Electronic Transactions on Numerical Analysis.
Volume 46, pp. 89–106, 2017.

ISSN 1068–9613.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

SOLUTION OF COUPLED DIFFERENTIAL EQUATIONS ARISING FROM IMBALANCE PROBLEMS*

JENNY NIEBSCH†, RONNY RAMLAU†, AND KIRK M. SOODHALTER†

Abstract. We investigate the efficient solution of a set of coupled ordinary differential equations arising from a model describing vibrations of a wind turbine induced by imbalances of its spinning blades. The Forward Problem (computing vibrations from imbalances) admits a coupled integral equation formulation. Each integral equation is solved over the same underlying Hilbert space $\mathbb{H}$. We observe that these coupled integral equations can be represented as one compact operator acting on the tensor product space $\mathbb{R}^N \otimes \mathbb{H}$, where N is the number of coupled equations. A Galerkin discretization leads to a linear system of dimension nN with corresponding Kronecker product structure, where n is the number of basis elements used to discretize $\mathbb{H}$. Such systems can be solved using a variety of techniques which exploit the Kronecker structure. We demonstrate the effectiveness of exploiting the tensor structure with numerical experiments and show that our results agree with data recorded from actual wind turbines.

Key words. mathematical modeling, couple differential equations, integral equation, Kronecker product, tensor product, Hilbert space

AMS subject classifications. 15A69, 34A55, 45D05, 65R20, 65R30

1. Introduction and motivation. Any type of rotating machinery, from small devices such as vacuum pumps to large ones such as wind turbines, endures the effects of imbalances. A mass imbalance p_0 can be described as an additional mass m that is located at a distance r from the center of rotation and at an angle φ from a chosen zero angle, $p_0 = mre^{i\varphi}$. If such an eccentric mass rotates with a certain rotational frequency f or angular velocity $\omega = 2\pi f$, it induces forces that cause displacements in the form of vibrations of the same frequency. Provided that measurements for the vibrations are available, one can attempt to identify the imbalance and eliminate or reduce it through the placement of counterweights. It is clear that the reduction or removing of such imbalances is quite important in terms of secure and economic operation of rotating machinery.

The detection and elimination of imbalances in rotating machinery gives rise to two mathematical problems:

1. The Forward Problem of computing the vibrations for known imbalances and
2. The Inverse Problem of computing the imbalances from the vibrations which are usually measured at positions where sensors can be mounted.

This paper focuses on the first problem, the computation of vibrations for known imbalance distributions. This is of particular interest if one wants to determine the influence of imbalances on given rotating systems, e.g., to use simulations to define bounds of acceptable imbalances that still allow the safe operation of the system. The fast computation of an accurate solution to the Forward Problem is also an important ingredient for the Inverse Problem: as the Inverse Problem is ill-posed, it needs the application of regularization techniques for its solution. Popular iterative solvers require multiple applications of the forward operator. However, the Inverse Problem will be considered in a forthcoming paper.

Mathematically, the connection between the load $p(x,t)$ from an imbalance and the displacement $u(x,t)$ can be described by a partial differential equation (PDE). Since it is rarely explicitly solvable, a Finite Element Method can be used to transform it into a system

*Received February 27, 2017. Accepted March, 24, 2017. Published online on April, 13, 2017. Recommended by L. Reichel. The first author is supported by the Austrian Research Promotion Agency (FFG) under FFG project number 841288.

†Johann Radon Institute for Computational and Applied Mathematics, Altenbergerstraße 69, 4040 Linz, Austria {jenny.niebsch, ronny.ramlau, kirk.soodhalter}@ricam.oeaw.ac.at.

of ordinary differential equations (ODEs) of second order; cf. [5]:

$$(L\mathbf{u})(t,\omega(t)) := \mathbf{M}\mathbf{u}''(t,\omega(t)) + \mathbf{D}\mathbf{u}'(t,\omega(t)) + \mathbf{S}\mathbf{u}(t,\omega(t)) = \mathbf{p}(t,\omega(t)), \tag{1.1}$$

where $\omega(t)$ denotes the angular velocity of the rotation. Each node in the Finite Element model has the same number of degrees of freedom (DOF). $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{S}$ are square matrices of the dimension N, where N is the number of DOF in the model. They represent mass, damping, and stiffness properties of the rotating machine under consideration. The vector $\mathbf{u}$ is the vector of the displacements of every DOF; $\mathbf{p}$ is the vector of loads or forces induced by imbalances. In most applications there are only a few nodes where imbalances and thus the forces they induce can occur. Hence, the vector $\mathbf{p}$ is usually sparse.

Due to the unboundedness of the differential operator L in (1.1) and the fact that $\mathbf{u}$ is only given by measurements, usually at discrete time points, the problem cannot be solved directly. Even small deviations of the measurements, $\|\mathbf{u}-\mathbf{u}^\delta\| < \delta$, may result in arbitrary bad approximations $L\mathbf{u}^\delta$ of $L\mathbf{u}$. Additionally, the angular velocity $\omega(t)$ is usually also known from measurements only, which adds more instability to the evaluation of L. Instead of directly dealing with (1.1), we transform it into an equivalent operator equation of the form

$$(T\mathbf{p})(t) = \mathbf{u}(t). \tag{1.2}$$

This represents the direct problem as the operator T maps an imbalance or rather the load $\mathbf{p}$ induced by an imbalance $\mathbf{p}_0$ onto the vibrations $\mathbf{u}$.

In case the angular velocity ω is constant or quasi-static, the transformation from (1.1) into (1.2) is simple. We can use the fact that the load induced by an imbalance at the kth DOF, $p_0^k = m_k r_k e^{i\varphi_k}$, is a harmonic centrifugal force computed by $p_k(t) = p_0^k \omega^2 e^{i\omega t}$. The load vector for all DOF is then given by

$$\mathbf{p}(t) = \mathbf{p}_0 \cdot \omega^2 e^{i\omega t}, \quad \mathbf{p}_0 = [p_0^1, \cdots, p_0^N].$$

Assuming $\mathbf{u}(t) = \mathbf{u}_0 e^{i\omega t}$ and inserting this into the ODE system (1.1) results in an algebraic system for the amplitude vector $\mathbf{u}_0$ and the imbalance vector $\mathbf{p}_0$

$$\mathbf{T}\mathbf{p}_0 := \left(-\mathbf{M} + \frac{i}{\omega}\mathbf{D} + \frac{1}{\omega^2}\mathbf{S}\right)^{-1} \mathbf{p}_0 = \mathbf{u}_0. \tag{1.3}$$

This system can be reduced in dimension by taking into account the sparse structure of $\mathbf{p}_0$ and the fact that $\mathbf{u}_0$ can usually only be measured at a few DOF. This case has been treated previously and applied to several industrial applications, [3, 17, 18].

However, in practice, time measurements of the displacement often only are available for time dependent ω. For example, modern wind turbines operate with variable speed; a test run of an aircraft engine is so expensive that it saves a large amount of money if the measurements of the vibrations can be taken during an idle to maximum cycle or vice versa when the rotating frequency is continually increasing or decreasing. Such constraints motivate our investigation of the problem of reconstructing an imbalance distribution from measurements that are taken with time dependent angular velocity $\omega(t)$.

Mathematically the transformation of (1.1) to (1.2) becomes more challenging when ω is time-dependent. Since the load from an imbalance is now given by

$$\mathbf{p}(t) = \Re(\mathbf{p_0}[\omega^2(t) + i\omega'(t)]e^{i\,t\,\omega(t)}), \tag{1.4}$$

the above used transformation for the case of a constant frequency or angular velocity is no longer applicable. Additionally, as the frequency is usually measured, the differentiation of the frequency data induces another instability in the computation of the load.

There are several approaches to tackle the problem of solving (1.1) with general right-hand side $\mathbf{p}(t)$.

1. The standard method is to transform the system (1.1) of second-order ODEs into a system of first-order ODEs

$$\mathbf{x}' = \mathbf{B}\mathbf{x} + \mathbf{b},$$

   where

$$\mathbf{x} := \begin{bmatrix} \mathbf{u} \\ \mathbf{u}' \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} \mathbf{0} & \mathrm{I} \\ -\mathbf{M}^{-1}S & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{p} \end{bmatrix}.$$

   Formally, the solution of this system is given by

$$\mathbf{x}(t) = e^{\mathbf{B}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{B}(t-\tau)}\mathbf{b}(\tau)d\tau,$$

   cf., e.g., [10], with the initial value vector $\mathbf{x_0} := \begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}'(0) \end{bmatrix}$. Disadvantages of this approach are that the dimension of the problem is twice as large as the dimension of the original problem increasing numerical expense; the approach involves differentiation of the data $\mathbf{u}$ which is not stable for measured data, and the evaluation of matrix functions can be quite expensive.
2. Alternatively, numerical ODE solvers based on time step methods can be used. This also leads to high computational costs and suffers from problems of stability. Another disadvantage is that the operator T is not explicitly known. Thus, the direct computation of its adjoint and its derivative is not possible, potentially leading to further difficulties. Furthermore, the right hand side of the ODE now also depends on the derivative of the angular velocity, $\omega'(t)$, which can only be computed from noisy measurements and thus leads to unstable results for the ODE solver. Investigations with this approach did not lead to useful results for this application.
3. A third method is based on ideas presented in [24]. In this thesis, a one-dimensional initial value problem of the form

$$(pf')'(s) + q(s)f(s) = x(s), \quad f(a) = \alpha, f'(a) = \beta,$$

   is solved, with given measurements for $f \in C^2[a, b]$, known functions $p, q \in C[a, b]$, and $x \in C[a, b]$ being the unknown quantity. The equation is transformed into an integral equation of the first kind. With this approach, one preserves the original dimension of the problem and avoids differentiation of the data. If the problem is formulated in suitable Hilbert spaces, then the adjoint operators are easily derived, and Tikhonov regularization can be employed to solve the Inverse Problem of finding x in a stable fashion.

We expand this last approach to the N-dimensional setting of (1.1) but without the damping term $\mathbf{D}\mathbf{u}'(t)$. Including the damping term would again require the evaluations of computationally expensive matrix functions of the form $\exp(\mathbf{M}^{-1}\mathbf{D})$. Our main application is the area of wind turbines where the damping is in almost all cases negligible. The approach leads to an equivalent system of coupled integral equations of the same dimension as the original problem, which will be presented for known and unknown initial values in Section 2. As we will see in (2.7), the solution of the Forward Problem (i.e., the computation of the vibrations from known imbalances) itself requires the solution of an Inverse Problem. In Section 3 we show how the involved integral operator can be described using a tensor product formulation.

We give a brief summary of tensor products and results that will be useful for the further handling of our coupled integral equation system. We also show that if we discretize a tensor product operator through a Galerkin scheme we can write the resulting matrix equations in terms of Kronecker products. This approach simplifies not only the representation of our problem but also the computation of adjoint operators and most important the implementation using Kronecker products. In Section 4 we present the discretization of the Forward Problem operator and numerical examples.

2. Integral equation formulation of the Forward Problem. In this section we transform the ODE system (1.1) without the damping term into an equation of the form (1.2). We consider the system of differential equations with initial conditions

$$\begin{aligned} &\mathbf{M}\mathbf{u}''(t) + \mathbf{S}\mathbf{u}(t) = \mathbf{p}(t), \\ &\mathbf{u}(0) = \alpha,\ \mathbf{u}'(0) = \beta, \quad \alpha, \beta \in \mathbb{R}^N,\ t \in [0, T_e]. \end{aligned} \tag{2.1}$$

Let $\mathbf{M} = (m_{ij})_{i,j=1}^N, \mathbf{S} = (s_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times N}$ and

$$\mathbf{u}(t) = [u_1(t) \ \dots \ u_N(t)] := (u_1(t), \dots, u_N(t))^T$$

and $\mathbf{p}(t) = [p_1(t) \ \dots \ p_N(t)]$ accordingly be N-dimensional column vectors with functions as entries. Let $u_i \in C^2([0, T_e])$ and $p_i \in C([0, T_e])$ for $i = 1, \dots, N$.

We define the Volterra integral operators K mapping from $C[0, T_e] \to C[0, T_e]$ or $C^2[0, T_e] \to C[0, T_e]$, respectively, by

$$(Kx)(t) := \int_0^t (t - \theta) x(\theta) d\theta, \tag{2.2}$$

and $\tilde{\mathcal{K}}$ and $\mathcal{A}$ mapping from $(C[0, T_e])^N \to (C[0, T_e])^N$ by

$$\tilde{\mathcal{K}}\mathbf{p} := \mathbf{M}^{-1}[Kp_1 \ \dots \ Kp_N], \tag{2.3}$$

$$\mathcal{A}\mathbf{u} := \mathbf{M}^{-1}\mathbf{S}\,[-Ku_1 \ \dots \ -Ku_N], \tag{2.4}$$

where the action of a matrix on a vector of functions is the standard matrix-vector product. Let $\mathbf{I}$ be the identity operator on $(C[0, T_e])^N$.

PROPOSITION 2.1. *The equivalent integral equation formulation of* (2.1) *is given by*

$$(\tilde{\mathcal{K}}\mathbf{p})(t) = [(\mathbf{I} - \mathcal{A})\mathbf{u}](t) - \beta t - \alpha \qquad \textit{for } t \in [0, T_e]. \tag{2.5}$$

Proof. With $\int \mathbf{u}(t)dt$ we denote the vector of integrals taken of each component of $\mathbf{u}$. Thus, integration is commutable with a matrix multiplication. We have

$$\mathbf{u}(t) = \int_0^t \mathbf{u}'(\tau) d\tau + \alpha, \quad \mathbf{u}'(t) = \int_0^\tau \mathbf{u}''(\theta) d\theta + \beta.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Therefore,

$$
\begin{aligned}
\mathbf{u}(t) &= \int_0^t \left(\int_0^\tau \mathbf{u}''(\theta) d\theta + \beta \right) d\tau + \alpha \\
&=_{(2.1)} \int_0^t \left(\int_0^\tau [\mathbf{M}^{-1}(\mathbf{p} - \mathbf{S}\mathbf{u})](\theta) d\theta \right) d\tau + \int_0^t \beta d\tau + \alpha \\
&= \int_0^t \int_\theta^t [\mathbf{M}^{-1}(\mathbf{p} - \mathbf{S}\mathbf{u})](\theta) d\tau d\theta + \beta t + \alpha \\
&= \mathbf{M}^{-1} \int_0^t \int_\theta^t d\tau (\mathbf{p} - \mathbf{S}\mathbf{u})(\theta) d\theta + \beta t + \alpha \\
&= \mathbf{M}^{-1} \int_0^t (t-\theta)\mathbf{p}(\theta) d\theta + \mathbf{M}^{-1}\mathbf{S} \int_0^t (\theta - t)\mathbf{u}(\theta) d\theta + \beta t + \alpha \\
&= (\tilde{\mathcal{K}}\mathbf{p})(t) + (\mathcal{A}\mathbf{u})(t) + \beta t + \alpha,
\end{aligned}
$$

and (2.5) follows.

The kernel $k(t,\theta) = t - \theta$ is continuous on $[0,T]^2$. Hence the induced integral operator K is mapping from $C[0,T_e] \to C[0,T_e]$ or $C^2[0,T_e] \to C[0,T_e]$, respectively. The continuity properties are transferred to $\tilde{\mathcal{K}}$ and $\mathcal{A}$. □

The application of Tikhonov regularization requires the formulation of the problem in Hilbert spaces.

COROLLARY 2.2. *Let $\mathbb{H} = L_2[0,T_e]$, $K : \mathbb{H} \to \mathbb{H}$ be given as in (2.2). Then the operator $K : \mathbb{H} \to \mathbb{H}$ is well defined and continuous. Moreover, K is compact. Also the operators $\tilde{\mathcal{K}}, \mathcal{A} : \mathbb{H}^N \to \mathbb{H}^N$ as in (2.3) and (2.4) are well defined, continuous, and compact.*

Proof. Since the kernel of the integral operator K is continuous on the square $[0,T_e]^2$, $K(L_2[0,T_e]) \subset L_2[0,T_e]$, and $K : L_2[0,T_e] \to L_2[0,T_e]$ is compact according to the basic theorems of integral equations; cf., e.g., [4]. The components of the operators $\tilde{\mathcal{K}}$ and $\mathcal{A}$ are linear combinations of K and hence also continuous and compact. □

So far the integral formulation is presented for known initial values α, β. In most applications the initial conditions are unknown, and we modify the integral operator $\tilde{\mathcal{K}}$ by concatenating the unknown initial values to the vector of unknown loads $\mathbf{p}$:

$$\mathcal{K}[\mathbf{p},\ \alpha,\ \beta] := \tilde{\mathcal{K}}\mathbf{p} + \alpha + \beta t.$$

Obviously, $\mathcal{K}$ inherits the properties of $\tilde{\mathcal{K}}$. Therefore we state without a proof the following corollary.

COROLLARY 2.3. *Let $\mathbb{H} = L_2([0,T_e])$. Then*

$$
\begin{aligned}
\mathcal{K} : \mathbb{H}^N \times \mathbb{R}^{2N} &\to \mathbb{H}^N \\
[\mathbf{p},\ \alpha,\ \beta] &\mapsto \tilde{\mathcal{K}}\mathbf{p} + \alpha + \beta t,
\end{aligned}
\tag{2.6}
$$

is well defined, continuous, and compact.

We can now state the problem of computing $\mathbf{u}$ in (2.1) in the form

$$((\mathbf{I} - \mathcal{A})^{-1}\mathcal{K}\,[\mathbf{p},\ \alpha,\ \beta])(t) = \mathbf{u}(t), \tag{2.7}$$

for given $\mathbf{p} \in (L_2[0, T_e])^N$ provided that $\tilde{\mathcal{K}}\mathbf{p} + \alpha + \beta t$ is in the range of $(\mathbf{I} - \mathcal{A})$. This is in particular true if the spectrum of $\mathcal{A}$ fulfills $\sigma(\mathcal{A}) = \{0\}$. As K is a Volterra operator, we have $\sigma(K) = \{0\}$, and we will show in Proposition 3.4 that this property is inherited by $\mathcal{A}$.

We see that (2.7) has the form of (1.2) with $T = (\mathbf{I} - \mathcal{A})^{-1}\mathcal{K}$, with the difference being that the function vector $\mathbf{p}$ is concatenated with the initial values α and β.

3. Tensor product formulation. We now show that the operators $\mathcal{K}$, $\mathcal{A}$, and $(\mathbf{I} - \mathcal{A})$ can be elegantly described using tensor products. First we will give a brief description of the formal definition of a tensor product space between two Hilbert spaces and the tensor product of two operators. Then we demonstrate that the operators $\mathcal{K}$ and $\mathcal{A}$ can be formulated as tensor product operators on the space $\mathbb{R}^N \otimes \mathbb{H}$ with $\mathbb{H} = L_2[0, T_e]$ and that they are in $\mathcal{B}(\mathbb{R}^N \otimes \mathbb{H})$, where $\mathcal{B}(\cdot)$ denotes the bounded operators on a given space. We will also present a tensor product formulation of the equation

$$(\mathbf{I} - \mathcal{A})\mathbf{x} = \mathbf{y} \tag{3.1}$$

for an unknown $\mathbf{x}$, which must be solved during the determination of the solution $\mathbf{u}$ in (2.7). From this, we can relate the eigenvalues of $\mathcal{A}$ and those of $\mathbf{M}^{-1}\mathbf{S}$ and $-K$, allowing us to determine $\mathbf{x}$ when (3.1) is solvable. We then show that when this problem is discretized through a Galerkin scheme, the discretized matrix equation inherits this tensor product structure, admitting a Kronecker products representation. We note here that the Kronecker product is the tensor product in the special case that our underlying Hilbert spaces are real or complex Euclidean spaces; see Example 3.3.

In this section, let $\mathbb{H} = L_2(\Omega)$, and define $L : \mathbb{H} \to \mathbb{H}$ to be a compact integral operator induced by the kernel $k(s,t) \in \mathbb{H}(\Omega^2)$ with $(Lx)(s) = \int_\Omega k(s,t)x(t)\,dt$ for $x \in \mathbb{H}$. Let $\mathbb{H}^N$ denote the space of N-tuples of elements of $\mathbb{H}$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{H}^N$ denote two such N-tuples with

$$\mathbf{x} = [x_1 \ \ldots \ x_N] \ \text{ and } \ \mathbf{y} = [y_1 \ \ldots \ y_N]$$

for $x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N \in \mathbb{H}$, essentially vectors of Hilbert space elements. Let $\mathbf{V} = (v_{ij}) \in \mathbb{R}^{N\times N}$. We define an action of $\mathbb{R}^{N\times N}$ on $\mathbb{H}^N$ as matrix multiplication, i.e.,

$$\mathbf{V}\mathbf{x} = \left[\sum_{i=1}^N v_{1i}x_i \ \ldots \ \sum_{i=1}^N v_{Ni}x_i\right],$$

and the operator $\mathcal{L} : \mathbb{H}^N \to \mathbb{H}^N$ by

$$\mathcal{L}\mathbf{x} = \mathbf{V}\left[Lx_1 \ \ldots \ Lx_N\right] = \left[\sum_{i=1}^N v_{1i}Lx_i \ \ldots \ \sum_{i=1}^N v_{Ni}Lx_i\right]. \tag{3.2}$$

In this setting, $\tilde{\mathcal{K}}$ in (2.3) is represented by $\mathbf{V} = \mathbf{M}^{-1}$ and $L = K$, and $\mathcal{A}$ in (2.4) by $\mathbf{V} = \mathbf{M}^{-1}\mathbf{S}$ and $L = -K$.

3.1. The tensor product between Hilbert spaces. Here we give a brief description of the formal definition of the tensor product space between two Hilbert spaces, denoted $\mathbb{H}_1$ and $\mathbb{H}_2$ over the same underlying field (without loss of generality, assumed to be $\mathbb{C}$). The following summary is derived from [2, 13, 14, 15] and a more complete description of Hilbert space tensor products can be found in these sources and the references therein. Consider the Cartesian product set

$$\mathbb{H}_1 \times \mathbb{H}_2 = \{(h,k) | h \in \mathbb{H}_1,\ k \in \mathbb{H}_2\}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

For $x \in \mathbb{H}_1$ and $y \in \mathbb{H}_2$, we define the tensor product $x \otimes y$ to be the conjugate bilinear functional acting upon $\mathbb{H}_1 \times \mathbb{H}_2$ with $x \otimes y : \mathbb{H}_1 \times \mathbb{H}_2 \to \mathbb{C}$ defined by

$$(x \otimes y)(h, k) = \langle x, h \rangle_{\mathbb{H}_1} \langle y, k \rangle_{\mathbb{H}_2} \quad \forall \ h \in \mathbb{H}_1 \quad \text{and} \quad k \in \mathbb{H}_2.$$

To define the tensor product space $\mathbb{H}_1 \otimes \mathbb{H}_2$, we begin by building the space of finite sums (linear combinations)

$$\mathcal{S} = \left\{ \sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) \mid \alpha_i \in \mathbb{C}, \ (x_i, y_i) \in \mathbb{H}_1 \times \mathbb{H}_2, \quad \text{and} \quad n \in \mathbb{N} \right\}$$

with the operations of addition and scalar multiplication defined by

$$\begin{aligned} x_1 \otimes y_1 + x_2 \otimes y_2 &= (x_1 + x_2) \otimes (y_1 + y_2) - x_1 \otimes y_2 - x_2 \otimes y_1 \quad \text{and} \\ \alpha(x_1 \otimes y_1) &= ((\alpha x_1) \otimes y_1) = (x_1 \otimes (\alpha y_1)) \end{aligned}$$

with $x_1, x_2 \in \mathbb{H}_1$, $y_1, y_2 \in \mathbb{H}_2$, and $\alpha \in \mathbb{C}$. Furthermore, the tensor product is distributive with

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2 \quad \text{and} \quad (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$$

This is a linear space, and the tensor product space is an inner product space when paired with the inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathbb{H}_1 \otimes \mathbb{H}_2} = \langle x_1, x_2 \rangle_{\mathbb{H}_1} \langle y_1, y_2 \rangle_{\mathbb{H}_2},$$

with the understanding that this implies $x \otimes 0_{\mathbb{H}_2} = 0_{\mathbb{H}_1} \otimes y = 0_{\mathbb{H}_1} \otimes 0_{\mathbb{H}_2} = 0_{\mathbb{H}_1 \otimes \mathbb{H}_2}$, which must hold for this to be an inner product.

DEFINITION 3.1. *We define the tensor product space $\mathbb{H}_1 \otimes \mathbb{H}_2$ to be the completion of the space of finite sums $\mathcal{S}$ under $\langle \cdot, \cdot \rangle_{\mathbb{H}_1 \otimes \mathbb{H}_2}$.*

We can now define the tensor product of two operators.

DEFINITION 3.2. *The tensor product $A \otimes B : \mathbb{H}_1 \otimes \mathbb{H}_2 \to \mathbb{H}_1 \otimes \mathbb{H}_2$ of two operators $A \in \mathcal{B}(\mathbb{H}_1)$ and $B \in \mathcal{B}(\mathbb{H}_2)$ is defined through the action*

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

and we have $(A \otimes B) \in \mathcal{B}(\mathbb{H}_1 \otimes \mathbb{H}_2)$.

EXAMPLE 3.3. Let $\mathbf{A} = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} = (b_{ij})_{i,j=1}^{m} \in \mathbb{R}^{m \times m}$. The Kronecker product between $\mathbf{A}$ and $\mathbf{B}$, defined by

$$\mathbf{A} \otimes \mathbf{B} = (a_{ij} \cdot \mathbf{B})_{i,j=1}^{n} \in \mathbb{R}^{nm \times nm}$$

is a matrix representation of an operator acting upon the finite-dimensional tensor product space $\mathbb{R}^n \otimes \mathbb{R}^m = \mathbb{R}^{nm}$. Note that if $\mathbf{x} = (x_i)_{i=1}^{n} \in \mathbb{R}^n$ and $\mathbf{y} = (y_i)_{i=1}^{m} \in \mathbb{R}^m$, then $\mathbf{x} \otimes \mathbf{y} = (x_i \mathbf{y})_{i=1}^{n} \in \mathbb{R}^{nm}$.

As we will see in the next section, when we study the tensor product between a finite-dimensional space and an infinite-dimensional one, $\mathbb{R}^N \otimes \mathbb{H}$ retains the structure seen in Example 3.3, consisting of length-N vectors with scalar multiples of (now) infinite-dimensional elements of $\mathbb{H}$ as entries. Elements of $\mathcal{B}(\mathbb{R}^N \otimes \mathbb{H})$ consist of $N \times N$ matrices with scalar multiples of elements of $\mathcal{B}(\mathbb{H})$ as entries, i.e., $\mathcal{B}(\mathbb{R}^N \otimes \mathbb{H}) = \mathbb{R}^{N \times N} \otimes \mathcal{B}(\mathbb{H})$.

Further algebraic properties of tensor product spaces and the tensor product between operators (some of which we will later use) can be found, e.g., in [2, 13, 14, 15]. We will highlight, though, one interesting and useful property relating to eigenvalues of the tensor product between two bounded operators, which was shown in [2] for the full spectra of the two operators in the case that $\mathbb{H}_1 = \mathbb{H}_2$, and in [11] for the case $\mathbb{H}_1 = \mathbb{R}^{M_1}$ and $\mathbb{H}_2 = \mathbb{R}^{M_2}$.

PROPOSITION 3.4. *Let $A \in \mathcal{B}(\mathbb{H}_1)$ and $B \in \mathcal{B}(\mathbb{H}_2)$ be bounded linear operators for the Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, defined as above. Let $\lambda(A)$ and $\lambda(B)$ denote the eigenvalues of A and B, if indeed any exist. Then the eigenvalues of the tensor product operator $A \otimes B$ are precisely*

$$\lambda(A \otimes B) = \{\alpha \cdot \beta \mid \alpha \in \lambda(A), \quad \textit{and} \quad \beta \in \lambda(B)\}.$$

Proof. Let $\{(\lambda_i, v_i)\}$ and $\{(\gamma_i, w_i)\}$ be, respectively, the eigenpairs of A and B. Then we have that

$$(A \otimes B)(v_k \otimes w_\ell) = \lambda_k v_k \otimes \gamma_\ell w_\ell = \lambda_k \gamma_\ell (v_k \otimes w_\ell),$$

and it holds that $v_k \otimes w_\ell$ is an eigenvector of $A \otimes B$ with eigenvalue $\lambda_k \gamma_\ell$ for all $k, \ell \in \mathbb{N}$. Conversely, suppose that

$$(A \otimes B)(x \otimes y) = \beta(x \otimes y).$$

Then we have that

$$Ax \otimes By = \beta(x \otimes y)$$

for some $\beta \in \mathbb{C}$, and it must hold that

$$Ax = \beta_1 x \quad \text{and} \quad By = \beta_2 y \quad \text{such that} \quad \beta = \beta_1 \beta_2.$$

This then implies that $x = v_k$, $\beta_1 = \lambda_k$, $y = w_\ell$, and $\beta_2 = \gamma_\ell$ for some $k, \ell \in \mathbb{N}$. Thus every such β is of the form $\lambda_k \gamma_\ell$ for some pair $k, \ell \in \mathbb{N}$. ∎

Furthermore, it has been shown in [14] that operator compactness also carries over to the tensor product space.

PROPOSITION 3.5. *Let $A \in \mathcal{B}(\mathbb{H}_1)$ and $B \in \mathcal{B}(\mathbb{H}_2)$ be compact operators on $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Then $A \otimes B$ is a compact operator on $\mathbb{H}_1 \otimes \mathbb{H}_2$.*

3.2. Tensor product formulation of the coupled integral equations. In this section we demonstrate that each operator $\mathcal{L}$, defined as in (3.2), can be represented by tensor products, that $\mathcal{L} \in \mathcal{B}(\mathbb{R}^N \otimes \mathbb{H})$ and that (3.1) can be formulated as a tensor product operator equation on the space $\mathbb{R}^N \otimes \mathbb{H}$. This is applied to $\mathcal{K}$ from (2.6) and $\mathcal{A}$ from (2.4). We begin by observing that there is a one-to-one correspondence between $\mathbb{H}^N$ and the tensor product space $\mathbb{R}^N \otimes \mathbb{H}$, .

Let $\{\mathbf{e}_i\}_{i=1}^N$ be the canonical Euclidean basis of $\mathbb{R}^N$, and let $\{y_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathbb{H}$. Then it follows that for all pairs (i,j), with $1 \le i \le N$ and $1 \le j$, $\{\mathbf{e}_i \otimes y_j\}_{i,j}$ is an orthonormal basis for $\mathbb{R}^N \otimes \mathbb{H}$. If we choose $\mathbf{a} = \sum_{i=1}^N \alpha_i \mathbf{e}_i \in \mathbb{R}^N$ and $z = \sum_{j=1}^\infty \widetilde{\beta}_j y_j \in \mathbb{H}$ arbitrarily, then $\mathbf{a} \otimes z$ can be represented as

$$\begin{aligned}\mathbf{a} \otimes z = \left(\sum_{i=1}^N \alpha_i \mathbf{e}_i\right) \otimes \left(\sum_{j=1}^\infty \widetilde{\beta}_j y_j\right) &= \sum_{j=1}^\infty \sum_{i=1}^N \widetilde{\beta}_j \alpha_i (\mathbf{e}_i \otimes y_j) \\ &= \sum_{1 \le i \le N, 1 \le j} \widetilde{\beta}_j \alpha_i (\mathbf{e}_i \otimes y_j).\end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Since $\left\{\widetilde{\beta}_j\right\}_j \in \ell^2$ and $\left(\sum_{i=1}^N \alpha_i\right)$ is finite, we have that the series converges in the norm induced by the tensor inner product. Conversely, we have the mapping from the tensor product space to $\mathbb{H}^N$.

$$\eta : \sum_{1\leq i\leq N, 1\leq j} \beta_{ij}\left(\mathbf{e}_i \otimes y_j\right) \mapsto \left[\sum_{j=1}^\infty \beta_{1j} y_j \quad \sum_{j=1}^\infty \beta_{2j} y_j \quad \cdots \quad \sum_{j=1}^\infty \beta_{Nj} y_j\right] \in \mathbb{H}^N.$$

LEMMA 3.6. *Let $\mathbf{x} \in \mathbb{H}^N$ and $\mathcal{L}$ be defined as in (3.2). Then the action $\mathcal{L}$ on $\mathbf{x}$ can be represented as the action of the tensor product operator $\mathbf{V} \otimes L \in \mathcal{B}(\mathbb{R}^N \otimes \mathbb{H})$ on $\sum_{i=1}^N \mathbf{e}_i \otimes x_i$.*

Proof. This follows directly from the linearity of the tensor product operator and from the one-to-one correspondence between $\mathbb{H}^n$ and $\mathbb{R}^N \otimes \mathbb{H}$. We have that

$$(\mathbf{V} \otimes L) \sum_{i=1}^N \mathbf{e}_i \otimes x_i = \sum_{i=1}^N \mathbf{V}\mathbf{e}_i \otimes L x_i.$$

The vector $\mathbf{V}\mathbf{e}_i$ is simply the ith column of the matrix $\mathbf{V}$ and therefore can be decomposed in the canonical basis as $\mathbf{V}\mathbf{e}_i = \sum_{j=1}^N v_{ji}\mathbf{e}_j$. It follows then that this can be mapped backed to an N-tuple in $\mathbb{H}^N$ as

$$\sum_{i=1}^N \sum_{j=1}^N v_{ji}(\mathbf{e}_j \otimes L x_i) \mapsto \left[\sum_{i=1}^N v_{1i} L x_i \quad \sum_{i=1}^N v_{2i} L x_i \quad \cdots \quad \sum_{i=1}^N v_{Ni} L x_i\right] = \mathcal{L}\mathbf{x}. \qquad \square$$

We can now state the representation of $\tilde{\mathcal{K}}$ and $\mathcal{A}$ in the following corollary.

COROLLARY 3.7. *Let $\tilde{\mathcal{K}}$ be defined as in (2.3) and $\mathcal{A}$ as in (2.4). Let $\mathbf{B} := \mathbf{M}^{-1}$ and $\mathbf{C} := \mathbf{M}^{-1}\mathbf{S}$. Then we have*

$$\begin{aligned} \tilde{\mathcal{K}} &= \mathbf{B} \otimes K, \\ \mathcal{A} &= \mathbf{C} \otimes (-K), \\ \mathbf{I} - \mathcal{A} &= \mathbf{I}_N \otimes \mathbf{I}_{\mathbb{H}} - \mathbf{C} \otimes (-K), \end{aligned} \tag{3.3}$$

and all operators are in $\mathcal{B}(\mathbb{R}^N \otimes \mathbb{H})$.

We note from Proposition 3.5 that $\mathcal{A}$ is compact since $\mathbf{C}$ is a matrix and K is an integral operator with continuous kernel. The next corollary provides the tensor representation of $\mathcal{K}$:

COROLLARY 3.8. *Let $\mathcal{K}$ be defined as in (2.6). Then the tensor representation of $\mathcal{K}$ is given by*

$$\mathcal{K}[\mathbf{p},\ \alpha,\ \beta] = (\mathbf{B} \otimes K) \sum_{i=1}^N \mathbf{e}_i \otimes p_i + \sum_{i=1}^N \alpha_i \mathbf{e}_i \otimes 1 + \sum_{i=1}^N \beta_i \mathbf{e}_i \otimes t.$$

The identity (3.3) leads to a concise statement about the solvability of (3.1), which we get as a consequence of Fredholm's theorem. In this context, recall that for a compact operator $\mathcal{Q} \in \mathcal{L}(\mathbb{H})$, Fredholm's theorem states that for all $\lambda \in \mathbb{R}$, $(\lambda\mathbf{I} - \mathcal{Q})x = y$ is solvable for $y \in \mathbb{H}$ if and only if $y \perp \mathcal{N}(\lambda\mathbf{I} - \mathcal{Q}^*)$, i.e., if (λ, v) is an eigenpair of $\mathcal{Q}^*$ then it must hold that $\langle y, v\rangle = 0$.

COROLLARY 3.9. *The coupled integral equations (3.1) are solvable if*

1. *for every eigenvalue γ of $\mathbf{C}$, there does **not** exist an eigenvalue $\delta = 1/\gamma$ of $-K$, or*
2. *such pairs $(\gamma, 1/\gamma)$ do exist, and it holds that for all eigenpairs $(\gamma, \mathbf{a})$ of $\mathbf{C}^*$ and $(1/\gamma, v)$ of $-K^*$ (i.e., $(\mathbf{a} \otimes v) \in \mathcal{N}(\mathbf{I} - \mathcal{A}^*)$) we have $\left\langle \sum_{j=1}^N a_j y_j, v\right\rangle = 0$, where a_j denotes the jth component of $\mathbf{a}$, and y_j is the jth component function of the right-hand side of (3.1).*

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Proof. In this discussion, we use the fact that $-K$ is compact and thus has a spectrum consisting only of a discrete set of eigenvalues and their limit point 0. The matrix $\mathbf{C}$ is assumed to be nonsingular and thus has N nonzero eigenvalues with multiplicity. Therefore,

$$(\mathbf{I}_N \otimes I_{\mathbb{H}} - \mathbf{C} \otimes (-K)) \sum_{i=1}^{N} \mathbf{e}_i \otimes x_i = \sum_{i=1}^{N} \mathbf{e}_i \otimes y_i$$

is certainly solvable if 1 is not an eigenvalue of $\mathbf{C} \otimes (-K)$, proving Statement 1. From Proposition 3.5, we know that $\mathcal{A}$ is a compact operator. Thus, if 1 is an eigenvalue of $\mathcal{A}$, then (3.1) is solvable if $\mathbf{y} \perp \mathcal{N}(I - \mathcal{A}^*)$. If $\mathbf{a} \otimes v$ is an eigenvector of $\mathcal{A}^*$ with eigenvalue 1 then (according to the Fredholm alternative) for (3.1) to be solvable, we must have $\langle \mathbf{y}, \mathbf{a} \otimes v \rangle = 0$, and this must hold for all such $\mathbf{a} \otimes v$. Inserting $\mathbf{y} = \sum_{j=1}^{N} e_j \otimes y_j$ and $\mathbf{a} = (a_j)_j$ yields the result after some algebraic manipulation of the expression

$$\langle \mathbf{y}, \mathbf{a} \otimes v \rangle = \sum_{j=1}^{N} \langle e_j, \mathbf{a} \rangle \langle y_j, v \rangle . \qquad \square$$

What Corollary 3.9, Case 2, demonstrates is that using the tensor structure of the compact operator $\mathcal{A}$, the solvability of (3.1) can be characterized as a property in the Hilbert space $\mathbb{H}$ using eigenvectors of $\mathbf{C}^T$. If an eigenvector $\mathbf{a} \otimes v$ of $\mathcal{A}^*$ with eigenvalue 1 exists, we have already seen in the proof of Proposition 3.4 that v is an eigenvector of $-K^*$ and $\mathbf{a}$ is an eigenvector of $\mathbf{C}^T$. Let γ be an eigenvalue of $\mathbf{C}^T$ with geometric multiplicity k_1 with linearly independent eigenvectors $\{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k_1)}\}$, with $\mathbf{a}^{(i)} = \left(a_j^{(i)}\right)_j \in \mathbb{R}^N$, such that $1/\gamma$ is an eigenvalue of $-K^*$ with geometric multiplicity k_2 with distinct eigenvectors $\{v_1, v_2, \ldots, v_{k_2}\}$. It follows then that for all valid pairs (i_1, i_2) the element $\mathbf{a}_{i_1} \otimes v_{i_2}$ is an eigenvector of $\mathcal{A}^*$ with eigenvalue 1. What Corollary 3.9 tells us is that for all such eigenvalue pairs $(\gamma, 1/\gamma)$, we have that $\sum_{j=1}^{N} a_j^{(i_1)} y_j \perp v_{i_2}$ must hold for all valid pairs (i_1, i_2) in order for (3.1) to be solvable. Thus for each eigenvalue pair $(\gamma, 1/\gamma)$ of $\mathbf{C}^T$ and $-K^*$, respectively, the coordinates of the eigenvectors of $\mathbf{C}^T$ associated to γ determine which linear combinations of the coordinate functions of $\mathbf{y}$ must satisfy a Fredholm-like orthogonality condition with elements of the nullspace of the resolvent $\frac{1}{\gamma} I + K^*$. Since for each eigenvector $\mathbf{a}^{(i)}$ associated to γ Corollary 3.9 must hold for all eigenvectors of $-K^*$ associated to $1/\gamma$, we can restate Statement 2 of Corollary 3.9.

COROLLARY 3.10. *For any eigenvalue pair $(\gamma, 1/\gamma)$ of $\mathbf{C}^T$ and $-K^*$, respectively, for* (3.1) *to be solvable, it must hold that for each eigenvector $\mathbf{a}^{(i)}$ of $\mathbf{C}^T$ associated to γ*

$$\sum_{j=1}^{N} a_j^{(i)} y_i \perp \mathcal{N}\left(I + \frac{1}{\gamma} K^*\right).$$

For the sake of clarity, we will from this point forward explicitly identify

$$\tilde{\mathcal{K}} = \mathbf{B} \otimes K, \quad \mathcal{A} = \mathbf{C} \otimes (-K), \quad \mathbf{x} = \sum_{i=1}^{N} \mathbf{e}_i \otimes x_i, \quad \mathbf{y} = \sum_{i=1}^{N} \mathbf{e}_i \otimes y_i, \tag{3.4}$$
$$\text{and} \quad \mathbf{I} = \mathbf{I}_N \otimes I_{\mathbb{H}}.$$

3.3. Galerkin discretization of the tensor product operator is a Kronecker product. In this section, we discuss the discretization of (3.1), $(\mathbf{I} - \mathcal{A})\mathbf{x} = \mathbf{y}$, yielding a finite-dimensional matrix equation. We demonstrate that for a particular choice of a discretization strategy, the resulting matrix equation will have Kronecker product structure, similar to the observations made; see, e.g., [7, 9, 12, 23]. More generally, approximation theory in tensor product spaces has been discussed in the literature; see. e.g., [16]. A more general and thorough discussion of tensor product spaces can be found in, e.g., [8].

Consider the n-dimension subspace $\mathbb{H}_n \subset \mathbb{H}$ with the basis $\{\psi_1, \ldots, \psi_n\}$, which implicitly induces a finite-dimensional subspace of the tensor product space, $\mathbb{R}^N \otimes \mathbb{H}_n \subset \mathbb{R}^N \otimes \mathbb{H}$ with basis $\{\mathbf{e}_i \otimes \psi_j\}_{i=1,\ldots,N}^{j=1,\ldots,n}$ implying that $\dim(\mathbb{R}^N \otimes \mathbb{H}_n) = nN$. We index this basis with an integer pair, where the (k, ℓ)th element refers to $\mathbf{e}_k \otimes \psi_\ell$. We approximate the solution of (3.1) by the solution of matrix equation using the Galerkin method. We define the approximation of x_i as

$$\widetilde{x}_i = \sum_{j=1}^{n} a_j^{(i)} \psi_j \in \mathbb{H}_n.$$

We then have the approximate solution

$$\widetilde{\mathbf{x}} = \sum_{i=1}^{N} \mathbf{e}_i \otimes \widetilde{x}_i \in \mathbb{R}^N \otimes \mathbb{H}_n.$$

We further define $\underline{\mathbf{x}}_i = \left(\langle \psi_j, x_i \rangle\right)_{j=1}^{n}$ and $\mathbf{D} = \left(\langle \psi_i, \psi_j \rangle\right)_{ij=1}^{n}$. Then $\left(a_j^{(i)}\right)_{j=1}^{n} = \mathbf{D}^{-1}\underline{\mathbf{x}}_i$. Using the notation introduced in (3.4), we compute an approximation $\widetilde{\mathbf{x}}$ of the solution of (3.1) with a Galerkin method with $\{\mathbf{e}_i \otimes \psi_j\}_{i=1,\ldots,N}^{j=1,\ldots,n}$ serving as the test functions. Testing against the (k, ℓ)th test function results in the equation

$$\left\langle \mathbf{e}_k \otimes \psi_\ell, \sum_{i=1}^{N} \mathbf{e}_i \otimes y_i \right\rangle = \left\langle \mathbf{e}_k \otimes \psi_\ell, (\mathbf{I} - \mathcal{A}) \sum_{i=1}^{N} \mathbf{e}_i \otimes \widetilde{x}_i \right\rangle. \tag{3.5}$$

This leads to a system of nN equations with nN unknowns which, when simplified, has a convenient Kronecker structure.

LEMMA 3.11. *The computation of an approximate solution $\widetilde{\mathbf{x}}$ of $(\mathbf{I} - \mathcal{A})\mathbf{x} = \mathbf{y}$ using the Galerkin method with (3.5) leads to the solution of a Kronecker product linear system of the form*

$$\sum_{i=1}^{N} \mathbf{e}_i \otimes \underline{\mathbf{y}}_i = \left[(\mathbf{I}_N \otimes \mathbf{D}) - (\mathbf{C} \otimes (-\mathbf{F}))\right] \sum_{i=1}^{N} \mathbf{e}_i \otimes \left(a_j^{(i)}\right)_{j=1}^{n}, \tag{3.6}$$

where

$$\mathbf{F} = \left(\langle \psi_i, K\psi_j \rangle\right)_{ij=1}^{n} \qquad \textit{and} \qquad \underline{\mathbf{y}}_i = \left(\langle \psi_j, y_i \rangle\right)_{j=1}^{n}.$$

Proof. We show that (3.5) can be manipulated, directly leading to (3.6). To begin, we transform the inner product equation (3.5),

$$\left\langle \mathbf{e}_k \otimes \psi_\ell, \sum_{i=1}^{N} \mathbf{e}_i \otimes y_i \right\rangle = \left\langle \mathbf{e}_k \otimes \psi_\ell, (\mathbf{I} - \mathcal{A}) \sum_{i=1}^{N} \mathbf{e}_i \otimes \widetilde{x}_i \right\rangle$$

$$\sum_{i=1}^{N} \langle \mathbf{e}_k \otimes \psi_\ell, \mathbf{e}_i \otimes y_i \rangle = \left\langle \mathbf{e}_k \otimes \psi_\ell, \sum_{i=1}^{N} \left[\mathbf{e}_i \otimes \widetilde{x}_i - (\mathbf{C} \otimes (-K))\, \mathbf{e}_i \otimes \widetilde{x}_i\right] \right\rangle$$

$$\sum_{i=1}^{N} \langle \mathbf{e}_k \otimes \psi_\ell, \mathbf{e}_i \otimes y_i \rangle = \left\langle \mathbf{e}_k \otimes \psi_\ell, \sum_{i=1}^{N} \left[\mathbf{e}_i \otimes \sum_{j=1}^{n} a_j^{(i)} \psi_j - \mathbf{C}\mathbf{e}_i \otimes \sum_{j=1}^{n} a_j^{(i)} (-K) \psi_j \right] \right\rangle.$$

From the definition of the inner product of $\mathbb{R}^N \otimes \mathbb{H}$, we have

$$\begin{aligned}
\sum_{i=1}^{N} \langle \mathbf{e}_k, \mathbf{e}_i \rangle \langle \psi_\ell, y_i \rangle &= \sum_{i=1}^{N} \left\langle \mathbf{e}_k \otimes \psi_\ell, \left[\mathbf{e}_i \otimes \sum_{j=1}^{n} a_j^{(i)} \psi_j - \mathbf{C}\mathbf{e}_i \otimes \sum_{j=1}^{n} a_j^{(i)} (-K) \psi_j \right] \right\rangle \\
\sum_{i=1}^{N} \delta_{ki} \langle \psi_\ell, y_i \rangle &= \sum_{i=1}^{N} \left[\left\langle \mathbf{e}_k \otimes \psi_\ell, \mathbf{e}_i \otimes \sum_{j=1}^{n} a_j^{(i)} \psi_j \right\rangle \right. \\
&\qquad \left. - \left\langle \mathbf{e}_k \otimes \psi_\ell, \mathbf{C}\mathbf{e}_i \otimes \sum_{j=1}^{n} a_j^{(i)} (-K) \psi_j \right\rangle \right] \\
\langle \psi_\ell, y_k \rangle &= \sum_{i=1}^{N} \left[\delta_{ki} \left\langle \psi_\ell, \sum_{j=1}^{n} a_j^{(i)} \psi_j \right\rangle - \langle \mathbf{e}_k, \mathbf{C}\mathbf{e}_i \rangle \left\langle \psi_\ell, \sum_{j=1}^{n} a_j^{(i)} (-K) \psi_j \right\rangle \right] \\
\langle \psi_\ell, y_k \rangle &= \left\langle \psi_\ell, \sum_{j=1}^{n} a_j^{(k)} \psi_j \right\rangle - \sum_{i=1}^{N} c_{ki} \left\langle \psi_\ell, \sum_{j=1}^{n} a_j^{(i)} (-K) \psi_j \right\rangle \\
\langle \psi_\ell, y_k \rangle &= \sum_{j=1}^{n} \left[a_j^{(k)} \langle \psi_\ell, \psi_j \rangle - \sum_{i=1}^{N} c_{ki} a_j^{(i)} \langle \psi_\ell, (-K) \psi_j \rangle \right].
\end{aligned}$$

There are nN such equations, and combining them (for all pairs (k, ℓ)) into a vectorized set of equations yields the same system as that which results from the expansion and subsequent multiplication of the Kronecker products of (3.6). □

REMARK 3.12. We want to emphasize that this Kronecker product structure allows us to discretize the full problem through the much simpler discretization of $\mathbb{H}$. As we have shown, it suffices (and is computationally cheaper) to construct the matrices $\mathbf{D}$ and $\mathbf{F}$ which would result from the Galerkin discretization of a 1-dimensional problem and then to construct a discretization of the full problem using Kronecker products.

REMARK 3.13. It should be observed that the Kronecker-product system (3.6) can be further transformed into generalized Sylvester equations (and in some cases Sylvester equations) which may be useful for efficient solution when the level of discretization leads to really large-scale problems [6, 20, 21]. This follows from the fact that for large-scale problems, we must consider technical limitations, e.g., those on memory or the speed with which the computer can move data to and from the processor. In our current setting, a Sylvester formulation would allow iterative methods with a smaller coefficient matrix to be considered and these methods can take advantage of efficiencies arising from applying the matrix simultaneously to a block of vectors; see, e.g., [1, 19]. *However, this is far beyond the scope of this paper, and we do not pursue such strategies here*.

4. Numerical algorithm for the solution of the Forward Problem. As stated in Remark 3.12, with $\{\mathbf{e}_i\}_{i=1}^{N}$ the Cartesian basis of $\mathbb{R}^N$, the discretization of (2.7) only requires the discretization of $\mathbb{H} = L_2([0, T_e])$.

4.1. Basis functions. Let $\tau = \{t_1, \cdots, t_n\}$ be an equidistant n-partition of the time interval $[0, T_e]$ with width $h = t_k - t_{k-1}$. We chose a basis of normalized functions

$$\Psi = \{\psi_1(t), \cdots, \psi_n(t)\}, \;\; t \in [0, T_e], \quad \|\psi_i\|_2 = 1, \;\; i = 1, \cdots, n.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Then the space $L_2([0,T_e])$ is approximated by $L_2^n([0,T_e]) = \overline{\mathrm{span}\{\psi_1,\cdots,\psi_\mathrm{n}\}}$. Each function $f \in L_2([0,T_e])$ is approximated by

$$f(t) \approx f_n(t) = \sum_{j=1}^{n} f_j\psi_j(t), \quad \text{with} \quad (f_j)_{j=1}^n = \mathbf{D}^{-1}(\langle f, \psi_j\rangle)_{j=1}^n,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L_2([0,T_e])$. If an orthonormal basis (ONB) is chosen, then the Gramian matrix $\mathbf{D} := (\langle\psi_i,\psi_j\rangle)_{i,j=1..n}$ will be the identity matrix, and

$$f_j = \langle f, \psi_j\rangle.$$

Otherwise $\mathbf{D}$ can be computed either explicitly or numerically depending on the chosen basis. For example, for the basis of normed hat functions, $\mathbf{D}$ can be computed explicitly and is given by the tridiagonal matrix

$$\mathbf{D} = \begin{bmatrix} 1 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2\sqrt{2}} & 1 & \frac{1}{4} & 0 & 0 & \ddots & 0 \\ 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 \\ 0 & \ddots & \ddots & 0 & \frac{1}{4} & 1 & \frac{1}{2\sqrt{2}} \\ 0 & \cdots & \ddots & 0 & 0 & \frac{1}{2\sqrt{2}} & 1 \end{bmatrix}.$$

4.2. Discretization of the Forward Problem. For discretizing the operator equation $(\mathbf{I}-\mathcal{A})^{-1}\mathcal{K}\,[\mathbf{p},\ \alpha,\ \beta] = \mathbf{u}$ from (2.7), we have to

1. project the function $p_i(t), u_i(t), \alpha_i, \beta_i t, (i = 1,\cdots,n)$ onto L_2^n, and represent them by the n-dimensional vectors of coefficients of the basis representation,
2. discretize $\mathbf{y} = \mathcal{K}[\mathbf{p},\ \alpha,\ \beta]$, and
3. solve the discretized version of $(\mathbf{I}-\mathcal{A})\mathbf{u} = \mathbf{y}$.

In detail, these steps are achieved as follows:

1. Given a function vector $\mathbf{p} = [p_1,\cdots,p_N]$ represented by $\sum_{i=1}^N \mathbf{e}_i \otimes p_i$, and $\alpha = \sum_{i=1}^N \alpha_i\mathbf{e}_i \otimes 1$, $\beta t = \sum_{i=1}^N \beta_i\mathbf{e}_i \otimes t$, we can approximate the functions $p_i(t)$, 1, and t in L_2^n by

$$p_i(t) \approx p_{i,n}(t) = \sum_{j=1}^{n} p_j^{(i)}\psi_j(t), \;\; 1 \approx 1_n = \sum_{j=1}^{n} c_j\psi_j(t), \;\; t \approx t_n = \sum_{j=1}^{n} d_j\psi_j(t).$$

Using the abbreviations

$$\underline{\mathbf{p}}_i := (\langle p_i,\psi_j\rangle)_{j=1}^n, \quad \underline{\mathbf{c}} := (\langle 1,\psi_j\rangle)_{j=1}^n, \quad \underline{\mathbf{d}} := (\langle t,\psi_j\rangle)_{j=1}^n, \tag{4.1}$$

the coefficient vectors of the basis representation are given by

$$(p_j^{(i)})_{j=1}^n = \mathbf{D}^{-1}\underline{\mathbf{p}}_i, \quad (c_j)_{j=1}^n = \mathbf{D}^{-1}\underline{\mathbf{c}}, \quad (d_j)_{j=1}^n = \mathbf{D}^{-1}\underline{\mathbf{d}}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Hence the approximations of $\mathbf{p}, \alpha 1, \beta t \in \mathbb{R}^N \otimes \mathbb{H}$ are given by the sums of the Kronecker products

$$\mathbf{p} \approx \sum_{i=1}^{N} \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{p}}_i, \quad \alpha \approx \sum_{i=1}^{N} \alpha_i \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{c}}, \quad \beta t \approx \sum_{i=1}^{N} \beta_i \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{d}}.$$

In case the hat functions are chosen as basis, the vectors $\underline{\mathbf{c}}$ and $\underline{\mathbf{d}}$ can be computed explicitly. The vectors of L_2-functions $\mathbf{u}$ and $\mathbf{y}$ are approximated analogously:

$$\mathbf{u} \approx \sum_{i=1}^{N} \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{u}}_i, \quad \mathbf{y} \approx \sum_{i=1}^{N} \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{y}}_i.$$

2. With the definition

$$\mathbf{F} := \left(\langle \psi_i, K\psi_j \rangle \right)_{i,j=1,\cdots,n}, \tag{4.2}$$

as in Lemma 3.11 (with analogous proof), the discretization of $\mathbf{y} = \mathcal{K}[\mathbf{p},\ \alpha,\ \beta]$ is given in terms of Kronecker products as

$$\sum_{i=1}^{N} \mathbf{e}_i \otimes \underline{\mathbf{y}}_i = (\mathbf{B} \otimes \mathbf{F}) \sum_{i=1}^{N} \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{p}}_i + (\mathbf{I}_N \otimes \mathbf{D}) \left(\sum_{i=1}^{N} \alpha_i e_i \otimes \mathbf{D}^{-1}\underline{\mathbf{c}} \right) + \\ + (\mathbf{I}_N \otimes \mathbf{D}) \left(\sum_{i=1}^{N} \beta_i \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{d}} \right)$$

$$\sum_{i=1}^{N} \mathbf{e}_i \otimes \underline{\mathbf{y}}_i = (\mathbf{B} \otimes \mathbf{F}) \left(\sum_{i=1}^{N} \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{p}}_i \right) + \sum_{i=1}^{N} \alpha_i \mathbf{e}_i \otimes \underline{\mathbf{c}} + \sum_{i=1}^{N} \beta_i \mathbf{e}_i \otimes \underline{\mathbf{d}}.$$

Here $\mathbf{I}_N \otimes \mathbf{D}$ is the identity on $\mathbb{R}^N \otimes L_2^n$.
Depending on the basis, $\mathbf{F}$ will be computed either explicitly or numerically, e.g., for the hat functions, $K\psi_i, j = 1, \cdots, n$, is determined explicitly while the inner product is computed numerically using the trapezoidal rule; for the trigonometric functions, $\mathbf{F}$ can be computed explicitly, while for wavelets both $K\psi_i$ and the inner product are computed numerically. This last mentioned computation is quite time consuming but can be done in advance.

3. As stated in Section 3. we get the approximate solution of $(\mathbf{I} - \mathcal{A})\mathbf{u} = \mathbf{y}$ by solving the system with Kronecker products

$$\sum_{i=1}^{N} \mathbf{e}_i \otimes \underline{\mathbf{y}}_i = [(\mathbf{I}_N \otimes \mathbf{D}) - (\mathbf{C} \otimes (-\mathbf{F}))] \sum_{i=1}^{N} \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{u}}_i.$$

Thus, with $(u_j^{(i)})_{j=1}^n = \mathbf{D}^{-1}\underline{\mathbf{u}}_i$ the approximate solution of $\mathbf{u}$ in (2.7) is given by

$$u_{i,n}(t) = \sum_{j=1}^{n} u_j^{(i)} \psi_j(t), \quad i = 1, \cdots, N.$$

4.3. Algorithm for the solution of the Forward Problem. We can now summarize in Algorithm 4.1 the algorithm for computing the solution $\mathbf{u}(t)$ of the Forward Problem for a given imbalance load vector $\mathbf{p}(t)$ and initial values α and β.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Algorithm 4.1: Algorithm for the solution of the Forward Problem.

1: Given T_e, n, τ and $\psi_j(t), j = 1, \cdots, n$.
2: Define $\mathbf{B} = \mathbf{M}^{-1}$, $\mathbf{C} = \mathbf{M}^{-1}\mathbf{S}$, $\mathbf{D}$.
3: Compute $\underline{\mathbf{p}}_i$, $\underline{\mathbf{c}}$ and $\underline{\mathbf{d}}$ from (4.1), and $\mathbf{F}$ from (4.2).
4: Compute $\sum_{i=1}^{N} \mathbf{e}_i \otimes \underline{\mathbf{y}}_i = (\mathbf{B} \otimes \mathbf{F}) \left(\sum_{i=1}^{N} \mathbf{e}_i \otimes \mathbf{D}^{-1}\underline{\mathbf{p}}_i\right) + \sum_{i=1}^{N} \alpha_i \mathbf{e}_i \otimes \underline{\mathbf{c}} + \sum_{i=1}^{N} \beta_i \mathbf{e}_i \otimes \underline{\mathbf{d}}$.
5: Solve $\sum_{i=1}^{N} \mathbf{e}_i \otimes \underline{\mathbf{y}}_i = [(\mathbf{I}_N \otimes \mathbf{D}) - (\mathbf{C} \otimes (-\mathbf{F}))] \sum_{i=1}^{N} \mathbf{e}_i \otimes \left(u_j^{(i)}\right)_{j=1}^{n}$ for $\left(u_j^{(i)}\right)_{j=1}^{n}$
6: Compute the approximate solution as $\tilde{u}_i(t) = \sum_{j=1}^{n} u_j^{(i)} \psi_j(t), \quad i = 1, \cdots, N.$

4.3.1. A numerical example with fixed frequency. To verify the algorithm we chose system matrices $\mathbf{M}$ and $\mathbf{S}$ from the model of a wind turbine; cf. [18]. The simple structure of a wind turbines allows to use models with few nodes. In this case the model has only 4 nodes, each node is considered to have 2 degrees of freedom (DOF), i.e., the displacement in radial direction and the cross section slope. Hence the matrices are of dimension 8×8. The matrices $\mathbf{B} = \mathbf{M}^{-1}$ and $\mathbf{C} = \mathbf{M}^{-1}\mathbf{S}$ were computed. The imbalance vector was chosen as $\mathbf{p}_0 = [0, 0, 0, 0, 20, 0, 250e^{i\pi/6}, 0]$, which means that there is an imbalance at node 4 of 250 kgm at an angle of 30° and a second at node 3 of 20 kgm at an angle of 0°. In order to compute a reference solution $\mathbf{u}$ we chose a constant angular velocity $\omega = 0.7 \cdot \pi$ rad/s. This corresponds to a revolution frequency of 0.35 Hz. Now the load vector is given by $\mathbf{p}(t) = \mathbf{p}_0\omega^2 \exp(i\omega t)$. As the frequency is constant, the solution $\mathbf{u}$ can be computed explicitly via (1.3) for comparison, and we have the initial values $\alpha = \mathbf{u}(0) = \Re(\mathbf{u}_0)$ and $\beta = \mathbf{u}'(0) = -\omega\Im(\mathbf{u}_0)$ as input parameters.

The time interval $[0, 1]$ was discretized equidistantly with $\tau = [0 : 0.005 : 1], n = 201$, and hat-functions as well as $\mathbf{D}$ were set up accordingly. The vectors $\underline{\mathbf{c}}$ and $\underline{\mathbf{d}}$ were computed explicitly while $\underline{\mathbf{p}}_i, i = 1, \cdots, n$, and the matrix $\mathbf{F}$ was computed using numerical integration. Using the MATLAB function `kron()` for the Kronecker-product, the implementation for the computation of $\underline{\mathbf{y}}_i$ is quite simple. The algorithm produced very good approximation results especially for the uneven numbered DOF that represent the displacement which is usually the measurable quantity; cf. Figure 4.1. Similar results were achieved choosing wavelets or trigonometric functions as basis functions. The overall relative error is $\|\mathbf{u} - \tilde{\mathbf{u}}\|_2/\|\mathbf{u}\|_2 \approx 2.25 \cdot 10^{-4}$.

4.3.2. A numerical example with variable frequency. In a second example, we chose the same matrices and imbalance $\mathbf{p}_0$ but a variable angular velocity $\omega(t) = 0.7\pi + 0.3\sin(t)$ on a time interval $[0, 10]s$. This choice simulates the situation of a wind turbine when the frequency of 0.35 Hz is disturbed. The disturbance is modeled by a sine function. The load vector changes according to (1.4) to $\mathbf{p}(t) = \mathbf{p}_0[\omega^2(t) + i0.3\cos(t)] \exp(it\omega(t))$. In Figure 4.2 the load vector is shown for DOF 7.

The computation of a reference solution is no longer possible. But for comparison we have the solution for the constant frequency case from the example above with a revolution frequency of 0.35 Hz. As initial values for the solution we chose the values that were computed in the same constant frequency case. The computed solution at the DOF 7 is shown in Figure 4.3 for both cases, variable and constant frequency. Although the load looks more disturbed by the variable frequency, the deviation in the displacement is less significant.

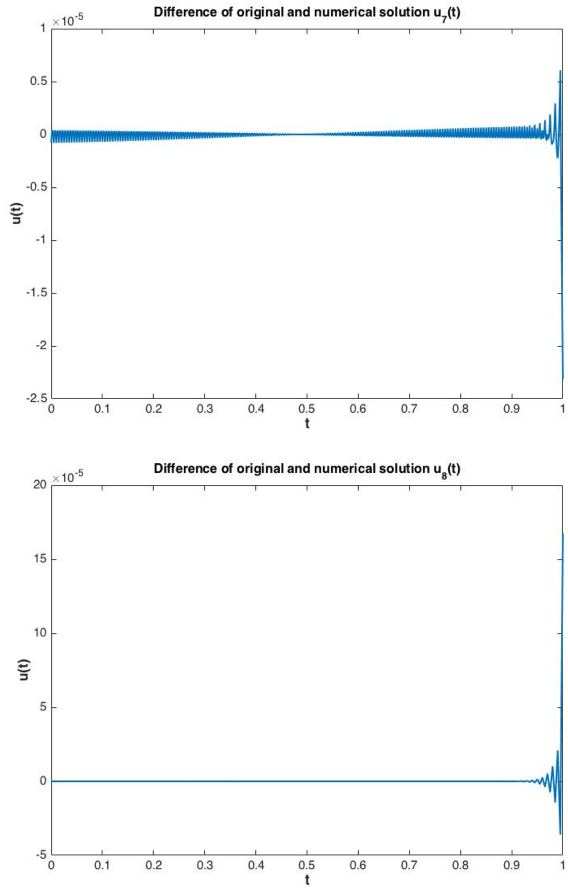

FIG. 4.1. *Difference of the exact and numerical solution of the Forward Problem for fixed frequency displayed for DOF 7 (displacement of the uppermost node) and DOF 8 (cross section slope of the uppermost node).*

5. Summary and outlook towards the Inverse Problem. The goal of this paper was the computation of the vibrations of a rotating systems from a given imbalance distribution, assuming that the angular velocity is time dependent. We showed that the governing mathematical model, an ordinary differential equation system of the form (2.1) with unknown initial conditions can be transformed into an equivalent integral equation presented in (2.7). The operators involved in the integral equation can be written in terms of tensor products of a matrix and a one-dimensional integral operator. Besides the elegant notation, the tensor product representation has the advantage that the discretized version of equation (2.7) only requires a discretization in the one-dimensional function space $L_2[0, T_e]$. Additionally, the tensor products correspond to Kronecker products in the discretized setting. A numerical algorithm for the approximate solution of the Forward Problem, i.e., to compute $\mathbf{u}(t)$ in (2.7) for given $[\mathbf{p}, \alpha, \beta]$, is presented and tested with an example inspired by an actual application for wind turbines.

As a next step, we plan to solve the Inverse Problem of finding $\mathbf{p}$ from measurements of $\mathbf{u}$. Since the Inverse Problem is ill-posed, regularization techniques have to be employed to solve it. Iterative methods like Tikhonov regularization with a discrepancy principle for the choice of the regularization parameter require the solution of the Forward Problem several times.

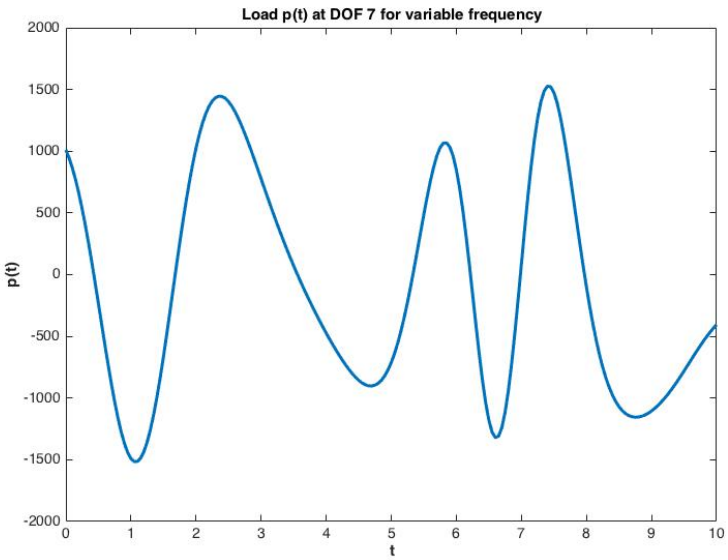

FIG. 4.2. *Load at DOF 7 for variable frequency.*

Hence the formulation (2.7) and the derived algorithm for handling the Forward Problem are important steps towards a reconstruction algorithm.

REFERENCES

[1] A. H. BAKER, J. M. DENNIS, AND E. R. JESSUP, *On improving linear solver performance: a block variant of GMRES*, SIAM J. Sci. Comp., 27 (2006), pp. 1608–1626.
[2] A. BROWN AND C. PEARCY, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc., 17 (1966), pp. 162–166.
[3] V. DICKEN, I. MENZ, P. MAASS, J. NIEBSCH, AND R. RAMLAU, *Nonlinear inverse unbalance reconstruction in rotor dynamics*, Inverse Probl. Sci. Eng., 13 (2005), pp. 507–543.
[4] H. W. ENGL, *Integralgleichungen*, Springer, Vienna, 1997.
[5] R. GASCH AND K. KNOTHE, *Strukturdynamik 2*, Springer, Berlin, 1989.
[6] A. EL GUENNOUNI, K. JBILOU, AND A. J. RIQUET, *Block Krylov subspace methods for solving large Sylvester equations*, Numer. Algorithms, 29 (2002), pp. 75–96.
[7] L. GRASEDYCK, *Existence and computation of low Kronecker-rank approximations for large linear systems of tensor product structure*, Computing, 72 (2004), pp. 247–265.
[8] W. HACKBUSCH, *Tensor Spaces and Numerical Tensor Calculus*, Springer, Heidelberg, 2012.
[9] W. HACKBUSCH AND B. N. KHOROMSKIJ, *Tensor-product approximation to operators and functions in high dimensions*, J. Complexity, 23 (2007), pp. 697–714.
[10] H. HEUSER *Gewöhnliche Differentialgleichungen*, Teubner, Stuttgart, 1991.
[11] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
[12] B. N. KHOROMSKIJ AND V. KHOROMSKAIA, *Low rank Tucker-type tensor approximation to classical potentials*, Cent. Eur. J. Math., 5 (2007), pp. 523–550.
[13] C. S. KUBRUSLY, *A concise introduction to tensor product*, Far East J. Math. Sci. (FJMS), 22 (2006), pp. 137–174.
[14] C. S. KUBRUSLY AND N. LEVAN, *Preservation of tensor sum and tensor product*, Acta Math. Univ. Comenian. (N.S.), 80 (2011), pp. 133–142.
[15] C. S. KUBRUSLY AND P. C. M. VIEIRA, *Convergence and decomposition for tensor products of Hilbert space operators*, Oper. Matrices, 2 (2008), pp. 407–416.
[16] W. A. LIGHT AND E. W. CHENEY, *Approximation Theory in Tensor Product Spaces*, Springer, Berlin, 1985.
[17] J. NIEBSCH AND R. RAMLAU, *Mathematical imbalance determination from vibrational measurements and industrial applications*, in ASME 2008 International Design Engineering, Technical Conferences and

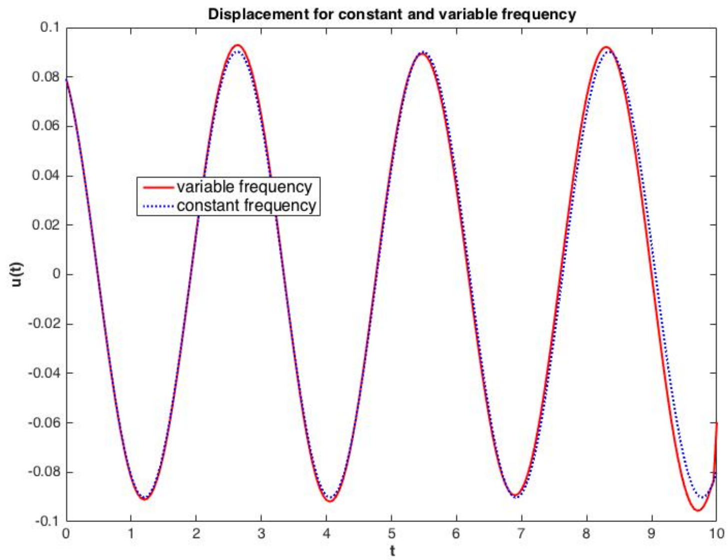

FIG. 4.3. *Numerical solution of the Forward Problem with variable and constant frequency displayed for DOF 7 (displacement of the uppermost node).*

Computers and Information in Engineering Conference, ASME, New York, 2008, pp. 1035–1042.

[18] R. RAMLAU, AND J. NIEBSCH, *Imbalance estimation without test masses for wind turbines*, J. Sol. Energy Eng., 131 (2009), Art. 011010 (7 pages).

[19] M. L. PARKS, K. M. SOODHALTER, AND D. B. SZYLD, *A block Recycled GMRES method with investigations into aspects of solver performance*, Preprint on arXiv, 2016. https://arxiv.org/abs/1604.01713

[20] M. ROBBÉ AND M. SADKANE, *A convergence analysis of GMRES and FOM methods for Sylvester equations*, Numer. Algorithms, 30 (2002), pp. 71–89.

[21] V. SIMONCINI, *On the numerical solution of* $AX - XB = C$, BIT, 36 (1996), pp. 814–830.

[22] ———, *Computational methods for linear matrix equations*. SIAM Rev., 58 (2016), pp. 377–441.

[23] L. VENKATARAMANAN, Y.-Q. SONG, AND M. D. HÜRLIMANN, *Solving Fredholm integrals of the first kind with tensor product structure in 2 and 2.5 dimensions*, IEEE Trans. Signal Process., 50 (2002), pp. 1017–1026.

[24] M. YUDYTSKIY, *Stable Parameter Identification for a Second Order Ordinary Differential Equation - Analysis and Application from Biochemistry*. Master Thesis, Industrial Mathematics Institute, Johannes Kepler University Linz, 2010.