Electronic Transactions on Numerical Analysis.
Volume 48, pp. 15–39, 2018.

ISSN 1068–9613.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

QUADRATIC SPLINE WAVELETS WITH SHORT SUPPORT SATISFYING HOMOGENEOUS BOUNDARY CONDITIONS*

DANA ČERNÁ† AND VÁCLAV FINĚK†

Abstract. In this paper, we construct a new quadratic spline-wavelet basis on the interval and on the unit square satisfying homogeneous Dirichlet boundary conditions of the first order. The wavelets have one vanishing moment and the shortest support among quadratic spline wavelets with at least one vanishing moment adapted to the same type of boundary conditions. The stiffness matrices arising from the discretization of the second-order elliptic problems using the constructed wavelet basis have uniformly bounded condition numbers, and the condition numbers are small. We present some quantitative properties of the constructed basis. We provide numerical examples to show that the Galerkin method and the adaptive wavelet method using our wavelet basis require fewer iterations than methods with other quadratic spline wavelet bases. Moreover, due to the small support of the wavelets, when using these methods with the new wavelet basis, the system matrix is sparser, and thus one iteration requires a smaller number of floating point operations than for other quadratic spline wavelet bases.

Key words. wavelet, quadratic spline, homogeneous Dirichlet boundary conditions, condition number, elliptic problem

AMS subject classifications. 46B15, 65N12, 65T60

1. Introduction. Wavelets are powerful tools in signal analysis, image processing, and engineering applications. They are also used for the numerical solution of various types of equations. Wavelet methods are used especially for preconditioning systems of linear algebraic equations arising from the discretization of elliptic problems [9], adaptive solution of operator equations [6, 7], solution of certain types of partial differential equations with a dimension-independent convergence rate [12], and sparse representation of operators [2].

The quantitative properties of any wavelet method strongly depends on the used wavelet basis, namely on its condition number, the length of the support of the wavelets, the number of vanishing wavelet moments, and the smoothness of the basis functions. Therefore, the construction of appropriate wavelet bases is an important issue.

In this paper, we construct a quadratic spline wavelet basis on the interval and on the unit square that is well-conditioned and adapted to homogeneous Dirichlet boundary conditions of the first order. The wavelets have one vanishing moment, and we show that the support is the shortest among all quadratic spline wavelets with one vanishing moment. The condition numbers of the stiffness matrices arising from the discretization of elliptic problems using the constructed basis are uniformly bounded and small. Let $\Omega_d = (0,1)^d$, $d = 1, 2$. The wavelet basis of the space $H_0^1(\Omega_2)$ is then obtained by an isotropic tensor product. More precisely, our aim is to propose a wavelet basis on Ω_d that satisfies the following properties:

– *Riesz basis property.* We construct Riesz bases for the space $H_0^1(\Omega_d)$.
– *Locality.* The primal basis functions are local in the sense of Definition 2.1.
– *Vanishing moments.* The wavelets have one vanishing moment.
– *Polynomial exactness.* Since the scaling basis functions are quadratic B-splines, the primal multiresolution analysis has polynomial exactness of order three.
– *Short support.* The wavelets have the shortest possible support among quadratic spline wavelets with one vanishing moment.
– *Closed form.* The primal scaling functions and wavelets have an explicit expression.

*Received June 25, 2015. Accepted January 19, 2018. Published online on February 16, 2018. Recommended by G. Teschke. This work was supported by the SGS project “Wavelets” financed by the Technical University of Liberec.

†Department of Mathematics and Didactics of Mathematics, Technical University in Liberec, Studentská 2, 461 17 Liberec, Czech Republic ({dana.cerna,vaclav.finek}@tul.cz).

– *Homogeneous Dirichlet boundary conditions.* The wavelet basis satisfies homogeneous Dirichlet boundary conditions of the first order.
– *Well-conditioned basis.* The wavelet basis is well-conditioned with respect to the $H^1(\Omega_d)$-seminorm.

In [8, 10], a construction of a biorthogonal spline-wavelet basis on the interval was proposed. Both the primal and dual wavelets are local. A disadvantage of these bases is their relatively large condition number. Therefore many modifications of this construction were proposed [1, 3, 4, 15]. The construction in [20] outperforms the previous constructions for the linear and quadratic spline-wavelet bases with respect to the conditioning of the wavelet bases. In [11, 22, 23] the construction was significantly improved also for cubic spline wavelet bases.

Spline wavelet bases with nonlocal duals were also constructed and adapted to various types of boundary conditions [5, 13, 17, 18, 19, 25, 26, 27]. The main advantage of these types of bases in comparison to bases with local duals are usually the shorter support of the wavelets, the lower condition number of the basis and the corresponding stiffness matrices, and the simplicity of the construction. The cubic spline multiwavelet basis from [13] has the additional advantage that the discretization of the second-order elliptic equations with constant coefficients leads to truly sparse matrices, i.e., the number of all nonzero entries in any row is bounded by some constant c independent of the matrix size, whereas the discretization matrices for other wavelet bases have typically $\mathcal{O}(N \log N)$ nonzero entries, where $N \times N$ is the matrix size. This allows a simplification and improvement of adaptive wavelet methods because a routine called APPLY for the multiplication of the discretization matrix with a vector can be avoided.

The constructed basis can be used in many applications, e.g., the wavelet Galerkin method and an adaptive wavelet method for solving second-order elliptic equations, parabolic equations, and partial integro-differential equations on tensor product domains and domains that are images of tensor product domains under continuous mappings. These problems arise, for example, in financial mathematics for the valuation of options under the Black–Scholes model, stochastic volatility models, and the Lévy model; see [16]. Wavelet methods seem to be superior to classical methods especially for the solution of partial integro-differential equations because they make it possible to represent the integral term by sparse or almost-sparse matrices while the classical methods typically lead to full matrices. Due to the short support and the small condition number, the constructed basis can lead to improved efficiency of these methods.

Wavelet bases of the same type as in this paper are those from [11, 20, 22]. The constructions in [11, 20, 22] are based on the constructions of boundary dual scaling functions that are linear combinations of restrictions of dual functions on the real line to $[0, 1]$ such that the boundary dual scaling functions preserve the polynomial exactness. Then boundary wavelets are constructed by the method of stable completion. In this paper the construction is much simpler because we construct boundary wavelets directly without using dual scaling functions. The constructions from [22] and [20] lead to the same basis in the case of quadratic spline wavelet bases adapted to homogeneous Dirichlet boundary conditions of the first order. Therefore in Section 5 we compare our basis with bases from [11, 20]. Furthermore, we adapt bases from [3, 5] to homogeneous boundary conditions and compare the resulting bases with ours.

2. Construction of quadratic-spline wavelets. In this section we propose a construction of a new quadratic spline wavelet basis on the unit interval and on the unit square. The proposed wavelets have one vanishing moment, and we show that their support is the smallest possible. First, we briefly review a definition of a wavelet basis; for more details about wavelet bases see [21]. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set, and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j,k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale*. We define

$$\|\mathbf{v}\|_2 := \sqrt{\sum_{\lambda\in\mathcal{J}} v_\lambda^2}, \qquad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\, v_\lambda \in \mathbb{R}$$

and

$$l^2(\mathcal{J}) := \left\{\mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\, v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_2 < \infty\right\}.$$

Our aim is to construct a wavelet basis for H in the sense of the following definition.

DEFINITION 2.1. *A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a wavelet basis for H, if*

(i) Ψ is a Riesz basis for H, i.e., the closure of the span of Ψ is H and there exist constants $c, C \in (0,\infty)$ such that

$$c\,\|\mathbf{b}\|_2 \le \left\|\sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda\right\|_H \le C\,\|\mathbf{b}\|_2\,, \tag{2.1}$$

for all $\mathbf{b} := \{b_\lambda\}_{\lambda\in\mathcal{J}} \in l^2(\mathcal{J})$.

(ii) The functions are local in the sense that $\operatorname{diam}\operatorname{supp}\psi_\lambda \le C2^{-|\lambda|}$ *for all $\lambda \in \mathcal{J}$, and at a given level j, the supports of only finitely many wavelets overlap at any point x.*

For the two countable sets of functions $\Gamma, \Theta \subset H$, the symbol $\langle\Gamma,\Theta\rangle_H$ denotes the matrix

$$\langle\Gamma,\Theta\rangle_H := \{\langle\gamma,\theta\rangle_H\}_{\gamma\in\Gamma,\theta\in\Theta}.$$

REMARK 2.2. The constants

$$c_\Psi := \sup\{c : c \text{ satisfies (2.1)}\} \qquad \text{and} \qquad C_\Psi := \inf\{C : C \text{ satisfies (2.1)}\}$$

are called *Riesz bounds*, and the number $\operatorname{cond}\Psi = C_\Psi/c_\Psi$ is called the *condition number* of Ψ. It is known that the constants c_Ψ and C_Ψ satisfy:

$$c_\Psi = \sqrt{\lambda_{\min}\left(\langle\Psi,\Psi\rangle_H\right)}, \qquad C_\Psi = \sqrt{\lambda_{\max}\left(\langle\Psi,\Psi\rangle_H\right)},$$

where $\lambda_{\min}(\langle\Psi,\Psi\rangle_H)$ and $\lambda_{\max}(\langle\Psi,\Psi\rangle_H)$ are the smallest and the largest eigenvalues of the matrix $\langle\Psi,\Psi\rangle_H$, respectively.

We define a scaling basis as a basis for quadratic B-splines in the same way as in [5, 20, 22]. Let ϕ be a quadratic B-spline defined on the knots $[0,1,2,3]$. It can be written explicitly as

$$\phi(x) = \begin{cases} \frac{x^2}{2}, & x \in [0,1], \\ -x^2 + 3x - \frac{3}{2}, & x \in [1,2], \\ \frac{x^2}{2} - 3x + \frac{9}{2}, & x \in [2,3], \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

The function ϕ satisfies the scaling equation [5]

$$\phi(x) = \frac{\phi(2x)}{4} + \frac{3\phi(2x-1)}{4} + \frac{3\phi(2x-2)}{4} + \frac{\phi(2x-3)}{4}. \tag{2.3}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Let ϕ_b be a multiple of the quadratic B-spline defined on the knots $[0,0,1,2]$ such that $\|\phi_b\|_{L^1} = \|\phi\|_{L^1}$, i.e.,

$$\phi_b(x) = \begin{cases} -\frac{9x^2}{4} + 3x, & x \in [0,1], \\ \frac{3x^2}{4} - 3x + 3, & x \in [1,2], \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

The function ϕ_b satisfies the scaling equation [5]

$$\phi_b(x) = \frac{\phi_b(2x)}{2} + \frac{9\phi(2x)}{8} + \frac{3\phi(2x-1)}{8}. \tag{2.5}$$

The graphs of the functions ϕ_b and ϕ are displayed in Figure 2.1.

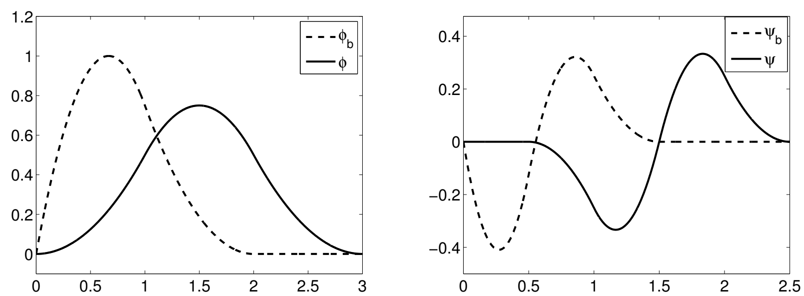


FIG. 2.1. *The scaling functions ϕ and ϕ_b and the wavelets ψ and ψ_b.*

For $j \geq 2$ and $x \in [0,1]$ we set

$$\begin{aligned} \phi_{j,k}(x) &= 2^{j/2}\phi(2^j x - k + 2), \qquad k = 2, \ldots, 2^j - 1, \\ \phi_{j,1}(x) &= 2^{j/2}\phi_b(2^j x), \qquad \phi_{j,2^j}(x) = 2^{j/2}\phi_b(2^j(1-x)). \end{aligned} \tag{2.6}$$

We define a wavelet ψ and a boundary wavelet ψ_b as

$$\psi(x) = -\frac{1}{2}\phi(2x-1) + \frac{1}{2}\phi(2x-2) \quad \text{and} \quad \psi_b(x) = \frac{-\phi_b(2x)}{2} + \frac{\phi(2x)}{2}. \tag{2.7}$$

Due to the normalization of ϕ_b, the coefficients in these two equations are the same which will simplify the proofs in the next section. Then $\operatorname{supp}\psi = [0.5, 2.5]$, $\operatorname{supp}\psi_b = [0, 1.5]$, and both wavelets have one vanishing moment, i.e.,

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \psi_b(x)dx = 0. \tag{2.8}$$

The graphs of the wavelet ψ and the boundary wavelet ψ_b are displayed in Figure 2.1. In the following lemma we show that the support of the wavelet ψ is the shortest among all quadratic spline wavelets with one vanishing moment.

LEMMA 2.3. *Let ϕ be defined by (2.2). If $\psi \in \overline{\operatorname{span}}\{\phi(2\cdot -k), k \in \mathbb{Z}\}$ and ψ satisfies (2.8), then the length of the support of ψ is at least 2.*

Proof. Since $\psi \in \overline{\operatorname{span}}\{\phi(2\cdot -k), k \in \mathbb{Z}\}$ we have

$$\psi(x) = \sum_{k \in \mathbb{Z}} a_k \phi(2x - k),$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

for some coefficients $a_k \in \mathbb{R}$. Let us suppose that the length of the support of ψ is at most 2. Then $\operatorname{supp}\psi \subset [j/2,(j+4)/2]$ for some $j \in \mathbb{Z}$. Since $\psi(x) = 0$ for all $x \in [k/2,(k+1)/2]$, where $k \in \mathbb{Z}\setminus\{j,j+1,j+2,j+3\}$, the coefficients satisfy $a_k = 0$ for all $k \in \mathbb{Z}\setminus\{j,j+1\}$. Due to (2.8) we have $a_j + a_{j+1} = 0$. Thus up to a multiplication by a constant and shifting by $k/2$, $k \in \mathbb{Z}$, there is only one wavelet that has the length of support at most 2 and this wavelet is a wavelet defined by (2.7). □

Using a similar argument as in the proof of Lemma 2.3 it is easy to see that also the boundary wavelet ψ_b has the shortest possible support among all boundary wavelets with one vanishing moment corresponding to scaling functions defined by (2.6).

For $j \geq 2$ and $x \in [0,1]$ we define

$$\begin{aligned} \psi_{j,k}(x) &= 2^{j/2}\psi(2^j x - k + 2), \qquad\qquad k = 2,\ldots,2^j-1, \\ \psi_{j,1}(x) &= 2^{j/2}\psi_b(2^j x), \qquad \psi_{j,2^j}(x) = -2^{j/2}\psi_b(2^j(1-x)). \end{aligned} \tag{2.9}$$

We denote the index sets by

$$\mathcal{I}_j = \left\{k \in \mathbb{Z} : 1 \leq k \leq 2^j\right\}.$$

We define

$$\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}, \qquad \Psi_j = \{\psi_{j,k}, k \in \mathcal{I}_j\},$$

and

$$\Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j, \qquad \Psi^s = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j, \quad j_0 = 2. \tag{2.10}$$

In Section 4 we prove that Ψ, when normalized with respect to the H^1-seminorm, forms a wavelet basis for the Sobolev space $H_0^1(0,1)$.

A basis on $\Omega_d = (0,1)^d$ is built from the univariate wavelet basis by a tensor product [21]. Let $j \geq 2$, $\mathbf{k} = (k_1,\ldots,k_d)$, $\mathbf{k} \in \mathcal{I}_j^d := \mathcal{I}_j \times \cdots \times \mathcal{I}_j$, and $\mathbf{x} = (x_1,\ldots,x_d) \in \Omega_d$. We define the multivariate scaling functions by

$$\phi_{j,\mathbf{k}}^d(\mathbf{x}) = \prod_{l=1}^{d} \phi_{j,k_l}(x_l),$$

and for any $\mathbf{e} = (e_1,\ldots,e_d) \in E^d := \{0,1\}^d \setminus (0,\ldots,0)$, we define the multivariate wavelet

$$\psi_{j,\mathbf{e},\mathbf{k}}^d(\mathbf{x}) = \prod_{l=1}^{d} \psi_{j,e_l,k_l}(x_l),$$

where

$$\psi_{j,e_l,k_l} = \begin{cases} \phi_{j,k_l}, & e_l = 1, \\ \psi_{j,k_l}, & e_l = 0. \end{cases}$$

The basis on the unit cube Ω_d is then given by

$$\Psi^{dD} = \left\{\phi_{2,\mathbf{k}}^d, \mathbf{k} \in \mathcal{I}_j^d\right\} \cup \left\{\psi_{j,\mathbf{e},\mathbf{k}}^d, \mathbf{e} \in E^d, \mathbf{k} \in \mathcal{I}_j^d, j \geq 2\right\}.$$

This approach is called an isotropic approach. It preserves the regularity and polynomial exactness. Another approach is an anisotropic approach. The anisotropic basis on the unit square is $\Psi \otimes \Psi$.

3. Refinement matrices. In this section we present refinement matrices $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ corresponding to primal scaling functions and wavelets. We show that the matrix $\mathbf{M}_j = (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$ is invertible, and thus there exist matrices $\tilde{\mathbf{M}}_{j,0}$ and $\tilde{\mathbf{M}}_{j,1}$ of the same sizes as $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$, respectively, such that

$$(\tilde{\mathbf{M}}_{j,0}, \tilde{\mathbf{M}}_{j,1}) = \mathbf{M}_j^{-1}. \tag{3.1}$$

We derive an explicit form of the matrix $\tilde{\mathbf{M}}_{j,0}$ and an estimate for the norm of the product $\tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \ldots \tilde{\mathbf{M}}_{n,0}^T$ because this estimate is crucial for the proof of the Riesz basis property that will be presented in Section 4.

By (2.3), (2.5), (2.6), (2.7), and (2.9), there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \qquad \Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}. \tag{3.2}$$

In these formulas we view the sets Φ_j and Ψ_j as column vectors with entries $\phi_{j,k}$ and $\psi_{j,k}$, $k \in \mathcal{I}_j$, respectively.

By the Riesz representation theorem there exist dual functions $\tilde{\phi}_{j,k}$ and $\tilde{\psi}_{j,k}$ such that

$$\langle \phi_{j,k}, \tilde{\phi}_{j,m} \rangle = \delta_{k,m}, \quad \langle \phi_{j,k}, \tilde{\psi}_{l,m} \rangle = 0, \quad \langle \psi_{l,m}, \tilde{\phi}_{j,k} \rangle = 0, \quad \langle \psi_{j,k}, \tilde{\psi}_{l,m} \rangle = \delta_{j,k}\delta_{k,m},$$

for all $j, l \geq 2$, $l \geq j$, $k \in \mathcal{I}_j$, $m \in \mathcal{I}_l$. Let us denote

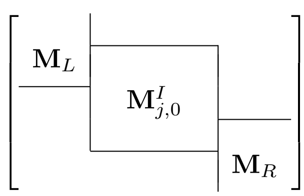
$$\tilde{\Phi}_j = \{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\}, \qquad \tilde{\Psi}_j = \{\tilde{\psi}_{j,k}, k \in \mathcal{I}_j\},$$

and view these sets as column vectors. Then $\tilde{\Phi}_j, \tilde{\Psi}_j \subset \operatorname{span} \tilde{\Phi}_{j+1}$, and the matrices $\tilde{\mathbf{M}}_{j,0}$ and $\tilde{\mathbf{M}}_{j,1}$ defined by (3.1) are the refinement matrices for the dual system, i.e.,

$$\tilde{\Phi}_j = \tilde{\mathbf{M}}_{j,0} \tilde{\Phi}_{j+1}, \qquad \tilde{\Psi}_j = \tilde{\mathbf{M}}_{j,1} \tilde{\Phi}_{j+1}.$$

Due to Remark 2.2, the Riesz bounds for the multiscale systems are related to the spectral norms of refinement matrices and products of these matrices.

Due to (2.3) and (2.5), the refinement matrix $\mathbf{M}_{j,0}$ has the following structure:

$\mathbf{M}_{j,0} =$ 

,

where $\mathbf{M}_{j,0}^I$ is a $2^{j+1} \times 2^j$ matrix given by

$$\left(\mathbf{M}_{j,0}^I\right)_{m,n} = \begin{cases} \frac{h_{m+2-2n}}{\sqrt{2}}, & n = 1, \ldots, 2^j,\ 1 \leq m + 2 - 2n \leq 4, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\mathbf{h} = [h_1, h_2, h_3, h_4] = \left[\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right]$$

is a vector of coefficients from the scaling equation (2.3). The matrix $\mathbf{M}_L$ is given by

$$\mathbf{M}_L = \frac{1}{\sqrt{2}} \mathbf{h}_b^T, \qquad \text{where} \quad \mathbf{h}_b = [h_1^b, h_2^b, h_3^b] = \left[\frac{1}{2}, \frac{9}{8}, \frac{3}{8}\right]$$

is a vector of coefficients from the scaling equation (2.5). The matrix $\mathbf{M}_R$ is obtained from a matrix $\mathbf{M}_L$ by reversing the ordering of the rows.

It follows from (2.7) that the matrix $\mathbf{M}_{j,1}$ is of the size $2^{j+1} \times 2^j$ and has the structure

$$\mathbf{M}_{j,1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & & & 0 & 0 \\ \vdots & \vdots & & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^T. \tag{3.3}$$

The following lemmas are crucial for the proof of a Riesz basis property.

LEMMA 3.1. *Let $j \geq 2$ and the entries $\tilde{M}^{j,0}_{k,l}$, $k \in \mathcal{I}_{j+1}$, $l \in \mathcal{I}_j$, of the matrix $\tilde{\mathbf{M}}_{j,0}$ be given by:*

$$\tilde{M}^{j,0}_{2,l} = \tilde{M}^{j,0}_{1,l} = \frac{d^j_1}{a^{|1-l|}} + \frac{d^j_n}{a^{|n-l|}},$$
$$\tilde{M}^{j,0}_{2^{j+1},l} = \tilde{M}^{j,0}_{2^{j+1}-1,l} = \frac{d^j_1}{a^{|n-l|}} + \frac{d^j_n}{a^{|1-l|}},$$

where $n = 2^j$, $a = -3 - 2\sqrt{2}$,

$$d^j_1 = \frac{6\alpha_n}{3+\sqrt{2}}, \qquad d^j_n = \frac{-36b\,\alpha_n\,a^{2-n}}{11+6\sqrt{2}}, \tag{3.4}$$
$$\alpha_n = \left(1 - \frac{36\,b^2\,a^{4-2n}}{11+6\sqrt{2}}\right)^{-1}, \qquad b = \frac{13-9\sqrt{2}}{6},$$

and for $k = 2, \ldots, n-1$ and $l \in \mathcal{I}_j$, let

$$\tilde{M}^{j,0}_{2k,l} = \tilde{M}^{j,0}_{2k-1,l} = \frac{1}{a^{|k-l|}} + \frac{d^j_k}{a^{|1-l|}} + \frac{d^j_{n+1-k}}{a^{|n-l|}},$$

where

$$d^j_k = \frac{-6b\,\alpha_n\,a^{2-k}}{3+\sqrt{2}} - \frac{36b\,\alpha_n\,a^{k+3-2n}}{11+6\sqrt{2}}. \tag{3.5}$$

Then

$$\mathbf{M}^T_{j,0}\tilde{\mathbf{M}}_{j,0} = \mathbf{I}_j, \qquad \text{and} \qquad \mathbf{M}^T_{j,1}\tilde{\mathbf{M}}_{j,0} = \mathbf{0}_j, \tag{3.6}$$

where $\mathbf{I}_j$ denotes the identity matrix and $\mathbf{0}_j$ denotes the zero matrix of the appropriate size.

Proof. By a similar approach as in [25, 26] we derive the explicit form of the entries $\tilde{M}^{j,0}_{k,l}$, $k \in \mathcal{I}_{j+1}$, $l \in \mathcal{I}_j$, of the matrix $\tilde{\mathbf{M}}_{j,0}$ such that (3.6) is satisfied. From (3.3) we obtain

$$\tilde{M}_{2k-1,l} = \tilde{M}_{2k,l}, \qquad \text{for } k = 1, \ldots, 2^j. \tag{3.7}$$

We substitute (3.7) into (3.6), and we obtain a new system $\mathbf{A}_j\mathbf{B}_j = \mathbf{I}_j$, where

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{13}{8} & \frac{3}{8} & 0 & \cdots & & & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & & \cdots & 0 & \frac{3}{8} & \frac{13}{8} \end{bmatrix} = \frac{\mathbf{H}_j}{\sqrt{2}} \begin{bmatrix} \frac{13}{12} & \frac{1}{4} & 0 & \cdots & & & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & & \cdots & 0 & \frac{1}{4} & \frac{13}{12} \end{bmatrix},$$

with

$$(\mathbf{H}_j)_{k,l} = \begin{cases} \frac{3}{2}, & (k,l) = (1,1)\,,\ (k,l) = \left(2^j, 2^j\right), \\ 1, & k = l, k \neq 1, k \neq 2^j, \\ 0, & \text{otherwise}, \end{cases}$$

and $\mathbf{B}_j$ is the $2^j \times 2^j$ matrix with entries $B^j_{k,l} = \tilde{M}^{j,0}_{2k,l}$, $k,l \in \mathcal{I}_j$. We factorize the matrix $\mathbf{A}_j$ as $\mathbf{A}_j = \mathbf{H}_j \mathbf{C}_j \mathbf{D}_j$, where

$$\mathbf{C}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{3+2\sqrt{2}}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & \cdots & 0 & 0 & \frac{1}{4} & \frac{3+2\sqrt{2}}{4} \end{bmatrix}$$

and

$$\mathbf{D}_j = \begin{bmatrix} \frac{3+\sqrt{2}}{6} & 0 & 0 & \cdots & 0 & 0 & \frac{b}{a^{n-2}} \\ b & 1 & 0 & & 0 & 0 & \frac{b}{a^{n-3}} \\ \frac{b}{a} & 0 & 1 & & 0 & 0 & \frac{b}{a^{n-4}} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \frac{b}{a^{n-4}} & 0 & 0 & & 1 & 0 & \frac{b}{a} \\ \frac{b}{a^{n-3}} & 0 & 0 & & 0 & 1 & b \\ \frac{b}{a^{n-2}} & 0 & 0 & \cdots & 0 & 0 & \frac{3+\sqrt{2}}{6} \end{bmatrix}.$$

More precisely, the entries $D^j_{k,l}$ of the matrix $\mathbf{D}_j$ are given by

$$\begin{aligned} D^j_{1,1} &= D^j_{n,n} = \frac{3+\sqrt{2}}{6}, && \\ D^j_{k,1} &= D^j_{n+1-k,n} = \frac{b}{a^{k-2}}, && \text{for } k = 2,\dots,n, \\ D^j_{k,k} &= 1, && \text{for } k = 2,\dots,n-1, \\ D^j_{k,l} &= 0, && \text{otherwise.} \end{aligned}$$

It is easy to verify that $\tilde{\mathbf{C}}_j = \mathbf{C}_j^{-1}$ has entries $\tilde{C}^j_{k,l} = a^{-|k-l|}$, and the matrix $\mathbf{D}_j^{-1}$ has the structure

$$\mathbf{D}_j^{-1} = \begin{bmatrix} d^j_1 & 0 & \cdots & 0 & d^j_n \\ d^j_2 & 1 & & 0 & d^j_{n-1} \\ \vdots & & \ddots & & \vdots \\ d^j_{n-1} & 0 & & 1 & d^j_2 \\ d^j_n & 0 & \cdots & 0 & d^j_1 \end{bmatrix}$$

with d^j_k given by (3.4) and (3.5). Since the matrices $\mathbf{C}_j$, $\mathbf{D}_j$ and $\mathbf{H}_j$ are invertible, we can define

$$\mathbf{B}_j = \mathbf{A}_j^{-1} = \mathbf{D}_j^{-1}\mathbf{C}_j^{-1}\mathbf{H}_j^{-1}. \tag{3.8}$$

By substituting (3.8) into (3.7), the lemma is proved. □

LEMMA 3.2. *There exist unique matrices* $\tilde{\mathbf{M}}_{j,1}$, $j \geq 2$, *such that*

$$\mathbf{M}_{j,0}^T \tilde{\mathbf{M}}_{j,1} = \mathbf{0}_j \qquad \text{and} \qquad \mathbf{M}_{j,1}^T \tilde{\mathbf{M}}_{j,1} = \mathbf{I}_j. \tag{3.9}$$

Proof. For $l \in \mathcal{I}_{j+1}$ and $k \in \mathcal{I}_j$ the entries $\tilde{M}_{k,l}^{j,1}$ of the matrix $\tilde{\mathbf{M}}_{j,1}$ satisfy

$$\tilde{M}_{2k-1,l}^{j,1} = 2\delta_{2k-1,2l-1} + \tilde{M}_{2k,l}^{j,1}.$$

Using these relations we obtain a system of equations with the matrix $\mathbf{A}_j$ defined in the proof of Lemma 3.4. Since the matrix $\mathbf{A}_j$ is invertible, the matrix $\tilde{\mathbf{M}}_{j,1}$ exists and is unique. □

LEMMA 3.3. *We have* $\Phi_{j+1} = \tilde{\mathbf{M}}_{j,0}\Phi_j + \tilde{\mathbf{M}}_{j,1}\Psi_j$ *for all* $j \geq 2$.

Proof. Due to (3.2) we have

$$\begin{bmatrix} \Phi_j \\ \Psi_j \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{j,0}^T \\ \mathbf{M}_{j,1}^T \end{bmatrix} \Phi_{j+1}, \qquad j \geq 2.$$

By multiplying this equation by the matrix $[\tilde{\mathbf{M}}_{j,0}, \tilde{\mathbf{M}}_{j,1}]$ from the left-hand side and using (3.6) and (3.9), the lemma is proved. □

For any matrix $\mathbf{M}$ of the size $m \times n$ we set

$$\|\mathbf{M}\|_2 = \sup_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}\neq\mathbf{0}} \frac{\|\mathbf{M}\mathbf{v}\|_2}{\|\mathbf{v}\|_2}$$

and

$$\|\mathbf{M}\|_1 = \max_{l=1,\ldots,n} \sum_{k=1}^{m} |M_{k,l}|, \qquad \|\mathbf{M}\|_\infty = \max_{k=1,\ldots,m} \sum_{l=1}^{n} |M_{k,l}|.$$

It is well-known that

$$\|\mathbf{M}\|_2 \leq \sqrt{\|\mathbf{M}\|_1 \|\mathbf{M}\|_\infty}. \tag{3.10}$$

LEMMA 3.4. *The matrices* $\tilde{\mathbf{M}}_{j,0}$, $j \geq 2$, *have uniformly bounded norms, i.e., there exists* $C \in \mathbb{R}$ *independent of* j *such that* $\|\tilde{\mathbf{M}}_{j,0}\|_2 \leq C$ *for all* $j \geq 2$.

Proof. Since the matrices $\tilde{\mathbf{M}}_{j,0}$ are known in the explicit form, they have a regular structure, and the entries in each column and row are exponentially decreasing, we compute upper bounds for the 1-norm and ∞-norm by computing several of the largest entries in each row and column and estimating the sum of the remaining entries. We obtain

$$\|\tilde{\mathbf{M}}_{j,0}\|_1 \leq 1.42, \qquad \|\tilde{\mathbf{M}}_{j,0}\|_\infty \leq 2.91,$$

and due to (3.10) we have $\|\tilde{\mathbf{M}}_{j,0}\|_2 \leq 2.04$. □

For comparison we computed the norms of the matrices $\tilde{\mathbf{M}}_{j,0}$ numerically and found that $\|\tilde{\mathbf{M}}_{j,0}\|_2 \leq 2$ for $j = 3, \ldots, 12$, and $\|\tilde{\mathbf{M}}_{12,0}\|_2 = 1.9999997$.

LEMMA 3.5. *Let* $\mathbf{S}_j = \tilde{\mathbf{M}}_{j,0}^T \tilde{\mathbf{M}}_{j+1,0}^T$, $j \geq 3$, *and* $\tilde{\mathbf{S}}_j$ *be the matrix given by*

$$(\tilde{\mathbf{S}}_j)_{k,l} = (\mathbf{S}_j)_{2k-1,l} + (\mathbf{S}_j)_{2k,l}, \qquad k \in \mathcal{I}_{j-1}, l \in \mathcal{I}_{j+2}.$$

Then there exists a constant C *independent of* j *such that* $\|\tilde{\mathbf{S}}_j\|_2 < C < 2\sqrt{2}$.

Proof. Let $\mathbf{K}_j$ be a $2^j \times 2^{j+1}$ matrix with entries

$$(\mathbf{K}_j)_{k,2l-1} = (\mathbf{K}_j)_{k,2l} = a^{-|k-l|}, \qquad k, l \in \mathcal{I}_j, \quad a = -3 - 2\sqrt{2}, \tag{3.11}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

and let $\mathbf{L}_j = \tilde{\mathbf{M}}_{j,0}^T - \mathbf{K}_j$. We know the explicit expression of the matrix $\mathbf{L}_j$ because they are know for both $\tilde{\mathbf{M}}_{j,0}$ and $\mathbf{K}_j$. We have

$$\mathbf{S}_j = \tilde{\mathbf{M}}_{j,0}^T \tilde{\mathbf{M}}_{j+1,0}^T = \mathbf{K}_j \mathbf{K}_{j+1} + \mathbf{K}_j \mathbf{L}_{j+1} + \mathbf{L}_j \mathbf{K}_{j+1} + \mathbf{L}_j \mathbf{L}_{j+1}.$$

Let us denote

$$\mathbf{N}_j = \mathbf{K}_j \mathbf{K}_{j+1}, \quad \mathbf{O}_j = \mathbf{K}_j \mathbf{L}_{j+1}, \quad \mathbf{P}_j = \mathbf{L}_j \mathbf{K}_{j+1}, \quad \mathbf{Q}_j = \mathbf{L}_j \mathbf{L}_{j+1},$$

and let $\tilde{\mathbf{N}}_j$, $\tilde{\mathbf{O}}_j$, $\tilde{\mathbf{P}}_j$, and $\tilde{\mathbf{Q}}_j$ be derived from $\mathbf{N}_j$, $\mathbf{O}_j$, $\mathbf{P}_j$, and $\mathbf{Q}_j$ in a similar way as $\tilde{\mathbf{S}}_j$ from $\mathbf{S}_j$. Then $\tilde{\mathbf{S}}_j = \tilde{\mathbf{N}}_j + \tilde{\mathbf{O}}_j + \tilde{\mathbf{P}}_j + \tilde{\mathbf{Q}}_j$. From (3.11) we have for $k \in \mathcal{I}_j$, $l \in \mathcal{I}_{j+1}$

$$(\mathbf{N}_j)_{k,2l-1} = (\mathbf{N}_j)_{k,2l} = \mathbf{u}_k^T \mathbf{v}_l,$$

where

$$\mathbf{u}_k = \left[\frac{1}{a^{k-1}}, \frac{1}{a^{k-1}}, \frac{1}{a^{k-2}}, \ldots, \frac{1}{a}, \frac{1}{a}, 1, 1, \frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a^{n-k}}, \frac{1}{a^{n-k}}\right]^T,$$
$$\mathbf{v}_l = \left[\frac{1}{a^{l-1}}, \frac{1}{a^{l-2}}, \ldots, \frac{1}{a}, 1, \frac{1}{a}, \ldots, \frac{1}{a^{2n-l}}\right]^T,$$

$n = 2^j$. Due to the structure of the vector $\mathbf{u}_k$ we can write

$$(\mathbf{N}_j)_{k,l} = \frac{a+1}{a}\, \tilde{\mathbf{u}}_k^T \tilde{\mathbf{v}}_l,$$

where

$$\tilde{\mathbf{u}}_k = \left[\frac{1}{a^{k-1}}, \frac{1}{a^{k-2}}, \ldots, \frac{1}{a}, 1, \frac{1}{a}, \ldots, \frac{1}{a^{n-k}}\right]^T,$$
$$\tilde{\mathbf{v}}_l = \begin{cases} \left[\frac{1}{a^{l-2}}, \frac{1}{a^{l-4}}, \ldots, \frac{1}{a^2}, 1, \frac{1}{a}, \frac{1}{a^3} \ldots, \frac{1}{a^{2n-l-1}}\right]^T, & l \text{ even}, \\ \left[\frac{1}{a^{l-2}}, \frac{1}{a^{l-4}}, \ldots, \frac{1}{a}, 1, \frac{1}{a^2}, \frac{1}{a^4} \ldots, \frac{1}{a^{2n-l-1}}\right]^T, & l \text{ odd}. \end{cases}$$

For $k > \frac{l}{2}$, $l \in \mathcal{I}_{j+1}$, l even, we have

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{\frac{l}{2}} a^{3m-k-l} + \sum_{m=\frac{l}{2}+1}^{k} a^{l+1-k-m} + \sum_{m=k+1}^{n} a^{l+k+1-3m}\right) \\ &= \frac{a+1}{a}\left(a^{\frac{l}{2}-k}\frac{1-\left(\frac{1}{a^3}\right)^{\frac{l}{2}}}{1-\frac{1}{a^3}} + a^{\frac{l}{2}-k}\frac{1-\left(\frac{1}{a}\right)^{k-\frac{l}{2}}}{1-\frac{1}{a}} + a^{l-2-2k}\frac{1-\left(\frac{1}{a^3}\right)^{n-k}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

Similarly for $k > \frac{l-1}{2}$, $l \in \mathcal{I}_{j+1}$, l odd, we obtain

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{\frac{l-1}{2}} a^{3m-k-l} + \sum_{m=\frac{l+1}{2}}^{k} a^{l+1-k-m} + \sum_{m=k+1}^{n} a^{l+k+1-3m}\right) \\ &= \frac{a+1}{a}\left(a^{\frac{l}{2}-k-\frac{3}{2}}\frac{1-\left(\frac{1}{a^3}\right)^{\frac{l-1}{2}}}{1-\frac{1}{a^3}} + a^{\frac{l+1}{2}-k}\frac{1-\left(\frac{1}{a^3}\right)^{k-\frac{l-1}{2}}}{1-\frac{1}{a}} + a^{l-2-2k}\frac{1-\left(\frac{1}{a^3}\right)^{n-k}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

If $k \le \frac{l}{2}$, $l \in \mathcal{I}_{j+1}$, l even, then we have

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{k} a^{3m-k-l} + \sum_{k+1}^{\frac{l}{2}} a^{m+k-l} + \sum_{m=\frac{l}{2}+1}^{n} a^{l+k+1-3m}\right)\\ &= \frac{a+1}{a}\left(a^{2k-l}\frac{1-\left(\frac{1}{a^3}\right)^k}{1-\frac{1}{a^3}} + a^{k-\frac{l}{2}}\frac{1-\left(\frac{1}{a}\right)^{\frac{l}{2}-k}}{1-\frac{1}{a}} + a^{k-\frac{l}{2}-2}\frac{1-\left(\frac{1}{a^3}\right)^{n-\frac{l}{2}}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

If $k \le \frac{l-1}{2}$, $l \in \mathcal{I}_{j+1}$, l odd, then we have

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{k} a^{3m-k-l} + \sum_{k+1}^{\frac{l-1}{2}} a^{m+k+2-l} + \sum_{m=\frac{l+1}{2}+1}^{n} a^{l+k+1-3m}\right)\\ &= \frac{a+1}{a}\left(a^{2k-l}\frac{1-\left(\frac{1}{a^3}\right)^k}{1-\frac{1}{a^3}} + a^{k-\frac{l}{2}-\frac{1}{2}}\frac{1-\left(\frac{1}{a}\right)^{\frac{l-1}{2}-k}}{1-\frac{1}{a}} + a^{k-\frac{l}{2}-\frac{1}{2}}\frac{1-\left(\frac{1}{a^3}\right)^{n-\frac{l-1}{2}}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

To compute an upper bound for the norm of the matrix $\tilde{\mathbf{S}}_j$, we compute bounds for the sums of the absolute values of the entries in the rows and columns for the matrices $\tilde{\mathbf{N}}_j$, $\tilde{\mathbf{O}}_j$, $\tilde{\mathbf{P}}_j$, and $\tilde{\mathbf{Q}}_j$. Since the entries in the columns of the matrix $\tilde{\mathbf{N}}_j$ are exponentially decreasing, we can compute several of the largest entries in each column and estimate the sum of the absolute values of the remaining entries. We define

$$\bar{\mathcal{I}}_{j+2} = \left\{1,2,3,4,2^{j+2}-3,2^{j+2}-2,2^{j+2}-1,2^{j+2}\right\}, \qquad \check{\mathcal{I}}_{j+2} = \mathcal{I}_{j+2}\backslash\bar{\mathcal{I}}_{j+2},$$

and we set

$$(\tilde{\mathbf{N}}_j)_{k,l} = 0, \qquad \text{for } k \notin \mathcal{I}_{j-1}.$$

For l such that $l \bmod 8 \in \{0,1,6,7\}$ and $l \in \check{\mathcal{I}}_{j+2}$, we obtain

$$\begin{aligned}\sum_{k=1}^{2^{j-1}} \left|(\tilde{\mathbf{N}}_j)_{k,l}\right| &\le \left|(\tilde{\mathbf{N}}_j)_{\lfloor\frac{l}{8}\rfloor-1,l}\right| + \left|(\tilde{\mathbf{N}}_j)_{\lfloor\frac{l}{8}\rfloor,l}\right| + \left|(\tilde{\mathbf{N}}_j)_{\lfloor\frac{l}{8}\rfloor+1,l}\right|\\ &\quad + \sum_{k=1}^{\lfloor\frac{l}{8}\rfloor-2}\left|(\tilde{\mathbf{N}}_j)_{k,l}\right| + \sum_{k=\lfloor\frac{l}{8}\rfloor+2}^{2^{j-1}}\left|(\tilde{\mathbf{N}}_j)_{k,l}\right|\\ &\le 0.018+0.727+0.239+0.007+0.001 \le 1.\end{aligned}$$

For l such that $l \bmod 8 \in \{2,3,4,5\}$ and $l \in \check{\mathcal{I}}_{j+2}$, we obtain

$$\begin{aligned}\sum_{k=1}^{2^{j-1}} \left|(\tilde{\mathbf{N}}_j)_{k,l}\right| &\le \left|(\tilde{\mathbf{N}}_j)_{\lfloor\frac{l}{8}\rfloor-1,l}\right| + \left|(\tilde{\mathbf{N}}_j)_{\lfloor\frac{l}{8}\rfloor,l}\right| + \left|(\tilde{\mathbf{N}}_j)_{\lfloor\frac{l}{8}\rfloor+1,l}\right|\\ &\quad + \sum_{k=1}^{\lfloor\frac{l}{8}\rfloor-2}\left|(\tilde{\mathbf{N}}_j)_{k,l}\right| + \sum_{k=\lfloor\frac{l}{8}\rfloor+2}^{2^{j-1}}\left|(\tilde{\mathbf{N}}_j)_{k,l}\right|\\ &\le 0.101+0.566+0.037+0.002+0.004 \le 1.\end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

For $l \in \bar{\mathcal{I}}_{j+2}$ we have

$$\sum_{k=1}^{2^{j-1}} \left| (\tilde{\mathbf{N}}_j)_{k,l} \right| \leq 0.5.$$

We use a similar approach for computing the sums of absolute values of the entries in the rows. We obtain

$$\sum_{k=1}^{2^{j-1}} \left| (\tilde{\mathbf{N}}_j)_{k,l} \right| \leq \begin{cases} 0.73, & l \in \bar{\mathcal{I}}_{j+2}, \\ 1.00, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \qquad \sum_{l=1}^{2^{j+2}} \left| (\tilde{\mathbf{N}}_j)_{k,l} \right| \leq \begin{cases} 5.95, & k = 1, 2^{j-1}, \\ 6.80, & \text{otherwise}. \end{cases}$$

Similarly, we obtain

$$\sum_{k=1}^{2^{j-1}} \left| (\tilde{\mathbf{O}}_j)_{k,l} \right| \leq \begin{cases} 0.13, & l \in \bar{\mathcal{I}}_{j+2}, \\ 0.04, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \qquad \sum_{l=1}^{2^{j+2}} \left| (\tilde{\mathbf{O}}_j)_{k,l} \right| \leq \begin{cases} 0.30, & k = 1, 2^{j-1}, \\ 0.02, & \text{otherwise}, \end{cases}$$

$$\sum_{k=1}^{2^{j-1}} \left| (\tilde{\mathbf{P}}_j)_{k,l} \right| \leq \begin{cases} 0.15, & l \in \bar{\mathcal{I}}_{j+2}, \\ 0.05, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \qquad \sum_{l=1}^{2^{j+2}} \left| (\tilde{\mathbf{P}}_j)_{k,l} \right| \leq \begin{cases} 0.68, & k = 1, 2^{j-1}, \\ 0.04, & \text{otherwise}, \end{cases}$$

$$\sum_{k=1}^{2^{j-1}} \left| (\tilde{\mathbf{Q}}_j)_{k,l} \right| \leq \begin{cases} 0.03, & l \in \bar{\mathcal{I}}_{j+2}, \\ 0.01, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \qquad \sum_{l=1}^{2^{j+2}} \left| (\tilde{\mathbf{Q}}_j)_{k,l} \right| \leq \begin{cases} 0.06, & k = 1, 2^{j-1}, \\ 0.01, & \text{otherwise}. \end{cases}$$

Therefore using (3.10) we have

$$\|\tilde{\mathbf{S}}_j\|_2 \leq \sqrt{1.1 \cdot 7} < 2\sqrt{2}. \qquad \square$$

For comparison we computed the norms of the matrices $\tilde{\mathbf{S}}_j$ numerically, and we found that $\|\tilde{\mathbf{S}}_j\|_2 \leq 2.27$ for $j = 1, \dots, 12$, $\|\tilde{\mathbf{S}}_{12}\|_2 \approx 2.2623$, and it seems that this value does not further increase with increasing j.

LEMMA 3.6. *Let $m, n \geq 2$, $m < n$. Then there exists a constant $C < 2$ such that*

$$\left\|\tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \dots \tilde{\mathbf{M}}_{n,0}^T \tilde{\mathbf{M}}_{n+1,0}^T\right\|_2 \leq C \left\|\tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \dots \tilde{\mathbf{M}}_{n-1,0}^T\right\|_2.$$

Proof. For m and n fixed such that $m, n \geq 2$, $m < n$, we use the notation:

$$\mathbf{R} = \tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \dots \tilde{\mathbf{M}}_{n-1,0}^T, \qquad \mathbf{S} = \tilde{\mathbf{M}}_{n,0}^T \tilde{\mathbf{M}}_{n+1,0}^T.$$

Due to the structure of the matrices $\tilde{\mathbf{M}}_{j,0}$ given in Lemma 3.4 we have

$$\mathbf{R}_{k,2l} = \mathbf{R}_{k,2l-1}, \qquad k \in \mathcal{I}_m, l \in \mathcal{I}_{n-1}.$$

Therefore, we can write $\mathbf{RS} = \tilde{\mathbf{R}}\tilde{\mathbf{S}}$, where the matrix $\tilde{\mathbf{R}}$ is a $2^m \times 2^{n-1}$ matrix containing the even columns of the matrix $\mathbf{R}$, i.e., $\tilde{\mathbf{R}}_{k,l} = \mathbf{R}_{k,2l}$, and the matrix $\tilde{\mathbf{S}}$ is given by

$$\tilde{\mathbf{S}}_{k,l} = \mathbf{S}_{2k-1,l} + \mathbf{S}_{2k,l}, \qquad k \in \mathcal{I}_{n-1}, l \in \mathcal{I}_{n+2}.$$

We have

$$\|\tilde{\mathbf{R}}\|_2 = \sup_{\mathbf{x} \in \mathbb{R}, \mathbf{x} \neq 0} \frac{\|\tilde{\mathbf{R}}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sup_{\mathbf{x} \in \mathbb{R}, \mathbf{x} \neq 0} \frac{\left(\sum_{k \in \mathcal{I}_m} \left(\sum_{l \in \mathcal{I}_{n-1}} \tilde{\mathbf{R}}_{k,l} \mathbf{x}_l \right)^2 \right)^{1/2}}{\|\mathbf{x}\|_2}.$$

Let $\tilde{\mathbf{x}}$ be a vector of length $q = 2^n$ such that $\tilde{x}_{2j-1} = \tilde{x}_{2j} = x_j$, and let

$$\tilde{X} = \{\tilde{\mathbf{x}} \in \mathbb{R}^q : \tilde{x}_{2j-1} = \tilde{x}_{2j}, \tilde{\mathbf{x}} \neq \mathbf{0}\}.$$

Then $\|\tilde{\mathbf{x}}\|_2 = \sqrt{2}\,\|\mathbf{x}\|_2$, and we have

$$\|\tilde{\mathbf{R}}\|_2 = \sup_{\tilde{\mathbf{x}} \in \tilde{X}} \frac{\left(\sum_{k \in \mathcal{I}_m} \left(\sum_{l \in \mathcal{I}_n} 2^{-1} \mathbf{R}_{k,l} \tilde{\mathbf{x}}_l \right)^2 \right)^{1/2}}{2^{-1/2}\,\|\tilde{\mathbf{x}}\|_2}$$

$$\leq \sup_{\tilde{\mathbf{x}} \in \mathbb{R}^q, \tilde{\mathbf{x}} \neq 0} \frac{2^{-1} \left(\sum_{k \in \mathcal{I}_m} \left(\sum_{l \in \mathcal{I}_n} \mathbf{R}_{k,l} \tilde{\mathbf{x}}_l \right)^2 \right)^{1/2}}{2^{-1/2}\,\|\tilde{\mathbf{x}}\|_2} = \frac{\|\mathbf{R}\|_2}{\sqrt{2}}.$$

Using Lemma 3.5 we obtain

$$\|\mathbf{R}\mathbf{S}\|_2 = \|\tilde{\mathbf{R}}\tilde{\mathbf{S}}\|_2 \leq \|\tilde{\mathbf{R}}\|_2\,\|\tilde{\mathbf{S}}\|_2 \leq C\,\|\mathbf{R}\|_2$$

with $C < 2$. ∎

LEMMA 3.7. *There exist constants $C \in \mathbb{R}$ and $p < 0.5$ such that for all $m, n \geq 2$, $m < n$, we have*

$$\left\| \tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \ldots \tilde{\mathbf{M}}_{n-1,0}^T \right\|_2 \leq C\,2^{p(n-m)}. \tag{3.12}$$

Proof. The assertion of the lemma is a direct consequence of Lemma 3.4 and Lemma 3.6. ∎

4. Riesz basis on Sobolev spaces. In this section, we prove that Ψ is a Riesz basis for $H_0^1(\Omega_1)$ and Ψ^{2D} is a Riesz basis for $H_0^1(\Omega_2)$. The proof is based on the lemmas from Section 3 and on the theory developed in [19] that is summarized in the following theorem.

THEOREM 4.1. *Let H be a Hilbert space and let V_j, $j \geq J$, be closed subspaces of $L_2(\Omega)$ such that $V_j \subset V_{j+1}$ and $\bigcup_{j=J}^{\infty} V_j$ is dense in H. Let H_q for fixed $q > 0$ be a linear subspace of H that is itself a normed linear space and assume that there exist positive constants A_1 and A_2 such that*

(a) If $f \in H_q$ has decomposition $f = \sum_{j \geq J} f_j$, $f_j \in V_j$, then

$$\|f\|_{H_q}^2 \leq A_1 \sum_{j \geq J} 2^{qj}\,\|f_j\|_H^2. \tag{4.1}$$

(b) For each $f \in H_q$ there exists a decomposition $f = \sum_{j \geq J} f_j$, $f_j \in V_j$, such that

$$\sum_{j \geq J} 2^{qj}\,\|f_j\|_H^2 \leq A_2\,\|f\|_{H_q}^2. \tag{4.2}$$

Furthermore, suppose that P_j is a linear projection from V_{j+1} onto V_j, W_j is the kernel space of P_j, $\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}$ are Riesz bases of V_j with respect to the L_2-norm with uniformly bounded condition numbers, and $\Psi_j = \{\psi_{j,k}, k \in \mathcal{I}_j\}$ are Riesz bases of W_j with uniformly bounded condition numbers. If there exist constants C and p such that $0 < p < q$ and

$$\|P_m P_{m+1} \ldots P_{n-1}\| \leq C\,2^{p\,(n-m)}, \tag{4.3}$$

then

$$\left\{2^{-Jq}\phi_{J,k}, k \in \mathcal{I}_J\right\} \cup \left\{2^{-jq}\psi_{j,k}, j \geq J, k \in \mathcal{I}_j\right\} \tag{4.4}$$

is a Riesz basis for H_q.

Now we define suitable projections P_j from V_{j+1} onto V_j and show that these projections satisfy (4.3). Then we show that Ψ—which differs from (4.4) only by scaling—is also a Riesz basis for $H_0^1(0,1)$. For $j \geq 2$ we define

$$\Gamma_j = \{\phi_{j,k}\}_{k\in\mathcal{I}_j} \cup \{\psi_{j,k}\}_{k\in\mathcal{I}_j} \qquad \text{and} \qquad \mathbf{F}_j = \langle \Gamma_j, \Gamma_j \rangle .$$

Let a set

$$\hat{\Gamma}_j = \{\hat{\phi}_{j,k}\}_{k\in\mathcal{I}_j} \cup \{\hat{\psi}_{j,k}\}_{k\in\mathcal{J}_j} \tag{4.5}$$

be given by

$$\hat{\Gamma}_j = \mathbf{F}_j^{-1}\Gamma_j. \tag{4.6}$$

Since obviously $\langle \Gamma_j, \hat{\Gamma}_j \rangle = \mathbf{I}_j$, functions from $\hat{\Gamma}_j$ are duals to functions from Γ_j in the space V_{j+1}. Since $\mathbf{F}_j^{-1}$ is not a sparse matrix, these duals are not local. We define a projection P_j from V_{j+1} onto V_j by

$$P_j f = \sum_{k\in\mathcal{I}_j} \langle f, \hat{\phi}_{j,k} \rangle \, \phi_{j,k}.$$

LEMMA 4.2. *There exist* $p < 0.5$ *such that a projection* P_j *satisfies*

$$\|P_m P_{m+1} \dots P_{n-1}\| \leq C\, 2^{p\,(n-m)} \tag{4.7}$$

for all $2 \leq m < n$ *and a constant* C *independent of* m *and* n.

Proof. Let $f \in V_{j+1}$, $a_k^j = \langle f, \hat{\phi}_{j,k} \rangle$, $\mathbf{a}_j = \left\{a_k^j\right\}_{k\in\mathcal{I}_j}$, $j \geq 2$, and $\mathbf{S}_j : \mathbf{a}_{j+1} \mapsto \mathbf{a}_j$. Then

$$P_j f = \sum_{k\in\mathcal{I}_j} a_k^j \phi_{j,k} = \sum_{k\in\mathcal{I}_j} \langle f, \hat{\phi}_{j,k} \rangle \, \phi_{j,k} = \sum_{k\in\mathcal{I}_j} \sum_{l\in\mathcal{I}_{j+1}} a_l^{j+1} \langle \phi_{j+1,l}, \hat{\phi}_{j,k} \rangle \, \phi_{j,k}.$$

Therefore

$$a_k^j = \sum_{l\in\mathcal{I}_{j+1}} a_l^{j+1} \langle \phi_{j+1,l}, \hat{\phi}_{j,k} \rangle .$$

Let us denote

$$S_{l,k}^j = \langle \hat{\phi}_{j,k}, \phi_{j+1,l} \rangle , \qquad \mathbf{S}_j = \left\{S_{l,k}^j\right\}_{l\in\mathcal{I}_{j+1}, k\in\mathcal{I}_j}.$$

Then we can write $\mathbf{a}_j = \mathbf{S}_j \mathbf{a}_{j+1}$, and due to Lemma 3.3 we have

$$\mathbf{S}_j = \langle \hat{\Phi}_j, \Phi_{j+1} \rangle = \langle \hat{\Phi}_j, \tilde{\mathbf{M}}_{j,0}\Phi_j + \tilde{\mathbf{M}}_{j,1}\Psi_j \rangle = \tilde{\mathbf{M}}_{j,0}.$$

Now, let us consider $f_n \in V_n$ and $f_m = P_m P_{m+1} \dots P_{n-1} f_n$. Then f_j can be represented by $f_j = \sum_{k\in\mathcal{I}_j} a_k^j \phi_j$ for $j = m, n$, and we set $\mathbf{a}_j = \left\{a_k^j\right\}_{k\in\mathcal{I}_j}$. Since Φ_j is a Riesz basis for V_j, see [22], there exist constants C_1 and C_2 independent of j such that

$$C_1 \left\|\mathbf{a}_j\right\|_2 \leq \left\| \sum_{k\in\mathcal{I}_j} a_k^j \phi_{j,k} \right\| \leq C_2 \left\|\mathbf{a}_j\right\|_2 .$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Due to Lemma 3.7 we have

$$\begin{aligned}\|f_m\| &\le C_2\,\|\mathbf{a}_m\|_2 \le C_2\,\|\mathbf{S}_m\,\mathbf{S}_{m+1}\dots\mathbf{S}_{n-1}\|_2\,\|\mathbf{a}_n\|_2\\ &= C_2\,\left\|\tilde{\mathbf{M}}_{m,0}^T\,\tilde{\mathbf{M}}_{m+1,0}^T\dots\tilde{\mathbf{M}}_{n-1,0}^T\right\|_2\,\|\mathbf{a}_n\|_2\\ &\le C_2\,2^{p(n-m)}\;\|\mathbf{a}_n\|_2 \le C_1^{-1}\,C_2\,2^{p(n-m)}\;\|f_n\|\,.\end{aligned}$$

Thus (4.7) is proved. ∎

THEOREM 4.3. *The sets Ψ_j are Riesz bases of the spaces $W_j = \operatorname{span}\Psi_j$, $j \ge 2$, with the condition numbers bounded independently of j, namely* $\operatorname{cond}\Psi_j \le 2$.

Proof. The matrix $\mathbf{U}_j = \langle\Psi_j, \Psi_j\rangle$ is tridiagonal with entries

$$\begin{aligned}(\mathbf{U}_j)_{1,1} = (\mathbf{U}_j)_{2^j,2^j} &= \frac{27}{320},\\ (\mathbf{U}_j)_{2,1} = (\mathbf{U}_j)_{1,2} = (\mathbf{U}_j)_{2^j-1,2^j} = (\mathbf{U}_j)_{2^j,2^j-1} &= \frac{47}{1920},\\ (\mathbf{U}_j)_{k,k} &= \frac{1}{12}, && k = 2,\dots,2^j-1,\\ (\mathbf{U}_j)_{k,k+1} = (\mathbf{U}_j)_{k+1,k} &= -\frac{1}{40}, && k = 2,\dots,2^j-2,\\ (\mathbf{U}_j)_{k,l} &= 0, && \text{otherwise.}\end{aligned}$$

Thus, $\mathbf{U}_j$ is strictly diagonally dominant, and using the Gershgorin circle theorem we obtain $\lambda_{\min}(\mathbf{U}_j) \ge \frac{1}{30} \approx 0.0333$, $\lambda_{\max}(\mathbf{U}_j) \le \frac{2}{15} \approx 0.1333$, and $\operatorname{cond}\Psi_j \le 2$. ∎

We also computed the eigenvalues of the matrix $\mathbf{U}_j$ numerically and the numerical values $\lambda_{\min} \approx 0.0333$ and $\lambda_{\max} \approx 0.1333$ correspond to the values from Gershgorin's theorem. Thus the inequality in Theorem 4.3 seems to be sharp.

THEOREM 4.4. *The set*

$$\left\{2^{-2}\phi_{2,k}, k \in \mathcal{I}_2\right\} \cup \left\{2^{-j}\psi_{j,k}, j \ge 2, k \in \mathcal{I}_j\right\}$$

is a Riesz basis for $H_0^1(0,1)$.

Proof. Using the same argument as in [19], we conclude that (4.1) and (4.2) follows from the polynomial exactness of the scaling basis and the smoothness of the basis functions, and these inequalities are satisfied for $H = L^2(0,1)$ and $H_q = H_0^q(0,1)$, $0 < q < 1.5$. Due to Lemma 4.2 the condition (4.3) is fulfilled. Therefore by Theorem 4.1 the assertion of Theorem 4.4 is proved. ∎

THEOREM 4.5. *The set*

$$\left\{\phi_{2,k}/\left|\phi_{2,k}\right|_{H_0^1(0,1)}, k \in \mathcal{I}_2\right\} \cup \left\{\psi_{j,k}/\left|\psi_{j,k}\right|_{H_0^1(0,1)}, j \ge 2, k \in \mathcal{I}_j\right\},$$

where $|\cdot|_{H_0^1(0,1)}$ denotes the $H_0^1(0,1)$-seminorm, is a Riesz basis for $H_0^1(0,1)$.

Proof. We follow the proof of Lemma 2 in [25]. From (2.9), there exist constants C_1 and C_2 such that

$$C_1 2^j \le \left|\psi_{j,k}\right|_{H_0^1(\Omega)} \le C_2 2^j, \qquad \text{for } j \ge 2, \qquad k \in \mathcal{I}_j, \tag{4.8}$$

and

$$C_1 2^2 \le \left|\phi_{2,k}\right|_{H_0^1(\Omega)} \le C_2 2^2, \qquad \text{for } k \in \mathcal{I}_2. \tag{4.9}$$

Theorem 4.4 implies that there exist constants C_3 and C_4 such that

$$C_3 \left\|\mathbf{b}\right\|_2 \leq \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} 2^{-2} \phi_{2,k} + \sum_{k \in \mathcal{I}_j, j \geq 2} b_{j,k} 2^{-j} \psi_{j,k} \right\|_{H_0^1(0,1)} \leq C_4 \left\|\mathbf{b}\right\|_2, \tag{4.10}$$

for any $\mathbf{b} = \{a_{2,k},\ k \in \mathcal{I}_2\} \cup \{b_{j,k}, j \geq 2,\ k \in \mathcal{J}_j\}$. Using (4.8), (4.9), and (4.10) we obtain

$$\left\|\mathbf{b}\right\|_2 \leq \frac{C_2}{C_3} \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} \frac{\phi_{2,k}}{\left|\phi_{2,k}\right|_{H_0^1(\Omega)}} + \sum_{k \in \mathcal{J}_j, j \geq 2} b_{j,k} \frac{\psi_{j,k}}{\left|\psi_{j,k}\right|_{H_0^1(\Omega)}} \right\|_{H_0^1(0,1)}$$

and

$$\left\|\mathbf{b}\right\|_2 \geq \frac{C_1}{C_4} \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} \frac{\phi_{2,k}}{\left|\phi_{2,k}\right|_{H_0^1(\Omega)}} + \sum_{k \in \mathcal{J}_j, j \geq 2} b_{j,k} \frac{\psi_{j,k}}{\left|\psi_{j,k}\right|_{H_0^1(\Omega)}} \right\|_{H_0^1(0,1)}. \qquad \square$$

REMARK 4.6. By Theorem 4.1 and the proof of Lemma 4.2, if p satisfies (3.12), then the norm equivalence (2.1) for Ψ from Section 2 normalized with respect to the H^s-norm is satisfied for $H = H^s$, where $s \in (p, 1.5)$. Since we proved in Section 3 that there exists p satisfying (3.12) such that $p < 0.5$, we proved the norm equivalence (2.1) for H^s with $s \in (0.5, 1.5)$. We computed the norms in (3.12) also numerically, and we found that this theoretical estimate of p is not sharp. It seems that (3.12) holds also for any $p > 0$.

THEOREM 4.7. *The set Ψ^{2D} normalized with respect to the H^1-seminorm is a Riesz basis for $H_0^1\left((0,1)^2\right)$.*

Proof. Recall that $\hat{\phi}_{j,k}$ are defined by (4.5) and (4.6). For $\mathbf{k} = (k_1, k_2)$ let us define $\hat{\phi}^2_{j,\mathbf{k}} = \hat{\phi}_{j,k_1} \otimes \hat{\phi}_{j,k_2}$. Then for $\mathbf{k} = (k_1, k_2)$ and $\mathbf{l} = (l_1, l_2)$ we have

$$\left\langle \phi^2_{j,\mathbf{k}}, \hat{\phi}^2_{j,\mathbf{l}} \right\rangle = \delta_{k_1,l_1} \delta_{k_2,l_2},$$

and P_j^{2D} defined by

$$P_j^{2D} f = \sum_{\mathbf{k} \in \mathcal{I}_j \times \mathcal{I}_j} \left\langle f, \hat{\phi}^2_{j,\mathbf{k}} \right\rangle \phi^2_{j,\mathbf{k}}$$

is a projection from V^2_{j+1} onto V^2_j, where $V^2_j = V_j \otimes V_j$ for $j \geq 2$. We define the matrix $\mathbf{S}^{2D}_j = \tilde{\mathbf{M}}^T_{j,0} \otimes \tilde{\mathbf{M}}^T_{j,0}$. It is well-known that for any matrix $\mathbf{B}$ we have $\left\|\mathbf{B} \otimes \mathbf{B}\right\|_2 = \left\|\mathbf{B}\right\|_2^2$. Using this relation and the same arguments as in the proof of Lemma 4.2, we obtain for $f_n \in V^2_n$ and $f_m = P^{2D}_m P^{2D}_{m+1} \ldots P^{2D}_{n-1} f_n$ the estimate

$$\begin{aligned}
\left\|f_m\right\| &\leq C_1 \left\|\mathbf{a}_m\right\|_2 \leq C_2 \left\|\mathbf{S}^{2D}_m \mathbf{S}^{2D}_{m+1} \ldots \mathbf{S}^{2D}_{n-1}\right\|_2 \left\|\mathbf{a}_n\right\|_2 \\
&= C_2 \left\|\left(\tilde{\mathbf{M}}^T_{m,0} \ldots \tilde{\mathbf{M}}^T_{n-1,0}\right) \otimes \left(\tilde{\mathbf{M}}^T_{m,0} \ldots \tilde{\mathbf{M}}^T_{n-1,0}\right)\right\|_2 \left\|\mathbf{a}_n\right\|_2 \\
&\leq C_3\, 2^{2p(n-m)} \left\|\mathbf{a}_n\right\|_2 \leq C_4\, 2^{2p(n-m)} \left\|f_n\right\|
\end{aligned}$$

with $2p < 1$. Hence by Theorem 4.1 the assertion of the theorem is proved. $\square$

5. Quantitative properties of the constructed bases. In this section, we present the condition numbers of the stiffness matrices for the Helmholtz equation

$$-\epsilon \Delta u + a u = f \text{on } \Omega_d, \qquad u = 0 \quad \text{on} \quad \partial\Omega_d, \tag{5.1}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

where Δ is the Laplace operator and ϵ and a are positive constants. We also study the case $\epsilon = 1$ and $a = 0$, i.e., the Poisson equation, and the case $\epsilon = 0$ and $a = 1$.

The variational formulation is

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \tag{5.2}$$

where

$$\mathbf{A} = \epsilon \langle \nabla\Psi, \nabla\Psi \rangle + a \langle \Psi, \Psi \rangle, \qquad u = (\mathbf{u})^T \Psi, \qquad \mathbf{f} = \langle f, \Psi \rangle .$$

An advantage of discretizing the elliptic equation (5.1) using a wavelet basis is that the system (5.2) can be simply preconditioned by a diagonal preconditioner [9]. Let $\mathbf{D}$ be the matrix of the diagonal elements of the matrix $\mathbf{A}$, i.e., $\mathbf{D}_{\lambda,\mu} = \mathbf{A}_{\lambda,\mu}\delta_{\lambda,\mu}$, where $\delta_{\lambda,\mu}$ denotes the Kronecker delta. Setting

$$\tilde{\mathbf{A}} = (\mathbf{D})^{-1/2}\mathbf{A}(\mathbf{D})^{-1/2}, \qquad \tilde{\mathbf{u}} = (\mathbf{D})^{1/2}\mathbf{u}, \qquad \tilde{\mathbf{f}} = (\mathbf{D})^{-1/2}\mathbf{f},$$

we obtain the preconditioned system

$$\tilde{\mathbf{A}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}. \tag{5.3}$$

It is known [9] that there exists a constant C such that $\operatorname{cond} \tilde{\mathbf{A}} \leq C < \infty$.

Let Ψ^s be defined by (2.10) for $d = 1$ and similarly for $d > 1$. We define

$$\mathbf{A}_s = \epsilon \langle \nabla\Psi^s, \nabla\Psi^s \rangle + a \langle \Psi^s, \Psi^s \rangle, \qquad u_s = (\mathbf{u}_s)^T \Psi^s, \qquad \mathbf{f}_s = \langle f, \Psi^s \rangle .$$

Let $\mathbf{D}_s$ be the matrix of the diagonal elements of the matrix $\mathbf{A}_s$, i.e., $(\mathbf{D}_s)_{\lambda,\mu} = (\mathbf{A}_s)_{\lambda,\mu}\delta_{\lambda,\mu}$. We set

$$\tilde{\mathbf{A}}_s = (\mathbf{D}_s)^{-1/2}\mathbf{A}_s(\mathbf{D}_s)^{-1/2}, \qquad \tilde{\mathbf{u}}_s = (\mathbf{D}_s)^{1/2}\mathbf{u}_s, \qquad \tilde{\mathbf{f}}_s = (\mathbf{D}_s)^{-1/2}\mathbf{f}_s,$$

and we obtain the preconditioned finite-dimensional system

$$\tilde{\mathbf{A}}_s\tilde{\mathbf{u}}_s = \tilde{\mathbf{f}}_s. \tag{5.4}$$

Since $\tilde{\mathbf{A}}_s$ is a part of the matrix $\tilde{\mathbf{A}}$ that is symmetric and positive definite, we also have

$$\operatorname{cond} \tilde{\mathbf{A}}_s \leq C.$$

The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ for $\epsilon = 1$, $a = 0$, and $d = 1, 2$, are shown in Table 5.1. By Remark 2.2 these numbers correspond to the squares of the condition numbers of Ψ^s with respect to the H^1-seminorm. We also computed the condition numbers of Ψ^s with respect to the H^1-norm. The values were very close to the values presented in Table 5.1 (the difference was less than 1%).

For comparison, we also provide the condition numbers for other wavelet bases and display them in Figure 5.1 and Figure 5.2. The bases CF_2 and CF_3 refer to the wavelet bases from this paper with the coarsest level 2 and 3, respectively. D_{j_0} and P_{j_0} refer to the quadratic spline wavelet basis with 3 vanishing moments and the coarsest level j_0 from [11] and [20], respectively. We modify the construction from [3] to homogeneous boundary conditions. The resulting quadratic spline wavelet basis with three vanishing wavelet moments with the coarsest level j_0 is denoted as B_{j_0}. We found that the bases D_{j_0}, P_{j_0}, and B_{j_0} lead to the same results and realized that they contain the same wavelets up to a multiplication by a constant factor. Semi-orthogonal quadratic spline wavelets with three vanishing moments

TABLE 5.1
The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets for the one- (left), two- (right), and three-dimensional (bottom) Poisson equation.

s	N	$\lambda_{\min}$	$\lambda_{\max}$	cond $\tilde{\mathbf{A}}_s$
1	8	0.50	1.38	2.77
2	16	0.50	1.41	2.83
3	32	0.50	1.42	2.83
4	64	0.50	1.42	2.84
5	128	0.50	1.42	2.84
6	256	0.50	1.42	2.84
7	512	0.50	1.42	2.84
8	1024	0.50	1.42	2.84

s	N	$\lambda_{\min}$	$\lambda_{\max}$	cond $\tilde{\mathbf{A}}_s$
1	64	0.25	1.88	7.5
2	256	0.19	2.08	11.1
3	1 024	0.16	2.17	13.7
4	4 096	0.14	2.20	15.4
5	16 384	0.13	2.22	16.6
6	65 536	0.13	2.23	17.4
7	262 144	0.12	2.23	17.9
8	1 048 576	0.12	2.23	18.3

s	N	$\lambda_{\min}$	$\lambda_{\max}$	cond $\tilde{\mathbf{A}}_s$
1	512	0.15	3.23	47.4
2	4 096	0.04	3.69	85.0
3	32 768	0.03	3.83	113.8
4	262 144	0.03	3.87	132.9
5	2 097 152	0.03	3.89	145.3

on the interval were constructed in [5]. In Appendix A we show that the semi-orthogonal quadratic spline wavelet basis corresponding to scaling functions that are B-splines on the Schoenberg sequence of knots such that wavelets have three vanishing moments and the basis is adapted to homogeneous boundary conditions do not exist. Therefore, we adapt this basis such that semi-orthogonality is preserved and $2^j - 2$ wavelets on the level j have three vanishing moments and 2 wavelets on the level j are without vanishing moments. We denote the resulting basis by CQ. We also tested wavelet bases from [11, 20] with 5 vanishing moments, but the condition numbers were larger than for bases with 3 vanishing moments. All wavelets used in the numerical experiments are presented in Appendix A.

Although it was not proved in this paper that by appropriate tensorising the 1D wavelet basis we obtain the wavelet basis in 3D, we list the condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ for the 3D case in Table 5.1. The condition numbers for several constructions of quadratic spline wavelet bases and various values of parameters ϵ and a are compared in Table 5.2.

TABLE 5.2
The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size 65536×65536 for several choices of ϵ and a for our bases and the bases from [11, 20].

ϵ	a	CF_2	CF_3	CF_2^{ort}	CF_3^{ort}	CQ	D_2	D_3
1000	1	17.4	16.3	17.1	16.4	62.0	116.3	98.4
1	0	17.4	16.7	17.1	16.4	62.0	116.3	98.4
1	1	17.4	16.7	17.1	16.4	62.0	116.6	98.5
10^{-3}	1	72.1	35.9	35.6	22.5	61.1	328.1	139.2
10^{-6}	1	746.0	577.0	425.7	287.6	46.3	1878.0	1115.4
0	1	872.6	687.4	511.0	351.5	46.4	2034.6	1251.4

We also provide the condition numbers for the discretization matrices $\tilde{\mathbf{A}}_s$ corresponding to $\epsilon = 0$, $a = 1$, and $d = 1$. By Remark 4.6 these condition numbers represent the squares of the L^2 condition numbers of Ψ^s normalized with respect to the L^2-norm. The results

ETNA
Kent State University and
Johann Radon Institute (RICAM)

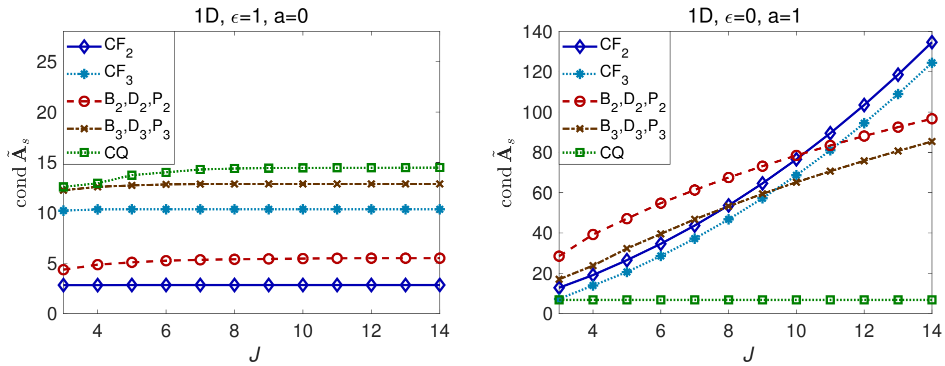


FIG. 5.1. *The condition numbers of the matrices $\tilde{\mathbf{A}}_s$, $s = J - j_0 + 1$, for the one-dimensional problem* (5.1) *with parameters $\epsilon = 1$, $a = 0$, and $\epsilon = 0$, $a = 1$. The parameter J denotes the finest level, and j_0 denotes the coarsest level.*

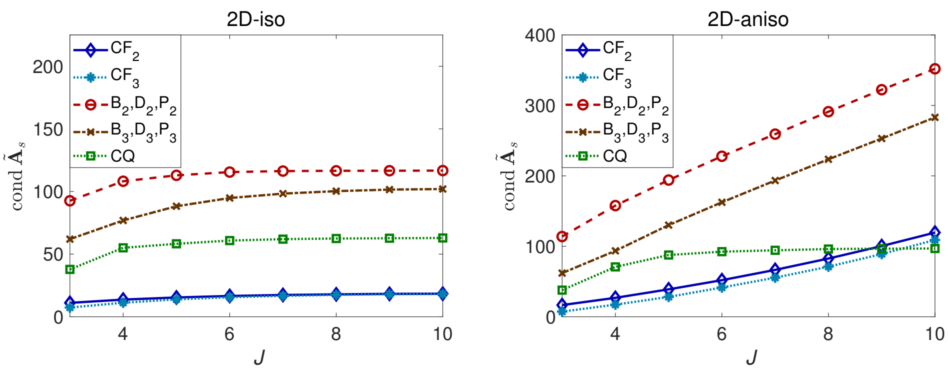


FIG. 5.2. *The condition numbers of the matrices $\tilde{\mathbf{A}}_s$, $s = J - j_0 + 1$, for $\epsilon = 1$, $a = 0$ and two-dimensional wavelet bases constructed using an isotropic approach and an anisotropic approach. The parameter J denotes the finest level, and j_0 denotes the coarsest level.*

are displayed in Figure 5.1. In this paper, we have proved that the constructed basis is a Riesz basis for $H_0^1(0,1)$. The condition numbers of matrices $\tilde{\mathbf{A}}_s$ corresponding to $\epsilon = 0$ and $a = 1$ for the new basis seem to be unbounded, and thus it seems that the new basis is not a Riesz basis in $L^2(0,1)$; see also Remark 4.6. Since the condition numbers of the matrices $\tilde{\mathbf{A}}_s$ for $\epsilon = 1$ and $a = 0$ corresponding to the anisotropic basis $\Psi \otimes \Psi$ with respect to the H^1-seminorm depend on the condition numbers of Ψ^s both with respect to the L^2-norm and the H^1-seminorm, they are also increasing; see Figure 5.2. Thus in our case an isotropic wavelet basis from Section 2 has bounded and significantly smaller condition number than an anisotropic basis. We performed numerical experiments with both types of bases, but since the isotropic system lead to significantly better results we present in Section 6 only the experiments with the isotropic wavelet bases.

6. Numerical examples. In this section we use the constructed wavelet basis in the wavelet-Galerkin method and the adaptive wavelet method.

6.1. Multilevel Galerkin method. We consider the problem (5.1) with Ω_2, $\epsilon = 1$, and $a = 0$. The right-hand side f is such that the solution u is given by

$$u(x,y) = v(x)\, v(y), \qquad v(x) = x\left(1 - e^{50x-50}\right).$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

We discretize the equation using the Galerkin method with the wavelet basis constructed in this paper, and we obtain the discrete problem $\tilde{\mathbf{A}}_s \tilde{\mathbf{u}}_s = \tilde{\mathbf{f}}_s$. We solve it by the conjugate gradient method using a simple multilevel approach similarly to [19, 27]:

1. Compute $\tilde{\mathbf{A}}_s$ and $\tilde{\mathbf{f}}_s$, choose $\mathbf{v}_0$ of length 4^2.
2. For $j = 0, \ldots, s$ find the solution $\tilde{\mathbf{u}}_j$ of the system $\tilde{\mathbf{A}}_j \tilde{\mathbf{u}}_j = \tilde{\mathbf{f}}_j$ by the conjugate gradient method with initial vector $\mathbf{v}_j$ defined for $j \geq 1$ by

$$(\mathbf{v}_j) = \begin{cases} \tilde{\mathbf{u}}_{j-1}, & i = 1, \ldots, k_j, \\ 0, & i = k_j, \ldots, k_{j+1}, \end{cases}$$

   where $k_j = 2^{2(j+1)}$.

Let u be the exact solution of (5.1) and

$$u_s^* = (\tilde{\mathbf{u}}_s^*)^T (\mathbf{D}_s)^{-1/2} \Psi^s,$$

where $\tilde{\mathbf{u}}_s^*$ is the exact solution of the discrete problem (5.4). It is known [21] that due to the polynomial exactness of the spaces $\operatorname{span} \Psi^s$, there exists a constant C independent of s such that

$$\|u - u_s^*\| \leq C 2^{-3s}, \qquad \|u - u_s^*\|_{H^1(\Omega_d)} \leq C 2^{-2s}, \tag{6.1}$$

for $u \in H^3(\Omega_d)$. Let u_s be an approximate solution obtained by the multilevel Galerkin method with s levels of wavelets. It was shown in [27] that if we use as criterion for terminating iterations $\|\mathbf{r}_s\|_2 \leq C 2^{-2s}$, where $\mathbf{r}_s := \tilde{\mathbf{A}}_s \tilde{\mathbf{u}}_s - \tilde{\mathbf{f}}_s$, then we achieve for u_s the same convergence rate as for u_s^*. In our example, for the given number of levels s we use the criterion $\|\mathbf{r}_j\|_2 \leq 10^{-4} 2^{-2s}$, $j = 0, \ldots, s$, for terminating the iterations in each level.

We denote the number of iterations on the level j by M_j. It is known [21] that employing the discrete wavelet transform, one CG iteration can be performed with a complexity of order $\mathcal{O}(N)$, where $N \times N$ is the size of the matrix. Therefore the number of operations needed to compute one CG iteration on level j requires about one quarter of the operations needed to compute one CG iteration on level $j+1$. We compute the total number of equivalent iterations by

$$M = \sum_{j=0}^{s} \frac{M_j}{4^{s-j}}.$$

The results are listed in Table 6.1. It can be seen that the number of conjugate gradient iterations is quite small and that

$$\frac{\|u_s - u\|_\infty}{\|u_{s+1} - u\|_\infty} \approx \frac{\|u_s - u\|}{\|u_{s+1} - u\|} \approx \frac{1}{8},$$

i.e., that the order of convergence is 3. It corresponds to (6.1). The parameters r_2 and r_∞ in Table 6.1 are the experimental rates of convergence, i.e.

$$(r_2)_s = \frac{\log\left(\|u_{s-1} - u\| \,/\, \|u_s - u\|\right)}{\log 2}, \qquad (r_\infty)_s = \frac{\log\left(\|u_{s-1} - u\|_\infty \,/\, \|u_s - u\|_\infty\right)}{\log 2}.$$

We present also the wall clock time in Table 6.1. It includes the computation of the right-hand side, the system matrix, iterations, and evaluation of the solution on the grid with the step size 2^{-j_0-s}, where j_0 is the coarsest level.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

TABLE 6.1
Number of iterations and error estimates for the multilevel conjugate gradient method.

CF_2

s	N	M	$\|u_s - u\|_\infty$	r_∞	$\|u_s - u\|$	r_2	time $[s]$
1	64	18.50	3.19e-1		4.54e-2		0.04
2	256	21.63	1.32e-1	1.27	1.26e-3	5.17	0.05
3	1 024	23.66	2.60e-2	2.34	2.02e-3	2.64	0.06
4	4 096	23.00	2.91e-3	3.16	2.45e-4	3.04	0.09
5	16 384	20.89	4.06e-4	2.84	2.89e-5	3.08	0.16
6	65 536	18.37	5.35e-5	2.92	3.41e-6	3.08	0.30
7	262 144	15.68	6.82e-6	2.97	4.23e-7	3.01	0.99
8	1 048 576	13.02	8.63e-7	2.98	5.28e-8	3.00	3.89
9	4 194 304	10.35	1.08e-7	3.00	6.59e-9	3.00	14.87
10	16 777 216	8.85	1.41e-8	2.94	8.25e-10	3.00	58.12

D_2, P_2, B_2

s	N	M	$\|u_s - u\|_\infty$	r_∞	$\|u_s - u\|$	r_2	time $[s]$
1	64	27.50	3.19e-1		4.54e-2		0.04
2	256	48.88	1.32e-1	1.27	1.26e-3	5.17	0.07
3	1 024	59.22	2.60e-2	2.34	2.02e-3	2.64	0.11
4	4 096	59.38	2.91e-3	3.16	2.45e-4	3.04	0.19
5	16 384	50.76	4.06e-4	2.84	2.89e-5	3.08	0.33
6	65 536	39.44	5.35e-5	2.92	3.41e-6	3.08	0.68
7	262 144	29.92	6.84e-6	2.97	4.23e-7	3.01	2.20
8	1 048 576	21.50	8.64e-7	2.98	5.29e-8	3.00	9.53
9	4 194 304	17.66	1.09e-7	2.99	6.73e-9	2.97	47.39
10	16 777 216	15.79	1.38e-8	2.98	9.43e-10	2.84	248.41

CQ

s	N	M	$\|u_s - u\|_\infty$	r_∞	$\|u_s - u\|$	r_2	time $[s]$
0	64	13.00	3.19e-1		4.54e-2		0.03
1	256	30.25	1.32e-1	1.27	1.26e-3	5.17	0.05
3	1 024	35.06	2.60e-2	2.34	2.02e-3	2.64	0.07
4	4 096	33.82	2.91e-3	3.16	2.45e-4	3.04	0.14
5	16 384	30.30	4.06e-4	2.84	2.89e-5	3.08	0.21
6	65 536	25.32	5.35e-5	2.92	3.41e-6	3.08	0.41
7	262 144	20.74	6.84e-6	2.97	4.23e-7	3.01	1.39
8	1 048 576	17.87	8.64e-7	2.98	5.29e-8	3.00	5.55
9	4 194 304	14.82	1.08e-7	3.00	6.73e-9	2.97	21.62
10	16 777 216	12.36	1.36e-8	2.99	8.56e-10	2.97	83.54

6.2. Adaptive wavelet method. We compare the quantitative behavior of the adaptive wavelet method with our wavelet basis, the wavelet basis from [11], and the wavelet basis that is a modification of the basis from [5]; see Appendix A. We consider the equation (5.1) with $d = 1$, $\epsilon = 1$, $a = 0$, and the solution

$$u(x) = e^{-\left|\frac{x}{4} - \frac{1}{8}\right|} - e^{-\frac{1}{8}} + \sin 3\pi x, \qquad x \in [0, 1].$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Note that u is the sum of an infinitely differentiable function and the function

$$g(x) = e^{-\left|\frac{x}{4}-\frac{1}{8}\right|}$$

which does not have a derivative at the point 0.5. Let $\hat{g}$ be the Fourier transform of g, i.e.,

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x)\, e^{-ix\xi} dx.$$

Since

$$\int_{\mathbb{R}} |\xi|^{2\mu} |\hat{g}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{64\,|\xi|^{2\mu}}{(16\xi^2+1)^2} d\xi$$

is finite for $\mu < 3/2$ and it is not finite for $\mu \geq 3/2$, the solution u belongs to the Sobolev space $u \in H^s(0,1)$ only for $s < 3/2$. Therefore it is not guaranteed that (6.1) holds and that the Galerkin method converges with the optimal rate. Since u is continuous and piecewise smooth, it can be shown that u belongs to the Besov space $B^s_{\tau,\tau}(0,1)$ for any $s > 0$ and $\tau = (s+1/2)^{-1}$. It is therefore convenient to solve this problem with the adaptive wavelet method proposed in [6, 7] because it is proved that this method converges with the optimal rate for functions from such spaces. More precisely, let u_j be the approximate solution in the jth step, and let ρ_j denote the error in the energy norm which is in this example the same as the H^1-seminorm, i.e., $\rho_j = |u-u_j|_{H^1}$. Let $\mathbf{u}_j$ be the vector of coefficients corresponding to u_j, and let N_j be the number of nonzero entries of $\mathbf{u}_j$. It follows from the theory developed in [7] that if the used basis is a quadratic spline wavelet basis, then there exists a constant C independent of j such that

$$\rho_j \leq C N_j^{-r} \qquad \text{for any } r < 2. \tag{6.2}$$

The method consists in solving the infinite preconditioned system (5.3) with Richardson iterations. The algorithm contains the routine **COARSE** that is based on thresholding the coefficients and the routine **RHS** that approximates the infinite right-hand side vector by a finite vector with a prescribed accuracy. For details about these two routines we refer to [7]. It is possible to modify the algorithm such that the routine **COARSE** is avoided; see [14]. Furthermore, it is necessary to have a routine that allows multiplication of the bi-infinite matrix $\tilde{\mathbf{A}}$ with a finitely supported vector. This routine called **APPLY** was proposed in [7] and modified in [12, 24]. We use the version from [24]. We use a similar version of the method and notations that is presented as **CDD02SOLVE** in [14]. We compute the relaxation parameter ω and the error reduction factor ρ by

$$\omega = \frac{2}{\lambda_{\max}(\tilde{\mathbf{A}}) + \lambda_{\min}(\tilde{\mathbf{A}})}, \qquad \rho = \frac{\operatorname{cond}\tilde{\mathbf{A}} - 1}{\operatorname{cond}\tilde{\mathbf{A}} + 1},$$

and we set $\theta = 0.3$ and $K \in \mathbb{N}$ such that $2\rho^K/\theta < 0.6$.

We use the following version of the method:

ETNA
Kent State University and
Johann Radon Institute (RICAM)

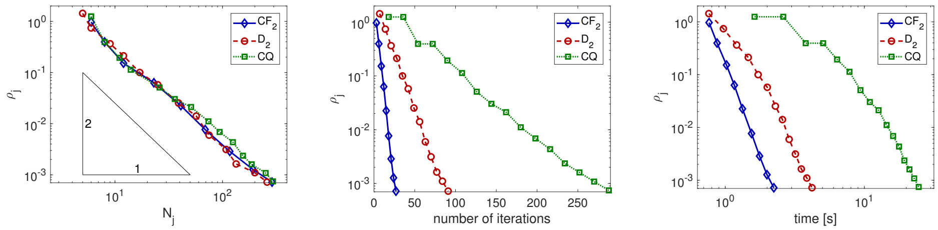


FIG. 6.1. *The convergence history for adaptive wavelet scheme with various wavelet bases.*

ALGORITHM 6.1. ***SOLVE*** **[$\tilde{\mathbf{A}}$, f, ϵ] $\to \mathbf{u}_\epsilon$**

Set $j := 0$, $\mathbf{u}_0 := 0$, *and* $\epsilon_0 \geq \|\tilde{\mathbf{u}}\|_2$*;*
while $\epsilon_j > \epsilon$ **do**
  $\mathbf{z}_0 := \mathbf{u}_j$*;*
  for $l = 1, \ldots, K$ **do**
    $\mathbf{z}_l := \mathbf{z}_{l-1} + \omega \left(\mathbf{RHS}[\mathbf{f}, \frac{\epsilon_j \rho^l}{2\omega K}] - \mathbf{APPLY}[\tilde{\mathbf{A}}, \mathbf{z}_{l-1}, \frac{\epsilon_j \rho^l}{2\omega K}] \right)$*;*
  end
  $j := j + 1$*;*
  $\epsilon_j := \frac{2\rho^K \epsilon_{j-1}}{\theta}$*;*
  $\mathbf{u}_j := \mathbf{COARSE}[\mathbf{z}_K, (1-\theta)\,\epsilon_j]$*;*
end
$\mathbf{u}_\epsilon := \mathbf{u}_j$*;*

We use the following parameters in the numerical experiments:

CF_2: $\omega = 1.04$, $\rho = 0.48$, $K = 4$,
D_2: $\omega = 0.89$, $\rho = 0.70$, $K = 7$,
CQ_3: $\omega = 0.95$, $\rho = 0.87$, $K = 18$.

The convergence history is shown in Figure 6.1. Since the entries of the matrix $\tilde{\mathbf{A}}$, the estimates of eigenvalues of $\tilde{\mathbf{A}}$, and the parameters ω, ρ, and K were precomputed for every basis, the wall clock time includes the computation of the right-hand side and the computation of the iterations. The experimental convergence rate, i.e., the parameter r from (6.2) estimated for the observed values (N_j, ρ_j) by the least-squares method, for the bases CF_2, D_2, and CQ was $r \approx 1.87$, $r \approx 1.95$, and $r \approx 1.77$, respectively. It can be seen that the number of iterations and the computational time needed to solve the problem with the desired accuracy is significantly smaller for the new wavelet basis. Moreover, due to the shorter support of the wavelets, the stiffness matrix is sparser, and thus, one iteration requires a smaller number of operations.

Appendix A. Quadratic spline wavelet bases.

In this section we present inner and boundary scaling functions and wavelets that were used in the numerical experiments in Section 6. The wavelet bases are generated from these functions in a way similar to (2.6) and (2.9). Let ϕ be given by (2.2) and $\check{\phi}_b = 2\phi^b/3$, where ϕ_b is given by (2.4). Since diagonal preconditioning (5.4) is similar to the normalization of the basis with respect to the energy norm, the multiplication of ϕ_b with a constant has no effect on the resulting condition numbers presented in Section 5 and the numerical results in Section 6. The wavelets are given by

$$\check{\psi}(x) = \sum_{k=0}^{7} g_k \phi(2x-k), \qquad \check{\psi}^i = g_{-1}^i \check{\phi}^b(2x) + \sum_{k=0}^{5} g_k^i \phi(2x-k),$$

for $i = 1, 2$. The values of the parameters g_k and g_k^i are presented for several constructions below.

A.1. The Primbs wavelet basis. The parameters for the construction from [20] are given by

$$[g_0, \ldots, g_7] = [-3, -9, 7, 45, -45, -7, 9, 3]\,/64,$$
$$[g_{-1}^1, \ldots, g_6^1] = \left[-10, \frac{65}{6}, -\frac{9}{14}, -\frac{31}{7}, -\frac{11}{21}, \frac{15}{14}, \frac{5}{14}\right]/64,$$
$$[g_{-1}^2, \ldots, g_6^2] = \left[-\frac{10}{3}, -\frac{5}{6}, \frac{65}{6}, -\frac{25}{3}, -\frac{13}{9}, \frac{3}{2}, \frac{1}{2}\right]/64.$$

More precisely, in [20] the parameters are multiples of these parameters, but as we already mentioned, different normalizations do not play a role because we use diagonal preconditioning (5.4) in our experiments.

A.2. The Dijkema wavelet basis. There are several constructions in [11]. We used the parameters that are listed in the file mats.zip attached to [11], but we found that in the case of quadratic spline wavelets with three vanishing moments and homogeneous boundary conditions, these parameters are multiples of the parameters from [20] and thus lead to the same results.

A.3. Modification of the Chui-Quak wavelet basis. In [5] the semi-orthogonal quadratic spline wavelets with three vanishing moments were adapted to the interval. We adapt these wavelets to homogeneous boundary conditions. Since wavelets on the level j are linear combinations of scaling functions on the level $j+1$, they are given by 2^{j+1} parameters. We want to preserve semi-orthogonality, therefore we have 2^j conditions of orthogonality for the scaling functions on the level j. Furthermore, we want to preserve three vanishing moments. We obtain a homogeneous system with $2^j + 2$ independent equations with 2^{j+1} variables that has only $2^j - 2$ independent solutions. Therefore there exist only $2^j - 2$ wavelets with three vanishing moments that are semi-orthogonal. We add two wavelets on each level that are semi-orthogonal but without vanishing moments. We obtain wavelets with the parameters

$$[g_0, \ldots, g_7] = [-1, 29, -147, 303, -303, 147, -29, 1]\,/480,$$
$$\left[g_{-1}^1, \ldots, g_6^1\right] = [450, -332, 148, -29, 1, 0, 0]\,/480,$$
$$\left[g_{-1}^2, \ldots, g_6^2\right] = \left[\frac{780}{11}, -\frac{1949}{11}, \frac{3481}{11}, -\frac{3362}{11}, \frac{1618}{11}, -29, 1\right]/480.$$

A.4. Modification of the Bittner wavelet basis. In [3] spline wavelet bases on the interval were constructed. We use a similar approach as in [3], but for quadratic spline wavelets with three vanishing moments satisfying homogeneous boundary conditions. The inner wavelet is the third derivative of the sixth-order B-spline on the knots $[0, 1, 2, 5/2, 3, 4, 5]$. The boundary wavelets are the third derivatives of the sixth-order B-splines on the knots $[0, 0, 1/2, 1, 2, 3, 4]$ and $[0, 0, 1, 3/2, 2, 3, 4]$, respectively. We found that by this approach we again obtain the same wavelets up to a constant factor as in [11, 20].

REFERENCES

[1] T. BARSCH, *Adaptive Multiskalenverfahren für elliptische partielle Differentialgleichungen—Realisierung, Umsetzung und numerische Ergebnisse*, PhD. Thesis, RWTH Aachen, Aachen 2001.

[2] G. BEYLKIN, R. COIFMAN, AND V. ROKHLIN, *Fast wavelet transforms and numerical algorithms. I*, Comm. Pure Appl. Math., 44 (1991), pp. 141–183.

[3] K. BITTNER, *Biorthogonal spline wavelets on the interval*, in Wavelets and Splines: Athens 2005, G. Chen and M.-J. Lai, eds., Mod. Methods Math., Nashboro Press, Brentwood, 2006, pp. 93–104.
[4] C. BURSTEDDE, *Fast Optimized Wavelet Methods for Control Problems Constrained by Elliptic PDEs*, PhD. Thesis, Universität Bonn, Bonn, 2005.
[5] C. K. CHUI AND E. QUAK, *Wavelets on a bounded interval*, in Numerical Methods in Approximation Theory Vol. 9, D. Braess and L. L. Schumaker, eds., vol. 105 of Internat. Ser. Numer. Math., Birkhäuser, Basel, 1992, pp. 53–75.
[6] A. COHEN, W. DAHMEN, AND R. DEVORE, *Adaptive wavelet methods for elliptic operator equations: convergence rates*, Math. Comp., 70 (2001), pp. 27–75.
[7] ———, *Adaptive wavelet methods. II. Beyond the elliptic case*, Found. Comput. Math., 2 (2002), pp. 203–245.
[8] W. DAHMEN, B. HAN, R.-Q. JIA, AND A. KUNOTH, *Biorthogonal multiwavelets on the interval: cubic Hermite splines*, Constr. Approx., 16 (2000), pp. 221–259.
[9] W. DAHMEN AND A. KUNOTH, *Multilevel preconditioning*, Numer. Math., 63 (1992), pp. 315–344.
[10] W. DAHMEN, A. KUNOTH, AND K. URBAN, *Biorthogonal spline wavelets on the interval – stability and moment conditions*, Appl. Comput. Harmon. Anal., 6 (1999), pp. 132–196.
[11] T. J. DIJKEMA, *Adaptive Tensor Product Wavelet Methods for the Solution of PDEs*, PhD. Thesis, Universiteit Utrecht, Utrecht, 2009.
[12] T. J. DIJKEMA, C. SCHWAB, AND R. STEVENSON, *An adaptive wavelet method for solving high-dimensional elliptic PDEs*, Constr. Approx., 30 (2009), pp. 423–455.
[13] T. J. DIJKEMA AND R. STEVENSON, *A sparse Laplacian in tensor product wavelet coordinates*, Numer. Math., 115 (2010), pp. 433–449.
[14] T. GANTUMUR, H. HARBRECHT, AND R. STEVENSON, *An optimal adaptive wavelet method without coarsening of the iterands*, Math. Comp., 76 (2007), pp. 615–629.
[15] S. GRIVET-TALOCIA AND A. TABACCO, *Wavelets on the interval with optimal localization*, Math. Models Methods Appl. Sci., 10 (2000), pp. 441–462.
[16] N. HILBER, O. REICHMANN, C. SCHWAB, AND C. WINTER, *Computational Methods for Quantitative Finance*, Springer Finance, Springer, Heidelberg, 2013.
[17] R.-Q. JIA, *Spline wavelets on the interval with homogeneous boundary conditions*, Adv. Comput. Math., 30 (2009), pp. 177–200.
[18] R.-Q. JIA AND S.-T. LIU, *Wavelet bases of Hermite cubic splines on the interval*, Adv. Comput. Math., 25 (2006), pp. 23–39.
[19] R.-Q. JIA AND W. ZHAO, *Riesz bases of wavelets and applications to numerical solutions of elliptic equations*, Math. Comp., 80 (2011), pp. 1525–1556.
[20] M. PRIMBS, *New stable biorthogonal spline-wavelets on the interval*, Results Math., 57 (2010), pp. 121–162.
[21] K. URBAN, *Wavelet Methods for Elliptic Partial Differential Equations*, Oxford University Press, Oxford, 2009.
[22] D. ČERNÁ AND V. FINĚK, *Construction of optimally conditioned cubic spline wavelets on the interval*, Adv. Comput. Math., 34 (2011), pp. 219–252.
[23] ———, *Cubic spline wavelets with complementary boundary conditions*, Appl. Math. Comput., 219 (2012), pp. 1853–1865.
[24] ———, *Approximate multiplication in adaptive wavelet methods*, Cent. Eur. J. Math., 11 (2013), pp. 972–983.
[25] ———, *Cubic spline wavelets with short support for fourth-order problems*, Appl. Math. Comput., 243 (2014), pp. 44–56.
[26] ———, *Quadratic spline wavelets with short support for fourth-order problems*, Results Math., 66 (2014), pp. 525–540.
[27] ———, *Wavelet basis of cubic splines on the hypercube satisfying homogeneous boundary conditions*, Int. J. Wavelets Multiresolut. Inf. Process., 13 (2015), Art. 1550014 (21 pages).