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BDDC AND FETI-DP ALGORITHMS WITH A CHANGE OF BASIS FORMULATION ON ADAPTIVE PRIMAL CONSTRAINTS*

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Abstract. BDDC (Balancing Domain Decomposition by Constraints) and FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithms with adaptively enriched primal constraints are considered. The coarse component of the two algorithms is built on the set of primal unknowns consisting of those at subdomain vertices and those from the adaptive primal constraints after a change of basis. For the FETI-DP algorithm, a more general form of a preconditioner is proposed to extend the algorithm to the set of primal unknowns including those from the adaptive primal constraints. In addition, it can be shown that the two algorithms share the same spectra except those equal to one or zero when the same set of adaptive primal constraints are employed. Numerical results are included for both two and three dimensional model problems.

Key words. FETI-DP, BDDC, adaptive primal constraints, change of basis, condition numbers

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction.

A finite element discretization of the following model elliptic problem is considered,

$$\int_\Omega \rho(x)\nabla u(x)\cdot\nabla v(x)\,dx = \int_\Omega f(x)v(x)\,dx, \quad \forall v(x)\in H_0^1(\Omega), \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $H_0^1(\Omega)$ is the space of square integrable functions up to the first weak derivatives with trace equal to zero. The coefficient $\rho(x)$ can be highly heterogeneous across the finite element boundaries. The discrete problem of the above model problem can be efficiently solved iteratively by utilizing domain decomposition preconditioners. In this work, BDDC (Balancing Domain Decomposition by Constraints) and FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithms with adaptively enriched primal constraints are analyzed under a change of basis formulation. The adaptive primal constraints are introduced to enhance the robustness of the BDDC and FETI-DP preconditioners to the heterogeneous coefficients in the model elliptic problems. We refer to [4, 19, 7, 2, 18] for a general introduction to standard BDDC and FETI-DP algorithms and their connections. For the variants of the methods enriched with adaptive primal constraints we refer to [20, 21, 22, 5, 23, 15, 16, 14, 3, 11, 13]. We note that for all such algorithms certain generalized eigenvalue problems are solved to select the set of adaptive primal constraints under a given tolerance value. Estimates for the condition number by the given tolerance value were provided in [14] for the FETI-DP algorithm and in [3, 23] for the BDDC algorithm, both in three dimensions. In [13] both algorithms are considered and generalized eigenvalue problems are proposed on face and edge nodal equivalence classes in three dimensions. We refer to [8, 9, 24, 6, 25] for adaptive algorithms in a different framework of domain decomposition methods.

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In the authors' previous studies [12, 13] and other related works [14], the adaptive primal constraints in the FETI-DP algorithm are enforced by using a projection. For the proposed algorithm it was shown that the condition numbers are controlled by a user-defined tolerance value, which is used to select the adaptive primal constraints from generalized eigenvalue problems on each equivalence classes, i.e., on edges and faces. For the BDDC algorithm in [12, 13] the adaptive primal constraints can be transformed to explicit unknowns and added to the set of primal unknowns. The standard form of the BDDC algorithm can then be generalized to such a set of primal unknowns with the estimate of condition numbers controlled by the user defined tolerance value. On the other hand, the FETI-DP algorithm with a change of basis formulation to the adaptive primal constraints has a limitation in the analysis of condition numbers. For the proposed preconditioner, one can not obtain the identity $P_D = I - E_D$ which is used to show the analysis of condition numbers in the standard FETI-DP and BDDC methods; see (4.1) for definitions of E_D and P_D, and Lemma 5.2 for their relation. In [13], it was observed that the FETI-DP algorithm presents numerical instability and needs considerable cost for projection when the number of adaptive constraints becomes significant.

In this work, we propose a more general form of a FETI-DP preconditioner which makes it possible to extend the FETI-DP algorithm to the change of basis formulation for the adaptive primal constraints. The change of basis formulation can give a more stable and efficient FETI-DP algorithm. For the proposed preconditioner, we can obtain the identity $E_D + P_D = I$, after the change of basis, for the FETI-DP algorithms as well, and thus, show that the condition numbers of the adaptive BDDC and FETI-DP algorithms with the change of basis formulation are identical. Different from the standard FETI-DP preconditioners, the blocks of subdomain matrices and scaling matrices corresponding to the adaptive primal unknowns will appear in the proposed preconditioner. We note that part of this work was presented at the 24th International Conference on Domain Decomposition Methods and at the same conference an adaptive FETI-DP algorithm with a change of basis formulation was presented by Axel Klawonn, where different generalized eigenvalue problems are introduced and different tools are used in the analysis of condition numbers.

This paper is organized as follows. In Sections 2 and 3, finite element spaces and adaptive BDDC and FETI-DP algorithms are introduced, where a more general form of the FETI-DP preconditioner is proposed. In Section 4, generalized eigenvalue problems are formed on faces and edges to select the effective adaptive primal constraints which make the resulting algorithms robust to the heterogeneous coefficients. In Section 5, the operators E_D and P_D are shown to satisfy the property $E_D + P_D = I$, and thus the two algorithms share the same set of eigenvalues except zeros and ones. In Section 6 numerical results are included for both two and three dimensional model problems to show that the two algorithms give the same nonzero extreme eigenvalues.

2. Model problem and finite element spaces. We recall the model problem in (1.1). The domain Ω can be in two or three dimensions. We introduce a finite element space $\widehat{X}$ for the given domain Ω. For a presentation of BDDC and FETI-DP algorithms, we introduce a non-overlapping subdomain partition $\{\Omega_i\}$, where we assume that the subdomain boundaries do not cut the triangles in the finite element mesh. We use the notation X_i to denote the restriction of $\widehat{X}$ to Ω_i. Each subdomain Ω_i is then equipped with the finite element space X_i.

We further introduce W_i as the restriction of X_i to the subdomain interface unknowns, W, and X as the product of local finite element spaces W_i and X_i, respectively. We note that functions in W or X are decoupled across the subdomain interfaces. We then select some primal unknowns among the decoupled unknowns on the subdomain interfaces and enforce that finite element functions are continuous on them. We denote the corresponding spaces by

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$\widetilde{W}$ and $\widetilde{X}$, respectively. We also introduce $\widehat{W}$ as the subspace of W, where the unknowns are fully coupled on the subdomain interface.

The preconditioners in BDDC and FETI-DP algorithms will be developed based on the partially coupled space $\widetilde{W}$ and appropriate scaling matrices. In our adaptive methods, we will select primal unknowns from each nodal equivalence classes of subdomain interfaces. In more detail, edges in $2D$ and faces in $3D$ are nodal equivalence classes shared by two subdomains, edges in $3D$ are nodal equivalence classes shared by more than two subdomains, and vertices are end points of edges in both $2D$ and $3D$. We refer to [17] for these definitions.

In our approach, we first include the unknowns at subdomain vertices in the set of primal unknowns. Adaptive primal constraints will then be selected from eigenvectors of certain generalized eigenvalue problems on faces and edges using a given tolerance value. The corresponding adaptive primal unknowns are then obtained by applying a change of basis on the adaptively selected primal constraints. These explicit unknowns can then be assembled strongly just as primal unknowns at subdomain vertices in the standard BDDC and FETI-DP algorithms. We note that in our previous study a change of basis formulation is only considered and analyzed for the BDDC algorithm. In this work, we will extend the change of basis formulation to the FETI-DP algorithm by introducing a more general form of the FETI-DP preconditioner.

3. Adaptive BDDC and FETI-DP algorithms. We introduce matrices K_i and S_i for each subdomain Ω_i. The matrices K_i are obtained from a Galerkin approximation of

$$a_i(u,v) = \int_{\Omega_i} \rho(x) \nabla u \cdot \nabla v \, dx$$

on finite element spaces X_i. S_i are the Schur complements of K_i, obtained from K_i by eliminating unknowns interior to Ω_i. Let $\widetilde{R}_i : \widetilde{W} \rightarrow W_i$ be the restriction operator corresponding to $\partial\Omega_i$, and let $\widetilde{S}$ be a partially coupled matrix defined by

$$\widetilde{S} = \sum_{i=1}^{N} \widetilde{R}_i^T S_i \widetilde{R}_i.$$

Let $\widetilde{R}$ be the extension from $\widehat{W}$ to $\widetilde{W}$. The discrete problem of (1.1) from the finite element space $\widehat{X}$ can be reduced to the following interface problem,

$$\widetilde{R}^T \widetilde{S} \widetilde{R} u_\Gamma = \widetilde{R}^T \widetilde{g},$$

where u_Γ, in $\widehat{W}$, denotes the restriction of u to the subdomain interface, and $\widetilde{g}$, in $\widetilde{W}$, is the vector obtained from the right-hand side $f(x)$.

In the BDDC algorithm the above matrix equation is solved iteratively by using the following preconditioner,

$$M_{BDDC}^{-1} = \widetilde{R}^T \widetilde{D} \widetilde{S}^{-1} \widetilde{D}^T \widetilde{R},$$

where $\widetilde{D}$ is a scaling matrix of the form

$$\widetilde{D} = \sum_{i=1}^{N} \widetilde{R}_i^T D_i \widetilde{R}_i.$$

Here the matrices D_i are defined for unknowns in W_i and they are introduced to resolve heterogeneity in $\rho(x)$ across the subdomain interface. In more details, D_i consists of blocks

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$D_F^{(i)}$, $D_E^{(i)}$, $D_V^{(i)}$, where F denotes corresponding blocks to unknowns in faces, E to unknowns in edges, and V to unknowns at vertices, respectively. In addition, those blocks satisfy the partition of unity for a given F, E, and V, respectively.

The FETI-DP algorithm [7, 18, 2] is a dual form of the BDDC algorithm. After the change of basis for the unknowns corresponding to the adaptively selected constraints, we can obtain the resulting FETI-DP algebraic system

$$B\widetilde{S}^{-1}B^T\lambda = d, \tag{3.1}$$

where $\widetilde{S}$ is the partially coupled matrix at subdomain vertices and adaptive primal unknowns, and B is the matrix with entries 0, -1, and 1, which is used to enforce continuity for the remaining decoupled interface unknowns. We introduce the notation M for the set of Lagrange multipliers λ. The dimension of M is identical to the number of continuity constraints enforced on the remaining decoupled interface unknowns. The algebraic system (3.1) is then solved by an iterative method with the following preconditioner

$$M_{FETI}^{-1} = \sum_{i=1}^{N} B_{D,\Delta}^{(i)} S_i (B_{D,\Delta}^{(i)})^T,$$

where $(B_{D,\Delta}^{(i)})^T : M \to W_i$ is defined by

$$(B_{D,\Delta}^{(i)})^T\lambda|_F = \text{sign}(i,\lambda_{ij}) D_{F,\Delta}^{(j)}\lambda_{ij} \text{ on each } F \in F(i), \tag{3.2}$$

$$(B_{D,\Delta}^{(i)})^T\lambda|_E = \sum_{l\in n(E,i)} \text{sign}(i,\lambda_{il}) D_{E,\Delta}^{(l)}\lambda_{il} \text{ on each } E \in E(i), \tag{3.3}$$

and

$$(B_{D,\Delta}^{(i)})^T\lambda|_V = 0 \text{ on each } V \in V(i). \tag{3.4}$$

Here $F(i)$, $E(i)$, and $V(i)$ denote the set of faces, edges, and vertices of subdomain Ω_i, respectively, $n(E,i)$ denotes the set of neighboring subdomain indices sharing the edge E with Ω_i, and λ_{ij} denotes the part of Lagrange multipliers λ used to enforce continuity on the decoupled unknowns across Ω_i and Ω_j. In addition, sign(i,λ_{il}) are 1 or -1 depending on the sign of $B_\Delta^{(i)}$ in B for the corresponding location to λ_{il}. The matrices $D_{F,\Delta}^{(j)}$ and $D_{E,\Delta}^{(l)}$ are obtained from blocks of $D_F^{(j)}$ and $D_E^{(l)}$ as follows:

$$D_{F,\Delta}^{(j)} = \begin{bmatrix} D_{F,\Delta\Delta}^{(j)} \\ D_{F,\Pi\Delta}^{(j)} \end{bmatrix}, \quad D_{E,\Delta}^{(l)} = \begin{bmatrix} D_{E,\Delta\Delta}^{(l)} \\ D_{E,\Pi\Delta}^{(l)} \end{bmatrix},$$

where the subscripts Δ and Π correspond to blocks in $D_F^{(j)}$ and $D_E^{(l)}$ related to the decoupled unknowns and the adaptive primal unknowns, respectively, i.e.,

$$D_F^{(j)} = \begin{bmatrix} D_{F,\Delta\Delta}^{(j)} & D_{F,\Delta\Pi}^{(j)} \\ D_{F,\Pi\Delta}^{(j)} & D_{F,\Pi\Pi}^{(j)} \end{bmatrix}, \; D_E^{(l)} = \begin{bmatrix} D_{E,\Delta\Delta}^{(l)} & D_{E,\Delta\Pi}^{(l)} \\ D_{E,\Pi\Delta}^{(l)} & D_{E,\Pi\Pi}^{(l)} \end{bmatrix}.$$

Different from the standard FETI-DP preconditioner, the proposed preconditioner contains blocks of the scaling matrices and local Schur complement matrices involving the adaptive primal unknowns. With this new form of the FETI-DP preconditioner, we can show that the adaptive FETI-DP algorithm with the change of basis formulation has the same spectra as the corresponding BDDC algorithm except those equal to zero or one. We can thus obtain the same condition number bound as that of the BDDC algorithm. We note that when no adaptive primal unknowns are chosen, this preconditioner is identical to that considered in the standard FETI-DP algorithm.

4. Adaptively enriched primal unknowns. The adaptive constraints will be selected by considering a generalized eigenvalue problem on each nodal equivalence class. The idea is originated from the upper bound estimate of BDDC and FETI-DP preconditioners. We note that the lower bound can be obtained from the partition of unity property of the scaling matrices and it can be shown that the minimum eigenvalue is one [2, 18, 26]. In the estimate of condition numbers of BDDC and FETI-DP algorithms, the average and jump operators are defined as

$$E_D = \widetilde{R}\widetilde{R}^T\widetilde{D}, \quad P_D = B_D^T B, \tag{4.1}$$

where $B = (B_\Delta \quad 0)$ and $B_D^T = (B_{D,\Delta}^{(1)} \ \cdots \ B_{D,\Delta}^{(N)})^T$. We note that $B : \widetilde{W} \to M$ and $B_D^T : M \to W$; see the definition of $(B_{D,\Delta}^{(i)})^T$ in (3.2)-(3.4).

The adaptive constraints are treated just as unknowns at subdomain vertices after change of basis formulation in both BDDC and FETI-DP algorithms, i.e., the continuities of them are enforced explicitly. We note that in our previous work one can not get $E_D + P_D = I$ when the standard FETI-DP preconditioner is considered for the change of basis formulation, i.e., without the blocks from the adaptive primal unknowns in B_D^T.

In the following, we review the generalized eigenvalue problems and the estimate of condition numbers for the adaptive BDDC algorithm proposed in [13]. We first form generalized eigenvalue problems for faces F, which are nodal equivalence classes shared by two subdomains. For that we introduce $S_F^{(i)}$ to represent the block in S_i corresponding to the unknowns interior to F. $\widetilde{S}_F^{(i)}$ represents the Schur complement of S_i obtained by eliminating unknowns except those interior to F. $\widetilde{S}_F^{(i)}$ satisfies the following minimal energy property,

$$v_F^T \widetilde{S}_F^{(i)} v_F \leq v^T S_i v, \text{ for any } v|_F = v_F, \tag{4.2}$$

where $v|_F$ denotes the restriction of v to the unknowns interior to F. The notation $A : B$ is a parallel sum for symmetric and semi-positive definite matrices A and B defined as, see [1],

$$A : B = A(A+B)^+ B,$$

where $(A+B)^+$ denotes a pseudo inverse. The parallel sum satisfies the following properties

$$A : B = B : A, \quad A : B \leq A, \quad A : B \leq B, \tag{4.3}$$

and it was first used in [5] when forming generalized eigenvalues problems.

In $3D$, for a face F, the following generalized eigenvalue problem is considered

$$A_F v_F = \lambda \widetilde{A}_F v_F, \tag{4.4}$$

where

$$A_F = (D_F^{(j)})^T S_F^{(i)} D_F^{(j)} + (D_F^{(i)})^T S_F^{(j)} D_F^{(i)}, \ \widetilde{A}_F = \widetilde{S}_F^{(i)} : \widetilde{S}_F^{(j)}.$$

The eigenvalues are all positive. For a given tolerance λ_{TOL}, we select eigenvectors $v_{F,l},\ l \in N(F)$, corresponding to eigenvalues λ_l larger than λ_{TOL}. The following constraints will then be enforced on the unknowns in F,

$$(A_F v_{F,l})^T (w_F^{(i)} - w_F^{(j)}) = 0,\ l \in N(F).$$

After a change of basis, the above constraints can be transformed into explicit unknowns; see [13] for more implementation details of the change of basis. The explicit unknowns are denoted by $w_{F,\Pi}^{(i)}$ and they are then added to the initial set of primal unknowns. The remaining unknowns are called dual unknowns and are denoted by $w_{F,\Delta}^{(i)}$. In $2D$, for an edge we can form the generalized eigenvalue problem as in the case of a face in $3D$. We will use the same notation F to denote an edge in $2D$.

For the $2D$ case with only edge and vertex nodal equivalence classes, we can obtain that

$$\begin{aligned}
&\langle \widetilde{S}(I - E_D)\widetilde{w}, (I - E_D)\widetilde{w}\rangle \le C \sum_F (\langle A_F \widetilde{w}_{F,\Delta}^{(i)}, \widetilde{w}_{F,\Delta}^{(i)}\rangle + \langle A_F \widetilde{w}_{F,\Delta}^{(j)}, \widetilde{w}_{F,\Delta}^{(j)}\rangle) \\
&\le C\lambda_{TOL} \sum_F (\langle \widetilde{A}_F \widetilde{w}_{F,\Delta}^{(i)}, \widetilde{w}_{F,\Delta}^{(i)}\rangle + \langle \widetilde{A}_F \widetilde{w}_{F,\Delta}^{(j)}, \widetilde{w}_{F,\Delta}^{(j)}\rangle) \\
&\le C\lambda_{TOL} \sum_F (\langle S_i w_i, w_i\rangle + \langle S_j w_j, w_j\rangle) \\
&\le C\lambda_{TOL} \langle \widetilde{S}\widetilde{w}, \widetilde{w}\rangle,
\end{aligned}$$

where the estimate on the dual unknowns are bounded by λ_{TOL} in the second inequality, and (4.3) and the property of $\widetilde{S}_F^{(i)}$ in (4.2) are used in the third inequality. Above, we use $\widetilde{w}_{F,\Delta}^{(i)}$ to represent the vector with value $w_{F,\Delta}^{(i)}$ at the location of dual unknowns and value zero at the location of adaptive primal unknowns.

For an edge E in $3D$, shared by more than two subdomains, we introduce the following generalized eigenvalue problem,

$$A_E v_E = \lambda \widetilde{A}_E v_E, \tag{4.5}$$

where

$$A_E = \sum_{m \in I(E)} \sum_{l \in I(E)\setminus\{m\}} (D_E^{(l)})^T S_E^{(m)} D_E^{(l)}, \quad \widetilde{A}_E = \prod_{m \in I(E)} \widetilde{S}_E^{(m)}.$$

Here $I(E)$ denotes the set of subdomain indices sharing the edge E, and $\prod_{m \in I(E)} \widetilde{S}_E^{(m)}$ is the parallel sum of $\widetilde{S}_E^{(m)}$. The matrices $S_E^{(m)}$ and $\widetilde{S}_E^{(m)}$ are defined similarly as $S_F^{(i)}$ and $\widetilde{S}_F^{(i)}$. For a given λ_{TOL}, the eigenvectors with their eigenvalues larger than λ_{TOL} will be selected and denoted by $v_{E,l}$, $l \in N(E)$. The following constraints will be then enforced on the unknowns in E,

$$(A_E v_{E,l})^T (w_E^{(i)} - w_E^{(m)}) = 0,\ l \in N(E),\ m \in I(E) \setminus \{i\}.$$

Similarly to the face case, the above constraints can be transformed into explicit unknowns after a change of basis. We note that when we form matrices for generalized eigenvalue problems in (4.4) and (4.5), the local Schur complement matrices $S_F^{(l)}$, $\widetilde{S}_F^{(l)}$, $S_E^{(l)}$, and $\widetilde{S}_E^{(l)}$ are explicitly formed in our algorithm. This adds considerable computational cost to our algorithm compared to the standard BDDC or FEFI-DP methods. To reduce the cost one can form an economic version of the Schur complement matrices by solving local problems restricted on slabs of a face F or an edge E; see [13, Section 5.2] and [16]. In [14], generalized eigenvalue problems are defined on the closed faces shared by two subdomains and the selected adaptive primal constraints are restricted and enforced on the open faces and open edges. In their method, generalized eigenvalue problems are not needed for edges shared by three subdomains, but are still needed for edges shared by more than three subdomains to enhance the set of adaptive primal constraints.

By using the adaptively selected primal unknowns on each face F and edge E as above, in $3D$ we can obtain the following estimate

$$\langle \widetilde{S}(I-E_D)\widetilde{w},(I-E_D)\widetilde{w}\rangle \le C\lambda_{TOL}\langle \widetilde{S}\widetilde{w},\widetilde{w}\rangle, \tag{4.6}$$

where C is a constant depending on the maximum number of edges and faces per subdomain, and the maximum number of subdomains sharing an edge, but independent of the coefficient $\rho(x)$; see [13]. We note that (4.6) is the key estimate in the analysis of the BDDC algorithm.

5. Analysis of condition number bounds. Using the adaptively enriched primal unknowns described in Section 4 and the estimate in (4.6), we can obtain the following estimate of condition numbers for the BDDC algorithm; see [13].

THEOREM 5.1. *The BDDC algorithm with a change of basis formulation for the adaptively chosen set of primal unknowns with a given tolerance λ_{TOL} has the following bound for condition numbers,*

$$\kappa(M_{BDDC}^{-1}\widetilde{R}^T\widetilde{S}\widetilde{R}) \le C\lambda_{TOL},$$

where C is a constant depending only on $N_{F(i)}$, $N_{E(i)}$, $N_{I(E)}$, which are the number of faces per subdomain, the number of edges per subdomain, and the number of subdomains sharing an edge E, respectively.

For the FETI-DP algorithm with the same set of primal unknowns, those at subdomain vertices and those from the adaptive primal constraints after the change of basis, we can show that the FETI-DP algorithm shares the same spectra with the associate BDDC algorithm except zero and one. To obtain the result we first show the following properties of the P_D operator defined in (4.1).

LEMMA 5.2. *For any $\widetilde{w}$ in $\widetilde{W}$, $P_D\widetilde{w}$ in W has the same values at the location of primal unknowns and thus $P_D\widetilde{w}$ is in $\widetilde{W}$. In addition, $P_D\widetilde{w}$ satisfies that*

$$\widetilde{R}_i(\widetilde{w}-E_D\widetilde{w}) = R_i(P_D\widetilde{w}), \quad \forall \widetilde{w}\in\widetilde{W},$$

where $\widetilde{R}_i(\widetilde{w})$ and $R_i(w)$ are restrictions of $\widetilde{w}\in\widetilde{W}$ and $w\in W$ to W_i, respectively.

Proof. Recall the definition of $P_D = B_D^T B$ and $(B_{D,\Delta}^{(i)})^T$ in (3.2)-(3.4). In the following, we simply use the notation $(w)_i$ to denote $R_i(w)$ or $\widetilde{R}_i(w)$ for w in W or $\widetilde{W}$. On a face F of $\partial\Omega_i$, we assume that F is shared with its neighboring subdomain Ω_j. We then obtain

$$(P_D\widetilde{w})_i|_F = \begin{bmatrix} D_{F,\Delta\Delta}^{(j)}(w_{F,\Delta}^{(i)}-w_{F,\Delta}^{(j)}) \\ D_{F,\Pi\Delta}^{(j)}(w_{F,\Delta}^{(i)}-w_{F,\Delta}^{(j)}) \end{bmatrix}, \quad (P_D\widetilde{w})_j|_F = \begin{bmatrix} D_{F,\Delta\Delta}^{(i)}(w_{F,\Delta}^{(j)}-w_{F,\Delta}^{(i)}) \\ D_{F,\Pi\Delta}^{(i)}(w_{F,\Delta}^{(j)}-w_{F,\Delta}^{(i)}) \end{bmatrix},$$

where $(P_D\widetilde{w})_i$ denotes the restriction to W_i, $w_i|_F$ denotes the restriction of w_i to unknowns interior to F. From the partition of unity property $D_F^{(i)}+D_F^{(j)}=I$, we have

$$D_{F,\Pi\Delta}^{(i)} + D_{F,\Pi\Delta}^{(j)} = 0$$

and thus

$$D^{(j)}_{F,\Pi\Delta}(w^{(i)}_{F,\Delta}-w^{(j)}_{F,\Delta})=D^{(i)}_{F,\Pi\Delta}(w^{(j)}_{F,\Delta}-w^{(i)}_{F,\Delta}),$$

which means $P_D\widetilde{w}$ has the same value at the location of adaptive primal unknowns in the face F.

Similarly, for an edge E, we obtain

$$\begin{aligned}(P_D\widetilde{w})_i|_E &= \begin{bmatrix}\sum_{l\in I(E)} D^{(l)}_{E,\Delta\Delta}(w^{(i)}_{E,\Delta}-w^{(l)}_{E,\Delta})\\ \sum_{l\in I(E)} D^{(l)}_{E,\Pi\Delta}(w^{(i)}_{E,\Delta}-w^{(l)}_{E,\Delta})\end{bmatrix},\\ (P_D\widetilde{w})_k|_E &= \begin{bmatrix}\sum_{l\in I(E)} D^{(l)}_{E,\Delta\Delta}(w^{(k)}_{E,\Delta}-w^{(l)}_{E,\Delta})\\ \sum_{l\in I(E)} D^{(l)}_{E,\Pi\Delta}(w^{(k)}_{E,\Delta}-w^{(l)}_{E,\Delta})\end{bmatrix},\end{aligned}$$

where $I(E)$ denotes the set of subdomain indices sharing the edge E and the indices i and k are in $I(E)$. By subtracting the primal parts in the above two terms and using $\sum_{l\in I(E)} D^{(l)}_{E,\Pi\Delta}=0$, we obtain

$$\begin{aligned}&\sum_{l\in I(E)} D^{(l)}_{E,\Pi\Delta}(w^{(i)}_{E,\Delta}-w^{(l)}_{E,\Delta})-\sum_{l\in I(E)} D^{(l)}_{E,\Pi\Delta}(w^{(k)}_{E,\Delta}-w^{(l)}_{E,\Delta})\\ &=\left(\sum_{l\in I(E)} D^{(l)}_{E,\Pi\Delta}\right)(w^{(i)}_{E,\Delta}-w^{(k)}_{E,\Delta})\\ &=0,\end{aligned}$$

which shows that $P_D\widetilde{w}$ has the same value at the location of adaptive primal unknowns in the edge E. Above, we have also used the partition of unity property, $\sum_{l\in I(E)} D^{(l)}_E=I$ and thus $\sum_{l\in I(E)} D^{(l)}_{E,\Pi\Delta}=0$.

Since $(P_D\widetilde{w})_i|_V=0$ for the unknowns at subdomain vertices V, by combining the results on the adaptive primal unknowns in faces and edges, we conclude that $P_D\widetilde{w}$ has the same value at the location of primal unknowns in different subdomains.

We will now show that $(\widetilde{w}-E_D\widetilde{w})_i=(P_D\widetilde{w})_i$. For that, we first consider $(\widetilde{w}-E_D\widetilde{w})_i$ at dual unknowns on each F of subdomain Ω_i and obtain that

$$\begin{aligned}(\widetilde{w}-E_D\widetilde{w})_i|_{F,\Delta} &= w^{(i)}_{F,\Delta}-\sum_{k=i,j} D^{(k)}_{F,\Delta\Delta}w^{(k)}_{F,\Delta}-\sum_{k=i,j} D^{(k)}_{F,\Delta\Pi}w^{(k)}_{F,\Pi}\\ &= D^{(j)}_{F,\Delta\Delta}(w^{(i)}_{F,\Delta}-w^{(j)}_{F,\Delta})\\ &= (P_D\widetilde{w})_i|_{F,\Delta},\end{aligned}$$

where we have used $w^{(i)}_{F,\Pi}=w^{(j)}_{F,\Pi}$, $\sum_{k=i,j} D^{(k)}_{F,\Delta\Delta}=I$, and $\sum_{k=i,j} D^{(k)}_{F,\Delta\Pi}=0$.

Similarly for adaptive primal unknowns on each F of subdomain Ω_i, we obtain

$$\begin{aligned}(\widetilde{w}-E_D\widetilde{w})_i|_{F,\Pi} &= w^{(i)}_{F,\Pi}-\sum_{k=i,j} D^{(k)}_{F,\Pi\Delta}w^{(k)}_{F,\Delta}-\sum_{k=i,j} D^{(k)}_{F,\Pi\Pi}w^{(k)}_{F,\Pi}\\ &= D^{(j)}_{F,\Pi\Delta}(w^{(i)}_{F,\Delta}-w^{(j)}_{F,\Delta})\\ &= (P_D\widetilde{w})_i|_{F,\Pi},\end{aligned}$$

where we also have used $w^{(i)}_{F,\Pi}=w^{(j)}_{F,\Pi}$, $\sum_{k=i,j} D^{(k)}_{F,\Pi\Pi}=I$, and $D^{(i)}_{F,\Pi\Delta}=-D^{(j)}_{F,\Pi\Delta}$.

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For the dual unknowns on each edge E,

$$\begin{aligned}(\widetilde{w}-E_D\widetilde{w})_i|_{E,\Delta} &= w_{E,\Delta}^{(i)} - \sum_{l\in I(E)} D_{E,\Delta\Delta}^{(l)} w_{E,\Delta}^{(l)} - \sum_{l\in I(E)} D_{E,\Delta\Pi}^{(l)} w_{E,\Pi}^{(l)} \\ &= \sum_{l\in I(E)} D_{E,\Delta\Delta}^{(l)} (w_{E,\Delta}^{(i)} - w_{E,\Delta}^{(l)}) \\ &= (P_D\widetilde{w})_i|_{E,\Delta},\end{aligned}$$

where we have used that $w_{E,\Pi}^{(l)}$ have the same value, $\sum_{l\in I(E)} D_{E,\Delta\Pi}^{(l)} = 0$, and $\sum_{l\in I(E)} D_{E,\Delta\Delta}^{(l)} = I$.

For the adaptive primal unknowns on each edge E,

$$\begin{aligned}(\widetilde{w}-E_D\widetilde{w})_i|_{E,\Pi} &= w_{E,\Pi}^{(i)} - \sum_{l\in I(E)} D_{E,\Pi\Delta}^{(l)} w_{E,\Delta}^{(l)} - \sum_{l\in I(E)} D_{E,\Pi\Pi}^{(l)} w_{E,\Pi}^{(l)} \\ &= \sum_{l\in I(E)} D_{E,\Pi\Delta}^{(l)} (w_{E,\Delta}^{(i)} - w_{E,\Delta}^{(l)}) \\ &= (P_D\widetilde{w})_i|_{E,\Pi},\end{aligned}$$

where we have used that $w_{E,\Pi}^{(l)}$ have the same value, $\sum_{l\in I(E)} D_{E,\Pi\Delta}^{(l)} = 0$, and $\sum_{l\in I(E)} D_{E,\Pi\Pi}^{(l)} = I$.

□

By using Lemma 5.2, we can show that the two algorithms share the same set of spectra except one and zero; see [18, 2]. Combining with the result for the BDDC algorithm in Theorem 5.1, we have

THEOREM 5.3. *The FETI-DP algorithm with the change of basis formulation has the bound*

$$\kappa(M_{FETI}^{-1}F_{DP}) \le C\lambda_{TOL},$$

where C is a constant depending only on $N_{F(i)}$, $N_{E(i)}$, $N_{I(E)}$, which are the number of faces per subdomain, the number of edges per subdomain, and the number of subdomains sharing an edge E, respectively.

6. Numerical results. In our experiments, we consider a unit square or cubic domain Ω and divide it into a uniform mesh with grid size h. We partition the mesh into uniform square or cubic subdomains, or into irregular subdomains by the METIS mesh partitioner [10]. We use H to denote the size of the subdomains in the case of uniform square or cubic partition and N_d to denote the number of subdomains in a partition. In the conjugate gradient method for solving the system, the iteration is stopped when the relative residual norm is below 10^{-10}.

6.1. 2D examples. In this section, we consider two dimensional model problems. For a given subdomain partition and given tolerance value, we first include the unknowns at subdomain vertices to the set of primal unknowns and then enrich the initial set of primal unknowns by including adaptive primal unknowns selected from the generalized eigenvalue problem on each edge under the given tolerance value.

In the first example, we consider a model problem with $\rho(x)$ having both channels and inclusions of high contrast as shown in Figure 6.1. For this model, we present in Table 6.1 the results of the two algorithms when the domain is partitioned into uniform square subdomains with a given local problem size H/h. Consistent with our theory, the two algorithms have the same nonzero extreme eigenvalues. The two algorithms are quite robust with respect to the

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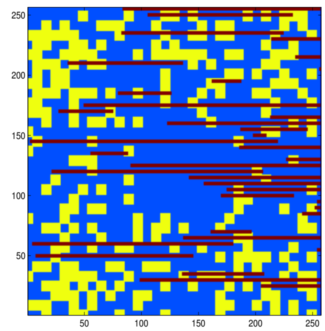

FIG. 6.1. $\rho(x)$ *with channels and inclusions: channel (*$\rho(x) = 10^3$*), inclusion (*$\rho(x) = 10^{-3}$*), and elsewhere (*$\rho(x) = 1$*).*

TABLE 6.1
Performance of adaptive BDDC (Bddc) and FETI-DP (Fdp) with $\lambda_{TOL} = 1 + \log(H/h)$ *for* $\rho(x)$ *as in Figure 6.1 with both channels and inclusions:* λ_{min} *(minimum eigenvalues),* λ_{max} *(maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns), and pF (number of adaptive primal unknowns per edge).*

$N_d\,(H/h)$	method	λ_{min}	λ_{max}	Iter	pnumF	pF
8^2	Bddc	1.00	3.11	13	178	1.58
$(H/h = 32)$	Fdp	1.00	3.11	13	178	1.58
16^2	Bddc	1.00	1.75	11	604	1.25
$(H/h = 16)$	Fdp	1.00	1.75	11	604	1.25
32^2	Bddc	1.00	1.47	8	2098	1.05
$(H/h = 8)$	Fdp	1.00	1.47	9	2098	1.05

coefficient $\rho(x)$, even with only less than two adaptive primal unknowns per edge. We can also observe that the minimum eigenvalues are identical to one and the maximum eigenvalues follow the tolerance value λ_{TOL}.

In the second example, we consider a more challenging case with random and high contrast value $\rho(x)$ in the range $(10^{-3}, 10^3)$. The value $\rho(x)$ is piecewise constant at each finite element. In Tables 6.2 and 6.3, the performance of the two algorithms is presented by increasing local problem size H/h for a given uniform subdomain partition and by increasing the number of subdomains in the uniform partition for a fixed local problem size H/h. As in the previous example, we observe that the minimum eigenvalues are all one and the two algorithms have the same maximum eigenvalues. In addition, for this highly heterogeneous case we can control the condition numbers by using only less than two adaptive primal unknowns per edge.

In Table 6.4, the results of the two algorithms are presented for the same $\rho(x)$ considered in the previous two tables, when irregular subdomain partitions as in Figure 6.2 are used. We again observe that the two algorithms have the same extreme eigenvalues and they perform well even for the quite irregular subdomain partitions, which shows the practicality of the proposed scheme in real applications.

6.2. 3D examples. We consider three dimensional model problems in this section. Unknowns at subdomain vertices are included in the set of primal unknowns first, and generalized eigenvalue problems defined in (4.4) and (4.5) are then used to introduce additional

TABLE 6.2

Performance of adaptive BDDC and FETI-DP with $\lambda_{TOL} = 1 + \log(H/h)$ for random $\rho(x)$ in $(10^{-3}, 10^3)$ by increasing H/h in a fixed subdomain partition $N_d = 3^2$: λ_{min} (minimum eigenvalues), λ_{max} (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns), and pF (number of adaptive primal unknowns per edge).

H/h	method	λ_{min}	λ_{max}	Iter	pnumF	pF
6	Bddc	1.00	1.30	7	17	1.41
	Fdp	1.00	1.30	7	17	1.41
12	Bddc	1.00	1.68	9	23	1.91
	Fdp	1.00	1.68	9	23	1.91
18	Bddc	1.00	1.81	9	21	1.75
	Fdp	1.00	1.81	9	21	1.75
24	Bddc	1.00	2.16	10	23	1.91
	Fdp	1.00	2.16	10	23	1.91
30	Bddc	1.00	2.63	10	20	1.66
	Fdp	1.00	2.63	10	20	1.66

TABLE 6.3

Performance of adaptive BDDC and FETI-DP with $\lambda_{TOL} = 1 + \log(H/h)$ for random $\rho(x)$ in $(10^{-3}, 10^3)$ by increasing N_d and with a fixed $H/h = 16$: λ_{min} (minimum eigenvalues), λ_{max} (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns), and pF (number of adaptive primal unknowns per edge).

N_d	method	λ_{min}	λ_{max}	Iter	pnumF	pF
4^2	Bddc	1.00	1.74	10	42	1.75
	Fdp	1.00	1.74	11	42	1.75
8^2	Bddc	1.00	3.11	16	189	1.68
	Fdp	1.00	3.11	16	189	1.68
16^2	Bddc	1.00	2.69	16	805	1.67
	Fdp	1.00	2.69	17	805	1.67

adaptive primal unknowns on faces and edges using the tolerance values $\lambda_{TOL}^F = 1 + \log(H/h)$, $\lambda_{TOL}^E = 1000$, respectively. We note that in [13] it was observed that eigenvalues from generalized eigenvalue problems on edges are much larger than those on faces and by choosing larger tolerance values for edges one can choose less and still effective adaptive constraints. Much larger eigenvalues are obtained for edges since the matrix $\widetilde{A}_E$ in the generalized eigenvalue problem on an edge, see (4.5), is involved by more subdomains than $\widetilde{A}_F$ on a face, see (4.4). This makes the right-hand side matrix less optimal for the edge case. In our experiment, we thus choose different tolerance values for face and edge cases.

We first consider the model with coefficient distribution $\rho(x)$ shown in Figure 6.3. The performance of adaptive BDDC and FETI-DP methods is presented in Table 6.5 by increasing H/h in a fixed subdomain partition $N_d = 3^3$. We can see that both methods have the same set of primal unknowns, maximum eigenvalues, nearly the same minimum eigenvalues (which are equal to one) and iteration counts except a small numerical perturbation. The number of adaptive primal unknowns per face and per edge are relatively robust with respect to H/h and in our test case about two per face and one per edge are chosen. The condition numbers and iteration counts are also robust given the high contrast in the coefficient $\rho(x)$.

We next consider highly varying and random coefficients $\rho(x) = 10^r$, where r is chosen randomly from $(-3, 3)$ for each finite element. The results are listed in Table 6.6 with an increasing $N_d = N^3$ and a fixed $H/h = 12$. We observe the same performance as in the previous case that both methods have the same set of primal unknowns, maximum eigenvalues,

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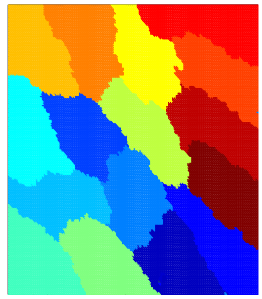

FIG. 6.2. *An example of irregular subdomain partition with $N_d = 16$ and $1/h = 256$.*

TABLE 6.4
Performance of adaptive BDDC and FETI-DP for random $\rho(x)$ in $(10^{-3}, 10^3)$ and on irregular subdomain partitions: N_d (number of subdomains), h (element size), λ_{min} (minimum eigenvalues), λ_{max} (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns), pF (number of adaptive primal unknowns per edge), n_F (maximum number of nodes per edge), and $\lambda_{TOL} = 1 + \log(n_F)$.

N_d	$1/h$	method	λ_{min}	λ_{max}	Iter	pnumF	pF	n_F
16	64	Bddc	1.00	2.69	13	44	1.37	39
		Fdp	1.00	2.69	13	44	1.37	39
64	129	Bddc	1.00	3.71	17	235	1.45	39
		Fdp	1.00	3.71	18	235	1.45	39
256	256	Bddc	1.00	3.14	17	1059	1.53	22
		Fdp	1.00	3.14	18	1059	1.53	22
16	256	Bddc	1.00	3.43	15	37	1.12	83
		Fdp	1.00	3.43	15	37	1.12	83
64	256	Bddc	1.00	3.17	17	213	1.33	45
		Fdp	1.00	3.17	17	213	1.33	45
256	256	Bddc	1.00	3.14	17	1059	1.53	22
		Fdp	1.00	3.14	18	1059	1.53	22

TABLE 6.5
Performance of adaptive BDDC (Bddc) and FETI-DP (Fdp) for the problem with channels and inclusions $\rho(x)$ as in Figure 6.3 by increasing H/h in a fixed subdomain partition $N_d = 3^3$ and with $\lambda_{TOL}^F = 1 + \log(H/h)$, $\lambda_{TOL}^E = 1000$: $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns on faces), and pnumE (total number of adaptive primal unknowns on edges). pF and pE are the number of adaptive primal unknowns per face and per edge, respectively.

H/h	method	$\lambda_{\min}$	$\lambda_{\max}$	Iter	pnumF	pnumE	pF	pE
4	Bddc	1.00	1.32	9	61	36	1.13	1.00
	Fdp	1.00	1.32	9	61	36	1.13	1.00
8	Bddc	1.00	1.61	11	84	36	1.56	1.00
	Fdp	1.01	1.61	11	84	36	1.56	1.00
12	Bddc	1.00	1.94	12	87	38	1.61	1.06
	Fdp	1.01	1.94	13	87	38	1.61	1.06
16	Bddc	1.00	2.24	14	88	40	1.63	1.11
	Fdp	1.01	2.24	14	88	40	1.63	1.11

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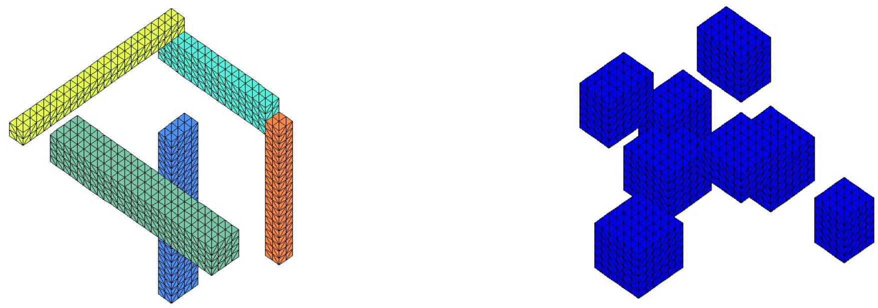

(a) Channels with $\rho(x) = 10^3$ (b) Inclusions with $\rho(x) = 10^{-3}$

FIG. 6.3. *Coefficient distribution with channels and inclusions: channels ($\rho(x) = 10^3$), small cubes ($\rho(x) = 10^{-3}$), and elsewhere ($\rho(x) = 1$).*

TABLE 6.6
Performance of adaptive BDDC and FETI-DP with $\lambda_{TOL}^F = 1 + \log(H/h)$ and $\lambda_{TOL}^E = 1000$ for highly varying and random $\rho(x)$ in $(10^{-3}, 10^3)$ by increasing N_d and with a fixed $H/h = 12$: $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns on faces), and pnumE (total number of adaptive primal unknowns on edges). pF and pE are the number of adaptive primal unknowns per face and per edge, respectively.

N_d	method	$\lambda_{\min}$	$\lambda_{\max}$	Iter	pnumF	pnumE	pF	pE
2^3	Bddc	1.01	3.15	15	46	18	3.83	3.00
	Fdp	1.00	3.15	16	46	18	3.83	3.00
3^3	Bddc	1.01	5.11	20	190	115	3.52	3.19
	Fdp	1.00	5.11	21	190	115	3.52	3.19
4^3	Bddc	1.01	5.83	23	533	320	3.70	2.96
	Fdp	1.00	5.83	24	533	320	3.70	2.96

nearly the same minimum eigenvalues (which are equal to one) and iteration counts. About four per face and three per edge are chosen as adaptive primal unknowns. The condition numbers and iteration counts increase very mildly as N_d increases. The performance of the two methods is presented in Table 6.7 with an increasing H/h in a fixed subdomain partition $N_d = 3^3$, which is similar to the performance in Table 6.5. We observe that the percentages of primal unknowns on face and edge both decrease despite the fact that the number of primal unknowns per face and edge increases. For example, for $H/h = 12$, the percentages of primal unknowns on face and edge are $3.52/121 = 2.91\%$ and $3.19/11 = 29\%$, respectively, and for $H/h = 16$, those are 1.95% and 27%.

Finally, for unstructured subdomain partitions as in Figure 6.4, we present the results for $\rho(x) = 1$ in Table 6.8, and in Table 6.9 the results for highly varying and random coefficients $\rho(x) = 10^r$, where r is chosen randomly from $(-3, 3)$ at each tetrahedron finite element. We also observe a similar performance as in the previous cases. We choose only one adaptive primal unknown per face and per edge for $\rho(x) = 1$ and a little more than one for random $\rho(x)$ under quite irregular subdomain partitions. It means that the two algorithms also perform well for both irregular subdomain partitions and highly heterogeneous coefficients in three dimensions.

7. Conclusion. In this paper, we consider BDDC and FETI-DP algorithms with adaptively enriched primal constraints and analyze their performance under a change of basis formulation. The adaptive primal constraints are introduced to enhance the robustness of

TABLE 6.7
Performance of adaptive BDDC and FETI-DP for the problem with random $\rho(x)$ in $(10^{-3}, 10^{3})$ by increasing H/h in a fixed subdomain partition $N_d = 3^3$ and with $\lambda_{TOL}^{F} = 1 + \log(H/h)$, $\lambda_{TOL}^{E} = 1000$: $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns on faces), and pnumE (total number of adaptive primal unknowns on edges). pF and pE are the number of adaptive primal unknowns per face and per edge, respectively.

H/h	method	$\lambda_{\min}$	$\lambda_{\max}$	Iter	pnumF	pnumE	pF	pE
4	Bddc	1.00	2.34	11	91	56	1.69	1.56
	Fdp	1.00	2.34	11	91	56	1.69	1.56
8	Bddc	1.00	2.53	15	147	83	2.72	2.31
	Fdp	1.00	2.53	16	147	83	2.72	2.31
12	Bddc	1.01	5.11	20	190	115	3.52	3.19
	Fdp	1.00	5.11	21	190	115	3.52	3.19
16	Bddc	1.01	5.42	22	237	146	4.39	4.06
	Fdp	1.00	5.42	23	237	146	4.39	4.06

the BDDC and FETI-DP preconditioners for solving elliptic problems with heterogeneous coefficients. The change of basis formulation can enhance the stability and efficiency of the algorithms. In particular, we show that the identity $E_D + P_D = I$ holds for both algorithms using the proposed change of basis. Hence, we show that the condition numbers of the adaptive BDDC and FETI-DP algorithms with the change of basis formulation are identical. Some numerical examples in both 2D and 3D with both structured and unstructured subdomain partitions are presented to demonstrate the robustness of the algorithms.

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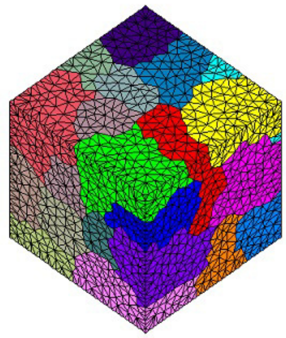
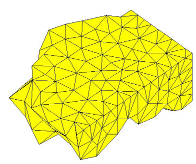


(a) METIS decomposition



(b) One of the METIS subdomains

FIG. 6.4. *Domain decomposition obtained by METIS for the unit cube,* $N_d = 27$ *and* $h = 1/20$.

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TABLE 6.8

Performance of adaptive BDDC and FETI-DP for $\rho(x) = 1$ on unstructured subdomain partitions (see Figure 6.4) and with $\lambda^F_{TOL} = 1+\log(\min_i\{H_i/h\})$, $\lambda^E_{TOL} = 1000$ (h (mesh size), $H_i = \max_{x_1,x_2\in\bar{\Omega}_i}\{|x_1-x_2|\}$): N_d(number of subdomains), $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns on faces), and pnumE (total number of adaptive primal unknowns on edges). pF and pE are the number of adaptive primal unknowns per face and per edge, respectively.

N_d	method	$1/h$	λ_{min}	λ_{max}	Iter	pnumF	pnumE	pF	pE
8	Bddc	10	1.00	1.15	7	12	7	1.00	1.00
		20	1.00	1.37	9	19	17	1.00	1.00
		40	1.00	1.57	10	21	22	1.00	1.00
		80	1.00	2.27	13	22	28	1.00	1.00
	Fdp	10	1.00	1.15	7	12	7	1.00	1.00
		20	1.00	1.37	9	19	17	1.00	1.00
		40	1.00	1.57	11	21	22	1.00	1.00
		80	1.00	2.27	14	22	28	1.00	1.00
27	Bddc	10	1.00	1.11	6	10	7	1.00	1.00
		20	1.00	1.26	8	75	53	1.00	1.00
		40	1.00	1.62	10	99	107	1.02	1.00
		80	1.00	1.92	12	108	134	1.00	1.00
	Fdp	10	1.01	1.11	7	10	7	1.00	1.00
		20	1.00	1.27	9	75	53	1.00	1.00
		40	1.01	1.62	11	99	107	1.02	1.00
		80	1.01	1.92	13	108	134	1.00	1.00
64	Bddc	10	1.00	1.04	4	2	1	1.00	1.00
		20	1.00	1.29	8	148	73	1.00	1.00
		40	1.00	1.46	9	271	279	1.00	1.00
		80	1.00	1.90	12	302	397	1.00	1.00
	Fdp	10	1.01	1.04	4	2	1	1.00	1.00
		20	1.01	1.29	9	148	73	1.00	1.00
		40	1.01	1.47	10	271	279	1.00	1.00
		80	1.01	1.90	13	302	397	1.00	1.00

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TABLE 6.9

Performance of adaptive BDDC and FETI-DP for random $\rho(x)$ in $(10^{-3}, 10^3)$ on unstructured subdomain partitions (see Figure 6.4) and with $\lambda_{TOL}^F = 1 + \log(\min_i\{H_i/h\}), \lambda_{TOL}^E = 1000$ (h (mesh size), $H_i = \max_{x_1,x_2\in\bar{\Omega}_i}\{|x_1 - x_2|\}$): N_d(number of subdomains), $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns on faces), and pnumE (total number of adaptive primal unknowns on edges). pF and pE are the number of adaptive primal unknowns per face and per edge, respectively.

N_d	method	$1/h$	λ_{min}	λ_{max}	Iter	pnumF	pnumE	pF	pE
8	Bddc	10	1.00	1.79	10	13	8	1.08	1.14
		20	1.00	2.10	12	24	17	1.26	1.00
		40	1.00	3.20	15	24	31	1.14	1.41
		80	1.00	4.33	19	28	34	1.27	1.21
	Fdp	10	1.00	1.79	10	13	8	1.08	1.14
		20	1.00	2.10	13	24	17	1.26	1.00
		40	1.00	3.20	15	24	31	1.14	1.41
		80	1.00	4.33	20	28	34	1.27	1.21
27	Bddc	10	1.00	1.51	8	12	8	1.20	1.14
		20	1.00	2.36	13	87	56	1.16	1.06
		40	1.01	4.91	18	123	126	1.27	1.18
		80	1.01	4.86	19	142	160	1.31	1.19
	Fdp	10	1.02	1.51	9	12	8	1.20	1.14
		20	1.00	2.36	13	87	56	1.16	1.06
		40	1.00	4.91	19	123	126	1.27	1.18
		80	1.00	4.86	20	142	160	1.31	1.19
64	Bddc	10	1.00	1.08	4	2	1	1.00	1.00
		20	1.00	1.99	12	158	78	1.07	1.07
		40	1.01	2.81	16	307	309	1.13	1.11
		80	1.01	5.38	19	359	471	1.19	1.19
	Fdp	10	1.00	1.08	4	2	1	1.00	1.00
		20	1.00	1.99	12	158	78	1.07	1.07
		40	1.00	2.81	17	307	309	1.13	1.11
		80	1.00	5.38	21	359	471	1.19	1.19