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# CIRCULANT PRECONDITIONERS FOR CONVOLUTION-LIKE INTEGRAL EQUATIONS WITH HIGHER-ORDER QUADRATURE RULES*

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**Abstract.** In this paper, we consider solving matrix systems arising from the discretization of convolution-like integral equations by preconditioned conjugate gradient (PCG) methods. Circulant integral operators as preconditioners have been proposed and studied. However, the discretization of these circulant preconditioned equations by employing higher-order quadratures leads to matrix systems that cannot be solved efficiently by using fast Fourier transforms (FFTs). The aim of this paper is to propose "inverted" circulant preconditioners for convolution-like integral equations. The discretization of these preconditioned integral equations by higher-order quadratures leads to matrix systems that involve only Toeplitz, circulant and diagonal matrix-vector multiplications, and hence can be computed efficiently by FFTs in each iteration. Numerical examples are given to illustrate the fast convergence of the method and the improvement of the accuracy of the computed solutions with using higher-order quadratures. We also apply our method to solve the convolution-like equation arising from the linear least squares estimation in signal processing.

**Key words.** Integral equations, displacement kernel, quadratures, circulant matrices, Toeplitz matrices, fast Fourier transforms, signal processing.

**AMS subject classifications.** 45E10, 45L10, 65R20, 65J10.

**1. Introduction.** In this paper, we study the numerical solution of integral equation of the form

$$y(t) + \int_0^\tau a(t,s)y(s)ds = g(t), \quad 0 \le t \le \tau \le \infty, \tag{1.1}$$

where $g(\cdot)$ and $a(\cdot,\cdot)$ are given functions in $L_2[0,\infty)$ and $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ respectively. We always assume that $a(t,s)$ is conjugate symmetric, i.e.,

$$a(t,s) = \overline{a(s,t)},$$

and $a(t,s)$ satisfies

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) a(t,s) = \sum_{j=1}^{\alpha} \gamma_j b_j(t)\overline{b_j(s)}, \tag{1.2}$$

for some numbers $\alpha$, $\gamma_j$ and some functions $b_j(\cdot)$. The following are examples of $a(t,s)$ with this property .

(i) If $a(\cdot,\cdot)$ is a displacement (or convolution or Toeplitz) kernel, we have

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) a(t-s) = 0,$$

and $a(t,s)$ satisfies (1.2) with $\alpha = 0$.

(ii) If $r(t,s)$ is the Fredholm resolvent associated with a displacement kernel $a(t-s)$, it has been shown in [10] that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) r(t,s) = u(t)\overline{u(s)} - v(t)\overline{v(s)},$$

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for some functions $u(\cdot)$ and $v(\cdot)$. Therefore, $r(t,s)$ satisfies (1.2) with $\alpha = 2$, $\gamma_1 = 1$, $\gamma_2 = -1$, $b_1(t) = u(t)$ and $b_2(t) = v(t)$.

(iii) The kernel function $a(t,s)$ is given by the covariance function of the output of a constant state-space model in the linear least squares estimation [8]:

$$\begin{aligned} \dot{w}(t) &= Fw(t) + G\eta(t), \quad w(0) = w_0, \\ z(t) &= Hw(t), \end{aligned}$$

where $E[\eta(t)\eta(s)^*] = \delta(t-s)$ and $E[w_0 w_0^*] = \Pi_0$. In this case, the covariance function $a(t,s)$ can be written as

$$a(t,s) = He^{F(t-s)}\Pi(s)H^*, \quad \forall t \geq s,$$

where $\Pi(s)$, the covariance of $w(t)$ is such that

$$\dot{\Pi}(t) = F\Pi(t) + \Pi(t)F^* + GG^*, \quad \Pi(0) = \Pi_0.$$

Then

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) a(t,s) = He^{Ft}\dot{\Pi}(0)e^{F^* s}H^*,$$

and the covariance function $a(t,s)$ satisfies (1.2) with

$$\alpha = \text{ rank } \dot{\Pi}(0).$$

We will consider this kind of kernel as an numerical example in §4.

Henceforth we will say that a kernel $a(t,s)$ which satisfies (1.2) with small $\alpha$ is a displacement-like kernel. For discussions of displacement-like kernels and their applications; see [7, 8].

In the following, we let $\mathcal{I}$ be the identity operator, and we let

$$(\mathcal{A}_\tau y)(t) = \int_0^\tau a(t,s)y(s)ds, \quad 0 \leq t \leq \tau. \tag{1.3}$$

We always assume that $\mathcal{A}_\tau$ is a positive definite operator. The operator equation $(\mathcal{I}+\mathcal{A}_\tau)y = g$ (cf. (1.1)) can be solved numerically by iterative methods. We remark that conjugate gradient methods, especially when combined with preconditioning, are known to be powerful methods for the solution of linear systems [6]. The convergence rate of the CG method can be speeded up by applying a preconditioner. Thus instead of solving (1.1), we solve the preconditioned operator equation

$$(\mathcal{I} + \mathcal{C}_\tau)^{-1}(\mathcal{I} + \mathcal{A}_\tau)y(t) = (\mathcal{I} + \mathcal{C}_\tau)^{-1}g(t), \quad 0 \leq t \leq \tau. \tag{1.4}$$

A good preconditioner $\mathcal{C}_\tau$ is an operator that is close to $\mathcal{A}_\tau$ in some norm and yet the operator equation

$$(\mathcal{I} + \mathcal{C}_\tau)x(t) = f(t), \quad 0 \leq t \leq \tau, \tag{1.5}$$

is easier to solve than (1.1) for an given function $f(t)$. A class of candidates is the class of operators of the form

$$\mathcal{C}_\tau x(t) = \int_0^\tau c_\tau(t-s)x(s)ds, \quad 0 \leq t \leq \tau,$$

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where the function $c_\tau$ is periodic in $[0,\tau]$. They are called *circulant integral operators* in [4]. The eigenfunctions and eigenvalues of the operator $\mathcal{C}_\tau$ are given by

$$u_m(t) = \frac{1}{\sqrt{\tau}} e^{2\pi imt/\tau}, \quad m \in \mathbb{Z},$$

and

$$\lambda_m(\mathcal{C}_\tau) = \sqrt{\tau}(c_\tau, u_m)_\tau = \sqrt{\tau}\int_0^\tau h_\tau(t)\bar{u}_m(t)dt, \quad m \in \mathbb{Z}. \tag{1.6}$$

Therefore, (1.5) can be solved efficiently by using the Fast Fourier Transforms (FFTs).

The paper is organized as follows. In §2, we will use circulant integral operators $\mathcal{C}_\tau$ to precondition convolution-like integral operators $\mathcal{A}_\tau$. We will show that if the functions $b_j$ in (1.2) are in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, the preconditioned operator equations will have clustered spectra for sufficiently large $\tau$. Hence the preconditioned conjugate gradient method will converge superlinearly for sufficiently large $\tau$. For the discretization, we will use higher-order quadrature rules such as the Simpson's rule to discretize the operator equations in order to obtain high order of accuracy of the solution. The corresponding discretization matrices of the circulant integral operators are $\mathbf{I} + \mathbf{CD}$ where $\mathbf{C}$ is a circulant matrix and $\mathbf{D}$ is a diagonal matrix. We note that they are, in general, not circulant, and therefore their inversion $(\mathbf{I} + \mathbf{CD})^{-1}$ cannot be computed by using FFTs. Hence the cost per iteration of the PCG method will exceed $O(n \log n)$ operations. In §3, we will propose and construct "inverted" circulant matrices for the discretization of (1.4) such that only $O(n \log n)$ operations are required in each iteration of the PCG method even when higher-order quadrature rule is employed. Finally, numerical examples are given in §4 to illustrate the effectiveness of the "inverted" circulant preconditioners and the improvement of accuracy by employing higher-order quadrature rules. Some convolution-like equations arising from the linear least squares estimation in signal processing are also tested.

**2. Convergence Analysis of Circulant Preconditioned Convolution-like Operators.** For the convolution-like integral equations considered in this paper, the operator equations will be solved by preconditioned conjugate gradient method. In [5], Gohberg and Koltracht showed that $a(t,s)$ satisfies (1.2) if and only if it can be represented in the form

$$a(t,s) = b_0(t-s) + \sum_{j=1}^{\alpha} \gamma_j \int_0^{\min\{t,s\}} b_j(t-u)\overline{b_j(s-u)}, \quad 0 \le t,s \le \tau,$$

for some functions $b_0(\cdot)$. Thus we can express the operator $\mathcal{A}_\tau$ as follows:

$$\mathcal{A}_\tau = \mathcal{B}_\tau^{(0)} + \sum_{j=1}^{\alpha} \mathcal{B}_\tau^{(j)}\mathcal{B}_\tau^{(j)*}, \tag{2.1}$$

where $\mathcal{B}_\tau^{(0)}$ and $\mathcal{B}_\tau^{(j)}$ are both convolution integral operators of the forms

$$(\mathcal{B}_\tau^{(0)})x(t) = \int_0^\tau b_0(t-s)x(s)ds, \quad 0 \le t \le \tau, \tag{2.2}$$

$$(\mathcal{B}_\tau^{(j)})x(t) = \int_0^\tau \tilde{b}_j(t-s)x(s)ds, \quad 0 \le t \le \tau, \tag{2.3}$$

with

$$\tilde{b}_j(t) = \begin{cases} b_j(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

The representation in (2.1) allows one to compute $\mathcal{A}_\tau x$ by a FFT. Hence assuming that only few iterations have to be carried out, solving (1.1) by iterative methods such as conjugate gradient (CG) methods will be less expensive than direct methods.

In [4], circulant integral operators are used to precondition convolution integral operators. A number of different circulant integral operators are proposed there. In this paper, we only focus on the "optimal" circulant integral operators considered in [4]. Given the integral operator $\mathcal{K}_\tau$ with kernel function $k(t,s)$, the corresponding "optimal" circulant integral operator $c(\mathcal{K}_\tau)$ is defined to be the circulant integral operator that minimizes the Hilbert-Schmidt norm

$$|||\mathcal{K}_\tau - \mathcal{C}_\tau|||^2 \equiv \int_0^\tau \int_0^\tau (k(t,s) - c_\tau(t-s))(\overline{k(t,s) - c_\tau(t-s)})dsdt \tag{2.4}$$

over all circulant integral operators $\mathcal{C}_\tau$. The "optimal" circulant integral operator $c(\mathcal{K}_\tau)$ is given by

$$(c(\mathcal{K}_\tau)y)(t) = \int_0^\tau c_\tau(t-s)x(s)ds, \quad 0 \leq t \leq \tau, \tag{2.5}$$

where $c_\tau(t)$ is

$$c_\tau(t) = \frac{1}{\tau}\int_{\tau-t}^{\tau} k(v+t-\tau, v)dv + \frac{1}{\tau}\int_0^{\tau-t} k(v+t, v)dv, \quad -\tau \leq t \leq \tau; \tag{2.6}$$

see [2, Lemma 1]. In view of (2.1), (2.2) and (2.3), we may construct the circulant approximation $\mathcal{P}_\tau$ to $\mathcal{A}_\tau$ by

$$\mathcal{P}_\tau = c(\mathcal{B}_\tau^{(0)}) + \sum_{j=0}^{\alpha} \gamma_j c(\mathcal{B}_\tau^{(j)})c(\mathcal{B}_\tau^{(j)*}), \tag{2.7}$$

where $c(\mathcal{B}_\tau^{(j)})$ are the corresponding "optimal" circulant integral operators of $\mathcal{B}_\tau^{(j)}$.

We recall the following two theorems which are useful in the analysis of the spectra of the preconditioned operators $(\mathcal{I}+\mathcal{P}_\tau)^{-1}(\mathcal{I}+\mathcal{A}_\tau)$. Their proofs can be found in [4] and [2] respectively.

THEOREM 2.1. *Let $\mathcal{K}$ be a convolution integral operator with kernel function $k(\cdot) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Let $c(\mathcal{K}_\tau)$ be the optimal circulant integral operator of $\mathcal{K}_\tau$. Then for any given $\epsilon > 0$, there exists a positive integer $N$ and a $\tau^* > 0$ such that for all $\tau > \tau^*$, there is a decomposition*

$$\mathcal{K}_\tau - c(\mathcal{K}_\tau) = \mathcal{R}_\tau + \mathcal{E}_\tau$$

*with operators $\mathcal{R}_\tau$ and $\mathcal{E}_\tau$ satisfying*

$$\text{rank } \mathcal{R}_\tau \leq N \quad \text{and} \quad ||\mathcal{E}_\tau||_2 \leq \epsilon.$$

THEOREM 2.2. *Let $\mathcal{K}$ be a self-adjoint positive integral operator with kernel function $k(\cdot,\cdot) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Let $c(\mathcal{K}_\tau)$ be the optimal circulant integral operator of $\mathcal{K}_\tau$. Then $c(\mathcal{K}_\tau)$ is a self-adjoint positive integral operator.*

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We prove the following results.

THEOREM 2.3. *Let $\mathcal{A}$ be a self-adjoint convolution-like integral operator with kernel function $a(\cdot,\cdot)$ satisfying (1.2). Let $\mathcal{P}_\tau$ be the preconditioner for $\mathcal{A}_\tau$. If $b_j(t) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ for $0 \le j \le \alpha$, then for any given $\epsilon > 0$, there exist a positive integer $N$ and a $\tau^* > 0$ such that for all $\tau > \tau^*$, there is a decomposition*

$$\mathcal{A}_\tau - \mathcal{P}_\tau = \mathcal{R}_\tau + \mathcal{E}_\tau$$

*with self-adjoint operators $\mathcal{R}_\tau$ and $\mathcal{E}_\tau$ satisfying*

$$\text{rank}\, \mathcal{R}_\tau \le N \quad \text{and} \quad ||\mathcal{E}_\tau||_2 \le \epsilon.$$

*Proof.* Using 2.1, we have

$$\begin{aligned} &\mathcal{B}_\tau^{(j)}\mathcal{B}_\tau^{(j)*} - c(\mathcal{B}_\tau^{(j)})c(\mathcal{B}_\tau^{(j)*}) \\ &= \mathcal{B}_\tau^{(j)}\mathcal{R}_\tau^{(j)*} + \mathcal{R}_\tau^{(j)}\mathcal{B}_\tau^{(j)*} + \mathcal{B}_\tau^{(j)}\mathcal{E}_\tau^{(j)*} + \mathcal{E}_\tau^{(j)}\mathcal{B}_\tau^{(j)*}. \end{aligned}$$

It is straightforward to see that the operators $(\mathcal{B}_\tau^{(j)}\mathcal{R}_\tau^{(j)*} + \mathcal{R}_\tau^{(j)}\mathcal{B}_\tau^{(j)*})$ and $(\mathcal{B}_\tau^{(j)}\mathcal{E}_\tau^{(j)*} + \mathcal{E}_\tau^{(j)}\mathcal{B}_\tau^{(j)*})$ are both self-adjoint with

$$\text{rank}\,(\mathcal{B}_\tau^{(j)}\mathcal{R}_\tau^{(j)*} + \mathcal{R}_\tau^{(j)}\mathcal{B}_\tau^{(j)*}) \le N$$

and

$$||\mathcal{B}_\tau^{(j)}\mathcal{E}_\tau^{(j)*} + \mathcal{E}_\tau^{(j)}\mathcal{B}_\tau^{(j)*}||_2 \le \epsilon$$

respectively. □

THEOREM 2.4. *Let $\mathcal{A}$ be a self-adjoint, positive convolution-like integral operator with kernel function $a(\cdot,\cdot)$ satisfying (1.2). Let $\mathcal{P}_\tau$ be the preconditioner for $\mathcal{A}_\tau$. If $b_j(t) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ for $0 \le j \le \alpha$, then there exists a $\tau^* > 0$ such that for all $\tau > \tau^*$, the integral operators $\mathcal{P}_\tau$ are positive.*

*Proof.* Since $\mathcal{A}_\tau$ is positive definite, $c(\mathcal{B}_\tau^{(0)}) + \sum_{j=1}^{\alpha} \gamma_j c(\mathcal{B}_\tau^{(j)}\mathcal{B}_\tau^{(j)*})$ is also positive definite. Therefore, the lemma can be proved if we can show that for each $1 \le j \le \alpha$,

$$\lim_{\tau\to\infty} ||c(\mathcal{B}_\tau^{(j)}\mathcal{B}_\tau^{(j)*}) - c(\mathcal{B}_\tau^{(j)})c(\mathcal{B}_\tau^{(j)*})||_2 = 0.$$

Since $b_j \in L_1(\mathbb{R})$, for each given $\epsilon > 0$, there is a $\tau_\epsilon^{(j)} > 0$ such that $\int_{\tau_\epsilon^{(j)}}^{\infty} |b_j(s)|ds < \epsilon$. Let

$$\tau^* \equiv \max_{1\le j\le\alpha} \left\{ \frac{\tau_\epsilon^{(j)}||b_j||_1}{\epsilon}, \frac{2\tau_\epsilon^{(j)}||b_j||_2^2}{\epsilon} \right\}.$$

For simplicity, we drop the subscripts and superscripts $j$ on functions and operators respectively when their meaning is apparent. For each $\tau > \tau^*$, we decompose the difference $\mathcal{B}_\tau - c(\mathcal{B}_\tau)$ as

$$\mathcal{B}_\tau - c(\mathcal{B}_\tau) = \mathcal{R}_\tau + \mathcal{E}_\tau,$$

where $\mathcal{R}_\tau$ and $\mathcal{E}_\tau$ are convolution operators with kernel functions

$$r_\tau(t) = \begin{cases} -\frac{t}{\tau} b(t+\tau), & -\tau \le t < -\tau + \tau_\epsilon, \\ 0, & -\tau + \tau_\epsilon \le t \le \tau \end{cases} \tag{2.8}$$

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and

$$e_\tau(t) = \begin{cases} 0, & -\tau \le t < -\tau + \tau_\epsilon, \\ -\frac{t}{\tau} b(t+\tau), & -\tau + \tau_\epsilon \le t \le 0, \\ \frac{t}{\tau} b(t), & 0 \le t \le \tau \end{cases} \tag{2.9}$$

respectively. Using the property of circulant integral operator [2, Lemma 3], we have

$$c(\mathcal{B}_\tau \mathcal{B}_\tau^*) - c(\mathcal{B}_\tau)c(\mathcal{B}_\tau^*) = c(\mathcal{E}_\tau\mathcal{E}_\tau^* + \mathcal{R}_\tau\mathcal{E}_\tau^* + \mathcal{R}_\tau^*\mathcal{E}_\tau) + c(\mathcal{R}_\tau\mathcal{R}_\tau^*). \tag{2.10}$$

Next we estimate the 2-norm of the two terms on the right hand side of (2.10). For the first term, we need estimates of $||\mathcal{E}_\tau||_2$ and $||\mathcal{R}_\tau||_2$. For $||\mathcal{E}_\tau||_2$, we obtain

$$\begin{aligned} ||\mathcal{E}_\tau||_2 &\le \int_{-\tau+\tau_\epsilon}^0 |\frac{s}{\tau} b(s+\tau)|ds + \int_0^\tau |\frac{s}{\tau} b(s)|ds \\ &\le \int_{-\tau+\tau_\epsilon}^0 |b(s+\tau)|ds + \int_0^{\tau_\epsilon} \frac{\tau_\epsilon}{\tau}|b(s)|ds + \int_{\tau_\epsilon}^\tau |b(s)|ds < 3\epsilon. \end{aligned}$$

Since $||c(\mathcal{B}_\tau)||_2 \le ||\mathcal{B}_\tau||_2$ (see [4]), we have

$$||\mathcal{R}_\tau||_2 = ||\mathcal{B}_\tau + c(\mathcal{B}_\tau) + \mathcal{E}_\tau||_2 \le 2||\mathcal{B}_\tau||_2 + ||\mathcal{E}_\tau||_2 \le 2||b||_1 + 3\epsilon.$$

Thus, we get

$$\begin{aligned} ||c(\mathcal{E}_\tau\mathcal{E}_\tau^* + \mathcal{R}_\tau\mathcal{E}_\tau^* + \mathcal{R}_\tau^*\mathcal{E}_\tau)||_2 &\le ||\mathcal{E}_\tau\mathcal{E}_\tau^* + \mathcal{R}_\tau\mathcal{E}_\tau^* + \mathcal{R}_\tau^*\mathcal{E}_\tau||_2 \\ &\le 27\epsilon^2 + 12||b||_1\epsilon. \end{aligned}$$

For the second term, we note by (2.6) that the kernel function of $c(\mathcal{R}_\tau\mathcal{R}_\tau)^*$ is given by

$$\begin{aligned} \hat{r}_\tau(t) = \frac{1}{\tau}\int_{\tau-t}^\tau \int_0^\tau r_\tau(v+t-\tau-w)\overline{r_\tau(v-w)}dwdv + \\ \frac{1}{\tau}\int_0^{\tau-t}\int_0^\tau r_\tau(v+t-w)\overline{r_\tau(v-w)}dwdv. \end{aligned}$$

Using (2.8) and the fact that $||r_\tau||_2^2 \le ||b||_2^2$, we can show that

$$\begin{cases} |\hat{r}_\tau(t)| \le \frac{\tau_\epsilon}{\tau}||b||_2^2, & 0 \le t \le \tau_\epsilon, \\ |\hat{r}_\tau(t)| = 0, & \tau_\epsilon < t \le \tau. \end{cases}$$

Therefore,

$$||c(\mathcal{R}_\tau\mathcal{R}_\tau^*)||_2 \le 2\int_0^\tau |\hat{r}_\tau(t)|dt \le \frac{2\tau_\epsilon}{\tau}||b||_2^2 \le \epsilon, \quad \text{for } \tau > \tau^*.$$

The result follows. □

Combining Theorems 2.3 and 2.4, we have our main result.

THEOREM 2.5. *Let $\mathcal{A}$ be a self-adjoint, positive convolution-like integral operator with kernel function $a(\cdot,\cdot)$ satisfying (1.2). Let $\mathcal{P}_\tau$ be the preconditioner for $\mathcal{A}_\tau$. If $b_j(t) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ for $0 \le j \le \alpha$, then for any given $\epsilon > 0$, there exist a positive integer $N$ and a $\tau^* > 0$ such that for all $\tau > \tau^*$, at most $N$ eigenvalues of the operator $(\mathcal{I}+\mathcal{P}_\tau)^{-1/2}(\mathcal{I}+\mathcal{A}_\tau)(\mathcal{I}+\mathcal{P}_\tau)^{-1/2}$ are at distance greater than $\epsilon$ from 1.*

It follows easily from Theorem 2.5 that the conjugate gradient method, when applied to solving preconditioned operator equation $(\mathcal{I}+\mathcal{C}_\tau)^{-1}(\mathcal{I}+\mathcal{A}_\tau)y = (\mathcal{I}+\mathcal{C}_\tau)^{-1}g$, converges superlinearly, see [3] and [4].

**3. Construction of "Inverted" Circulant Preconditioners.** Let us see when higher-order quadratures are applied to discretize the preconditioned equation (1.4). Suppose the interval $[0, \tau]$ be divided into $n$ subintervals of equal length $h$, i.e.

$$\tau = hn.$$

Given any convolution integral operator $\mathcal{K}_\tau$ defined on $[0, \tau]$ with the kernel function $k(t)$, its discretization matrix will be of the form $\mathbf{K}_n \mathbf{D}_n$ where $\mathbf{K}_n$ is a Toeplitz matrix whose first column given by

$$[\mathbf{K}_n]_{j0} = hk(jh), \quad j = 0, 1, \cdots, n-1 \tag{3.1}$$

and $\mathbf{D}_n$ is a diagonal matrix that depends only on the quadrature formula used. Some quadrature formulas can be found in [1, pp.16–17]. Thus after discretization on the operators $I + \mathcal{A}_\tau$ and $I + \mathcal{P}_\tau$, the corresponding matrices are

$$\mathbf{I}_n + \mathbf{A}_n \mathbf{D}_n = \mathbf{I}_n + \mathbf{B}_n^{(0)} \mathbf{D}_n + \sum_{j=1}^{\alpha} \gamma_j \mathbf{B}_n^{(j)} \mathbf{D}_n \mathbf{B}_n^{(j)*} \mathbf{D}_n$$

and

$$\mathbf{I}_n + \mathbf{P}_n \mathbf{D}_n = \mathbf{I}_n + \mathbf{C}_n^{(0)} \mathbf{D}_n + \sum_{j=1}^{\alpha} \gamma_j \mathbf{C}_n^{(j)} \mathbf{D}_n \mathbf{C}_n^{(j)*} \mathbf{D}_n$$

respectively. Since the evaluation of the matrix-vector products $(\mathbf{I}_n + \mathbf{P}_n \mathbf{D}_n)^{-1}\mathbf{z}$ is costly in each iteration of the preconditioned conjugate method, $\mathbf{I} + \mathbf{P}_n \mathbf{D}_n$ is not a good choice as a matrix preconditioner.

We consider using the inverse $I - \mathcal{Q}_\tau$ of $I + \mathcal{P}_\tau$ to precondition (1.1). Now the preconditioned equation becomes

$$(I - \mathcal{Q}_\tau)(I + \mathcal{A}_\tau)y(t) = (I - \mathcal{Q}_\tau)g(t). \tag{3.2}$$

For each iteration, we need to compute $(\mathcal{I} - \mathcal{Q}_\tau)x(t)$ for an given function $x(t)$. We note that

$$(I - \mathcal{Q}_\tau)x(t) = x(t) - \int_0^\tau q_\tau(t-s)x(s)ds$$

where

$$q_\tau(t) = \frac{\lambda_m(c(\mathcal{B}_\tau^{(0)})) + \sum_{j=1}^{\alpha} \gamma_j |\lambda_m(c(\mathcal{B}_\tau^{(j)}))|^2}{1 + \lambda_m(c(\mathcal{B}_\tau^{(0)})) + \sum_{j=1}^{\alpha} \gamma_j |\lambda_m(c(\mathcal{B}_\tau^{(j)}))|^2} e^{2\pi i m t/\tau}, \quad 0 \le t \le \tau. \tag{3.3}$$

We remark that $q_\tau$ is periodic in $[0, \tau]$ and $Q_\tau$ is also a circulant integral operator. We know from (3.1) that we only need the values of $q_\tau(jh)$ for $j = 0, 1, \cdots, n$ in order to construct matrix preconditioner. To approximate these values, we partition the interval $[-\tau/2, \tau/2]$ into $n$ equal subintervals of step size $h$ and approximate $\lambda_m(c(\mathcal{B}_\tau^{(j)}))$ by using quadrature formula into (1.6). We note that these approximate values of $\lambda_m(c(\mathcal{B}_\tau^{(j)}))$ can be computed by using FFTs in $O(n \log n)$ operations. Then the approximate values of $q_\tau(\cdot)$ in (3.3) can be calculated from $m = -n/2$ to $m = n/2$. Hence $(\mathcal{I} - \mathcal{Q})x(t)$ can be computed efficiently using FFTs.

The main feature of the preconditioner is that it is already inverted. Hence only circulant matrix-vector products (plus some inner products) are required in each step of PCG algorithm.

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In contrast, if circulant integral operators (see [4]) are used with higher-order quadrature rules, then one has to invert matrix of the form $\mathbf{I} + \mathbf{C}\mathbf{D}$ which, in general, has no fast inversion formula.

We see that the discretization of preconditioned equations are now given by

$$(\mathbf{I}_n - \mathbf{Q}_n\mathbf{D}_n)(\mathbf{I}_n + \mathbf{B}_n^{(0)}\mathbf{D}_n + \sum_{j=1}^{\alpha}\gamma_j\mathbf{B}_n^{(j)}\mathbf{D}_n\mathbf{B}_n^{(j)*}\mathbf{D}_n)\mathbf{y} = (\mathbf{I}_n - \mathbf{Q}_n\mathbf{D}_n)\mathbf{g}. \tag{3.4}$$

Using the transformation,

$$\tilde{\mathbf{y}} = \mathbf{D}_n^{1/2}\mathbf{y} \quad \text{and} \quad \tilde{\mathbf{g}} = \mathbf{D}_n^{1/2}\mathbf{g},$$

(3.4) can be symmetrized as

$$\left[\mathbf{I}_n - \mathbf{D}_n^{1/2}\mathbf{Q}_n\mathbf{D}_n^{1/2}\right]\left[\mathbf{I}_n + \mathbf{D}_n^{1/2}\mathbf{B}_n^{(0)}\mathbf{D}_n^{1/2} + \sum_{j=1}^{\alpha}\gamma_j\mathbf{D}_n^{1/2}\mathbf{B}_n^{(j)}\mathbf{D}_n\mathbf{B}_n^{(j)*}\mathbf{D}_n^{1/2}\right]\tilde{\mathbf{y}}$$
$$= \left[\mathbf{I} - \mathbf{D}_n^{1/2}\mathbf{Q}_n\mathbf{D}_n^{1/2}\right]\tilde{\mathbf{g}}$$

The discretized system can then solved by conjugate gradient method. In each iteration, we only need to compute Toeplitz, circulant and diagonal matrix-vector multiplications. All these matrix-vector products can be computed by using FFTs, and hence the cost per iteration is $O(n\log n)$ operations.

**4. Numerical Examples.** In this section, we test the effectiveness of our proposed preconditioners using the following function $a(t,s)$ that satisfies (1.2) with

$$b_0(t) = \frac{50}{(1+t^2)},\ b_1(t) = e^{-2t},\ b_2(t) = e^{-0.5t},$$

where $\gamma_1$ and $\gamma_2$ are both equal 1. We choose our right hand side function $g(t)$ such that the corresponding solution for the equation (1.1) is

$$x(t) = \begin{cases} (16-t)^2, & 0 \le t \le 16, \\ 0, & 16 < t \le \tau. \end{cases}$$

The stopping criterion of the PCG algorithm is : the residual of the PCG method less than $10^{-6}$. The initial guess is chosen to be the zero vector.

Tables 1-3 give the numbers of iterations required for convergence. The preconditioners are discretized according to the discussion in §2.2. The symbols $I$ and $Q$ indicate that (1.1) is solved without any preconditioner and with the preconditioner $\mathcal{I} - \mathcal{Q}_\tau$, respectively . The discretization rule used is listed in the caption. In Table 3, the symbol $P$ indicates that the preconditioner $\mathcal{I} + \mathcal{P}_\tau$ is discretized by using the rectangular rule. In this case, the inverse of the preconditioner can be computed easily using FFTs. From the tables, we see that when no preconditioner is used or when the preconditioner $\mathcal{I} + \mathcal{P}_\tau$ is discretized by rectangular rule, the method will converge very slowly especially for large $\tau$. However, our “inverted” circulant preconditioner works well.

To illustrate the usefulness of higher-order quadrature rules, we give in Table 4, the error of the numerical solutions. The error is computed as

$$\left\{h * \sum_{j=0}^{n}|y(jh) - x(jh)|^2\right\}^{1/2} \approx \left\{\int_0^\tau |y(t) - x(t)|^2 dt\right\}^{1/2},$$

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| | $\tau$=16 | | $\tau$=32 | | $\tau$=64 | | $\tau$=128 | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $Q$ | $I$ | $Q$ | $I$ | $Q$ | $I$ | $Q$ | $I$ |
| 512 | 8 | 44 | 9 | 65 | 8 | 87 | 8 | 98 |
| 1024 | 8 | 45 | 9 | 66 | 8 | 86 | 8 | 97 |
| 2048 | 10 | 50 | 8 | 68 | 8 | 85 | 8 | 97 |
| 4096 | 8 | 53 | 8 | 69 | 8 | 86 | 8 | 99 |
| 8192 | 8 | 55 | 8 | 71 | 8 | 90 | 8 | 101 |

TABLE 1
*The numbers of iterations for the rectangular rule.*

| | $\tau$=16 | | $\tau$=32 | | $\tau$=64 | | $\tau$=128 | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $Q$ | $I$ | $Q$ | $I$ | $Q$ | $I$ | $Q$ | $I$ |
| 512 | 8 | 44 | 8 | 67 | 9 | 86 | 8 | 97 |
| 1024 | 8 | 44 | 8 | 66 | 8 | 86 | 8 | 97 |
| 2048 | 8 | 46 | 8 | 69 | 8 | 85 | 8 | 97 |
| 4096 | 8 | 47 | 8 | 70 | 8 | 87 | 8 | 98 |
| 8192 | 8 | 49 | 8 | 71 | 8 | 86 | 8 | 98 |

TABLE 2
*The numbers of iterations for the trapezoidal rule.*

where $\{y(jh)\}_{j=0}^{n}$ is the computed solution and $x(t)$ is the true solution. We see from Table 4 that the error decreases like $O(h)$, $O(h^2)$ and $O(h^4)$ for the rectangular, trapezoidal and Simpson's rules respectively. The quadrature formulas we used in the test can be found in [1, pp.16–17].

**4.1. An Example in Signal Processing.** Next we solve the convolution-like equations arising from the least squares estimation in signal processing. Suppose that we have observations $u(t)$ of a signal process $z(t)$ with additive white noise $v(t)$

$$u(t) = z(t) + v(t),$$

where

$$E[z(t)z(s)^*] = a(t,s), \quad E[z(t)v(s)^*] = 0 \quad \text{and} \quad E[v(t)v(s)^*] = \delta(t-s).$$

The covariance function of the process $u(\cdot)$ is given by

$$E[u(t)u(s)^*] = \delta(t-s) + a(t,s).$$

In signal processing problems, it is often necessary to estimate the linear least squares filter for the given observed process $u(t)$. One way to do this is to choose $h_\tau(s)$ so as to minimize

$$E[(z(\tau) - \hat{z}(\tau))(z(\tau) - \hat{z}(\tau))^*],$$

where

$$\hat{z}(\tau) = \int_0^\tau h_\tau(s)v(s)ds.$$

In [7], Kailath has shown that $h_\tau(t)$ is the solution of the integral equation

$$h_\tau(t) + \int_0^\tau a(t,s)h_\tau(s)ds = a(\tau,t), \quad 0 \le t \le \tau. \tag{4.1}$$

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| | $\tau$=16 | | | $\tau$=32 | | | $\tau$=64 | | | $\tau$=128 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $Q$ | $P$ | $I$ | $Q$ | $P$ | $I$ | $Q$ | $P$ | $I$ | $Q$ | $P$ | $I$ |
| 512 | 8 | 38 | 44 | 8 | 47 | 65 | 9 | 53 | 85 | 9 | 56 | 98 |
| 1024 | 8 | 38 | 46 | 8 | 47 | 66 | 8 | 53 | 86 | 9 | 56 | 97 |
| 2048 | 9 | 38 | 50 | 8 | 48 | 69 | 8 | 52 | 86 | 8 | 57 | 98 |
| 4096 | 8 | 39 | 51 | 8 | 48 | 70 | 8 | 53 | 87 | 8 | 57 | 99 |
| 8192 | 8 | 39 | 52 | 8 | 47 | 72 | 8 | 54 | 86 | 8 | 57 | 98 |

TABLE 3
*The numbers of iterations for the Simpson's rule.*

| | $\tau = 64$ | | | $\tau = 128$ | | |
|---|---|---|---|---|---|---|
| $n$ | Rect. | Trap. | Simp. | Rect. | Trap. | Simp. |
| 512 | 31.8452 | 3.2306 | 0.046 | 60.2334 | 10.0034 | 1.4530 |
| 1024 | 17.4371 | 0.6993 | 0.0017 | 29.8999 | 2.5606 | 0.0411 |
| 2048 | 8.2301 | 0.1560 | 1.0123e-4 | 15.4156 | 0.6993 | 0.0018 |
| 4096 | 4.0057 | 0.0398 | 5.6782e-6 | 8.1231 | 0.1990 | 1.1265e-4 |
| 8192 | 1.9676 | 0.0098 | 7.8489e-7 | 4.1237 | 0.0288 | 5.7639e-6 |

TABLE 4
*Error in the computed solution.*

In the following, we compute the numerical solution of integral equation (4.1) and test the covariance function $a(t,s)$ of the process $z(t)$ generated by the output of a constant state-space model as discussed in §1. We remark that the displacement-like kernel $a(t,s)$ may be matrix-valued. The circulant integral operators can be constructed by matrix kernel functions with each element being periodic functions considered in (2.7). For discussion of preconditioned conjugate gradient methods for solving convolution integral equations with matrix-valued kernel functions, see [9].

In the numerical test, we consider a state-space model and use the matrices $F$, $G$, $H$ and $\Pi_0$ given by

$$F = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}, \; G = \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \end{bmatrix}, \; H = \begin{bmatrix} 4 \\ 0.01 \end{bmatrix}, \; \Pi_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

to test the performance of our proposed preconditioner. It is straightforward to show that

$$e^{Ft} = \begin{bmatrix} e^{-t} & 0 \\ -2te^{-t} & e^{-t} \end{bmatrix}, \quad \dot{\Pi}(0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix},$$

and that therefore the covariance function $a(t,s)$ satisfies (1.2) with $\alpha = 1$, $\gamma_1 = 1$ and $b_1(t) = 4.01e^{-t} - 0.02te^{-t}$. Table 5 shows the number of iterations required for convergence. We see that the number of iteration required for convergence without a preconditioner or with the preconditioner $\mathcal{I} + \mathcal{P}_\tau$ is greater than that with the preconditioner $\mathcal{I} - \mathcal{Q}_\tau$.

To conclude the paper, we remark that the accuracy of the computed solution depends only on the quadrature rule used to discretize $\mathcal{I} + \mathcal{A}_\tau$. However, the convergence rate of the preconditioned systems and the costs per iteration of the PCG method depend on how we discretize the preconditioning operators. From the numerical results, we see that it is advantageous to use a higher-order quadrature rule to discretize the operator equation because of the increased accuracy. But to speed up the convergence rate of the method and to minimize the costs per iteration, one may need to use our proposed preconditioner.

| $n$ | $\tau$=16 | | | $\tau$=32 | | | $\tau$=64 | | | $\tau$=128 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $Q$ | $P$ | $I$ | $Q$ | $P$ | $I$ | $Q$ | $P$ | $I$ | $Q$ | $P$ | $I$ |
| 512 | 7 | 19 | 27 | 7 | 19 | 32 | 9 | 21 | 38 | 11 | 25 | 42 |
| 1024 | 6 | 19 | 28 | 7 | 19 | 34 | 8 | 22 | 39 | 11 | 24 | 45 |
| 2048 | 6 | 18 | 28 | 6 | 18 | 35 | 8 | 23 | 41 | 10 | 27 | 46 |
| 4096 | 6 | 19 | 29 | 6 | 19 | 34 | 8 | 23 | 40 | 10 | 27 | 47 |
| 8192 | 6 | 20 | 28 | 6 | 19 | 35 | 7 | 24 | 42 | 9 | 26 | 46 |

TABLE 5
*The numbers of iterations for the example using the Simpson's rule.*

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