j$ that are continuous in the primal variables that the subdomains Ω_i and Ω_j have in common and by Π_{ij} the ℓ_2-orthogonal projection from $W_i \times W_j$ onto $\widetilde{W}_{ij}$. Another orthogonal projection from $W_i \times W_j$ to Range$(\Pi_{ij} S_{ij} \Pi_{ij} + \sigma(I - \Pi_{ij}))$ is denoted by $\overline{\Pi}_{ij}$, where σ is a positive constant, e.g., the maximum of the entries of the diagonal of S_{ij}. Let $v \in$ Ker$(\Pi_{ij} S_{ij} \Pi_{ij} + \sigma(I - \Pi_{ij}))$ with $v = w + z$ and $w \in$ Range Π_{ij} as well as $z \in$ Ker Π_{ij}. Then, we obtain

$$0 = v^T (\Pi_{ij} S_{ij} \Pi_{ij} + \sigma(I - \Pi_{ij})) v = w^T S_{ij} w + \sigma z^T z.$$

Thus, $z = 0$, and we find $v \in$ Range(Π_{ij}) since S_{ij} is positive semidefinite and $\sigma > 0$. For $v \in$ Ker$(S_{ij}) \cap$ Range(Π_{ij}), we get $v \in$ Ker$(\Pi_{ij} S_{ij} \Pi_{ij} + \sigma(I - \Pi_{ij}))$. Thus, we have

$$\text{Range}(\Pi_{ij}) \cap \text{Ker}(S_{ij}) = \text{Ker}(\Pi_{ij} S_{ij} \Pi_{ij} + \sigma(I - \Pi_{ij})).$$

Therefore, $(I - \overline{\Pi}_{ij})$ is an orthogonal projection onto Ker$(S_{ij}) \cap$ Range(Π_{ij}), e.g., in case of linear elasticity, the space of rigid body modes as functions in $W_i \times W_j$ that are continuous; cf. also [30]. This implies $\Pi_{ij}(I - \overline{\Pi}_{ij}) w_{ij} = (I - \overline{\Pi}_{ij}) w_{ij}$. Hence, $P_{D_{ij}} \Pi_{ij} (I - \overline{\Pi}_{ij}) w_{ij} = 0$ and thus

$$P_{D_{ij}} \Pi_{ij} \overline{\Pi}_{ij} w_{ij} = P_{D_{ij}} \Pi_{ij} w_{ij}. \tag{5.1}$$

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5.2. Generalized eigenvalue problem. We solve the eigenvalue problem

$$\begin{aligned}(5.2)\qquad &\overline{\Pi}_{ij}\Pi_{ij}P_{D_{ij}}^T S_{ij}P_{D_{ij}}\Pi_{ij}\overline{\Pi}_{ij}w_{ij,m}\\ &\qquad = \mu_{ij,m}(\overline{\Pi}_{ij}(\Pi_{ij}S_{ij}\Pi_{ij}+\sigma(I-\Pi_{ij}))\overline{\Pi}_{ij}+\sigma(I-\overline{\Pi}_{ij}))w_{ij,m},\end{aligned}$$

where $\mu_{ij,m}\geq$TOL, $m=k,\dots,n$. We then enforce the constraints

$$w_{ij,m}^T P_{D_{ij}}^T S_{ij}P_{D_{ij}}w_{ij}=0.$$

From (5.2), we obtain by using (5.1)

$$\begin{aligned}(5.3)\qquad &\Pi_{ij}P_{D_{ij}}^T S_{ij}P_{D_{ij}}\Pi_{ij}w_{ij,m}\\ &\qquad = \mu_{ij,m}(\overline{\Pi}_{ij}(\Pi_{ij}S_{ij}\Pi_{ij}+\sigma(I-\Pi_{ij}))\overline{\Pi}_{ij}+\sigma(I-\overline{\Pi}_{ij}))w_{ij,m}.\end{aligned}$$

From (5.3) using [30, Theorem 9] and [30, Theorem 11], we obtain the estimate

$$(5.4)\qquad w_{ij}^T\Pi_{ij}P_{D_{ij}}^T S_{ij}P_{D_{ij}}\Pi_{ij}w_{ij}\leq \mu_{ij,k-1}w_{ij}^T\Pi_{ij}S_{ij}\Pi_{ij}w_{ij}$$

for all w_{ij} in $W_i\times W_j$ with $w_{ij,m}^T P_{D_{ij}}^T S_{ij}P_{D_{ij}}w_{ij}=0$, $\mu_{ij,m}\geq$ TOL, $m=k,\dots,n$.

5.3. Condition number estimate of the coarse space in 2D.

THEOREM 5.1. *Let $N_{\mathcal{E}}$ be the maximum number of edges of a subdomain. The condition number $\kappa(\widehat{M}^{-1}F)$ of the FETI-DP algorithm with adaptive constraints defined in Section 5.2 either enforced by the projector preconditioner $\widehat{M}^{-1}=M_{PP}^{-1}$ or by the balancing preconditioner $\widehat{M}^{-1}=M_{BP}^{-1}$ satisfies*

$$\kappa(\widehat{M}^{-1}F)\leq N_{\mathcal{E}}^2\text{TOL}.$$

Proof. The local jump operator in the eigenvalue problems is

$$P_{D_{ij}}=\begin{bmatrix}B_{D,E_{ij}}^{(i)T}B_{E_{ij}}^{(i)} & B_{D,E_{ij}}^{(i)T}B_{E_{ij}}^{(j)}\\ B_{D,E_{ij}}^{(j)T}B_{E_{ij}}^{(i)} & B_{D,E_{ij}}^{(j)T}B_{E_{ij}}^{(j)}\end{bmatrix}.$$

Application to a vector yields

$$P_{D_{ij}}\begin{bmatrix}R^{(i)}w\\ R^{(j)}w\end{bmatrix}=\begin{bmatrix}I^h(\theta_{E_{ij}}D^{(i)}(w_i-w_j))\\ I^h(\theta_{E_{ij}}D^{(j)}(w_j-w_i))\end{bmatrix}.$$

For $w\in\widetilde{W}$, we have $\begin{bmatrix}R^{(i)}w\\ R^{(j)}w\end{bmatrix}\in\widetilde{W}_{ij}$, and therefore $\Pi_{ij}\begin{bmatrix}R^{(i)}w\\ R^{(j)}w\end{bmatrix}=\begin{bmatrix}R^{(i)}w\\ R^{(j)}w\end{bmatrix}$. All vertices are assumed to be primal. In the following, we use the notation $w_i=R^{(i)}w$ and $w_j=R^{(j)}w$. As before, we assume that w_i and w_j are first restricted to the edge $\mathcal{E}_{ij}$ and that the difference is then extended by zero to the rest of the interface Γ. Thus, for $w \in \widetilde{W}$, we obtain

$$\begin{aligned}
|P_D w|_{\widetilde{S}}^2 &= \sum_{i=1}^{N} |R^{(i)} P_D w|_{S^{(i)}}^2 \\
&\leq N_{\mathcal{E}} \sum_{i=1}^{N} \sum_{j \in N_i} |I^h(\theta_{\mathcal{E}_{ij}} D^{(i)}(w_i - w_j))|_{S^{(i)}}^2 \\
&= N_{\mathcal{E}} \sum_{\mathcal{E}_{ij} \subset \Gamma} |I^h(\theta_{\mathcal{E}_{ij}} D^{(i)}(w_i - w_j))|_{S^{(i)}}^2 + |I^h(\theta_{\mathcal{E}_{ij}} D^{(j)}(w_j - w_i))|_{S^{(j)}}^2 \\
&= N_{\mathcal{E}} \sum_{\mathcal{E}_{ij} \subset \Gamma} \begin{bmatrix} w_i \\ w_j \end{bmatrix}^T \Pi_{ij} P_{D_{ij}}^T \begin{bmatrix} S^{(i)} & \\ & S^{(j)} \end{bmatrix} P_{D_{ij}} \Pi_{ij} \begin{bmatrix} w_i \\ w_j \end{bmatrix} \\
&\overset{(5.4)}{\leq} N_{\mathcal{E}} \sum_{\mathcal{E}_{ij} \subset \Gamma} \mu_{ij,k-1} \begin{bmatrix} w_i \\ w_j \end{bmatrix}^T \Pi_{ij} \begin{bmatrix} S^{(i)} & \\ & S^{(j)} \end{bmatrix} \Pi_{ij} \begin{bmatrix} w_i \\ w_j \end{bmatrix} \\
&\leq N_{\mathcal{E}} \text{ TOL} \sum_{\mathcal{E}_{ij} \subset \Gamma} |w_i|_{S^{(i)}}^2 + |w_j|_{S^{(j)}}^2 \leq N_{\mathcal{E}}^2 \text{ TOL} \sum_{i=1}^{N} |R^{(i)} w|_{S^{(i)}}^2 = N_{\mathcal{E}}^2 \text{ TOL } |w|_{\widetilde{S}}^2.
\end{aligned}$$

∎

6. Third coarse space [23]. We now discuss our approach from [23], which is not based on the localization of the P_D-estimate, and also introduce some improvements. We denote the weighted mass matrix on the closure of the edge $\mathcal{E}_{ij}$ of Ω_l, $l = i, j$, by

$$m_{E_{ij}}^{(l)}(u, v) := \int_{\mathcal{E}_{ij}} \rho_l \, u \cdot v \, ds, \qquad \text{for } l = i, j,$$

and the corresponding matrix by $M_{E_{ij}}^{(l)}$. We introduce two generalized eigenvalue problems. The first is related to a replacement of a generalized Poincaré inequality on the edge in cases where the Poincaré inequality depends on the jumps in the diffusion coefficient, while the second is related to an extension theorem. For a detailed description of the algorithm including a proof of the condition number estimate, see [23].

We introduce the matrix $S_{E_{ij},c}^{(l)}$ which is obtained by eliminating from $S^{(l)}$ all degrees of freedom of $\Gamma^{(l)} \setminus \overline{\mathcal{E}_{ij}}$ where $\overline{\mathcal{E}_{ij}}$ is the closure of $\mathcal{E}_{ij}$. Thus, $S_{E_{ij},c}^{(l)}$ is a Schur complement of the Schur complement $S^{(l)}$.

6.1. First generalized eigenvalue problem. We solve

$$S_{E_{ij},c}^{(l)} x_m^{ij,l} = \mu_m^{ij,l} M_{E_{ij}}^{(l)} x_m^{ij,l}, \qquad \text{for } l = i, j, \tag{6.1}$$

where $\mu_m^{ij,l} \leq \text{TOL}_\mu$, $m = 1, \ldots, k$. We then build the vectors $M_{E_{ij}}^{(l)} x_m^{(l)}$ and discard the entries not associated with dual variables by removing those entries associated with the primal vertices at the endpoints. We denote the resulting vectors by $u_m^{(l)}$ and enforce the constraints $u_m^{(l)T}(w_i - w_j) = 0$, e.g., as described in Section 3.

6.2. Second generalized eigenvalue problem. Following [23], we introduce a second eigenvalue problem to ensure that we can extend a function from one subdomain to another without significantly increasing the energy. Depending on the kernels of the local Schur

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complements $S^{(l)}_{E_{ij},c}$, a generalized eigenvalue problem of the form

$$S^{(i)}_{E_{ij},c} x^{ij}_k = \nu^{ij}_k S^{(j)}_{E_{ij},c} x^{ij}_k, \tag{6.2}$$

can have arbitrary eigenvalues, e.g., if $x^{ij}_k \in \mathrm{Ker}(S^{(i)}_{E_{ij},c}) \cap \mathrm{Ker}(S^{(j)}_{E_{ij},c})$. This means that both matrices are singular and $\mathrm{Ker}(S^{(i)}_{E_{ij},c}) \cap \mathrm{Ker}(S^{(j)}_{E_{ij},c})$ is not trivial. We make use of the ℓ_2-orthogonal projection $\overline{\Pi}$, where $I - \overline{\Pi}$ is the orthogonal projection from $V^h(\mathcal{E}_{ij})$ to $\mathrm{Ker}(S^{(i)}_{E_{ij},c}) \cap \mathrm{Ker}(S^{(j)}_{E_{ij},c})$ and solve the generalized eigenvalue problem

$$\overline{\Pi}\, S^{(i)}_{E_{ij},c} \overline{\Pi}\, x^{ij}_m = \nu^{ij}_m (\overline{\Pi}\, S^{(j)}_{E_{ij},c} \overline{\Pi} + \sigma(I - \overline{\Pi}))x^{ij}_m,$$

where $\nu^{ij}_m \leq \mathrm{TOL}_\nu$, $m = 1, \ldots, k$, and $\sigma > 0$ is an arbitrary positive constant. In our computations, we use $\sigma = \max(\mathrm{diag}\,(S^{(j)}_{E_{ij},c}))$. Analogously to the first eigenvalue problem, we build $(\overline{\Pi}\, S^{(j)}_{E_{ij},c} \overline{\Pi} + \sigma(I - \overline{\Pi}))x^{ij}_m$ and discard the entries not associated with the dual variables. Denoting the resulting constraint vectors by r^{ij}_m, we enforce $r^{ij\,T}_m (w_i - w_j) = 0$.

6.3. Economic variant. Analogously to Section 4.6, we present an economic version in which the modified Schur complements, which are cheaper to compute, are used. This variant is new and was not considered in [23]. All Schur complements are only computed on slabs related to the edges they are associated with. We define $S^{(l)}_{E,c,\eta} = S^{\overline{\mathcal{E}},(l)}_{\mathcal{E}\mathcal{E},c,\eta} - S^{\overline{\mathcal{E}},(l)T}_{\overline{\mathcal{E}}^c\overline{\mathcal{E}},c,\eta} S^{\overline{\mathcal{E}},(l)-1}_{\overline{\mathcal{E}}^c\overline{\mathcal{E}}^c,c,\eta} S^{\overline{\mathcal{E}},(l)}_{\overline{\mathcal{E}}^c\overline{\mathcal{E}},c,\eta}$ as the Schur complement, which is obtained analogously to $S^{(l)}_{E,\eta}$ in Section 4.6.1 with the exception that the matrix $S^{(l)}_{E,c,\eta}$ is built with respect to the degrees of freedom on the closed edge $\overline{\mathcal{E}}$ and its complement in $\widetilde{\Omega}_{l\eta}$, respectively. The eigenvalue problems and constraints in Sections 6.1 and 6.2 are then computed with $S^{(l)}_{E,c,\eta}$ instead of $S^{(l)}_{E,c}$. The proof of the condition number in [23] can be extended to the economic case with the same arguments as in Lemma 4.15.

6.4. Extension with a modification of deluxe scaling. In the following, we construct a scaling for the extension which can be used as an alternative to the second eigenvalue problem (6.2). Thus, using this new scaling, we only need the eigenvalue problem (6.1).

DEFINITION 6.1 (Extension scaling). *For a pair of subdomains Ω_i and Ω_j sharing an edge $\mathcal{E}_{ij}$, let $D^{(i)}_{E_{ij},c}$ and $D^{(j)}_{E_{ij},c}$ be defined by*

$$D^{(i)}_{E_{ij},c} = (S^{(i)}_{E_{ij},c} + S^{(j)}_{E_{ij},c})^+ S^{(i)}_{E_{ij},c} + A_{ij},$$
$$D^{(j)}_{E_{ij},c} = (S^{(i)}_{E_{ij},c} + S^{(j)}_{E_{ij},c})^+ S^{(j)}_{E_{ij},c} + A_{ij},$$

where A_{ij} is defined by

$$A_{ij} = \frac{1}{2}\left(I - (S^{(i)}_{E_{ij},c} + S^{(j)}_{E_{ij},c})^+ (S^{(i)}_{E_{ij},c} + S^{(j)}_{E_{ij},c})\right).$$

By removing those columns and rows associated with the primal vertices at the endpoints of $\mathcal{E}_{ij}$ from the matrices $D^{(l)}_{E_{ij},c}$, $l = i, j$, we obtain the matrices $D^{(l)}_{E_{ij}}$. We define the subdomain scaling matrices by

$$D^{(i)} = \sum_{\mathcal{E}_{ij} \subset \Gamma^{(i)}} R^{(i)T}_{E_{ij}} D^{(j)}_{E_{ij}} R^{(i)}_{E_{ij}}.$$

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As in Section 4.4, the scaled jump operator B_D in the FETI-DP algorithm is then given by $B_D := [D^{(1)T}B^{(1)},\ldots,D^{(N)T}B^{(N)}]$, where the transpose is necessary since the matrices $D^{(i)}$ are not symmetric.

When using the scaling in Definition 6.1, we build the vectors $D^{(j)T}_{E_{ij},c}M^{(i)}_{E_{ij}}x^{(i)}_k$ and $D^{(i)T}_{E_{ij},c}M^{(j)}_{E_{ij}}x^{(j)}_k$ instead of $M^{(l)}_{E_{ij}}x^{(l)}_k$, $l=i,j$, where $x^{(l)}_k$ are the eigenvectors computed from (6.1). We then discard the entries not associated with dual variables to obtain our constraints $u^{(l)}_k$.

LEMMA 6.2. *For an edge $\mathcal{E}_{ij}$, let $I^{\mathcal{E}_{ij},(l)}_{L_l}$, for $l=i,j$, be defined by*

$$I^{\mathcal{E}_{ij},(l)}_{L_l}=\sum_{k=1}^{L_l}x^{(l)}_k x^{(l)T}_k M^{(l)}_{E_{ij}},$$

where $x^{(l)}_k$ are the eigenvectors from (6.1). Let $D^{(l)}_{E_{ij},c}$ be the scaling matrices in Definition 6.1. With the choice of the constraints $u^{(l)T}_k(w_i-w_j)=0$, $l=i,j$, where $u^{(i)}_k$ and $u^{(j)}_k$ are obtained by discarding the entries not associated with dual variables in the vectors $D^{(j)T}_{E_{ij},c}M^{(i)}_{E_{ij}}x^{(i)}_k$ and $D^{(i)T}_{E_{ij},c}M^{(j)}_{E_{ij}}x^{(j)}_k$ with $\mu^{ij,l}_k\le \mathrm{TOL}$ for $k=1,\ldots,L_l$, we have

$$I^{\mathcal{E}_{ij},(i)}_{L_i}D^{(j)}_{E_{ij},c}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}=0 \qquad \text{and} \qquad I^{\mathcal{E}_{ij},(j)}_{L_j}D^{(i)}_{E_{ij},c}(w_j-w_i)_{|\overline{\mathcal{E}}_{ij}}=0.$$

Proof. The entries not associated with dual variables in $w_{i|\overline{\mathcal{E}}_{ij}}-w_{j|\overline{\mathcal{E}}_{ij}}$ are zero since $w_l=R^{(l)}w$ with $w\in\widetilde{W}$. Therefore, we have

$$I^{\mathcal{E}_{ij},(i)}_{L_i}D^{(j)}_{E_{ij},c}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}=\sum_{k=1}^{L_i}x^{(i)}_k u^{(i)T}_k(w_{\Delta,i}-w_{\Delta,j})_{|\mathcal{E}_{ij}}=0,$$

where $w_{\Delta,l}$ denotes the dual part of w_l, $l=i,j$. By an analogous argument, we conclude that $I^{\mathcal{E}_{ij},(j)}_{L_j}D^{(i)}_{E_{ij},c}(w_j-w_i)_{|\overline{\mathcal{E}}_{ij}}=0$. ∎

The next two lemmas have been proven in [23]. We repeat them for the convenience of the reader.

LEMMA 6.3. *Let $\widetilde{\Omega}_{i\eta}\subset\Omega_i$ be a slab of width η such that $\mathcal{E}_{ij}\subset\partial\widetilde{\Omega}_{i\eta}$. Let $\omega_{ik}\subset\widetilde{\Omega}_{i\eta}$, $k=1,\ldots,n$, be a set of η-patches such that $\widetilde{\Omega}_{i\eta}=\bigcup_{k=1}^n\omega_{ik}$ and the coefficient function $\rho_{i|\omega_{ik}}=\rho_{ik}$ is constant on each ω_{ik}. Let $\omega_{ik}\cap\omega_{il}=\emptyset$, $k\neq l$, $\theta_{\mathcal{E}_{ij}}$ be the standard finite element cut-off function which equals 1 on the edge $\mathcal{E}_{ij}$ and is zero on $\partial\widetilde{\Omega}_{i\eta}\setminus\mathcal{E}_{ij}$, and let $\mathcal{H}^{(i)}_{\rho_i}$ be the ρ_i-harmonic extension. Then there exists a finite element function $\vartheta_{\mathcal{E}_{ij}}$ which equals $\theta_{\mathcal{E}_{ij}}$ on $\partial\widetilde{\Omega}_{i\eta}$ such that for $u\in W^h(\Omega_i)$*

$$\begin{aligned}|\mathcal{H}^{(i)}_{\rho_i}I^h(\theta_{\mathcal{E}_{ij}}u)|^2_{H^1_{\rho_i}(\Omega_i)}&\le|I^h(\vartheta_{\mathcal{E}_{ij}}u)|^2_{H^1_{\rho_i}(\widetilde{\Omega}_{i\eta})}\\&\le C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2\left(|u|^2_{H^1_{\rho_i}(\widetilde{\Omega}_{i\eta})}+\frac{1}{\eta^2}||u||^2_{L^2_{\rho_i}(\widetilde{\Omega}_{i\eta})}\right),\end{aligned}$$

where $C>0$ is a constant independent of H,h,η, and the value of ρ_i.

LEMMA 6.4 (weighted Friedrichs inequality). *For $u\in H^1(\widetilde{\Omega}_{i\eta})$, we have*

$$||u||^2_{L^2_{\rho_i}(\widetilde{\Omega}_{i\eta})}\le C\left(\eta^2|u|^2_{H^1_{\rho_i}(\widetilde{\Omega}_{i\eta})}+\eta||u||^2_{L^2_{\rho_i}(\mathcal{E}_{ij})}\right).$$

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For simplicity, we prove the next theorem only for the diffusion problem. In Definition 4.12, for a given edge, we have introduced the notion of a slab and assumed that it can be represented as a union of patches. In the proof of the next theorem, we have to make the assumption that the diffusion coefficient is constant on each patch in such a decomposition. This is due to the fact that we have to use an edge lemma (Lemma 6.3) on each patch; cf. also [23, Lemma 6.3] and the proof of the next theorem.

THEOREM 6.5. *The condition number for our FETI-DP method with a scaling as defined in Definition 6.1 with all vertices primal and the coarse space enhanced with solutions of the eigenvalue problem (6.1) satisfies*

$$\kappa(\hat{M}^{-1}F) \leq C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2\left(1+\frac{1}{\eta\mu_{L+1}}\right),$$

where $\hat{M}^{-1}=M_{PP}^{-1}$ or $\hat{M}^{-1}=M_{BP}^{-1}$. Here, $C>0$ is a constant independent of ρ, H, h, and η and

$$\frac{1}{\mu_{L+1}} = \max_{\substack{k=1,\dots,N\\ \mathcal{E}_{ij}\subset\Gamma^{(k)}}}\left\{\frac{1}{\mu_{L_k+1}^{ij,k}}\right\}.$$

Proof. The proof is modeled on the proof of Theorem 6.8 in [23] and uses the notation from [23]. By an application of Lemma 6.2, we obtain for each edge $\mathcal{E}_{ij}$ in $|P_Dw|_{\widetilde{S}}^2$ the term

$$\begin{aligned}
|I^h(\theta_{\mathcal{E}_{ij}}D^{(i)}(w_i-w_j))|_{S^{(i)}}^2 &= |I^h(\theta_{\mathcal{E}_{ij}}((I-I_{L_i}^{\mathcal{E}_{ij},(i)})D^{(i)}(w_i-w_j)))|_{S^{(i)}}^2\\
&\overset{\text{Lemma 6.3}}{\leq} C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2\Bigg(|\mathcal{H}_{\mathcal{E}_{ij},c}^{(i)}\left((I-I_{L_i}^{\mathcal{E}_{ij},(i)})D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}\right)|_{H_{\rho_i}^1(\widetilde{\Omega}_{i\eta})}^2\\
&\qquad + \frac{1}{\eta^2}||\mathcal{H}_{\mathcal{E}_{ij},c}^{(i)}\left((I-I_{L_i}^{\mathcal{E}_{ij},(i)})D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}\right)||_{L_{\rho_i}^2(\widetilde{\Omega}_{i\eta})}^2\Bigg)\\
&\overset{\text{Lemma 6.4}}{\leq} C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2\Bigg(|\mathcal{H}_{\mathcal{E}_{ij},c}^{(i)}\left((I-I_{L_i}^{\mathcal{E}_{ij},(i)})D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}\right)|_{H_{\rho_i}^1(\widetilde{\Omega}_{i\eta})}^2\\
&\qquad + \frac{1}{\eta}||\left((I-I_{L_i}^{\mathcal{E}_{ij},(i)})D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}\right)||_{L_{\rho_i}^2(\mathcal{E}_{ij})}^2\Bigg)\\
&\leq C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2\Bigg(\left|\left((I-I_{L_i}^{\mathcal{E}_{ij},(i)})D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}\right)\right|_{S_{E_{ij},c}^{(i)}}^2\\
&\qquad + \frac{1}{\eta}\left|\left((I-I_{L_i}^{\mathcal{E}_{ij},(i)})D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}\right)\right|_{M_{E_{ij}}}^2\Bigg)\\
&\leq C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2\left(1+\frac{1}{\eta\mu_{L_i+1}^{(i)}}\right)\left|D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}\right|_{S_{E_{ij},c}^{(i)}}^2\\
&\leq C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2\left(1+\frac{1}{\eta\mu_{L_i+1}^{(i)}}\right)\left(|w_{i|\overline{\mathcal{E}}_{ij}}|_{S_{E_{ij},c}^{(i)}}^2+|w_{j|\overline{\mathcal{E}}_{ij}}|_{S_{E_{ij},c}^{(j)}}^2\right).
\end{aligned}$$

Here, I denotes the identity operator. In the penultimate step, we have used Lemma 4.6 with $B=M_{E_{ij}}$, $x=D_{E_{ij},c}^{(j)}(w_i-w_j)_{|\overline{\mathcal{E}}_{ij}}$, $m=L_i$, $\Pi_M^B=I_{L_i}^{\mathcal{E}_{ij},(i)}$, and $A=S_{E_{ij},c}^{(i)}$. For the last step note that each column of A_{ij} in Definition 6.1 is in $\text{Ker}(S_{E_{ij},c}^{(i)}+S_{E_{ij},c}^{(j)})$ and with the same argument as in the proof of Lemma 4.4, we have

$$\text{Ker}(S_{E_{ij},c}^{(i)}+S_{E_{ij},c}^{(j)})=\text{Ker}(S_{E_{ij},c}^{(i)})\cap\text{Ker}(S_{E_{ij},c}^{(j)}).$$

Thus, we obtain $S^{(i)}_{E_{ij},c} A_{ij} = 0$ and $S^{(i)}_{E_{ij},c} D^{(j)}_{E_{ij},c} = S^{(i)}_{E_{ij},c} (S^{(i)}_{E_{ij},c} + S^{(j)}_{E_{ij},c})^{+} S^{(j)}_{E_{ij},c}$. Applying Lemma 4.4 with $A = S^{(i)}_{E_{ij},c}$, $B = S^{(j)}_{E_{ij},c}$, and $D_A = D^{(j)}_{E_{ij},c}$, we obtain the estimate. $\square$

REMARK 6.6. The constant in the condition number estimate for the third coarse space (cf. Theorem 6.5 and [23, Theorem 6.8]) depends on $N_{\mathcal{E}}^2$ in the same way as that for the first coarse space in Theorem 4.16 and that for the second coarse space in Theorem 5.1. Additionally, the constant depends on the constants in a weighted edge lemma (Lemma 6.3) and in a weighted Friedrichs inequality (Lemma 6.4). Therefore the constant C is not exactly determined.

REMARK 6.7. As in Section 6.3, we can replace the matrices $S^{(l)}_{E_{ij},c}$, $l = i, j$, by the economic version $S^{(l)}_{E,c,\eta}$ in the scaling in Definition 6.1 and in the generalized eigenvalue problem 6.1.

7. A brief comparison of computational costs. In the following, we give a short comparison of the costs of the algorithms described in this paper. For the algorithm in Section 4.4 (first coarse space), the matrices $S^{(l)}_{E_{ij}}$ and $S^{(l)}_{E_{ij},0}$, $l = i, j$, have to be computed. These matrices are usually dense. For their computation, a Cholesky factorization of a sparse matrix is required and usually needs $\mathcal{O}((H/h)^3)$ floating point operations in two space dimensions since the inverse involved in the Schur complement is of the order of $(H/h)^2 \times (H/h)^2$. If the Schur complements are computed explicitly, which might be necessary depending on the eigensolver that is used, a matrix-matrix multiplication, a matrix-matrix addition, and forward-backward substitutions for multiple right-hand sides with the Cholesky factorization have to be performed. If LAPACK (or MATLAB, which itself uses LAPACK) is used, the matrices $S^{(i)}_{E_{ij}} : S^{(j)}_{E_{ij}}$ and $S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0}$ are needed in explicit form. Otherwise, an application of the Schur complement needs a few matrix-vector multiplications and a forward-backward substitution. For $S^{(i)}_{E_{ij}} : S^{(j)}_{E_{ij}}$, depending on the kernel of $S^{(i)}_{E_{ij}} + S^{(j)}_{E_{ij}}$, a pseudoinverse or a Cholesky factorization is needed. For the scaling matrices, a factorization of $S^{(i)}_{E_{ij},0} + S^{(j)}_{E_{ij},0}$ has to be performed. If no deluxe scaling but ρ-scaling is used, the matrix $D^{(i)T}_{E_{ij}} S^{(j)}_{E_{ij},0} D^{(i)}_{E_{ij}} + D^{(j)T}_{E_{ij}} S^{(i)}_{E_{ij},0} D^{(j)}_{E_{ij}}$ has to be computed instead of $S^{(i)}_{E_{ij},0} : S^{(j)}_{E_{ij},0}$, which is much cheaper since no factorization of $S^{(i)}_{E_{ij},0} + S^{(j)}_{E_{ij},0}$ is needed. The computations of $S^{(l)}_{E_{ij},0,\eta}$ and $S^{(l)}_{E_{ij},\eta}$ need a $(\eta/H)^{3/2}$-fraction of floating point operations compared with the computations of $S^{(l)}_{E_{ij},0}$ and $S^{(l)}_{E_{ij}}$.

The eigenvalue problem in Section 5 (second coarse space) is larger but sparser. The left-hand side is not dense because of the structure of the local jump operator P_D, which contains only two non-zero entries for each row. The right-hand side consists of two dense blocks and two zero blocks in the dual part. The size of the eigenvalue problem is determined by the number of degrees of freedom on $\Gamma^{(i)} \times \Gamma^{(j)}$ while the other algorithms are determined by the number of degrees of freedom on an edge $\mathcal{E}_{ij}$, e.g., in two dimensions, it can be eight times larger. The computation of the left-hand side of the generalized eigenvalue problem in Section 5.2 also needs applications of the scaling matrices $D^{(i)}$ and $D^{(j)}$, which in case of deluxe scaling, is more expensive than for multiplicity- or ρ-scaling.

The generalized eigenvalue problem in Section 6.1 is completely local and needs no inter-subdomain communication but needs to be solved for two neighboring subdomains for each edge. For a chosen edge, the generalized eigenvalue problem (6.2) presented in Section 6.2 has to be solved once for each subdomain sharing that edge, but it needs inter-subdomain communication. While the algorithm in Section 4 needs to exchange the matrices $S^{(l)}_{E_{ij},0}$ and $S^{(l)}_{E_{ij}}$, $l = i, j$, and the scaling matrices, the algorithm in Section 5 needs to exchange $S^{(l)}$, the local jump matrices $B^{(l)}_{E}$, $l = i, j$, and the scaling matrices. Nonetheless, if ρ-scaling

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or deluxe scaling is used, the scaling data need to be communicated for the construction of B_D in the FETI-DP algorithm anyways. The algorithm in Section 6 only needs to exchange $S^{(l)}_{E_{ij}}$, $l = i, j$. However, locally, in two dimensions, a one-dimensional mass matrix has to be assembled for each edge of a subdomain. Note that this matrix has tridiagonal form if piecewise linear finite element functions are used. This makes a Cholesky factorization very cheap.

A disadvantage of the algorithm in Section 6 (third coarse space) compared to the other algorithms is that no ρ-scaling with varying scaling weights inside of a subdomain can be used. In Section 8 for our numerical results, we see that using multiplicity scaling can lead to a large number of constraints. However, if the extension constant is nicely bounded, e.g., for coefficient distributions which are symmetric with respect to the subdomain interfaces (at least on slabs) and have jumps only along but not across edges, only trivial local generalized eigenvalue problems with a tridiagonal mass matrix on the right-hand side need to be solved, and the number of constraints stays bounded independently of H/h. If the scaling in Section 6.4 is used, only the scaling matrices of neighboring subdomains have to be exchanged. The eigenvectors in the first eigenvalue problem can be computed completely locally. The constraints need an application of the mass matrix and the scaling matrix of a neighbor.

8. Numerical examples. In all our numerical computations, we have removed linearly dependent constraints by using a singular value decomposition of U. Constraints related to singular values less than a drop tolerance of 1e-6 were removed. In an efficient implementation this may not be feasible.

As a stopping criterion in the preconditioned conjugate gradient algorithm, we used $||r_k|| \leq 10^{-10}||r_0|| + 10^{-16}$, where r_0 is the preconditioned starting residual and r_k the preconditioned residual in the k-th iteration.

In our numerical experiments, whenever we need to compute a pseudoinverse of a symmetric matrix A, we first introduce the partition $A = \begin{bmatrix} A_{pp} & A_{rp}^T \\ A_{rp} & A_{rr} \end{bmatrix}$, where A_{pp} is an invertible submatrix of A and A_{rr} is a small submatrix of A with a size of at least the dimension of the kernel of A. Then we compute

$$A^+ = \begin{bmatrix} I & -A_{pp}^{-1}A_{rp}^T \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{pp}^{-1} & 0 \\ 0 & S_{rr}^{\dagger} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{rp}A_{pp}^{-1} & I \end{bmatrix}$$

with the Schur complement $S_{rr} = A_{rr} - A_{rp}A_{pp}^{-1}A_{rp}^T$. Here, $S_{rr}^{\dagger}$ denotes the Moore-Penrose pseudoinverse of S_{rr}. In the singular value decomposition of S_{rr}, we treat all singular values less than $(1\text{e-}3) \cdot \min(\mathrm{diag}(A))$ as zero.

We have considered different coefficient distributions. In Test Problem I (Figure 8.1), we consider the simple case of horizontal channels. In Test Problem II (Figure 8.2), we have a coefficient configuration that is symmetric in a small neighborhood of the vertical edges. In Test Problem III (Figure 8.3), the coefficient configuration is constructed with no symmetry with respect to the vertical edges. In Test Problem IV (Figure 8.5), we then have a challenging, randomly chosen coefficient distribution. Note that the coefficient distribution does not change when the meshes are refined. For our adaptive method, we therefore expect the number of constraints to stay bounded when H/h is increased.

We attempt a fair comparison of the adaptive methods by using suitable tolerances for the different eigenvalues, i.e., we attempt to choose tolerances such that all very large eigenvalues are removed but no more. For an illustration, we present detailed spectra for the Test Problem IV; see Figure 8.7.

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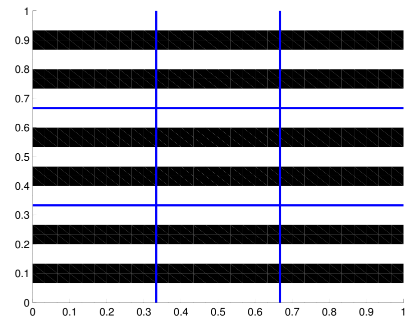

FIG. 8.1. *Test Problem I with a coefficient distribution consisting of two channels per subdomain for a 3x3 decomposition. In case of diffusion, the diffusion coefficient is* 10^6 *(black) and* 1 *(white). In case of elasticity, the Young modulus is* 10^3 *(black) and* 1 *(white).*

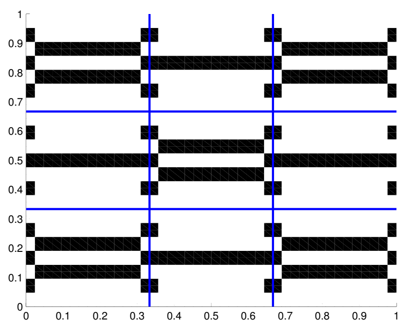

FIG. 8.2. *Test Problem II: symmetric in slabs with respect to the edges for a 3x3 decomposition. Diffusion coefficient* 10^6 *(black) and* 1 *(white). Domain decomposition in* 3×3 *subdomains,* $H/\eta = 14$.

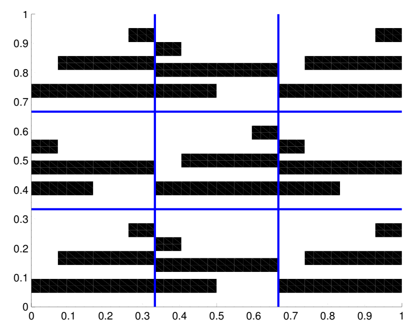

FIG. 8.3. *Test Problem III has a coefficient distribution which is unsymmetric with respect to the edges for a 3x3 decomposition. Diffusion coefficient* 10^6 *(black) and* 1 *(white).*

We give our numerical results for FETI-DP methods, but they are equally valid for BDDC methods. The adaptive constraints are incorporated into the balancing preconditioner M_{BP}^{-1} (cf. equation (3.1)), but other methods can also be used.

In Section 8.1, we present numerical results for a scalar diffusion equation and Problems I-IV using different adaptive coarse spaces. In Section 8.2, we consider the problem of almost incompressible elasticity.

8.1. Scalar diffusion. First, we perform a comparison for our scalar diffusion problem with a variable coefficient for the first (Section 4 [7, 21]), second (Section 5 [30]), and third (Section 6 [23]) coarse spaces. We use homogeneous Dirichlet boundary conditions on $\Gamma_D = \partial\Omega$ in all our experiments for scalar diffusion.

The first coefficient distribution is depicted in Figure 8.1 (Test Problem I; horizontal channels). This coefficient distribution is symmetric with respect to vertical edges. Since there are no jumps across the interface, the simple multiplicity scaling is sufficient, and ρ-scaling reduces to multiplicity scaling. The numerical results are shown in Table 8.1. The estimated condition numbers are identical for all cases and the number of constraints is similar.

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In Table 8.2, we consider the coefficient distribution depicted in Figure 8.2 (Test Problem II, horizontal channels on slabs). Here, the coefficient distribution is symmetric on slabs with respect to vertical edges. Again, there are no coefficient jumps across subdomains, and multiplicity scaling is equivalent to ρ-scaling. We note that in this test problem, e-deluxe scaling with $H/\eta = 14$ is not equivalent to multiplicity scaling since in the Schur complements $S^{(l)}_{E,0,\eta}$, $l = i, j$, the entries on $\partial\widetilde{\Omega}_{i\eta} \setminus (\partial\Omega_l \cap \partial\widetilde{\Omega}_{i\eta})$ are eliminated. The economic version of extension scaling in this case is equivalent to multiplicity scaling because the Schur complements $S^{(l)}_{E,\eta}$, $l = i, j$, are computed from local stiffness matrices on the slab. In Table 8.2, we report on multiplicity scaling, deluxe scaling, and e-deluxe scaling for the three cases. Using multiplicity scaling, the results are very similar, but not identical, for all three approaches to adaptive coarse spaces. The use of deluxe scaling can improve the results for the first two approaches. The use of extension scaling for the third approach has no significant impact. Where economic variants exist, e.g., versions on slabs, we also report on results using these methods. As should be expected, using the economic versions of the eigenvalue problems yields worse results.

Next, we use the coefficient distribution depicted in Figure 8.3 (Test Problem III, unsymmetric channel pattern). The results are collected in Table 8.3. In this problem, coefficient jumps across the interface are present in addition to the jumps inside subdomains. Therefore, multiplicity scaling is not sufficient, and none of the coarse space approaches are scalable with respect to H/h, i.e., the number of constraints increase when H/h is increased; cf. Table 8.3 (left). Using ρ-scaling or deluxe/extension scaling then yields the expected scalability in H/h, i.e., the number of constraints remains bounded when H/h is increased. Where deluxe scaling is available, it significantly reduces the size of the coarse problem; cf. Table 8.3 (middle and right). The smallest coarse problem is then obtained for the combination of the second coarse space with deluxe scaling. Using extension scaling in the third coarse space approach yields smaller condition numbers and iteration counts for ρ-, deluxe-, or extension scaling but at the price of a much larger coarse space. In Table 8.4, we display results for ρ-scaling and deluxe/extension scaling for Test Problem III and an increasing number of subdomains. As expected, since this results in a growing number of edges and discontinuities, this leads to a growing number of constraints for all three coarse spaces. Again, deluxe scaling yields the lowest number of constraints. Let us note that in these experiments, for each value of H, we solve a different problem since the discontinuities are determined on the level of the subdomains. This guarantees that the jumps within the subdomains and across and along the edges are of the same type; see Figure 8.6. Alternatively, one could consider a problem with a fixed number of discontinuities. Solving such a problem with a varying number of subdomains would lead to a different set of discontinuities for each value of H. In Table 8.5, we present results for Test Problem III using the slab variant of the first coarse space. Our results show that saving computational work by using the slab variants can increase the number of constraints significantly, i.e., the number of constraints grows with decreasing η/H. On the other hand, the condition numbers and iteration counts decrease. This implies that slab variants can be affordable if a good coarse space solver is available. The results may also indicate that scalability of the coarse space size with respect to H/h may be lost.

The results for Test Problem IV are collected in Table 8.6. Also for this difficult problem the number of constraints seems to remain bounded when H/h is increased although for ρ-scaling, the number of constraints increases slightly with H/h. The smallest coarse problem consisting of only four eigenvectors is obtained when the second coarse space approach is combined with deluxe scaling although the difference between ρ-scaling and deluxe scaling is not as large as in Test Problem III. The third coarse space using extension scaling is scalable in H/h but, in this current version, yields the largest number of constraints.

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TABLE 8.1
Scalar diffusion. Test Problem I (see Figure 8.1).

	First coarse space with mult. scaling TOL $= 1/10$			Second coarse space with mult. scaling TOL $= 10$			Third coarse space with mult. scaling TOL$_\mu = 1$, TOL$_\nu = -\inf$			
H/h	cond	its	#EV	cond	its	#EV	cond	its	#EV	#dual
10	1.0388	2	14	1.0388	2	12	1.0388	2	14	132
20	1.1509	3	14	1.1509	3	12	1.1509	3	14	252
30	1.2473	3	14	1.2473	3	12	1.2473	3	14	372
40	1.3274	3	14	1.3274	3	12	1.3274	3	14	492

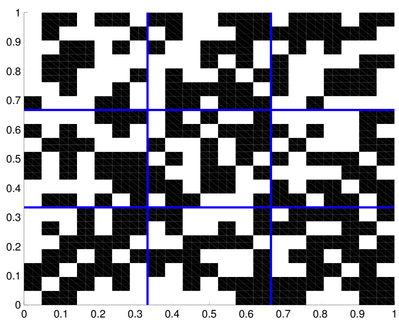

FIG. 8.4. *Plot of the square root of the condition number vs. H/h of the data given in Table 8.6 for the third coarse space with extension scaling using a logarithmic scale on the x-axis.*

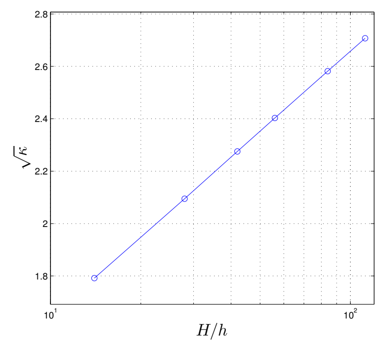

FIG. 8.5. *Test Problem IV has a random coefficient distribution which is constant on squares of size $1/21 \times 1/21$. Diffusion coefficient 10^6 (black) and 1 (white). Domain decomposition in 3×3 subdomains.*

For instance, the 50 largest eigenvalues appearing in the adaptive approaches for Test Problem IV using deluxe or extension scaling are presented in Figure 8.7. We can see that the tolerances chosen in Table 8.6 results in the removal of all large eigenvalues. We therefore believe that our comparison is fair.

For the third coarse space, we have also tested the combination of multiplicity scaling with both eigenvalue problems from [23], i.e., TOL$_\mu = 1/10$ and TOL$_\nu = 1/10$. As in the other cases where we use multiplicity scaling, see Table 8.3, this leads to a small condition number but at the cost of a large number of constraints, and the approach thus is not scalable with respect to H/h.

Results for the slab variant of the third coarse space are then presented in Table 8.1. In Table 8.6, we consider the distribution from Figure 8.5 for the different coarse space approaches. Note that we do not show the results for multiplicity scaling here since the coarse space grows significantly with H/h, and this approach is therefore not recommended.

8.2. Almost incompressible elasticity. In this section, we compare the algorithms for almost incompressible elasticity. First we consider a problem with a constant coefficient distribution. Mixed displacement-pressure $\mathbb{P}2$-$\mathbb{P}0$-elements are used for the discretization.

In the first test, we solve a problem with a Young modulus of 1 and a Poisson ratio of 0.499999. Zero Dirichlet boundary conditions are imposed on

$$\Gamma_D = \left\{(x, y) \in [0, 1]^2 | y = 0\right\}.$$

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TABLE 8.2
Scalar diffusion. For the slab variants of the algorithms, we only consider the first and the third coarse space; see Sections 4 and 6. Test Problem II (see Figure 8.2), $H/\eta = 14$, $1/H = 3$.

	First coarse space with mult. scaling TOL $= 1/10$			Second coarse space with mult. scaling TOL $= 10$			Third coarse space with mult. scaling TOL$_\mu = 1$, TOL$_\nu = -\infty$			
H/h	cond	its	#EV	cond	its	#EV	cond	its	#EV	#dual
14	1.0469	5	20	1.0498	5	18	1.0467	6	20	180
28	1.1680	5	20	1.1710	5	18	1.1678	6	20	348
42	1.2696	6	20	1.2728	5	18	1.2695	6	20	516
56	1.3531	6	20	1.3564	6	18	1.3529	6	20	684
70	1.4238	6	20	1.4272	6	18	1.4237	7	20	852
	First coarse space on slabs with mult. scaling TOL $= 1/10$			Second coarse space			Third coarse space on slabs with mult. scaling TOL$_\mu = 1$, TOL$_\nu = -\infty$			
H/h	cond	its	#EV	cond	its	#EV	cond	its	#EV	#dual
14	1.0466	5	26				1.0465	6	24	180
28	1.1678	6	26		no		1.1677	6	24	348
42	1.2695	6	26		slab		1.2694	6	24	516
56	1.3530	6	26		variant		1.3529	6	24	684
70	1.4237	6	26				1.4236	6	24	852
	First coarse space with deluxe scaling TOL $= 1/10$			Second coarse space with deluxe scaling TOL $= 10$			Third coarse space with extension scaling TOL$_\mu = 1$, TOL$_\nu = -\infty$			
H/h	cond	its	# EV	cond	its	# EV	cond	its	# EV	# dual
14	1.2319	5	8	1.2510	6	6	1.0798	5	20	180
28	1.2948	6	8	1.3222	6	6	1.2170	5	20	348
42	1.4024	6	8	1.4403	6	6	1.3285	5	20	516
56	1.4906	6	8	1.5372	6	6	1.4189	6	20	684
70	1.5652	7	8	1.6188	7	6	1.4950	6	20	852
	First coarse space on slabs with e-deluxe scaling TOL $= 1/10$			Second coarse space			Third coarse space on slabs with extension scaling on slabs TOL$_\mu = 1$, TOL$_\nu = -\infty$			
H/h	cond	its	# EV	cond	its	# EV	cond	its	# EV	# dual
14	1.0256	5	22				1.0465	6	24	180
28	1.1218	6	26		no		1.1677	6	24	348
42	1.2143	6	26		slab		1.2694	6	24	516
56	1.2945	6	26		variant		1.3529	6	24	684
70	1.3645	6	26				1.4236	6	24	852

The results for the approach in Section 4.5.2 with a tolerance $1/10$ and varying H/h are presented in Table 8.8. For constant $H/h = 20$ and varying Poisson ratio ν, see Table 8.9.

In the third case, we consider a distribution of Young's modulus as in Figure 8.1 and a Poisson ratio of $\nu = 0.4999$. The result for the approach in Section 4.5.2 can be found in Table 8.10. For the related results of the algorithm in [30], see Tables 8.8, 8.9, and 8.10.

Note that the third coarse space algorithm is not suitable for this problem since its eigenvalue problem is not based on a localization of the jump operator but designed to get constants in Korn-like inequalities and in an extension theorem that are independent of jumps in the coefficients. It therefore will not find the zero net flux condition which is necessary for a stable algorithm.

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TABLE 8.3
Scalar diffusion. Comparison of the coarse spaces for Test Problem III (see Figure 8.3). The numerical results for the first coarse space have already been published by the authors in [21, Section 3].

First coarse space, TOL $= 1/10$

scal.	multiplicity			ρ			deluxe			
H/h	cond	its	#EV	cond	its	#EV	cond	its	#EV	#dual
14	1.2911	8	44	1.3874	9	15	4.8937	11	5	180
28	1.4148	9	74	1.5782	11	15	4.8672	12	5	348
42	1.5167	10	104	1.7405	11	15	4.8891	12	5	516

Second coarse space, TOL $= 10$

scal.	multiplicity			ρ			deluxe			
H/h	cond	its	#EV	cond	its	#EV	cond	its	#EV	#dual
14	1.5013	9	42	1.5979	9	13	5.8330	11	3	180
28	1.6407	10	72	1.7351	11	13	5.8758	12	3	348
42	1.7413	10	102	1.8474	12	13	5.9296	13	3	516

Third coarse space, TOL$_\mu = 1$

scal.	multiplicity, TOL$_\nu = 1/10$			ρ			extension, TOL$_\nu = -\infty$			
H/h	cond	its	#EV	cond	its	#EV	cond	its	#EV	#dual
14	1.2870	8	55	no			1.2498	8	20	180
28	1.4154	9	88	rho			1.4238	9	20	348
42	1.5186	9	118	variant			1.5525	10	20	516

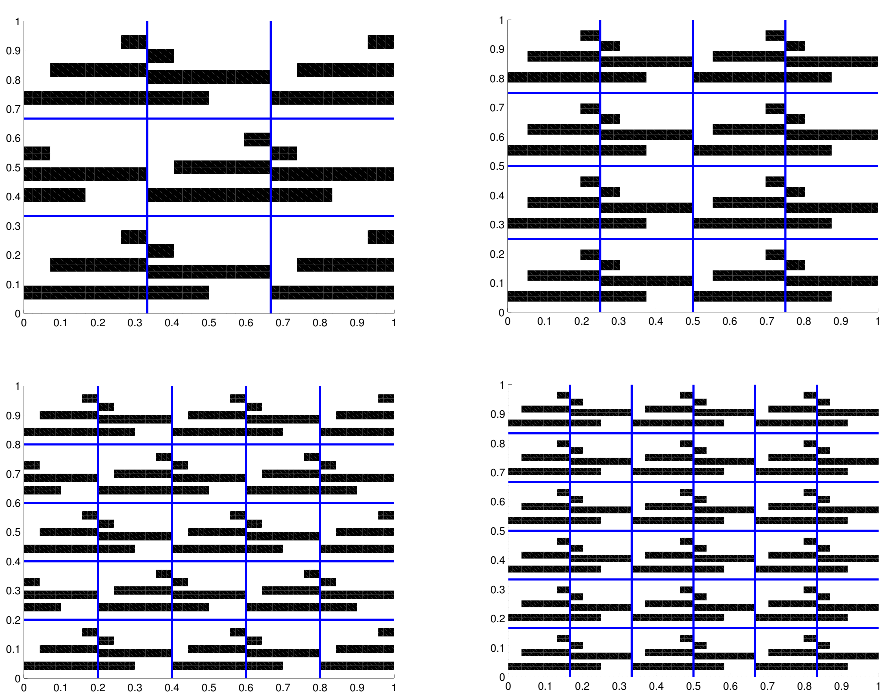

FIG. 8.6. *Variations of the coefficient distribution in Test Problem III for different numbers of subdomains.*

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TABLE 8.4
Scalar diffusion. Comparison of the coarse spaces for Test Problem III (see Figure 8.3) and an increasing number of subdomains, $H/h = 42$.

First coarse space, TOL $= 1/10$

scal.	ρ			deluxe			
$1/H$	cond	its	#EV	cond	its	#EV	#dual
3	1.7405	11	15	4.8891	12	5	516
4	1.8848	12	40	5.0022	14	14	1038
5	2.4267	13	58	6.7398	16	26	1736
6	2.5414	13	100	6.7185	17	43	2610
7	2.7905	14	129	6.7763	18	63	3660
8	2.8467	14	188	6.7437	19	88	4886
9	2.9341	15	228	6.7827	18	116	6288
10	2.9757	15	304	6.7539	20	149	7866
15	3.0681	16	693	6.7949	21	371	18396
20	3.1243	17	1304	6.7675	23	694	33326
25	3.1125	17	2028	6.7989	23	1116	52656

Second coarse space, TOL $= 10$

scal.	ρ			deluxe			
$1/H$	cond	its	#EV	cond	its	#EV	#dual
3	1.8474	12	13	5.9296	13	3	516
4	1.9186	12	34	10.0410	16	7	1038
5	3.4434	15	46	7.9252	18	14	1736
6	3.5331	16	80	10.0690	21	22	2610
7	3.9461	19	99	7.9624	22	33	3660
8	3.9765	19	146	10.0700	24	45	4886
9	4.4980	21	172	7.9768	25	60	6288
10	4.6656	21	232	10.0700	26	76	7866
15	5.7855	25	511	8.0252	28	189	18396
20	6.1949	26	962	10.0699	30	351	33326
25	6.4144	27	1476	8.1597	29	564	52656

Third coarse space, TOL$_\mu = 1$

scal.	ρ			extension, TOL$_\nu = -\infty$			
$1/H$	cond	its	#EV	cond	its	#EV	#dual
3				1.5525	10	20	516
4				1.5425	10	56	1038
5		no		1.5499	10	88	1736
6		rho		1.5465	10	152	2610
7		variant		1.5496	10	204	3660
8				1.5415	10	296	4886
9				1.5490	10	368	6288
10				1.5398	10	488	7866
15				1.5459	10	1148	18396
20				1.5366	10	2168	33326
25				1.5337	10	3408	52656

9. Conclusion. For the first and second coarse space, a condition number estimate is available for symmetric positive definite problems in two dimensions; see Sections 4.4.2 and 4.5.2 for the first coarse space and Section 5.3 for the second coarse space, respectively. A condition number estimate for the third coarse space applied to scalar diffusion problems can be found in [23] for constant ρ-scaling and in Section 6 for extension scaling. For this,

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TABLE 8.5
Test Problem III (see Figure 8.3). Results for the slab variant of the first coarse space; also cf. Table 8.3.

		First coarse space on slabs, e-deluxe scaling, TOL $= 1/10$			First coarse space on slabs, mult. scaling, TOL $= 1/10$			
η/h	H/h	cond	its	#EV	cond	its	#EV	#dual
1	14	1.1355	7	24	1.0776	6	52	180
1	28	1.1118	6	35	1.1831	7	89	348
1	42	1.1069	6	47	1.0881	7	133	516
2	14	1.1729	7	21	1.1552	7	48	180
2	28	1.2096	7	25	1.1867	7	82	348
2	42	1.1770	7	33	1.2897	8	116	516
3	14	1.3978	9	11	1.2912	8	44	180
3	28	1.2145	7	24	1.1865	7	82	348
3	42	1.3029	8	25	1.2941	8	112	516
5	14	1.4086	9	10	1.2911	8	44	180
5	28	1.3447	8	19	1.3021	8	78	348
5	42	1.2852	8	24	1.2927	8	112	516
10	14	2.6060	10	6	1.2911	8	44	180
10	28	1.4441	10	10	1.4148	9	74	348
10	42	1.5216	10	12	1.5171	9	105	516
14	14	4.8937	11	5	1.2911	8	44	180
28	28	4.8672	12	5	1.4148	9	74	348
42	42	4.8891	12	5	1.5167	10	104	516

a condition number estimate can be proven for linear elasticity using similar arguments and replacing the H^1-seminorms by the elasticity seminorms. For all three coarse spaces, to the best of our knowledge, no published theory exists yet for the three-dimensional case but the second coarse space has been successfully applied to three-dimensional problems in [31].

An advantage of the first and third coarse spaces is that the size of the eigenvalue problems depends only on the number of degrees of freedom on an edge. This has to be seen in comparison to the size of two local interfaces of two substructures in the second adaptive coarse space. In the eigenvalue problem for the first coarse space, the matrices involved are dense while in the second coarse space, the eigenvalue problem involves a sparse matrix on the left-hand side and a 2×2 block matrix with dense blocks on the right-hand side. However, the first coarse space needs the factorization of a matrix on the left-hand side and possibly matrix-matrix multiplications with Schur complements if a direct eigensolver is used. The third coarse space needs the solution of two eigenvalue problems for each of two substructures sharing an edge with a dense matrix on the left-hand side and a tridiagonal matrix in case of piecewise linear elements on the right-hand side. It may be advantageous that these eigenvalue problems can be computed locally on one substructure and that for building the constraints, only in case of extension scaling information of the neighboring substructure has to be used. Possibly another eigenvalue problem with two dense matrices needs to be solved for the extension if the extension constant is not small, e.g., if no extension scaling is used or if the coefficient is not symmetric with respect to an edge. A multilevel BDDC variant for the second coarse space can be found in [35].

All coarse spaces require an additional factorization with matrix-matrix multiplications or multiple forward-backward substitutions if deluxe scaling is used. In case of multiplicity scaling and a nonsymmetric coefficient, the size of all coarse spaces can depend on the size of the substructures H/h as can be seen in Section 8.

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TABLE 8.6
Scalar diffusion. Test Problem IV.

	First coarse space, TOL $= 1/10$						
scal.	ρ			deluxe			
H/h	cond	its	#EV	cond	its	#EV	#dual
14	7.3286	19	10	2.2748	11	7	180
28	8.8536	20	11	2.4667	10	9	348
42	6.4776	21	12	2.5994	10	9	516
56	7.0378	21	12	2.6947	11	9	684
84	7.8168	23	12	2.8302	11	9	1020
112	8.3651	24	13	2.9267	12	9	1356
	Second coarse space, TOL $= 10$						
scal.	ρ			deluxe			
H/h	cond	its	#EV	cond	its	#EV	#dual
14	7.4306	20	6	2.6263	12	4	180
28	9.0907	22	7	3.0782	13	4	348
42	8.3068	23	8	3.3914	13	4	516
56	9.0122	24	8	3.6362	14	4	684
84	7.8520	24	9	4.0141	15	4	1020
112	8.3651	24	10	4.3065	15	4	1356
	Third coarse space, TOL$_\mu = 1/10$, TOL$_\nu = -\infty$						
scal.	ρ			extension			
H/h	cond	its	#EV	cond	its	#EV	#dual
14				3.2109	12	19	180
28	no			4.3890	13	19	348
42	rho			5.1775	14	19	516
56	variant			5.7748	14	19	684
84				6.6648	15	19	1020
112				7.3288	16	19	1356

TABLE 8.7
Scalar diffusion. Test Problem IV (see Figure 8.5). Third coarse space uses extension scaling with different η/h (see Definition 6.1 and Remark 6.7) only eigenvalue problem 6.1 with TOL$_\mu = 1/10$*; cf. also the third coarse space in Table 8.6 for $H/h = 28$. On squares of four elements in each direction, the coefficient is constant. Consequently, the number of constraints is reduced if the slab size η/h is increased such that a multiple of four is exceeded.*

Third coarse space on slabs
Economic version of extension scaling

H/h	η/h	cond	its	#EV	#dual
28	1	2.2478	11	50	348
28	2	1.9102	9	50	348
28	3	1.7047	9	50	348
28	4	1.5663	9	50	348
28	5	8.7645	13	33	348
28	6	7.4083	13	33	348
28	7	6.8836	13	33	348
28	8	6.6282	13	33	348
28	9	5.3208	13	29	348
28	10	5.2299	13	29	348
28	28	4.3890	13	19	348

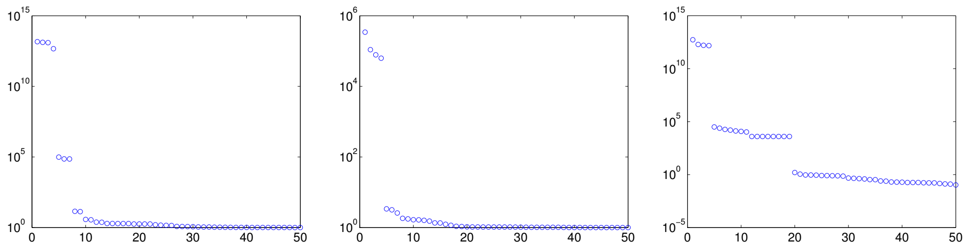

FIG. 8.7. *The 50 largest (inverse) eigenvalues of the generalized eigenvalue problems for Test Problem IV for* $H/h = 28$ *(see Figure 8.5). First coarse problem using deluxe scaling, second coarse problem using deluxe scaling, and third coarse space using extension scaling (from left to right); cf. Table 8.6.*

TABLE 8.8
Almost incompressible elasticity using a $\mathbb{P}2$-$\mathbb{P}0$*-finite elements discretization for* 3×3 *subdomains. Homogeneous coefficients with* $E = 1$ *and* $\nu = 0.499999$.

	First coarse space mult. scal., TOL $= 1/10$			Second coarse space mult. scal., TOL $= 10$			
H/h	cond	its	#EV	cond	its	#EV	#dual var.
10	2.2563	12	32	6.3215	22	10	516
20	2.1821	14	34	5.4016	20	16	996
30	2.4743	15	34	5.1404	20	20	1476
40	2.6969	15	34	5.4856	20	22	1956

TABLE 8.9
Almost incompressible elasticity using $\mathbb{P}2$-$\mathbb{P}0$*-finite elements and* 3×3 *subdomains. Homogeneous coefficients with* $E = 1$*. We vary* ν*,* $H/h = 20$*.*

	First coarse space mult. scal., TOL $= 1/10$			Second coarse space mult. scal., TOL $= 10$			
ν	cond	its	#EV	cond	its	#EV	#dual var.
0.3	1.9078	12	34	7.7232	24	4	996
0.49	2.6715	14	34	5.1983	20	16	996
0.499	2.2745	14	34	5.3790	20	16	996
0.4999	2.1915	14	34	5.3993	20	16	996
0.49999	2.1830	14	34	5.4014	20	16	996
0.499999	2.1821	14	34	5.4016	20	16	996

TABLE 8.10
Almost incompressible elasticity using a $\mathbb{P}2$-$\mathbb{P}0$*-finite elements discretization and* 3×3 *subdomains. Channel distribution with* $E_1 = 1e3$ *(black),* $E_2 = 1$ *(white) and* $\nu = 0.4999$*; cf. Figure 8.1.*

	First coarse space mult. scal., TOL $= 1/10$			Second coarse space mult. scal., TOL $= 10$			
H/h	cond	its	#EV	cond	its	#EV	#dual var.
10	11.5279	25	50	11.3414	25	50	516
20	11.9831	23	54	11.8391	26	50	996
30	12.0786	23	54	11.9445	26	50	1476

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