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ALMOST OPTIMAL CONVERGENCE OF FEM-FDM FOR A LINEAR PARABOLIC INTERFACE PROBLEM*

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Abstract. The solution of a second-order linear parabolic interface problem by the finite element method is discussed. Quasi-uniform triangular elements are used for the spatial discretization while the time discretization is based on a four-step implicit scheme. The integrals involved are evaluated by numerical quadrature, and it is assumed that the mesh cannot be fitted to the interface. With low regularity assumption on the solution across the interface, the stability of the method is established, and an almost optimal convergence rate of $O\left(k^4 + h^2\left(1 + \frac{1}{|\log h|}\right)\right)$ in the $L^2(\Omega)$-norm is obtained. In terms of matrices arising in the scheme, we show that the scheme preserves the maximum principle under certain conditions. Numerical experiments are presented to support the theoretical results.

Key words. finite element method, interface, almost optimal, parabolic equation, implicit scheme

AMS subject classifications. 65N06, 65N15, 65N30

1. Introduction.

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \setminus \bar{\Omega}_1$ be another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$. We consider the parabolic interface problem

$$u_t - \nabla \cdot (a(x,t)\nabla u) + b(x,t)u = f(x,t) \qquad \text{in } \Omega \times (0,T] \tag{1.1}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= u_0(x) && \text{in } \Omega \\ u(x,t) &= 0 && \text{on } \partial\Omega \times [0,T] \end{aligned} \tag{1.2}$$

and interface conditions

$$\begin{aligned} [u]_\Gamma &= 0 \\ \left[a(x,t)\frac{\partial u}{\partial n}\right]_\Gamma &= g(x,t), \end{aligned} \tag{1.3}$$

where $0 < T < \infty$, the symbol $[u]$ denotes the jump of a quantity u across the interface Γ, and n is the unit outward normal to the boundary $\partial\Omega_i$, $i = 1, 2$.

The interface conditions are defined as the differences of the limiting values from each side of the interface, i.e.,

$$[u]_{m\in\Gamma} := \lim_{x\to m^+} u_2(x,t) - \lim_{x\to m^-} u_1(x,t)$$

and

$$\left[a(x,t)\frac{\partial u}{\partial n}\right]_{m\in\Gamma} := \left[\lim_{x\to m^+} a_2\nabla u_2(x,t) - \lim_{x\to m^-} a_1\nabla u_1(x,t)\right] \cdot n\,.$$

The input functions $a(x,t)$, $b(x,t)$, and $f(x,t)$ are assumed continuous on each domain but discontinuous across the interface for $t \in [0,T]$.

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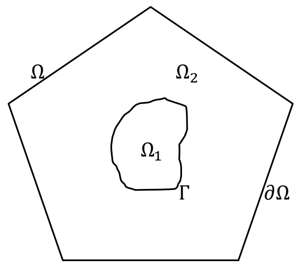

FIG. 1.1. *A polygonal domain* $\Omega = \Omega_1 \cup \Omega_2$ *with interface* Γ.

Time evolution equations (which often lead to parabolic PDEs) are considered to study and understand the dynamics of nature. The best-known linear parabolic PDE is the heat (or diffusion) equation, where an interface problem occurs when the heat transfer (or diffusion) involves more than one material medium, each having different properties such as conductivities, diffusion constants, etc. The solutions of interface problems may show higher regularities in each individual material region than in the entire physical domain because of discontinuities across the interface [3, 5]. Thus, achieving higher-order accuracy may be difficult using a classical method, hence, there is a need to find the solution to the problem by variational formulations. In what follows, we give a brief overview of existing works relevant to this research.

The study of interface problems by the FEM was first carried out by Babuska [3]. He studied finite element approximations to elliptic interface problems on smooth domains with a smooth interface. He formulated the problem as a minimization problem and defined and analyzed a quadratic functional which was used to obtain an error estimate of optimal order in the $H^1(\Omega)$-norm. For more works on linear elliptic interface problems, see [4, 7, 14, 16, 20]. The finite element approximation of nonlinear elliptic interface problems was discussed in [12, 17, 18, 28].

Using backward Euler time discretization, Chen and Zou [5] studied the convergence of the fully discrete solution to the exact solution using a fitted FEM. They obtained error estimates for clearly defined interpolation and elliptic projection operators, which were used to prove suboptimal error estimates in the $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$-norms when the global regularity of the solution is low. Sinha and Deka [24] proposed and analyzed an unfitted finite element discretization for both elliptic and parabolic problems with discontinuous coefficients. An optimal-order error estimate in the H^1-norm and an almost optimal-order error estimate in the L^2-norm were derived for elliptic interface problems. An extension to parabolic interface problems was also discussed, and estimates in the $L^2(H^1)$-norm and the $L^2(L^2)$-norm were derived for the spatially discrete scheme. A fully discrete scheme based on the backward Euler method was analyzed, and an optimal-order error estimate in the $L^2(H^1)$-norm was derived.

Sinha and Deka [25] studied FEMs for second-order semilinear elliptic and parabolic interface problems in two-dimensional convex polygonal domains. The approximation theory of Brezzi-Rappaz-Raviart was used to obtain an optimal error estimate in the H^1-norm for semilinear elliptic problems, and the linear theory of interface problems was used to obtain a similar estimate for semilinear parabolic problems. They assumed that the mesh can be fitted exactly to the arbitrary interface, which might not be so in practice.

Deka and Ahmed [8] improved on the works of [5, 23] and also confirmed the optimal error estimates in the $L^2(0,T;L^2(\Omega))$-norm. Optimal error estimates in the $L^2(L^2)$ and $L^2(H^1)$ norms were established for linear semi-discrete schemes, and a similar error estimate was also extended to semilinear interface problems.

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Recently, Chaoxia Yang [27] studied the convergence of the finite element solution of a nonlinear parabolic interface problem with a linear source term. A linearized two-step backward difference scheme was used for the time discretization, and convergence rates of almost optimal order in the L^2-norm were established for the fully discrete scheme.

It is known that spatial and time discretizations are sources of errors in the FEM, however, research has largely focused on the use of the FEM for linear parabolic interface problems with emphasis on the improvement on the spatial discretization, whereas not much work considered improvements on the time discretization. The most-widely used first-order backward Euler time discretization is of low accuracy in time. Therefore this work is designed to analyze and demonstrate (with relevant examples) the convergence rate of the finite element solution with a four-step time discretization to the exact solution under certain regularity assumptions on the data of the problem. The result of this work shows that almost optimal order of convergence in the $L^2(\Omega)$-norm can be obtained when the integrals involved are evaluated by numerical quadrature and in the case that the mesh cannot be fitted to the interface. In this study, the linear theories of interface and non-interface problems and the Sobolev imbedding inequality are used. Other tools utilized in this paper are approximation properties for linear interpolation operators and projection operators.

We employ the standard notation for Sobolev spaces and norms in this paper. For $m \geq 0$ and real p with $1 \leq p \leq \infty$, let $W^{m,p}$ denote the Sobolev space of order m. For the case $p = 2$, we write $W^{m,p} = H^m$. $H_0^m(\Omega)$ represents the closed subspace of $H^m(\Omega)$ that is the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^m(\Omega)$. We use the definition and notation in [1] when m is fractional.

For a given Banach space B, we define

$$u \in W^{m,p}(0,T;B) \Leftrightarrow$$
$$\begin{cases} u(t) \in B \text{ for a.e. } t \in (0,T) \text{ and } \sum_{i=0}^m \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p \, dt < 0 & 1 \leq p < \infty, \\ u(t) \in B \text{ for a.e. } t \in (0,T) \text{ and } \sum_{i=0}^m \ \operatorname{ess\,sup}_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B < 0 & p = \infty, \end{cases}$$

equipped with the norms

$$\|u\|_{W^{m,p}(0,T;B)} = \begin{cases} \left[\sum_{i=0}^m \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p \, dt \right]^{1/p} & 1 \leq p < \infty, \\ \sum_{i=0}^m \ \operatorname{ess\,sup}_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B & p = \infty. \end{cases}$$

We write $L^2(0,T;B) = W^{0,2}(0,T;B)$ and $H^m(0,T;B) = W^{m,2}(0,T;B)$. We shall use the following spaces

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2), \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norms

$$\begin{aligned} \|v\|_X &= \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} & \forall v \in X, \\ \|v\|_Y &= \|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)} & \forall v \in Y. \end{aligned}$$

The paper is organized as follows. In Section 2, we describe a finite element discretization of the problem, establish an error estimate for the interpolation operator, and state approximations across the interface. In Section 3, we establish the stability of the method and prove a convergence rate of almost optimal order for the fully discrete scheme. A discrete maximum principle of the scheme is established in Section 4, and numerical examples are presented in Section 5. A conclusion is given in Section 6. Throughout this paper, C is a generic positive constant (which is independent of the mesh parameter h and the time step size k) and may take on different values at different occurrences. Regarding the regularity of solutions of the interface problem (1.1)–(1.3), we have the following results:

THEOREM 1.1. *Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{1/2}(\Gamma))$, and $u_0 \in H_0^1(\Omega)$. Then the problem (1.1)–(1.3) has a unique solution*

$$u \in L^2(0,T;X) \cap H^1(0,T;Y)$$

and

$$\|u\|_{L^2(0,T;X)} \leq C\Big[\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|g(0)\|_{H^{1/2}(\Gamma)} + \|g(T)\|_{H^{1/2}(\Gamma)} + \|g\|_{L^2(0,T;H^{1/2}(\Gamma))}\Big]. \tag{1.4}$$

Proof. This follows from [22, Theorem 2.1, p. 736]. ∎

In what follows, we obtain the weak form by multiplying (1.1) by a test function $v \in H_0^1(\Omega)$ and using Green's identity yielding

$$(u_t, v) + A(u,v) = (f,v) + \langle g, v\rangle_\Gamma \quad \forall v(t) \in H_0^1(\Omega), \qquad \text{a.e. } t \in [0,T], \tag{1.5}$$

where

$$(\phi,\psi) = \int_\Omega \phi\psi\, dx \qquad A(\phi,\psi) = \int_\Omega [a(x,t)\nabla\phi\cdot\nabla\psi + b(x,t)\phi\psi]\ dx$$
$$\langle\phi,\psi\rangle_\Gamma = \int_\Gamma \phi\psi\, d\Gamma\,.$$

We recall that for $u \in H^1(\Omega)$, the boundary values of u (i.e. $u_{|\partial\Omega}$) are defined in $H^{1/2}(\partial\Omega)$, the trace space of $H^1(\Omega)$. Similarly, the trace space on the interface Γ is $H^{1/2}(\Gamma)$. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is continuous and satisfies the embedding

$$\|z\|_{H^{1/2}(\partial\Omega)} \leq C\|z\|_{H^1(\Omega)} \quad \forall z \in H^1(\Omega).$$

See Adams [1] and Evans [9] for more information on the trace operator.

2. The finite element discretization. $\mathcal{T}_h$ denotes a partition of Ω into disjoint triangles K (called elements) such that no vertex of any triangle lies in the interior of any side of another triangle. The domain Ω_1 is approximated by a domain Ω_1^h with a polygonal boundary Γ_h whose vertices all lie on the interface Γ. Ω_2^h represents the domain with $\partial\Omega$ and Γ_h as its exterior and interior boundaries, respectively.

Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{T}_h^\star$ denote the set of all elements that are intersected by the interface Γ:

$$\mathcal{T}_h^\star = \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}.$$

$K \in \mathcal{T}_h^\star$ is called an interface element, and we write $\Omega_h^\star = \bigcup_{K\in\mathcal{T}_h^\star} K$. The triangulation $\mathcal{T}_h$ of the domain Ω satisfies the following conditions:

(i) $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} \bar{K}$.

(ii) If $\bar{K}_1, \bar{K}_2 \in \mathcal{T}_h$ and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.

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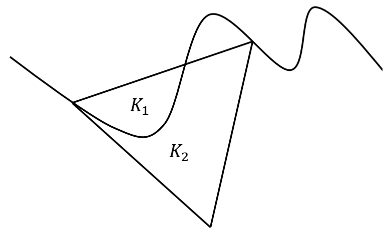

FIG. 2.1. *A typical interface element.*

(iii) Each $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h and has at most two vertices lying on Γ_h.
(iv) For each element $K \in \mathcal{T}_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles, respectively. It is assumed that, for some fixed $h_0 > 0$, there exist two positive constants C_0 and C_1, independent of h, such that

$$C_0 r_K \le h \le C_1 \bar{r}_K \qquad \forall h \in (0, h_0).$$

For any interface element $K \in \mathcal{T}_h^\star$, let $K_1 = K \cap \Omega_1$ and $K_2 = K \cap \Omega_2$. It was shown by Chen and Zou [5] that

$$\text{either} \qquad \text{meas}(K_1) \le C h_K^3 \qquad \text{or} \qquad \text{meas}(K_2) \le C h_K^3.$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise linear functions on $\mathcal{T}_h$ vanishing on $\partial\Omega$. The FE solution $u_h(x,t) \in S_h$ is represented as

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t) \phi_j(x) \ ,$$

where each basis function ϕ_j, $j = 1, 2, \ldots, N_h$, is a pyramid function with unit height. For the approximation $\hat{g}(t)$, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ be the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h. See [5, 6] for the construction of such finite element spaces. Let $\pi_h : C(\bar{\Omega}) \to S_h$ be the Lagrange interpolation operator corresponding to the space S_h. The standard interpolation theory cannot be applied because the solutions of interface problems are non-smooth or even discontinuous across the interface. We follow Chen and Zou [5] for the proof of the following result.

LEMMA 2.1. *For the linear interpolation operator $\pi_h : C(\bar{\Omega}) \to S_h$, we have, for $m = 0, 1$ and $0 < h < 1$,*

$$\|u - \pi_h u\|_{H^m(\Omega)} \le C h^{2-m} \left(1 + \frac{1}{|\ln h|}\right)^{1/2} \|u\|_X \qquad \forall u \in X. \tag{2.1}$$

Proof. By standard finite element interpolation theory [5, 6, 26], for any triangle $K \in \mathcal{T}_h \backslash \mathcal{T}_h^*$,

$$\|u - \pi_h u\|_{H^m(K)} \le C h^{2-m} \|u\|_{H^2(K)}, \qquad \text{for} \quad m = 0, 1. \tag{2.2}$$

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Now, for any element $K \in \mathcal{T}_h^*$, using Hölder's inequality and the fact that $\text{meas}(K_1) \leq Ch^3$, we have

$$\begin{aligned}
\|u - \pi_h u\|^2_{H^m(K_1)} &\leq \sum_{|\alpha|\leq m} \|D^\alpha(u - \pi_h u)\|^2_{L^2(K_1)} \\
&\leq [\text{meas}(K_1)]^{\frac{p-2}{p}} \left(\sum_{|\alpha|\leq m} \|D^\alpha(u - \pi_h u)\|^p_{L^2(K_1)} \right)^{2/p} \\
&\leq Ch^{\frac{3(p-2)}{p}} \left(\sum_{|\alpha|\leq m} \|D^\alpha(u - \pi_h u)\|^p_{L^2(K_1)} \right)^{2/p} \qquad \text{for } p > 2.
\end{aligned}$$

Therefore,

$$\|u - \pi_h u\|_{H^m(K_1)} \leq Ch^{\frac{3(p-2)}{2p}} \|u - \pi_h u\|_{W^{m,p}(K_i)} \leq Ch^{\frac{3(p-2)}{2p}} \|u - \pi_h u\|_{W^{m,p}(K)}.$$

Again by standard finite element interpolation theory,

$$\|u - \pi_h u\|_{H^m(K_1)} \leq Ch^{\frac{3p-6}{2p}+1-m} \|u\|_{W^{1,p}(K)} \quad \text{for any } p > 2,\ m = 0, 1.$$

Recall the Sobolev embedding inequality in two dimensions [21],

$$\|\phi\|_{L^p(\Omega_i)} \leq Cp^{1/2} \|\phi\|_{H^1(\Omega_i)} \quad \forall p > 2, \phi \in H^1(\Omega_i), \quad i = 1, 2.$$

Therefore,

$$\|u - \pi_h u\|_{H^m(K_1)} \leq Ch^{\frac{3p-6}{2p}+1-m} p^{1/2} \|u\|_{H^1(K)} \quad \text{for any } p > 2, \quad m = 0, 1. \tag{2.3}$$

By means of the extensions [5],

$$\|u - \pi_h u\|_{H^m(K_2)} \leq Ch^{2-m} \|u\|_{H^2(K)}, \quad m = 0, 1, \tag{2.4}$$

it follows from (2.3) and (2.4) that

$$\sum_{K\in\mathcal{T}_h^*} \|u - \pi_h u\|^2_{H^m(K)} \leq Ch^{4-2m} \left[1 + ph^{1-6/p}\right] \|u\|^2_X. \tag{2.5}$$

From (2.2) and (2.5), we have

$$\|u - \pi_h u\|^2_{H^m(\Omega)} \leq Ch^{4-2m} \|u\|^2_X + Ch^{4-2m} ph^{1-6/p} \|u\|^2_X, \qquad m = 0, 1, \quad p > 2.$$

Since $p > 2$, we take

$$p = 2\left(1 + \frac{1}{|\ln h|}\right) > 2 \qquad \text{for } 0 < h < 1,$$

and (2.1) follows. $\square$

For the approximation property of g_h to the interface function g, we have the following result (cf. [5]).

LEMMA 2.2. *Assume that $g \in H^2(\Gamma)$. Then we have*

$$|\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}| \leq Ch^{3/2} \|g\|_{H^2(\Gamma)} \|v_h\|_{H^1(\Omega_h^\star)} \qquad \forall v_h \in S_h.$$

We recall some results which will be used in our analysis; see [7, 24] for proofs.

LEMMA 2.3. *Let $\Omega_h^\star$ be the union of all interface triangles and $f \in H^2(\Omega)$ for $t \in [0, T]$. Then we have*

$$\begin{aligned}
\|v\|_{H^1(\Omega_h^\star)} &\leq Ch^{1/2} \|v\|_X \qquad\qquad \forall v \in X, \\
|(f, v) - (f, v)_h| &\leq Ch^2 \|f\|_{H^2(\Omega)} \|v\|_{H^1(\Omega)}.
\end{aligned}$$

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3. Error estimates. We discuss a fully discrete scheme based on a four-step backward difference approximation and analyze almost optimal order error estimates in the $L^2(\Omega)$-norm. The finite element analysis of parabolic non-interface problems is described in Thomee [26] and the references therein. The interval [0,T] is divided into M (for simplicity) equally-spaced subintervals:

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk$, $k = T/M$ being the time step. Let $I_n = (t_{n-1}, t_n]$ be the nth subinterval, and let

$$u^n = u(x, t_n), \quad f^n = f(x, t_n), \quad \text{and} \quad g^n = g(x, t_n).$$

For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the backward difference quotients defined by

$$\begin{aligned}
\partial^1 w^n &= \frac{w^n - w^{n-1}}{\tau_1} && n = 1, 2, \ldots, M,\\
\partial^2 w^n &= \frac{3w^n - 4w^{n-1} + w^{n-2}}{2\tau_2} && n = 2, 3, \ldots, M,\\
\partial^3 w^n &= \frac{11w^n - 18w^{n-1} + 9w^{n-2} - 2w^{n-3}}{6\tau_3} && n = 3, 4, \ldots, M,\\
\partial^4 w^n &= \frac{25w^n - 48w^{n-1} + 36w^{n-2} - 16w^{n-3} + 3w^{n-4}}{12k} && n = 4, 6, \ldots, M,
\end{aligned}$$

where τ_1, τ_2, and τ_3 are the time steps used to obtain U_h^1, U_h^2, and U_h^3, respectively. The fully discrete finite element approximation to (1.5) is defined as follows: with $U_h^0 = \pi_h u_0$, find $U_h^n \in S_h$ such that

$$\begin{aligned}
(\partial^1 U_h^1, v_h)_h + A_h(U_h^1, v_h) &= (f^1, v_h)_h + \langle g_h^1, v_h\rangle_{\Gamma_h} \quad \forall v_h \in S_h,\\
(\partial^2 U_h^2, v_h)_h + A_h(U_h^2, v_h) &= (f^2, v_h)_h + \langle g_h^2, v_h\rangle_{\Gamma_h} \quad \forall v_h \in S_h,\\
(\partial^3 U_h^3, v_h)_h + A_h(U_h^3, v_h) &= (f^3, v_h)_h + \langle g_h^3, v_h\rangle_{\Gamma_h} \quad \forall v_h \in S_h,\\
(\partial^4 U_h^n, v_h)_h + A_h(U_h^n, v_h) &= (f^n, v_h)_h + \langle g_h^n, v_h\rangle_{\Gamma_h} \quad \forall v_h \in S_h,\\
&\qquad n = 4, 5, \ldots, M,
\end{aligned} \tag{3.1}$$

where $(\psi, \phi)_h : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $A_h(\phi, \psi) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, and $\langle g(x,t), v_h\rangle_{\Gamma_h} : H^{1/2}(\Gamma) \times H^1(\Omega) \to \mathbb{R}$ are given by

$$\begin{aligned}
(\psi, \phi)_h &= \sum_{K \in \mathcal{T}_h} \int_K \psi\phi \, dx, \quad A_h(\phi, \psi) = \sum_{K \in \mathcal{T}_h} \int_K [a(x,t)\nabla\phi \cdot \nabla\psi + b(x,t)\phi\psi] \, dx\,,\\
\langle g(x,t), v_h\rangle_{\Gamma_h} &= \int_{\Gamma_h} g(x,t)\phi \, dx \qquad \forall \phi, \psi \in H^1(\Omega),\ g \in H^{1/2}(\Gamma),\ t \in [0, T].
\end{aligned}$$

Correspondingly, $(\psi, \phi)_h : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $A_h(\phi, \psi) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $\langle g(x,t), v_h\rangle_{\Gamma_h} : H^{1/2}(\Gamma) \times H^1(\Omega) \to \mathbb{R}$ are the discrete versions of (ψ, ϕ), $A(\phi, \psi)$, and $\langle g(x,t), v_h\rangle_\Gamma$, respectively, and are obtained numerically using quadrature schemes. See [13] and the reference therein for more information on numerical integration in the FEM.

The scheme (3.1) is zero-stable. To see this, we obtain the first characteristic polynomials

$$\begin{aligned}
\rho_1(y) &= y - 1\\
\rho_2(y) &= \frac{3}{2}y^2 - 2y + \frac{1}{2}\\
\rho_3(y) &= \frac{11}{6}y^3 - 3y^2 + \frac{3}{2}y - \frac{1}{3}\\
\rho_4(y) &= \frac{25}{12}y^4 - 4y^3 + 3y^2 - \frac{4}{3}y + \frac{1}{4}.
\end{aligned}$$

The roots of these polynomials have moduli less than one, and the roots with modulus one are simple. See [19] for more information on zero-stability of multistep methods. The analysis of this work is done with the assumption that $\frac{\partial^i u}{\partial t^i}$ exist for $i = 1, \ldots, 5$. It can be shown using a Taylor expansion that

$$\begin{aligned}
\|U_h^n - 2U_h^{n-1} + U_h^{n-2}\|_{L^2(\Omega)} &\le (\Delta t)^2 \lambda_0\\
\|U_h^n - 3U_h^{n-1} + 3U_h^{n-2} - U_h^{n-3}\|_{L^2(\Omega)} &\le (\Delta t)^3 \lambda_1\\
\|U_h^n - 4U_h^{n-1} + 6U_h^{n-2} - 4U_h^{n-3} + U_h^{n-4}\|_{L^2(\Omega)} &\le (\Delta t)^4 \lambda_2
\end{aligned} \tag{3.2}$$

for sufficiently small Δt and $\lambda_0, \lambda_1, \lambda_2 \ge 0$. We have the following stability result:

LEMMA 3.1. *Let $a_i(x,t)$, $b_i(x,t)$, and $f_i(x,t)$ be continuous on $\Omega_i \times (0,T]$, $i = 1, 2$, and suppose that $g(x,t) \in L^2(0,T; H^{1/2}(\Gamma))$. Then there exists a constant C independent of k and h such that*

$$\begin{aligned}
&\|U_h^n\|^2_{L^2(\Omega)} + k\|U_h^n\|^2_{H^1(\Omega)}\\
&\qquad \le C\left[\|U_h^0\|^2_{L^2(\Omega)} + k\sum_{j=1}^{n}\left(\|f^j\|^2_{L^2(\Omega)} + \|g_h^j\|^2_{H^{1/2}(\Gamma_h)}\right) + k^3\right],
\end{aligned} \tag{3.3}$$

for $n = 1\ldots$ and $0 < k \le k_0 < 1$.

Proof. We take $\tau_1 = \tau_2 = \tau_3 = k$. Let $v_h = U_h^1$ in the first equation in (3.1). Then

$$\begin{aligned}
\|U_h^1\|^2_{L^2(\Omega)} + c_1 k\|U_h^1\|^2_{H^1(\Omega)} \le \|U_h^1\|_{L^2(\Omega)}\|U_h^0\|_{L^2(\Omega)} + k\|f^1\|_{L^2(\Omega)}\|U_h^1\|_{L^2(\Omega)}\\
+ k\|g_h^1\|_{H^{1/2}(\Gamma_h)}\|U_h^1\|_{L^2(\Omega)}.
\end{aligned}$$

By Young's inequality,

$$\frac{1}{2}(1-k)\|U_h^1\|^2_{L^2(\Omega)} + c_1 k\|U_h^1\|^2_{H^1(\Omega)} \le \frac{1}{2}\|U_h^0\|^2_{L^2(\Omega)} + k\|f^1\|^2_{L^2(\Omega)} + k\|g_h^1\|^2_{H^{1/2}(\Gamma_h)}.$$

For $0 < k \le k_0 < 1$, there exists a constant $C_0 = \frac{1}{1-k_0}$ such that

$$1 < (1-k)^{-1} \le (1 + C_0 k) \le C_0.$$

Therefore,

$$\begin{aligned}
\|U_h^1\|^2_{L^2(\Omega)} + 2c_1 k\|U_h^1\|^2_{H^1(\Omega)} \le (1 + C_0 k)\|U_h^0\|^2_{L^2(\Omega)}\\
+ 2(1 + C_0 k)k\left(\|f^1\|^2_{L^2(\Omega)} + \|g_h^1\|^2_{H^{1/2}(\Gamma_h)}\right).
\end{aligned} \tag{3.4}$$

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Let $v_h = U_h^2$ in (3.1). Then

$$\begin{aligned}\frac{1}{k}\|U_h^2\|^2_{L^2(\Omega)} + c_1\|U_h^2\|^2_{H^1(\Omega)} \le \frac{1}{k}\|U_h^2\|_{L^2(\Omega)}\|U_h^1\|_{L^2(\Omega)} &+ \|f^2\|_{L^2(\Omega)}\|U_h^2\|_{L^2(\Omega)} \\ &+ \frac{1}{2k}\|U_h^2\|_{L^2(\Omega)}\|U_h^2 - 2U_h^1 + U_h^0\|_{L^2(\Omega)} \\ &+ \|g_h^2\|_{H^{1/2}(\Gamma_h)}\|U_h^2\|_{L^2(\Omega)}.\end{aligned}$$

By Young's inequality and (3.2), we have

$$\begin{aligned}&\frac{1}{2}(1-k)\|U_h^2\|^2_{L^2(\Omega)} + c_1 k\|U_h^2\|^2_{H^1(\Omega)} \\ &\qquad \le \frac{1}{2}\|U_h^1\|^2_{L^2(\Omega)} + 2k\|f^2\|^2_{L^2(\Omega)} + 2k\|g_h^2\|^2_{H^{1/2}(\Gamma_h)} + \frac{1}{4}\lambda_0^2 k^3.\end{aligned} \tag{3.5}$$

For $0 < k \le k_0$, (3.5) becomes

$$\begin{aligned}&\|U_h^2\|^2_{L^2(\Omega)} + 2c_1 k\|U_h^2\|^2_{H^1(\Omega)} \le (1+C_0k)\|U_h^1\|^2_{L^2(\Omega)} \\ &\qquad + 4(1+C_0k)k\left\{\|f^2\|^2_{L^2(\Omega)} + \|g_h^2\|^2_{H^{1/2}(\Gamma_h)}\right\} + \frac{1}{2}(1+C_0k)\lambda_0^2k^3.\end{aligned} \tag{3.6}$$

By a similar argument as that leading to (3.6), we obtain

$$\begin{aligned}\|U_h^3\|^2_{L^2(\Omega)} + 2c_1 k\|U_h^3\|^2_{H^1(\Omega)} \le (1+C_0k)&\|U_h^2\|^2_{L^2(\Omega)} \\ &+ 4(1+C_0k)k\left\{\|f^3\|^2_{L^2(\Omega)} + \|g_h^3\|^2_{H^{1/2}(\Gamma_h)}\right\} \\ &+ (1+C_0k)\lambda_0^2k^3 + \frac{4}{9}(1+C_0k)\lambda_1^2k^5.\end{aligned} \tag{3.7}$$

It follows from (3.4)–(3.7) that

$$\begin{aligned}&\|U_h^3\|^2_{L^2(\Omega)} + 2c_1 k\|U_h^3\|^2_{H^1(\Omega)} \\ &\qquad \le (1+C_0k)^3\|U_h^0\|^2_{L^2(\Omega)} \\ &\qquad\quad + 4k\sum_{j=1}^{3}\left\{(1+C_0k)^{4-j}\left(\|f^j\|^2_{L^2(\Omega)} + \|g_h^j\|^2_{H^{1/2}(\Gamma_h)}\right)\right\} \\ &\qquad\quad + k^3\lambda_0^2\sum_{j=1}^{2}(1+C_0k)^{2-j} + \frac{4}{9}k^5(1+C_0k)\lambda_1^2.\end{aligned} \tag{3.8}$$

For $n = 4, 5, \ldots$, a simple calculation shows that

$$\begin{aligned}\frac{1}{k}\|U_h^n\|^2_{L^2(\Omega)} + c_1\|U_h^n\|^2_{H^1(\Omega)} \le \frac{1}{k}\|U_h^{n-1}\|_{L^2(\Omega)}\|U_h^n\|_{L^2(\Omega)} &+ \frac{1}{4}k^3\lambda_2\|U_h^n\|_{L^2(\Omega)} \\ &+ \frac{1}{3}k^2\lambda_1\|U_h^n\|_{L^2(\Omega)} + \frac{1}{2}k\lambda_0\|U_h^n\|_{L^2(\Omega)} \\ &+ \|U_h^n\|_{L^2(\Omega)}\|f^n\|_{L^2(\Omega)} + \|U_h^n\|_{L^2(\Omega)}\|g_h^n\|_{H^{1/2}(\Gamma_h)}.\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}(1-k)\|U_h^n\|^2_{L^2(\Omega)} + c_1 k\|U_h^n\|^2_{H^1(\Omega)} \\ &\qquad\qquad \le \frac{1}{2}\|U_h^{n-1}\|^2_{L^2(\Omega)} + 2k\|f^n\|^2_{L^2(\Omega)} + 2k\|g_h^n\|^2_{H^{1/2}(\Gamma)} \\ &\qquad\qquad\quad + \frac{3}{16}k^7\lambda_2^2 + \frac{1}{3}k^5\lambda_1^2 + \frac{3}{4}k^3\lambda_0^2.\end{aligned} \tag{3.9}$$

For $0 < k \le k_0$ with $k_0 < 1$, (3.9) becomes

$$\begin{aligned}\|U_h^n\|_{L^2(\Omega)}^2 + 2c_1 k\|U_h^n\|_{H^1(\Omega)}^2 \le (1+C_0k)\|U_h^{n-1}\|_{L^2(\Omega)}^2 \\ + \; 4(1+C_0k)k\left(\|f^n\|_{L^2(\Omega)}^2 + \|g_h^n\|_{H^{1/2}(\Gamma)}^2\right) \\ + \; 2(1+C_0k)\left(\frac{3}{16}k^7\lambda_2^2 + \frac{1}{3}k^5\lambda_1^2 + \frac{3}{4}k^3\lambda_0^2\right).\end{aligned}$$

By iteration on n, we obtain

$$\begin{aligned}&\|U_h^n\|_{L^2(\Omega)}^2 + 2c_1k\|U_h^n\|_{H^1(\Omega)}^2 \\ &\le (1+C_0k)^{n-3}\|U_h^3\|_{L^2(\Omega)}^2 \\ &\quad + 4k\sum_{j=4}^{n}\left\{(1+C_0k)^{n-j+1}\left(\|f^j\|_{L^2(\Omega)}^2 + \|g_h^j\|_{H^{1/2}(\Gamma_h)}^2\right)\right\} \\ &\quad + 2\left(\frac{3}{16}k^7\lambda_2^2 + \frac{1}{3}k^5\lambda_1^2 + \frac{3}{4}k^3\lambda_0^2\right)\sum_{j=4}^{n}(1+C_0k)^{n-j+1}.\end{aligned} \tag{3.10}$$

Now the bound (3.3) follows from (3.8) and (3.10). □

The result below establishes the convergence of the fully discrete solution to the exact solution in the $L^2(\Omega)$-norm.

THEOREM 3.2. *Let u^n and U_h^n be the solutions of* (1.5) *and* (3.1)*, respectively. Suppose that $a_i(x,t)$, $b_i(x,t)$, and $f_i(x,t)$ are continuous on $\Omega_i \times (0,T]$, $i = 1,2$, and let $g(x,t) \in L^2(0,T;H^2(\Gamma))$. Then there exists a positive constant C independent of h and k such that*

$$\|u^n - U_h^n\|_{L^2(\Omega)} \le \left[k^4 + h^2\left(1 + \frac{1}{|\log h|}\right)\right]C.$$

The proof of this result requires some preparations:

LEMMA 3.3. *For all $\nu_h, \omega_h \in S_h$, we have*

$$|A(\nu_h,\omega_h) - A_h(\nu_h,\omega_h)| \le Ch\|\nu_h\|_{H^1(\Omega_h^\star)}\|\omega_h\|_{H^1(\Omega_h^\star)}\,. \tag{3.11}$$

Proof. Let $\tilde{K}$ denote either K_1 or K_2; see Figure 2.1.

$$\begin{aligned}|A(\nu_h,\omega_h) - A_h(\nu_h,\omega_h)| &\le C\sum_{K\in\mathcal{T}_h^\star}\int_{\tilde{K}}\{|\nabla\nu_h\cdot\nabla\omega_h| + |\nu_h\omega_h|\} \\ &\le C\sum_{K\in\mathcal{T}_h^\star}\left\{\|\nabla\nu_h\|_{L^2(\tilde{K})}\|\nabla\omega_h\|_{L^2(\tilde{K})} + \|\nu_h\|_{L^2(\tilde{K})}\|\omega_h\|_{L^2(\tilde{K})}\right\} \\ &\le C\sum_{K\in\mathcal{T}_h^\star}\left\{h\|\nabla\nu_h\|_{L^2(K)}\|\nabla\omega_h\|_{L^2(K)} + h^{-1}\|\nu_h\|_{L^2(K)}\|\omega_h\|_{L^2(K)}\right\}.\end{aligned}$$

We have made use of the fact that ν_h and ω_h are linear on $K \in \mathcal{T}_h$ and $\text{meas}(\tilde{K}) \le Ch^3$. Now, (3.11) follows using an inverse inequality [21]. □

Let $P_h : X \cap H^1(\Omega) \to S_h$ be the elliptic projection of the exact solution u in S_h defined by

$$A_h(P_h\nu,\phi) = A(\nu,\phi) \qquad \forall\phi \in S_h,\ t \in [0,T]. \tag{3.12}$$

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For this projection, we have the following result.

LEMMA 3.4. *Let $a_i(x,t)$, $b_i(x,t)$ be continuous on $\Omega_i \times (0,T]$ for $i = 1, 2$. Assume that $u \in X \cap H_0^1$, and let $P_h u$ be defined as in (3.12). Then*

$$\begin{aligned} \|P_h u - u\|_{H^1(\Omega)} &\le Ch\left(1 + \frac{1}{|\ln h|}\right)^{1/2} \|u\|_X, \\ \|P_h u - u\|_{L^2(\Omega)} &\le Ch^2\left(1 + \frac{1}{|\ln h|}\right) \|u\|_X. \end{aligned} \tag{3.13}$$

Proof. For $\rho > 0$, we have with an arbitrary $\phi \in S_h$

$$\begin{aligned} \rho\|P_h u - u\|^2_{H^1(\Omega)} &\le A_h(P_h u - u, P_h u - u) \\ &\le A_h(P_h u, P_h u - \phi) - A_h(u, P_h u - \phi) \\ &\quad + A_h(P_h u - u, \phi - u), \\ &\le |A(u, P_h u - \phi) - A_h(u, P_h u - \phi)| \\ &\quad + |A_h(P_h u - u, \phi - u)|. \end{aligned}$$

Using Hölder's inequality with $\text{meas}(\tilde{K}) \le Ch^3$, we obtain

$$\begin{aligned} \rho\|P_h u - u\|^2_{H^1(\Omega)} &\le Ch\|u\|_{H^1(\Omega)}\|P_h u - \phi\|_{H^1(\Omega)} + \|P_h u - u\|_{H^1(\Omega)}\|\phi - u\|_{H^1(\Omega)} \\ &\le Ch\|u\|_{H^1(\Omega)}\|P_h u - u\|_{H^1(\Omega)} + Ch\|u\|_{H^1(\Omega)}\|u - \phi\|_{H^1(\Omega)} \\ &\quad + \|P_h u - u\|_{H^1(\Omega)}\|\phi - u\|_{H^1(\Omega)} \\ &\le \varepsilon Ch^2\|u\|^2_{H^1(\Omega)} + \frac{3}{4\varepsilon}\|P_h u - u\|^2_{H^1(\Omega)} + \varepsilon\|\phi - u\|^2_{H^1(\Omega)}. \end{aligned}$$

Using (2.1) with $\varepsilon = 2/\rho$ and $\phi = \pi_h u$, we have

$$\|P_h u - u\|_{H^1(\Omega)} \le Ch\left(1 + \frac{1}{|\ln h|}\right)^{1/2} \|u\|_X \qquad \forall u \in X. \tag{3.14}$$

Now consider the dual problem

$$\begin{aligned} -\nabla \cdot (a(x,t)\nabla\psi) + b(x,t)\psi &= P_h u - u \qquad &&\text{in } \Omega, \\ \psi &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

whose weak form is

$$A(\psi, \phi) = (P_h u - u, \phi) \qquad \forall \phi \in H_0^1(\Omega). \tag{3.15}$$

By the Poincaré inequality [2],

$$\begin{aligned} \mu_1\|\psi\|^2_{H^1(\Omega)} &\le A(\psi,\psi) = (P_h u - u, \psi) \le \|P_h u - u\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \\ &\le C\|P_h u - u\|_{L^2(\Omega)}\|\psi\|_{H^1(\Omega)}. \end{aligned}$$

Thus, we arrive at

$$\|\psi\|_{H^1(\Omega)} \le C\|P_h u - u\|_{L^2(\Omega)}. \tag{3.16}$$

From the definition of X and the fact that $\psi \in H_0^1(\Omega)$,

$$\begin{aligned} \|\psi\|_X &\le C\|\psi\|_{H^2(\Omega)} \le C\|\Delta\psi\|_{L^2(\Omega)} \le C\|a\Delta\psi\|_{L^2(\Omega)} \\ &= C\|P_h u - u + \nabla a \cdot \nabla\psi - b\psi\|_{L^2(\Omega)} \\ &\le C\|P_h u - u\|_{L^2(\Omega)}. \end{aligned} \tag{3.17}$$

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In the last inequality, we have made use of (3.16) and the fact that a and b are bounded. Now, it follows from (3.15) that for all $\phi \in S_h$,

$$\begin{aligned}\|P_h u - u\|^2_{L^2(\Omega)} &= A(P_h u - u, \psi) = A(P_h u - u, \psi - \phi) + A(P_h u - u, \phi)\\ &\leq C\|P_h u - u\|_{H^1(\Omega)}\|\psi - \phi\|_{H^1(\Omega)} + |A(P_h u, \phi) - A_h(P_h u, \phi)|.\end{aligned}$$

By (2.1), (3.14), and Lemma 3.3 with $\phi = \pi_h \psi$, we obtain

$$\|P_h u - u\|^2_{L^2(\Omega)} \leq Ch^2\left(1 + \frac{1}{|\ln h|}\right)\|u\|_X\|\psi\|_X + Ch\|P_h u\|_{H^1(\Omega_h^\star)}\|\pi_h\psi\|_{H^1(\Omega_h^\star)}.$$

It follows from Lemma 2.3 that

$$\|P_h u - u\|^2_{L^2(\Omega)} \leq Ch^2\left(1 + \frac{1}{|\ln h|}\right)\|u\|_X\|\psi\|_X + Ch^2\|P_h u\|_{H^1(\Omega)}\|\pi_h\psi\|_{H^1(\Omega)}.$$

It is easy to see from (3.12) and the definition of π_h that $\|\pi_h\psi\|_{H^1(\Omega)} \leq C\|\psi\|_{H^1(\Omega)}$ and $\|P_h\psi\|_{H^1(\Omega)} \leq C\|\psi\|_{H^1(\Omega)}$ for $C > 0$, therefore

$$\|P_h u - u\|^2_{L^2(\Omega)} \leq Ch^2\left(1 + \frac{1}{|\ln h|}\right)\|u\|_X\|\psi\|_X. \tag{3.18}$$

Now (3.13) follows from (3.14), (3.17), and (3.18). $\square$

LEMMA 3.5. *Let $a_i(x,t)$, $b_i(x,t)$ be continuous on $\Omega_i \times (0,T]$, for $i = 1,2$. Assume that $u \in X \cap H^1_0$, and let $P_h u$ be defined as in (3.12). Then*

$$\begin{aligned}\|(P_h u - u)_t\|_{H^1(\Omega)} &\leq Ch\left(1 + \frac{1}{|\ln h|}\right)^{1/2}\left(\|u\|_X + \|u_t\|_X\right),\\ \|(P_h u - u)_t\|_{L^2(\Omega)} &\leq Ch^2\left(1 + \frac{1}{|\ln h|}\right)\left(\|u\|_X + \|u_t\|_X\right).\end{aligned} \tag{3.19}$$

Proof. Let $\xi = P_h u - u$, and assume that a_t is uniformly bounded. Following the argument of Thomee [26], we have for $\rho > 0$,

$$\begin{aligned}\rho\|\xi_t\|^2_{H^1(\Omega)} &\leq A(\xi_t, \xi_t) = A(\xi_t, \phi - u_t) + A(\xi_t, (P_h u)_t - \phi)\\ &= A(\xi_t, \phi - u_t) + \int_\Omega \left[\frac{\partial}{\partial t}(a\nabla\xi) - \frac{\partial a}{\partial t}\nabla\xi\right]\cdot\nabla((P_h u)_t - \phi)\,dx\\ &\qquad + \int_\Omega\left[\frac{\partial}{\partial t}(b\xi) - \frac{\partial b}{\partial t}\xi\right]((P_h u)_t - \phi)\,dx\\ &\leq \frac{\rho}{2}\|\xi_t\|^2_{H^1(\Omega)} + \frac{1}{2\rho}\|\phi - u_t\|^2_{H^1(\Omega)} + \|\xi\|^2_{H^1(\Omega)} + \|(P_h u)_t - \phi\|^2_{H^1(\Omega)}.\end{aligned}$$

The last inequality is obtained after some calculations using Young's inequality. From Lemma 2.1 and (3.13) with $\phi = \pi_h u_t$, we obtain

$$\|(P_h u - u)_t\|_{H^1(\Omega)} \leq Ch\left(1 + \frac{1}{|\ln h|}\right)^{1/2}(\|u\|_X + \|u_t\|_X).$$

Following the duality argument similar to the one that led to (3.18), it is not difficult to arrive at

$$\|(P_h u - u)_t\|_{L^2(\Omega)} \;\leq\; Ch^2\left(1 + \frac{1}{|\ln h|}\right)(\|u\|_X + \|u_t\|_X). \qquad \square$$

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Proof of Theorem 3.2. Letting $z^n = P_h u^n - U_h^n$ in (3.1) and using (3.12), we have

$$(\partial^4 z^n, v_h)_h + A_h(z^n, v_h) = B_1 + B_2, \tag{3.20}$$

where

$$\begin{aligned} B_1 &= (\partial^4(P_h u^n - u^n), v_h)_h + (\partial^4 u^n - u_t^n, v_h) + (\partial^4 u^n, v_h)_h - (\partial^4 u^n, v_h), \\ B_2 &= (f^n, v_h) - (f^n, v_h)_h + \langle g^n, v_h\rangle_\Gamma - \langle g_h^n, v_h\rangle_{\Gamma_h}. \end{aligned}$$

With $v_h = z^n$, we have

$$\begin{aligned} B_1 \le\, & \|\partial^4(P_h u^n - u^n)\|^2_{L^2(\Omega)} + \frac{1}{2}\|z^n\|^2_{L^2(\Omega)} + \|\partial^4 u^n - u_t^n\|^2_{L^2(\Omega)}, \\ & + \gamma C h^4 \|\partial^4 u^n\|^2_X + \frac{1}{4\gamma}\|z^n\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.21}$$

Using Lemma 2.2, Lemma 2.3, and the fact that $\|D^\alpha z^n\|_{L^2(\Omega)} = 0$ for $|\alpha| = 2$, we have

$$\begin{aligned} B_2 &\le Ch^2\|f^n\|_{H^2(\Omega)}\|z^n\|_{H^1(\Omega)} + Ch^2\|g^n\|_{H^2(\Gamma)}\|z^n\|_{H^1(\Omega)} \\ &\le C(\gamma)h^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right) + \frac{1}{2\gamma}\|z^n\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.20), we find, for $c_1 > 0$,

$$\begin{aligned} &\frac{1}{k}\|z^n\|^2_{L^2(\Omega)} + c_1\|z^n\|^2_{H^1(\Omega)} \\ &\qquad \le \frac{C}{k}\Big(\|z^n\|_{L^2(\Omega)}\|z^{n-1}\|_{L^2(\Omega)} + \|z^n\|_{L^2(\Omega)}\|z^{n-2}\|_{L^2(\Omega)} \\ &\qquad\qquad + \|z^n\|_{L^2(\Omega)}\|z^{n-3}\|_{L^2(\Omega)} + \|z^n\|_{L^2(\Omega)}\|z^{n-4}\|_{L^2(\Omega)}\Big) \\ &\qquad\quad + \|\partial^4(P_h u^n - u^n)\|^2_{L^2(\Omega)} + \frac{1}{2}\|z^n\|^2_{L^2(\Omega)} \\ &\qquad\quad + \|\partial_k u^n - u_t^n\|^2_{L^2(\Omega)} + Ch^4\|\partial^4 u^n\|^2_X \\ &\qquad\quad + Ch^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right) + \frac{3}{4\gamma}\|z^n\|^2_{H^1(\Omega)}. \end{aligned}$$

With $\gamma = \frac{3}{4c_1}$, we obtain

$$\begin{aligned} \left(1 - \frac{4}{7}k\right)\|z^n\|^2_{L^2(\Omega)} \le\, & C\Big(\|z^{n-1}\|^2_{L^2(\Omega)} + \|z^{n-2}\|^2_{L^2(\Omega)} \\ & \qquad + \|z^{n-3}\|^2_{L^2(\Omega)} + \|z^{n-4}\|^2_{L^2(\Omega)}\Big) \\ & + C\Big[k\|\partial^4(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^4 u^n - u_t^n\|^2_{L^2(\Omega)} \\ & + kh^4\|\partial^4 u^n\|^2_X + kh^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right)\Big]. \end{aligned} \tag{3.23}$$

For $k \in (0,1)$ it holds that $\left(1-\frac{4}{7}k\right)^{-1} < 1+\frac{4}{3}k < \frac{7}{3}$, and therefore (3.23) becomes

$$\begin{aligned}\|z^n\|^2_{L^2(\Omega)} \le C\Bigg[&\|z^{n-1}\|^2_{L^2(\Omega)} + \|z^{n-2}\|^2_{L^2(\Omega)} + \|z^{n-3}\|^2_{L^2(\Omega)} + \|z^{n-4}\|^2_{L^2(\Omega)}\\ &+ k\|\partial^4(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^4 u^n - u^n_t\|^2_{L^2(\Omega)} + kh^4\|\partial^4 u^n\|^2_X\\ &+ kh^4\left(\|f^n\|^2_{H^2(\Omega)} + \|g^n\|^2_{H^2(\Gamma)}\right)\Bigg],\end{aligned}$$

for $n = 4,\ldots,M$. By iteration on n, we have

$$\begin{aligned}\|z^n\|^2_{L^2(\Omega)} \le C&\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)} + \|z^2\|^2_{L^2(\Omega)} + \|z^3\|^2_{L^2(\Omega)}\right]\\ &+ Ck\sum_{j=4}^{n}\|\partial^4(u^j - P_h u^j)\|^2_{L^2(\Omega)} + Ch^4 k\sum_{j=4}^{n}(\|f^j\|^2_{H^2(\Omega)} + \|g^j\|^2_{H^2(\Gamma)})\\ &+ Ck\sum_{j=4}^{n}\|\partial^4 u^j - u^j_t\|^2_{L^2(\Omega)} + Ch^4 k\sum_{j=4}^{n}\|\partial^4 u^j\|^2_X.\end{aligned}$$

After a simple calculation, we obtain

$$\begin{aligned}\|z^n\|^2_{L^2(\Omega)} \le{}& C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)} + \|z^2\|^2_{L^2(\Omega)} + \|z^3\|^2_{L^2(\Omega)}\right]\\ &+ C\int_0^{t_n}\|(u-P_h u)_t\|^2_{L^2(\Omega)}\,dt + Ck^8\int_0^{t_n}\|\frac{\partial^5 u}{\partial t^5}\|^2_{L^2(\Omega)}\,dt\\ &+ Ch^4\int_0^{t_n}\left[\|u\|^2_X + \|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]\,dt\\ \le{}& C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)} + \|z^2\|^2_{L^2(\Omega)} + \|z^3\|^2_{L^2(\Omega)}\right]\\ &+ Ch^4\left(1+\frac{1}{|\log h|}\right)^2\times\\ &\qquad\int_0^{t_n}\left[\|u\|^2_X + \|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]\,dt\\ &+ Ck^8\int_0^{t_n}\|\frac{\partial^5 u}{\partial t^5}\|^2_{L^2(\Omega)}\,dt\end{aligned}\tag{3.24}$$

by applying (3.13) and (3.19) to the above inequalities. Let $z^1 = P_h u^1 - U^1_h$. From (3.12) and (3.1), we have

$$\begin{aligned}(\partial^1 z^1, v_h)_h + A_h(z^1, v_h) = {}&(\partial^1(P_h u^1 - u^1), v_h)_h + (\partial^1 u^1 - u^1_t, v_h)\\ &+ (\partial^1 u^1, v_h)_h - (\partial^1 u^1, v_h)\\ &+ (f^1, v_h) - (f^1, v_h)_h + \langle g^1, v_h\rangle_\Gamma - \langle g^1_h, v_h\rangle_{\Gamma_h}.\end{aligned}$$

With $v_h = z^1$, we have

$$\begin{aligned}\frac{1}{\tau_1}\|z^1\|^2_{L^2(\Omega)} + \mu_1\|z^1\|^2_{H^1(\Omega)} \le{}& \frac{1}{\tau_1}\|z^0\|_{L^2(\Omega)}\|z^1\|_{L^2(\Omega)} + \|\partial^1(P_h u^1 - u^1)\|^2_{L^2(\Omega)}\\ &+ \frac{1}{2}\|z^1\|^2_{L^2(\Omega)} + \|\partial^1 u^1 - u^1_t\|^2_{L^2(\Omega)} + Ch^4\|\partial^1 u^1\|^2_X\\ &+ Ch^4\left(\|f^1\|^2_{H^2(\Omega)} + \|g^1\|^2_{H^2(\Gamma)}\right) + \frac{3}{4\gamma}\|z^1\|^2_{H^1(\Omega)},\end{aligned}$$

where we have made use of Lemma 2.2 and Lemma 2.3 in the last inequality. With $\gamma = \frac{3}{4\mu_1}$, we obtain

$$\left(1-\frac{2}{3}\tau_1\right)\|z^1\|^2_{L^2(\Omega)} \le \|z^0\|^2_{L^2(\Omega)} + \tau_1\|\partial^1(P_h u^1 - u^1)\|^2_{L^2(\Omega)} + \tau_1\|\partial^1 u^1 - u^1_t\|^2_{L^2(\Omega)} + \tau_1 h^4\|\partial^1 u^1\|^2_X + \tau_1 C h^4\left(\|f^1\|^2_{H^2(\Omega)} + \|g^1\|^2_{H^2(\Gamma)}\right).$$

For $\tau_1 \in (0,1)$, there is a constant $C > 0$ such that $\left(1-\frac{2}{3}\tau_1\right)^{-1} \le C$, therefore,

$$\begin{aligned}
\|z^1\|^2_{L^2(\Omega)} &\le C\Big[\|z^0\|^2_{L^2(\Omega)} + \tau_1\|\partial^1(P_h u^1 - u^1)\|^2_{L^2(\Omega)} + \tau_1\|\partial^1 u^1 - u^1_t\|^2_{L^2(\Omega)} \\
&\quad + \tau_1 h^4\|\partial^1 u^1\|^2_X + \tau_1 h^4\left(\|f^1\|^2_{H^2(\Omega)} + \|g^1\|^2_{H^2(\Gamma)}\right)\Big] \\
&\le C\|z^0\|^2_{L^2(\Omega)} + C\int_0^{t_1}\|(u-P_h u)_t\|^2_{L^2(\Omega)}\,dt + C\tau_1^2\int_0^{t_1}\|u_{tt}\|^2_{L^2(\Omega)}\,dt \\
&\quad + Ch^4\int_0^{t_1}\left[\|u\|^2_X + \|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]\,dt \\
&\le C\|z^0\|^2_{L^2(\Omega)} + C\tau_1^2\int_0^{t_1}\|u_{tt}\|^2_{L^2(\Omega)}\,dt \\
&\quad + Ch^4\left(1+\frac{1}{|\log h|}\right)^2\int_0^{t_1}\left[\|u\|^2_X + \|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]\,dt.
\end{aligned} \tag{3.25}$$

Here, we employ (3.13) and (3.19) in the above inequalities. By similar arguments to the one that led to (3.25), we arrive at

$$\begin{aligned}
\|z^2\|^2_{L^2(\Omega)} &\le C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)}\right] + C\tau_2^4\int_0^{t_2}\|u_{ttt}\|^2_{L^2(\Omega)}\,dt \\
&\quad + Ch^4\left(1+\frac{1}{|\log h|}\right)^2 \times \\
&\qquad \int_0^{t_2}\left[\|u\|^2_X + \|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]dt,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\|z^3\|^2_{L^2(\Omega)} &\le C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)} + \|z^2\|^2_{L^2(\Omega)}\right] + C\tau_3^6\int_0^{t_3}\|u_{tttt}\|^2_{L^2(\Omega)}\,dt \\
&\quad + Ch^4\left(1+\frac{1}{|\log h|}\right)^2 \times \\
&\qquad \int_0^{t_3}\left[\|u\|^2_X + \|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]dt.
\end{aligned} \tag{3.27}$$

From (3.24)–(3.27) with $k \in (0,1)$, $\tau_1 \le k^4$, $\tau_2 \le k^2$, and $\tau_3 \le k^{4/3}$, we have

$$\begin{aligned}
\|z^n\|^2_{L^2(\Omega)} &\le C\|z^0\|^2_{L^2(\Omega)} + Ck^8\int_0^{t_n}\left\{\sum_{j=2}^{5}\|\frac{\partial^j u}{\partial t^j}\|^2_{L^2(\Omega)}\right\}\,dt \\
&\quad + Ch^4\left(1+\frac{1}{|\log h|}\right)^2 \times \\
&\qquad \int_0^{t_n}\left[\|u\|^2_X + \|u_t\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]\,dt.
\end{aligned} \tag{3.28}$$

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We obtain, from (3.13) and (3.28) with $U_h^0 = \pi_h u_0$,

$$\begin{aligned}\|u^n - U_h^n\|^2_{L^2(\Omega)} &\le 2\|u^n - P_h u^n\|^2_{L^2(\Omega)} + 2\|z^n\|^2_{L^2(\Omega)} \\ &\le Ch^4\left(1 + \frac{1}{|\log h|}\right)^2 \Bigg\{ \|u_0\|_X^2 + \|u^n\|_X^2 \\ &\qquad + \int_0^{t_n} \left[\|u\|_X^2 + \|u_t\|_X^2 + \|g\|^2_{H^2(\Gamma)}\right] dt \Bigg\} \\ &\quad + Ck^8 \int_0^{t_n} \left\{ \sum_{j=2}^{5} \|\partial_t^j u\|^2_{L^2(\Omega)} \right\} dt. \end{aligned} \tag{3.29}$$

The result follows from (3.29). ∎

REMARK 3.6. The initial three steps of the scheme are constructed using a low-order time discretization scheme. However, this does not affect the asymptotic convergence since the scheme is stable. Moreover, in the error analysis, the step sizes of these low-order discretization are chosen to be sufficiently small to guarantee the convergence rate.

REMARK 3.7. In the analysis, it was assumed that $\frac{\partial^5 u}{\partial t^5}$ exists. However, if the regularity of the solution with respect to time is very low, then the result obtained from the method may not be different from other low-order time discretization methods.

REMARK 3.8. The author has not investigated higher-order time discretization scheme which could lead to convergence rate of $O(k^5)$ and beyond. However, Thomee [26, p. 165] reported that the stability of higher-order time discretization scheme beyond order 6 is not guaranteed.

4. The discrete maximum principle (DMP). Here, we investigate the DMP of the proposed scheme. We show that the DMP is preserved under certain assumptions. See Farago et al. [10, 11] and the references therein for the DMP of parabolic non-interface problems. With $v_h = \phi_i$ in (3.1), we have

$$\mathbf{M}\,\frac{\frac{25}{12}\mathbf{u}^n - 4\mathbf{u}^{n-1} + 3\mathbf{u}^{n-2} - \frac{4}{3}\mathbf{u}^{n-3} + \frac{1}{4}\mathbf{u}^{n-4}}{k} + \mathbf{K}\mathbf{u}^n = \mathbf{l}^n, \tag{4.1}$$

where

$$M_{ij} = \int_\Omega \phi_j \phi_i \, dx \qquad K_{ij} = \int_\Omega [a\nabla\phi_j \cdot \nabla\phi_i + b\phi_j\phi_i]\, dx$$
$$l_i^n = \int_\Omega f(x,t_n)\phi_i \, dx \; + \int_{\Gamma_h} g_h(x,t_n)\phi_i \, d\Gamma_h.$$

Let $\mathbf{A} = \mathbf{M} + \frac{12}{25}k\mathbf{K}$. Then (4.1) becomes

$$\mathbf{A}\mathbf{u}^n \;=\; \mathbf{M}\left[\frac{48}{25}\mathbf{u}^{n-1} - \frac{36}{25}\mathbf{u}^{n-2} + \frac{16}{25}\mathbf{u}^{n-3} - \frac{3}{25}\mathbf{u}^{n-4}\right] + \frac{12}{25}k\,\mathbf{l}^n. \tag{4.2}$$

Let $\Omega_{ij} := \mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j)$. If $\mathrm{meas}(\Omega_{ij}) > 0$, then for regular meshes [11, p. 157],

$$\int_\Omega \phi_i\phi_j \, dx \le \mathrm{meas}(\Omega_{ij}), \qquad \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx = -K_0,$$

with some constants $K_0 > 0$ independent of i, j, h and $i \ne j$.

LEMMA 4.1. *Let $a(x,t) > 0$ and $b(x,t) > 0$ for $(x,t) \in \Omega \times (0,T)$. Let $\alpha = \sup a(x,t)$ and $\beta = \sup b(x,t)$ for $(x,t) \in \Omega \times (0,T)$. Let*

$$h < h_0 := \min\left\{1, \sqrt{\frac{\alpha K_0}{c\beta}}\right\} \qquad \textit{and} \qquad k \ge \frac{25ch^2}{12(\alpha K_0 - \beta ch^2)}. \tag{4.3}$$

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Then

$$A_{ij} \leq 0, \qquad \textit{for } i \neq j,\ i,j = 1,2,\ldots,N_h. \tag{4.4}$$

Proof.

$$\begin{aligned} A_{ij} &= \int_\Omega \phi_i \phi_j dx + \frac{12}{25} k \int_\Omega a \nabla \phi_i \cdot \nabla \phi_j + b \phi_i \phi_j \, dx \\ &\leq ch^2 \left[1 + \frac{12}{25} k \left(\beta - \frac{\alpha K_0}{ch^2} \right) \right]. \end{aligned} \tag{4.5}$$

Inequality (4.4) now follows from (4.3) and (4.5). ∎

We define the following

$$u^n_{\min} := \min\{u^n_1, u^n_2, \ldots, u^n_{N_h}\}, \qquad u^n_{\max} := \max\{u^n_1, u^n_2, \ldots, u^n_{N_h}\}$$

$$f^{(n-1,n)}_{\min} := \inf_{\substack{x \in \Omega \\ \tau \in ((n-1)k,\, nk)}} f(x,\tau), \qquad f^{(n-1,n)}_{\max} := \sup_{\substack{x \in \Omega \\ \tau \in ((n-1)k,\, nk)}} f(x,\tau),$$

for $n = 0, 1, \ldots, M$.

THEOREM 4.2. *Let the discretization be as in Section 3, and let*
(i) $A_{ij} \leq 0$ *for* $i \neq j,\ i,j = 1,2,\ldots,N_h$,
(ii) $M_{ii} \geq 0$ *for* $i = 1,2,\ldots,N_h$.
Then the scheme (3.1) *satisfies*

$$\underline{U}_n \leq u^n_i \leq \overline{U}_n, \tag{4.6}$$

with

$$\begin{aligned} \underline{U}_n &= \min\left\{0, \frac{48}{25} u^{n-1}_{\min} - \frac{36}{25} u^{n-2}_{\max} + \frac{16}{25} u^{n-3}_{\min} - \frac{3}{25} u^{n-4}_{\max}\right\} \\ &\qquad + k \min\left\{0, f^{(n-1,n)}_{\min} + \min_{\Gamma_{((n-1)k,nk)}} g_h\right\}, \\ \overline{U}_n &= \max\left\{0, \frac{48}{25} u^{n-1}_{\max} - \frac{36}{25} u^{n-2}_{\min} + \frac{16}{25} u^{n-3}_{\max} - \frac{3}{25} u^{n-4}_{\min}\right\} \\ &\qquad + k \max\left\{0, f^{(n-1,n)}_{\max} + \max_{\Gamma_{((n-1)k,nk)}} g_h\right\}, \end{aligned}$$

where $\Gamma_{((n-1)k,nk)} := \Gamma_h \times [(n-1)k, nk]$, $n = 4, \ldots, M$.

Proof. (i) and (ii) imply

$$\mathbf{A}^{-1} \geq 0 \qquad \text{and} \qquad \mathbf{A}^{-1}\mathbf{M} \geq 0\,. \tag{4.7}$$

From (4.2) we obtain

$$\begin{aligned} \mathbf{u}^n &= \mathbf{A}^{-1}\mathbf{M}\left[\frac{48}{25}\mathbf{u}^{n-1} - \frac{36}{25}\mathbf{u}^{n-2} + \frac{16}{25}\mathbf{u}^{n-3} - \frac{3}{25}\mathbf{u}^{n-4}\right] + \frac{12}{25} k \mathbf{A}^{-1}\mathbf{l}^n \\ &\leq \left[\frac{48}{25} u^{n-1}_{\max} - \frac{36}{25} u^{n-2}_{\min} + \frac{16}{25} u^{n-3}_{\max} - \frac{3}{25} u^{n-4}_{\min}\right] \mathbf{A}^{-1}\mathbf{M}\mathbf{e} \\ &\quad + \frac{12}{25} k \left[f^{(n-1,n)}_{\max} + \max_{\Gamma_{((n-1)k,nk)}} g_h \right] \mathbf{A}^{-1}\mathbf{e}, \end{aligned}$$

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where $\mathbf{e} = (1, 1, \ldots, 1)^T$. The lower bound in (4.6) is obtained by expressing the ith coordinate of $\mathbf{u}^n$ and using (4.7). The upper bound can be proved in a similar manner. □

REMARK 4.3. Following the same argument as above, it is not difficult to conclude from (3.1) that

$$\begin{aligned}
&\min\left\{0, u^0_{\min}\right\} + k\min\left\{0, f^{(0,1)}_{\min} + \min_{\Gamma_{(0,k)}} g_h\right\}\\
&\qquad\qquad \le u^1_i \le\\
&\qquad\qquad\qquad \max\left\{0, u^0_{\max}\right\} + k\max\left\{0, f^{(0,1)}_{\max} + \max_{\Gamma_{(0,k)}} g_h\right\},\\
&\min\left\{0, \frac{4}{3}u^1_{\min} - \frac{1}{3}u^0_{\max}\right\} + k\min\left\{0, f^{(1,2)}_{\min} + \min_{\Gamma_{(k,2k)}} g_h\right\}\\
&\qquad\qquad \le u^2_i \le\\
&\qquad\qquad\qquad \max\left\{0, \frac{4}{3}u^1_{\max} - \frac{1}{3}u^0_{\min}\right\} + k\max\left\{0, f^{(1,2)}_{\max} + \max_{\Gamma_{(k,2k)}} g_h\right\},\\
&\min\left\{0, \frac{18}{11}u^2_{\min} - \frac{9}{11}u^1_{\max} + \frac{2}{11}u^0_{\min}\right\} + k\min\left\{0, f^{(2,3)}_{\min} + \min_{\Gamma_{(2k,3k)}} g_h\right\}\\
&\qquad\qquad \le u^3_i \le\\
&\qquad\qquad\qquad \max\left\{0, \frac{18}{11}u^2_{\max} - \frac{9}{11}u^3_{\min} + \frac{2}{11}u^0_{\max}\right\} + k\max\left\{0, f^{(2,3)}_{\max} + \max_{\Gamma_{(2k,3k)}} g_h\right\},
\end{aligned}$$

with $\tau_1 = \tau_2 = \tau_3 = k$.

REMARK 4.4. (4.6) gives a DMP involving four steps. However, a one time-level DMP

$$\begin{aligned}
&\min\left\{0, u^{n-1}_{\min} - c_0k^2\right\} + k\min\left\{0, f^{(n-1,n)}_{\min} + \min_{\Gamma_{((n-1)k,nk)}} g_h\right\}\\
&\qquad\qquad \le u^n_i \le\\
&\qquad\qquad\qquad \max\left\{0, u^{n-1}_{\max} + c_0k^2\right\} + k\max\left\{0, f^{(n-1,n)}_{\max} + \max_{\Gamma_{((n-1)k,nk)}} g_h\right\}
\end{aligned} \tag{4.8}$$

could also be obtained. To see this, from (4.1), we have

$$\mathbf{B}\mathbf{u}^n = \mathbf{M}\left[-\frac{13}{12}\mathbf{u}^n + 4\mathbf{u}^{n-1} - 3\mathbf{u}^{n-2} + \frac{4}{3}\mathbf{u}^{n-3} - \frac{1}{4}\mathbf{u}^{n-4}\right] + k\mathbf{l}^n,$$

where $\mathbf{B} = \mathbf{M} + k\mathbf{K}$. If

$$h < h_0 := \min\left\{1, \sqrt{\frac{\alpha K_0}{c\beta}}\right\} \qquad \text{and} \qquad k \ge \frac{ch^2}{\alpha K_0 - \beta ch^2},$$

then

$$B_{ij} \le 0, \qquad \text{for } i \ne j,\ i, j = 1, 2, \ldots, N_h,$$

and therefore,

$$\begin{aligned}
\mathbf{u}^n &= \mathbf{B}^{-1}\mathbf{M}\left[-\frac{13}{12}\mathbf{u}^n + 4\mathbf{u}^{n-1} - 3\mathbf{u}^{n-2} + \frac{4}{3}\mathbf{u}^{n-3} - \frac{1}{4}\mathbf{u}^{n-4}\right] + k\mathbf{B}^{-1}\mathbf{l}^n\\
&\le \mathbf{B}^{-1}\mathbf{M}\left[\mathbf{u}^{n-1} + c_0k^2\mathbf{e}\right] + k\mathbf{B}^{-1}\mathbf{l}^n.
\end{aligned} \tag{4.9}$$

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TABLE 5.1
Numerical results for Example 5.1.

h	Error ($k = 0.125$)
0.291548	1.4308×10^{-2}
0.149147	3.10556×10^{-3}
0.0735402	7.78537×10^{-4}
0.046514	1.88909×10^{-4}
0.0197166	4.72251×10^{-5}

k	Error ($h = 0.0735402$)
0.5	7.98632×10^{-4}
0.25	7.79875×10^{-4}
0.125	7.78537×10^{-4}
0.0625	7.78525×10^{-4}

The inequality (4.9) follows from a Taylor expansion

$$-c_0 k^2 \mathbf{e} \leq -\frac{13}{12}\mathbf{u}^n + 3\mathbf{u}^{n-1} - 3\mathbf{u}^{n-2} + \frac{4}{3}\mathbf{u}^{n-3} - \frac{1}{4}\mathbf{u}^{n-4} \leq c_0 k^2 \mathbf{e}$$

for $c_0 \geq 0$. From (4.9) we find

$$\mathbf{u}^n \leq \left[u_{\max}^{n-1} + c_0 k^2\right] \mathbf{B}^{-1}\mathbf{M}\mathbf{e} + k\left[f_{\max}^{(n-1,n)} + \max_{\Gamma_{((n-1)k,nk)}} g_h\right] \mathbf{B}^{-1}\mathbf{e}.$$

The inequality on the right-hand side of (4.8) is obtained by expressing the ith coordinate of $\mathbf{u}^n$. The left-hand side can be proved in a similar manner.

5. Numerical results. For the numerical experiment, globally continuous piecewise linear finite element functions based on quasi-uniform triangulation described in Section 2 are used. The mesh generation and computation are done with FreeFEM++ [15].

EXAMPLE 5.1. We discuss the result of a two-dimensional linear parabolic interface problem in the domain $\Omega = (-1, 1) \times (-1, 1)$ where Ω_1 is a circle centered at $(0, 0)$ with radius $r = \sqrt{x^2 + y^2} = 0.5$, $\Omega_2 = \Omega \setminus \Omega_1$, and the interface Γ is a circle of radius 0.5 and therefore $\Gamma \neq \Gamma_h$.

Consider the problem (1.1) in $\Omega \times (0, 10]$. For the exact solution, we choose

$$u = \begin{cases} \frac{1}{8}(1 - 4r^2)t\cos(t) & \text{in } \Omega_1 \times (0, 10], \\ \frac{1}{4}(1 - x^2)(1 - y^2)(1 - 4r^2)\sin(t) & \text{in } \Omega_2 \times (0, 10]. \end{cases}$$

We choose a and b as

$$a = \begin{cases} x^2 + y^2 & \text{in } \Omega_1, \\ 4 & \text{in } \Omega_2, \end{cases} \qquad b = \begin{cases} 1 & \text{in } \Omega_1, \\ \exp(t) & \text{in } \Omega_2. \end{cases}$$

The source function f, the interface function g, and the initial data u_0 are determined from the choice of u. The errors in the L^2-norm at $t = 8$ for various step sizes k and mesh parameters h are presented in Table 5.1. It can be seen from Table 5.1 that

$$\text{Error} \;\approxeq\; O\left(k^{3.86} + h^{2.01}\left(1 + \frac{1}{|\ln h|}\right)\right).$$

The error estimate and the stability result also apply to a non-polygonal domain. We demonstrate this with the next example.

EXAMPLE 5.2. We consider a two-dimensional linear parabolic interface problem on a circular domain Ω with centre at $(0, 0)$ and radius 1 unit. Ω_1 is a circle centered at $(0, 0)$ with radius $r = \sqrt{x^2 + y^2} = 0.5$, $\Omega_2 = \Omega \setminus \Omega_1$, and the interface Γ is a circle of radius 0.5 and therefore $\Gamma \neq \Gamma_h$.

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TABLE 5.2
Numerical results for Example 5.2.

h	Error ($k = 0.125$)	k	Error ($h = 0.0373211$)
0.277835	3.375×10^{-3}	0.25	1.16381×10^{-4}
0.145117	8.16377×10^{-4}	0.125	5.2466×10^{-5}
0.0747083	2.07443×10^{-4}	0.0625	5.11201×10^{-5}
0.0373211	5.2466×10^{-5}	0.03125	5.10271×10^{-5}
0.0185029	1.43451×10^{-5}		

Consider the problem (1.1) in $\Omega \times (0, 10]$. For the exact solution, we choose

$$u = \begin{cases} (2 - 5x^2 - 5y^2)\sin t & \text{in } \Omega_1 \times (0, 10], \\ (1 - x^2 - y^2)\sin t & \text{in } \Omega_2 \times (0, 10]. \end{cases} \tag{5.1}$$

A piecewise constant value of a is used while $b = 0$:

$$a = \begin{cases} 1 & \text{in } \Omega_1, \\ 5 & \text{in } \Omega_2. \end{cases}$$

The source function f, the interface function g, and the initial data u_0 are determined from the choice of u. The errors in the L^2-norm at $t = 3$ for various step sizes k and mesh parameters h are presented in Table 5.2. It can be seen from the table that

$$\text{Error} \;\approxeq\; O\left(k^{5.53} + h^{1.94}\left(1 + \frac{1}{|\ln h|}\right)\right).$$

In the above examples, the mesh cannot be fitted exactly to the interface. However, convergence rate of optimal order could be obtained when the mesh captures the interface. We demonstrate this with the example below.

EXAMPLE 5.3. We consider a linear parabolic interface problem in the domain $\Omega = (-1, 1) \times (-1, 1)$ and where Ω_1 is a rectangle $(-1, 0) \times (-1, 1)$, $\Omega_2 = (0, 1) \times (-1, 1)$, and the interface Γ is the straight line $x = 0$.

Consider the problem (1.1) in $\Omega \times (0, 10]$. For the exact solution, we choose

$$u = \begin{cases} (1 - y^2)x(1 + x)\sin(t) & \text{in } \Omega_1 \times (0, 10], \\ (1 - y^2)\sin(4\pi x)t\exp(-t) & \text{in } \Omega_2 \times (0, 10]. \end{cases} \tag{5.2}$$

We choose $b = 0$ and a as

$$a = \begin{cases} t^2 + 1 & \text{in } \Omega_1, \\ 1 & \text{in } \Omega_2. \end{cases}$$

The source function f, the interface function g, and the initial data u_0 are determined from the choice of u. The errors in the L^2-norm at $t = 6$ for various step sizes k and mesh parameters h are presented in Table 5.3. It can be seen from the table that

$$\text{Error} \;\approxeq\; O\big(k^{3.94} + h^{1.99}\big).$$

TABLE 5.3
Numerical results for Example 5.3.

h	Error ($k = 0.3$)	k	Error ($h = 0.0329586$)
0.238603	2.91147×10^{-3}	0.6	5.58632×10^{-5}
0.127515	8.36758×10^{-4}	0.3	5.20176×10^{-5}
0.0653869	2.06193×10^{-4}	0.1	5.17526×10^{-5}
0.0329586	5.20176×10^{-5}	0.05	5.17489×10^{-5}
0.0170309	1.29882×10^{-5}		

6. Conclusion. In this paper, we have analyzed the convergence of a FEM for a parabolic interface problem with time discretization based on a four-step implicit scheme. It was shown that the method is numerically stable and that higher-order accuracy in time could be obtained for $k \in (0, k_0]$. To achieve the same accuracy using the backward Euler scheme, the step size k has to be very small which makes the latter computationally very time consuming. The scheme was shown to satisfy the DMP under certain assumptions on the mesh parameter h and time step k.

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