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ON A CONVERSE OF LAGUERRE'S THEOREM *

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Abstract. The problem of characterizing all real sequences $\{\gamma_k\}_{k=0}^{\infty}$ with the property that if $p(x) = \sum_{k=0}^{n} a_k x^k$ is any real polynomial, then $\sum_{k=0}^{n} \gamma_k a_k x^k$ has no more nonreal zeros than $p(x)$, remains open. Recently, the authors solved this problem under the additional assumption that the sequences $\{\gamma_k\}_{k=0}^{\infty}$, with the aforementioned property, can be interpolated by polynomials. The purpose of this paper is to extend this result to certain transcendental entire functions. In particular, the main result establishes a converse of a classical theorem of Laguerre for these transcendental entire functions.

Key words. Laguerre–Pólya class, entire functions, zero distribution, multiplier sequences.

AMS subject classifications. 26C10, 30D15, 30D10.

1. Introduction. In the theory of distribution of zeros of polynomials, the following open problem is of central interest. Let D be a subset of the complex plane. Characterize the linear transformations, T, taking complex polynomials into complex polynomials such that if p is a polynomial (either arbitrary or restricted to a certain class of polynomials), then the polynomial $T[p]$ has *at least* as many zeros in D as p has zeros in D. There is an analogous problem for transcendental entire functions. For special linear transformations, a complete characterization is given in [3] for the case when D is a convex region containing the origin. (For related questions and results see, for example, [2], [7, Ch. 2, Ch. 4], [8], [9, Ch. 7], [12, Ch. 3–5] and [14, Ch. 1–2].) In the classical setting ($D = \mathbb{R}$) the problem (solved by Pólya and Schur [17]) is to characterize all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in \mathbb{R}$, such that if a real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has only real zeros, then the polynomial

$$T[p(x)] = T\left[\sum_{k=0}^{n} a_k x^k\right] := \sum_{k=0}^{n} \gamma_k a_k x^k, \tag{1.1}$$

also has only real zeros (see (2.2) and (2.3) below). The purpose of this paper is to continue our investigation in [5] of the following more general problem. Characterize all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in \mathbb{R}$, such that if $p(x)$ is *any real polynomial*, then

$$Z_c(T[p(x)]) \le Z_c(p(x)), \tag{1.2}$$

where $Z_c(p(x))$ denotes the number of *nonreal* zeros of $p(x)$, counting multiplicities. In order to facilitate the description of our results, in Section 2 we first recall some definitions and terminology, and review some facts that will be needed in the sequel. The new results (Theorem 3.6, Theorem 3.9 and Theorem 3.10), which establish a converse of Laguerre's theorem for certain transcendental entire functions, are proved in Section 3. The techniques used in the proofs hinge on the properties of multiplier sequences (Definition 2.2), λ-sequences (Definition 2.8) and Schoenberg's celebrated theorem (see Theorem 3.3) on the representation of the reciprocal of functions in the Laguerre-Pólya class (Definition 2.1).

2. Background information and a review of previous results. Functions in the Laguerre-Pólya class are defined as follows.

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DEFINITION 2.1. *A real entire function* $\phi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ *is said to be in the* Laguerre-Pólya class, *denoted* $\phi(x) \in \mathcal{L}-\mathcal{P}$, *if* $\phi(x)$ *can be expressed in the form*

$$\phi(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1+\frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \tag{2.1}$$

where $c, \beta, x_k \in \mathbb{R}$, $c, x_k \neq 0$, $\alpha \geq 0$, n *is a nonnegative integer and* $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. *If* $-\infty \leq a < b \leq \infty$ *and if* $\phi(x) \in \mathcal{L}-\mathcal{P}$ *has all its zeros in* (a,b) *(or* $[a,b]$*), then we will use the notation* $\phi \in \mathcal{L}-\mathcal{P}(a,b)$ *(or* $\phi \in \mathcal{L}-\mathcal{P}[a,b]$*). If* $\gamma_k \geq 0$ *(or* $(-1)^k\gamma_k \geq 0$ *or* $-\gamma_k \geq 0$*) for all* $k = 0,1,2\ldots$*, then* $\phi \in \mathcal{L}-\mathcal{P}$ *is said to be of* type I in the Laguerre-Pólya class, *and we will write* $\phi \in \mathcal{L}-\mathcal{P}I$. *We will also write* $\phi \in \mathcal{L}-\mathcal{P}I^+$, *if* $\phi \in \mathcal{L}-\mathcal{P}I$ *and if* $\gamma_k \geq 0$ *for all* $k = 0,1,2\ldots$.

In order to clarify the terminology above, we remark that if $\phi \in \mathcal{L}-\mathcal{P}I$, then $\phi \in \mathcal{L}-\mathcal{P}(-\infty,0]$ or $\phi \in \mathcal{L}-\mathcal{P}[0,\infty)$, but that an entire function in $\mathcal{L}-\mathcal{P}(-\infty,0]$ need not belong to $\mathcal{L}-\mathcal{P}I$. Indeed, if $\phi(x) = \frac{1}{\Gamma(x)}$, where $\Gamma(x)$ denotes the gamma function, then $\phi(x) \in \mathcal{L}-\mathcal{P}(-\infty,0]$, but $\phi(x) \notin \mathcal{L}-\mathcal{P}I$. This can be seen, for example, by noting that functions in $\mathcal{L}-\mathcal{P}I$ are of exponential type, whereas $\frac{1}{\Gamma(x)}$ is an entire function of order one of maximal type (see [1, p. 8] or [11, Chapter 2] for the definition of the "type" of an entire function). Also, it is easy to see that $\mathcal{L}-\mathcal{P}I^+ = \mathcal{L}-\mathcal{P}I(-\infty,0]$.

DEFINITION 2.2. *A sequence* $T = \{\gamma_k\}_{k=0}^{\infty}$ *of real numbers is called a multiplier sequence if, whenever the real polynomial* $p(x) = \sum_{k=0}^{n} a_k x^k$ *has* only *real zeros, the polynomial* $T[p(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ *also has only real zeros.*

The following are well-known characterizations of multiplier sequences (cf. [17], [16, pp. 100–124] or [14, pp. 29–47]). A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$\phi(x) = T[e^x] := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}-\mathcal{P}I. \tag{2.2}$$

Moreover, the algebraic characterization of multiplier sequences asserts that a sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$g_n(x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j \in \mathcal{L}-\mathcal{P}I \text{ for all } n = 1,2,3\ldots. \tag{2.3}$$

DEFINITION 2.3. *We say that a sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *is a* complex zero decreasing sequence (CZDS), *if*

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k x^k\right), \tag{2.4}$$

for any real polynomial $\sum_{k=0}^{n} a_k x^k$. *(The acronym CZDS will also be used in the plural.)*

Now it follows from (2.4) that any complex zero decreasing sequence is also a multiplier sequence. If $T = \{\gamma_k\}_{k=0}^{\infty}$ is a sequence of *nonzero* real numbers, then inequality (2.4) is equivalent to the statement that for any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, $T[p]$ has at least as many real zeros as p has. There are, however, CZDS which have zero terms (cf. [5, Section 3]) and consequently it may happen that deg $T[p] <$ deg p. When counting the real zeros of p, the number generally increases with the application of T, but may in fact decrease due to a decrease in the degree of the polynomial. For this reason, we count nonreal zeros rather than real ones. The existence of a *nontrivial* CZDS is a consequence of the following theorem proved by Laguerre and extended by Pólya (see Pólya [15] or [16, pp. 314-321]).

THEOREM 2.4. (Laguerre [14, Satz 3.2])

1. *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n *and let* $h(x)$ *be a polynomial with only real zeros, none of which lie in the interval* $(0, n)$*. Then* $Z_c(\sum_{k=0}^{n} h(k) a_k x^k) \le Z_c(f(x))$*.*
2. *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n*, let* $\phi \in \mathcal{L}-\mathcal{P}$ *and suppose that none of the zeros of* ϕ *lie in the interval* $(0, n)$*. Then the inequality* $Z_c(\sum_{k=0}^{n} \phi(k) a_k x^k) \le Z_c(f(x))$ *holds.*
3. *Let* $\phi \in \mathcal{L}-\mathcal{P}(-\infty, 0]$*, then the sequence* $\{\phi(k)\}_{k=0}^{\infty}$ *is a complex zero decreasing sequence.*

REMARK 2.5. (a) *We remark that part* (2) *of Theorem 2.4 follows from (1) by a limiting argument. (b) For several analogues and extensions of* Theorem 2.4*, we refer the reader to S. Karlin [9, pp. 379–383], M. Marden [12, pp. 60–74], N. Obreschkoff [14, pp. 6–8, 42–47] and L. Weisner [20].*

One of the key results in [5] is the following converse of Laguerre's theorem in the case when ϕ (see part (3) of Theorem 2.4 above) is a polynomial.

THEOREM 2.6. ([5, Theorem 2.13]) *Let* $h(x)$ *be a real polynomial. The sequence* $T = \{h(k)\}_{k=0}^{\infty}$ *is a complex zero decreasing sequence (CZDS) if and only if either*

1. $h(0) \neq 0$ *and all the zeros of* h *are real and negative, or*
2. $h(0) = 0$ *and the polynomial* $h(x)$ *has the form*

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{i=1}^{p}(x-b_i), \tag{2.5}$$

where m *is a positive integer and* $b_i < m$ *for each* $i = 1, \ldots, p$*.*

Thus, Theorem 2.6 characterizes the class of all polynomials which interpolate CZDS. In contrast, the converse of Laguerre's theorem fails, in general, for transcendental entire functions, as the following example shows.

EXAMPLE 2.7. *Let* $p(x)$ *be a polynomial in* $\mathcal{L}-\mathcal{P}(-\infty, 0)$ *(so that the sequence* $\{p(k)\}_{k=0}^{\infty}$ *is a CZDS). Then*

$$\phi_1(x) := \frac{1}{\Gamma(-x)} + p(x) \qquad \text{and} \qquad \phi_2(x) := \sin(\pi x) + p(x)$$

are transcendental entire functions which interpolate the same sequence $\{p(k)\}_{k=0}^{\infty}$*, but these entire functions are* not *in* $\mathcal{L}-\mathcal{P}$*. Thus, in the transcendental case additional hypotheses are required in order that the converse of Laguerre's theorem should hold.*

Our main result (Theorem 3.10) shows that that the converse of Laguerre's theorem is valid for (transcendental) entire functions of the form $\phi(x)p(x)$, where $\phi(x) \in \mathcal{L}-\mathcal{P}I^+$ and $p(x)$ is a real polynomial which has no nonreal zeros in the left half-plane.

A sequence $\{\gamma_k\}_{k=0}^{\infty}$ which can be interpolated by a function $\phi \in \mathcal{L}-\mathcal{P}(-\infty, 0)$, that is, $\phi(k) = \gamma_k$ for $k = 0, 1, 2\cdots$, will be called a *Laguerre multiplier sequence* or a *Laguerre sequence*. It follows from Theorem 2.4 that a Laguerre sequence is a CZDS, and in particular, a multiplier sequence. The reciprocals of Laguerre sequences are examples of sequences which are termed in the literature (cf. Iliev [7, Ch. 4] or Kostova [10]) as λ-*sequences* and are defined as follows.

DEFINITION 2.8. *A sequence of nonzero real numbers,* $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$*, is called a* λ*-*

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sequence, if

$$\Lambda[p(x)] = \Lambda\left[\sum_{k=0}^{n} a_k x^k\right] := \sum_{k=0}^{n} \lambda_k a_k x^k > 0 \text{ for all } x \in \mathbb{R}, \tag{2.6}$$

whenever $p(x) = \sum_{k=0}^{n} a_k x^k > 0$ *for all* $x \in \mathbb{R}$.

REMARK 2.9. (1) *We remark that if* Λ *is a sequence of nonzero real numbers and if* $\Lambda[e^{-x}]$ *is an entire function, then a necessary condition for* Λ *to be a* λ*-sequence, is that* $\Lambda[e^{-x}] \geq 0$ *for all real* x. *(Indeed, if* $\Lambda[e^{-x}] < 0$ *for* $x = x_0$*, then continuity considerations show that there is a positive integer* n *such that* $\Lambda[(1 - \frac{x}{2n})^{2n} + \frac{1}{n}] < 0$ *for* $x = x_0$*.)*
(2) *In* [7, Ch. 4] (see also [10]) *it was pointed out by Iliev that* λ*-sequences are precisely the positive definite sequences. (There are several known characterizations of positive definite sequences* (see, for example, [13, Ch. 8] and [19, Ch. 3])*.)*

The importance of λ-sequences in our investigation stems from the fact that a *necessary condition* for a sequence $T = \{\gamma_k\}_{k=0}^{\infty},\ \gamma_k > 0$, to be a CZDS is that the sequence of reciprocals $\Lambda = \{\frac{1}{\gamma_k}\}_{k=0}^{\infty}$ be a λ-sequence. Thus, for example, the reciprocal of a Laguerre multiplier sequence is a λ-sequence. On the other hand, there are multiplier sequences whose reciprocals are not λ-sequences. For example, in [5, Example 1.8] we have demonstrated that the sequence $T := \{1 + k + k^2\}_{k=0}^{\infty}$ is a multiplier sequence, but that the sequence of reciprocals, $\{\frac{1}{1+k+k^2}\}_{k=0}^{\infty}$ is not a λ-sequence.

3. A converse of Laguerre's theorem for transcendental entire functions. It follows from Example 2.7 that Theorem 2.6 does not hold, in general, for transcendental entire functions. Here we will restrict our attention to entire functions of the form $\phi(x)p(x)$, where $\phi(x) \in \mathcal{L}-\mathcal{P}I^+$ and $p(x)$ is a real polynomial. Our goal is to show that if $p(x)$ has no nonreal zeros in the left half-plane $\Re z < 0$, then the sequence $T = \{p(k)\phi(k)\}_{k=0}^{\infty}$ is a CZDS if and only if $p(x)$ has only real negative zeros. To this end, we will state and prove several preliminary results.

PROPOSITION 3.1. *Let* $\beta \in \mathbb{R}$ *and let* $p(x)$ *be a real polynomial. If* $p(0) \neq 0$*, then the sequence* $\{p(k)e^{\beta k}\}_{k=0}^{\infty}$ *is a CZDS if and only if* $p(x)$ *has only real negative zeros.*

Proof. If $p(x)$ has only real negative zeros, then $T = \{p(k)e^{\beta k}\}_{k=0}^{\infty}$ is a CZDS by Laguerre's theorem (see part (3) of Theorem 2.4). To prove the converse, we first note that if $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ $(a_k, b_k > 0)$ are two CZDS, then their Hadamard product, that is $\{a_k b_k\}_{k=0}^{\infty}$, is also a CZDS. Since $\{e^{-\beta k}\}_{k=0}^{\infty}$ is a CZDS and $\{p(k)e^{\beta k}\}_{k=0}^{\infty}$ is a CZDS by assumption, it follows that the sequence $\{p(k)\}_{k=0}^{\infty}$ is a CZDS. Therefore, by part (1) of Theorem 2.6, $p(x)$ has only real negative zeros. □

In order to work with arbitrary functions in $\mathcal{L}-\mathcal{P}I$, we require some additional background information. In particular, in the proofs we will appeal to Schoenberg's theorem (Theorem 3.3) on the representation of the reciprocal of a function $\phi \in \mathcal{L}-\mathcal{P}I$ in terms of Pólya frequency functions. These functions are defined as follows.

DEFINITION 3.2. *A function* $K : \mathbb{R} \to \mathbb{R}$ *is a frequency function if it is a nonnegative measurable function such that*

$$0 < \int_{-\infty}^{\infty} K(s)\, ds < \infty.$$

A frequency function K *is said to be a* Pólya frequency function *if it satisfies the following condition. For every two sets of increasing real numbers* $s_1 < s_2 < \ldots < s_n$ *and* $t_1 < t_2 <$

$\ldots < t_n$ $(n = 1, 2, 3, \ldots)$, *the determinantal inequality*

$$\begin{vmatrix} K(s_1 - t_1) & K(s_1 - t_2) & \ldots & K(s_1 - t_n) \\ K(s_2 - t_1) & K(s_2 - t_2) & \ldots & K(s_2 - t_n) \\ & & \ldots & \\ K(s_n - t_1) & K(s_n - t_2) & \ldots & K(s_n - t_n) \end{vmatrix} \geq 0$$

holds.

THEOREM 3.3. (Schoenberg [18, p. 354]) *Suppose that $\phi(x) \in \mathcal{L} - \mathcal{P}I$, $\phi(x) > 0$ if $x > 0$, where $\phi(x)$ is not of the form $ce^{\beta x}$. Then the reciprocal of ϕ can be represented in the form*

$$\frac{1}{\phi(z)} = \int_0^\infty e^{-sz} K(s)\, ds, \qquad \Re z > 0,$$

where $K(s)$ is a Pólya frequency function such that $K(s) = 0$ if $s < 0$ and the integral converges up to the first pole of $\dfrac{1}{\phi(z)}$. Conversely, suppose that $K(s)$ is a Pólya frequency function such that $K(s) = 0$ for $s < 0$ and the integral converges for $\Re z > 0$. Then this integral represents, in the half-plane $\Re z > 0$, the reciprocal of a function $\phi(x) \in \mathcal{L} - \mathcal{P}I$, where $\phi(x)$ is not of the form $ce^{\beta x}$.

For the reader's convenience, we also include here some information concerning the asymptotic behavior of Pólya frequency functions.

THEOREM 3.4. ([6, p. 31 and p. 108]) *Suppose that $\phi(x) \in \mathcal{L} - \mathcal{P}I$ with $\phi(x) > 0$ if $x > 0$, where $\phi(x)$ is not of the form $ce^{\beta x}$. Let $K(s)$ denote the Pólya frequency function corresponding to ϕ. Then*

$$K(s) = e^{-|x_1|s} q(s) + O(e^{-|r|s}) \qquad (s \to \infty),$$

where x_1 is the largest (negative) zero of $\phi(x)$, $q(s)$ is a real polynomial and $|r| > |x_1|$.

REMARK 3.5. *In* [18, p. 358], *Schoenberg has also established results pertaining to the continuity properties of the Pólya frequency kernels. In particular, Schoenberg has shown that if $\phi(x) \in \mathcal{L} - \mathcal{P}I$ ($\phi(0) \neq 0$) has $n \geq 2$ nonzero roots, then the corresponding Pólya frequency function $K(s)$ is in $C^{n-2}(\mathbb{R})$. If $n = 1$, then $K(s)$ is discontinuous and $K(s)$ is essentially of the form $K(s) = e^{-s}$ if $s \geq 0$ and $K(s) = 0$ if $s < 0$.*

THEOREM 3.6. *Let $\phi(x) \in \mathcal{L} - \mathcal{P}I^+$ where $\phi(x)$ is not of the form $ce^{\beta x}$, $c, \beta \in \mathbb{R}$. Let $p(x)$ be a polynomial having only real zeros, and suppose that $\phi(0)p(0) = 1$. Then the sequence $T = \{\phi(k)p(k)\}_{k=0}^\infty$ is a CZDS if and only if p has only real negative zeros.*

Proof. If $p(x)$ has only real negative zeros, then $\phi(x)p(x) \in \mathcal{L} - \mathcal{P}I$ and T is a CZDS by Laguerre's theorem. Conversely, suppose that T is a CZDS. Assume that $p(x)$ has a positive zero; we shall show that this leads to a contradiction. We may assume, without loss of generality, that $p(x)$ has only positive zeros as the negative ones can be included in $\phi(x)$. Since T is a CZDS, it is, in particular, a multiplier sequence. By [4, Theorem 3.4], the sequence must either be of one sign or alternate in signs. Since we have $\phi(0)p(0) = 1$ and $\phi(x)p(x)$ has only finitely many positive zeros, it follows that $\phi(k)p(k) > 0$ for $k = 0, 1, 2, \ldots$. In particular, no nonnegative integer can be a zero of $\phi(x)p(x)$. Let $r_1, r_2, \ldots, r_n$ denote the zeros of $p(x)$. The argument below together with continuity considerations will show that we may assume that all of the zeros of $p(x)$ are simple and so we can write $0 < r_1 < r_2 < \ldots < r_n$. We may also assume that the largest zero, r_n, lies in an interval of the form $(2m, 2m+1)$ for some integer $m \geq 0$; this follows from the fact that if T is a CZDS, then so is the shifted sequence

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$\{\phi(k)p(k)\}_{k=1}^{\infty} = \{\phi(k+1)p(k+1)\}_{k=0}^{\infty}$. Since the zeros of $p(x)$ are simple, the partial fraction decomposition of $\dfrac{1}{p(x)}$ is of the form

$$\frac{1}{p(x)} = \frac{1}{\prod_{j=1}^{n}(x-r_j)} = \sum_{j=1}^{n} \frac{A_j}{x-r_j}, \tag{3.1}$$

where $A_j = \prod_{i \neq j} \frac{1}{r_j - r_i}$. Note, in particular, that $A_n > 0$ since $r_n > r_i$ for $i < n$. Now since T is a CZDS, the sequence $\{\frac{1}{\phi(k)p(k)}\}_{k=0}^{\infty}$ is a λ-sequence and so the application of this sequence to the positive function e^{-x} must give (see Remark 2.9(1))

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!\phi(k)p(k)} \geq 0$$

for all $x \in \mathbb{R}$. Since $\phi(x)$ is not of the form $ce^{\beta x}$, we may invoke Schoenberg's theorem (Theorem 3.3) and consequently we can write

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!p(k)} \int_0^{\infty} K(s)e^{-ks}\, ds, \tag{3.2}$$

where $K(s)$ is a Pólya frequency function such that $K(s) = 0$ for $s < 0$. Now, by the uniform convergence of the power series $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!p(k)}$ (on compact subsets of $\mathbb{C}$), we may interchange the summation with the integration, to obtain

$$\begin{aligned} F(x) &= \int_0^{\infty} K(s) \sum_{k=0}^{\infty} \frac{(-1)^k (xe^{-s})^k}{k!p(k)}\, ds \\ &= \textstyle\sum_{j=1}^{n} A_j \displaystyle\int_0^{\infty} K(s) \left(\sum_{k=0}^{\infty} \frac{(-1)^k (xe^{-s})^k}{k!(k-r_j)} \right) ds \\ &= \textstyle\sum_{j=1}^{n} A_j \displaystyle\int_0^{\infty} K(s) \frac{\gamma(-r_j, xe^{-s})}{x^{-r_j} e^{r_j s}}\, ds, \end{aligned}$$

where we have used (3.1) and where

$$\gamma(\alpha, x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+\alpha}}{k!(k+\alpha)},$$

for $x > 0$ and $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ is a representation (via analytic continuation) of the incomplete gamma function. For $\Re\alpha > 0$ and $x > 0$, the incomplete gamma function is also defined by $\gamma(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x)$, where $\Gamma(\alpha, x) = \int_x^{\infty} e^{-t} t^{\alpha-1}\, dt$. Since all $r_j > 0$ *and* $r_j \notin \mathbb{Z}$, we can express $F(x)$, for $x > 0$, in the form

$$\begin{aligned} F(x) &= \textstyle\sum_{j=1}^{n} A_j \displaystyle\int_0^{\infty} K(s) \left[\frac{\Gamma(-r_j) - \Gamma(-r_j, xe^{-s})}{x^{-r_j} e^{r_j s}} \right] ds \\ &= F_1(x) - F_2(x), \end{aligned} \tag{3.3}$$

where

$$F_1(x) = \sum_{j=1}^{n} A_j x^{r_j} \Gamma(-r_j) \int_0^{\infty} K(s) e^{-r_j s}\, ds \tag{3.4}$$

and

$$F_2(x) = \sum_{j=1}^{n} A_j x^{r_j} \int_0^\infty K(s) e^{-r_j s} \left(\int_{x e^{-s}}^\infty e^{-t} t^{-r_j - 1}\, dt \right) ds. \tag{3.5}$$

We will next show that $F_2(x) \to 0$ as $x \to \infty$. Now, the change of variables $t = ue^{-s}$ in (3.5) gives

$$\begin{aligned} x^{r_j} e^{-r_j s} \int_{x e^{-s}}^\infty e^{-t} t^{-r_j-1}\, dt &= x^{r_j} \int_x^\infty e^{-u e^{-s}} u^{-r_j - 1}\, du \\ &\le e^{-x e^{-s}} x^{r_j} \int_x^\infty u^{-r_j-1}\, du \\ &= \tfrac{e^{-x e^{-s}}}{r_j}. \end{aligned} \tag{3.6}$$

Inequality (3.6) yields the estimate

$$|F_2(x)| \le \sum_{j=1}^{n} \frac{|A_j|}{r_j} \int_0^\infty e^{-x e^{-s}} K(s)\, ds \le \sum_{j=1}^{n} \frac{|A_j|}{r_j} \int_0^\infty K(s)\, ds, \tag{3.7}$$

and so, using (3.7) and the dominated convergence theorem, we conclude that $F_2(x) \to 0$ as $x \to \infty$. Therefore, by (3.3), (3.4) and (3.5) we can express $F(x)$ in the form, as $x \to \infty$,

$$\begin{aligned} F(x) &= F_1(x) + o(1) \\ &= x^{r_n} \left[\textstyle\sum_{j=1}^{n-1} \frac{A_j \Gamma(-r_j)}{x^{r_n - r_j}} \displaystyle\int_0^\infty K(s) e^{-r_j s}\, ds + A_n \Gamma(-r_n) \int_0^\infty K(s) e^{-r_n s}\, ds \right] + o(1). \end{aligned}$$

Since $-r_n \in (-2m-1, -2m)$ for some integer $m \ge 0$ and since the real entire function $1/\Gamma(x)$ is negative on the interval $(-2m-1, -2m)$, it follows that $F(x) \to -\infty$ as $x \to \infty$. Consequently, $\{\frac{1}{\phi(k)p(k)}\}_{k=0}^\infty$ is not a λ-sequence and so we have obtained the desired contradiction. □

In the case of nonreal zeros, we need to first analyze the behavior of certain integrals involving Pólya frequency functions. To this end, we next prove the following preparatory result.

LEMMA 3.7. *Let* $\alpha = \frac{a}{2} + i\tau$ *and* $\tau = \frac{\sqrt{4b - a^2}}{2}$, *where* $a < 0,\ b \in \mathbb{R}$ *and* $4b - a^2 > 0$. *Suppose that* $\phi(x) \in \mathcal{L} - \mathcal{P}I$, *where* $\phi(x) > 0$ *if* $x > 0$ *and* $\phi(x)$ *is not of the form* $ce^{\beta x}$. *Let* $K(s)$ *denote the Pólya frequency function corresponding to* ϕ. *For* $s \ge 0$, $x \ge 1$, *set*

$$u(s,x,\alpha) := e^{as/2} \{ \Im[\Gamma(\alpha) - \Gamma(\alpha, xe^{-s})] \cos(\tau s) + \Re[\Gamma(\alpha) - \Gamma(\alpha, xe^{-s})] \sin(\tau s) \} \tag{3.8}$$

and set

$$E(x) := \int_0^\infty K(s) e^{as/2} [\cos(\tau s) \Im(\Gamma(\alpha, xe^{-s})) + \sin(\tau s) \Re(\Gamma(\alpha, xe^{-s}))]\, ds. \tag{3.9}$$

Then

$$\int_0^\infty K(s) |u(s,x,\alpha)|\, ds < \infty \tag{3.10}$$

and

$$\lim_{x \to \infty} E(x) = 0. \tag{3.11}$$

Proof. We first note that since $a < 0$, we have the elementary inequality

$$\begin{aligned}\left|\int_0^\infty K(s)(s,x,\alpha)\,ds\right| &\leq 2\int_0^\infty K(s)e^{as/2}|\Gamma(\alpha)|\,ds + 2\int_0^\infty K(s)e^{as/2}|\Gamma(\alpha,xe^{-s})|\,ds\\ &= 2\tfrac{|\Gamma(\alpha)|}{\phi(|a|/2)} + 2\int_0^\infty K(s)e^{as/2}|\Gamma(\alpha,xe^{-s})|\,ds.\end{aligned}$$

Since

$$|E(x)| \leq 2\int_0^\infty K(s)e^{as/2}|\Gamma(\alpha,xe^{-s})|\,ds,$$

it will suffice (for both (3.10) and (3.11)) to show that $\int_0^\infty K(s)e^{as/2}|\Gamma(\alpha,xe^{-s})|\,ds$ tends to zero as $x\to\infty$. To this end, we consider the estimates

$$\begin{aligned}\int_0^\infty K(s)e^{as/2}|\Gamma(\alpha,xe^{-s})|\,ds &= \int_0^\infty K(s)e^{as/2}\left|\int_{xe^{-s}}^\infty e^{-t}t^{\alpha-1}\,dt\right|\,ds\\ &\leq \int_0^\infty K(s)e^{as/2}\int_{xe^{-s}}^\infty e^{-t}t^{a/2-1}\,dt\,ds\\ &= \int_0^\infty K(s)e^{as/2}\int_x^\infty e^{-ue^{-s}}u^{a/2-1}e^{-as/2+s}e^{-s}\,du\,ds\\ &\leq \int_0^\infty K(s)\int_x^\infty u^{a/2-1}\,du\,ds\\ &= \tfrac{2}{|a|x^{|a|/2}}\int_0^\infty K(s)\,ds,\end{aligned}\tag{3.12}$$

where we have used the inequality $e^{-ue^{-s}} \leq 1$ for $u > 0$ and the assumption that $a = -|a| < 0$. Since $K(s)$ is a Pólya frequency function, $0 < \int_0^\infty K(s)\,ds < \infty$, and hence by (3.12), both (3.10) and (3.11) follow. □

PROPOSITION 3.8. *Let $\alpha = \frac{a}{2} + i\tau$ and $\tau = \frac{\sqrt{4b-a^2}}{2}$, where $a < 0,\ b \in \mathbb{R}$ and $4b - a^2 > 0$. Suppose that $\phi(x) \in \mathcal{L}-\mathcal{P}I$ with $\phi(x) > 0$ if $x \geq 0$ and ϕ is not of the form $ce^{\beta x}$. Then the function*

$$F(x,a,b) = \sum_{k=0}^\infty \frac{(-1)^k x^k}{k!(k^2+ak+b)\phi(k)}\tag{3.13}$$

changes sign infinitely often in the interval $(0,\infty)$.

Proof. By assumption, ϕ satisfies the hypotheses of Theorem 3.3 and thus there is a Pólya frequency function $K(s)$ such that $\dfrac{1}{\phi(x)} = \displaystyle\int_0^\infty K(s)e^{-xs}\,ds$, for $x \geq 0$. (Note that since $\phi(0) > 0$, this representation of $\dfrac{1}{\phi(x)}$ is valid for $x = 0$.) Using this representation of $\dfrac{1}{\phi(x)}$ in (3.13), for $x \geq 1$ we can express $F(x,a,b)$ in the form

$$\begin{aligned}F(x,a,b) &= \tfrac{1}{\bar\alpha-\alpha}\textstyle\sum_{k=0}^\infty \tfrac{(-1)^k x^k}{k!}\left(\left[\tfrac{1}{k+\alpha}-\tfrac{1}{k+\bar\alpha}\right]\displaystyle\int_0^\infty K(s)e^{-ks}\,ds\right)\\ &= \tfrac{1}{\bar\alpha-\alpha}\int_0^\infty K(s)\left(\sum_{k=0}^\infty \frac{(-1)^k(xe^{-s})^k}{k!}\left[\frac{1}{k+\alpha}-\frac{1}{k+\bar\alpha}\right]\right)\,ds\\ &= -\tfrac{1}{\tau}\int_0^\infty K(s)\Im\left(\frac{\gamma(\alpha,xe^{-s})}{x^\alpha e^{-\alpha s}}\right)\,ds,\end{aligned}\tag{3.14}$$

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where the interchanging of the integral with the summation is justified by the uniform convergence of the series in (3.13) and by (3.10) of Lemma 3.7. Now a calculation shows that for $x \geq 1$,

$$\Im\left(\frac{\gamma(\alpha, xe^{-s})}{x^{\alpha}e^{-\alpha s}}\right) = \frac{e^{as/2}}{x^{a/2}}\Im\left([\Gamma(\alpha) - \Gamma(\alpha, xe^{-s})]e^{i\tau(s-\log x)}\right). \tag{3.15}$$

Substituting (3.15) into (3.14), we obtain

$$F(x, a, b) = -\frac{1}{\tau x^{a/2}}[I_1(x, a, b) + I_2(x, a, b)], \tag{3.16}$$

where

$$\begin{aligned} I_1(x, a, b) &= \cos(\tau \log x) \int_0^{\infty} K(s)e^{as/2}\{\cos(\tau s)\Im[\Gamma(\alpha) - \Gamma(\alpha, xe^{-s})] \\ &\quad + \sin(\tau s)\Re[\Gamma(\alpha) - \Gamma(\alpha, xe^{-s})]\}\, ds \end{aligned}$$

and

$$\begin{aligned} I_2(x, a, b) &= \sin(\tau \log x) \int_0^{\infty} K(s)e^{as/2}\{\sin(\tau s)\Im[\Gamma(\alpha) - \Gamma(\alpha, xe^{-s})] \\ &\quad - \cos(\tau s)\Re[\Gamma(\alpha) - \Gamma(\alpha, xe^{-s})]\}\, ds, \end{aligned}$$

where the existence of these integrals (for $a < 0$) follows from (3.10) of Lemma 3.7.

Next we set

$$A = \Im\Gamma(\alpha) \int_0^{\infty} K(s)e^{as/2}\cos(\tau s)\, ds + \Re\Gamma(\alpha) \int_0^{\infty} K(s)e^{as/2}\sin(\tau s)\, ds$$

and

$$B = \Im\Gamma(\alpha) \int_0^{\infty} K(s)e^{as/2}\sin(\tau s)\, ds - \Re\Gamma(\alpha) \int_0^{\infty} K(s)e^{as/2}\cos(\tau s)\, ds$$

and observe that

$$iA - B \neq 0. \tag{3.17}$$

Indeed, since $\alpha \notin \mathbb{R}$ and $\Re(-\alpha) = -\frac{a}{2} > 0$, we have

$$0 \neq \frac{\Gamma(\alpha)}{\phi(-\alpha)} = \Gamma(\alpha) \int_0^{\infty} K(s)e^{-(-\alpha s)}\, ds = \int_0^{\infty} K(s)e^{as/2}(\Gamma(\alpha)e^{i\tau s})\, ds = iA - B.$$

Now suppose that $A \neq 0$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of positive numbers tending to infinity such that $\cos(\tau \log x_n) = (-1)^n$. Then $\sin(\tau \log x_n) = 0$, and so by (3.16) we obtain

$$F(x_n, a, b) = -\frac{1}{\tau x_n^{a/2}} I_1(x_n, a, b) = -\frac{(-1)^n}{\tau x_n^{a/2}}(A - E(x_n)),$$

where $E(x)$ is defined by (3.9) in Lemma 3.7. Also since by Lemma 3.7, $\lim_{n\to\infty} E(x_n) = 0$, and since $A \neq 0$, we conclude that $F(x, a, b)$ changes sign infinitely often in the interval $(0, \infty)$. If, on the other hand, $A = 0$, then by (3.17), $B \neq 0$ and the above argument, *mutatis mutandis*, shows that the conclusion of the proposition remains valid in this case as well. □

THEOREM 3.9. *Suppose that $\phi(x) \in \mathcal{L}-\mathcal{P}I$, $\phi(x) > 0$ if $x > 0$, where $\phi(x)$ is not of the form $ce^{\beta x}$. Let $p(x)$ be a real polynomial all of whose zeros lie in the right half-plane $\Re z > 0$. Let $h(x) = p(x)\phi(x)$. If the sequence $T = \{h(k)\}_{k=0}^{\infty}$ is a CZDS, then all the zeros of $p(x)$ are real.*

Proof. Assume the contrary so that $h(x)$ may be expressed in the form $h(x) = \tilde{g}(x)(x^2 + ax+b)\phi(x)$, where $x^2+ax+b = (x+\alpha)(x+\overline{\alpha})$ and $\alpha = \frac{a}{2}+i\tau$, $\tau = \frac{\sqrt{4b-a^2}}{2}$, $4b-a^2 > 0$ and $\Re\alpha = \frac{a}{2} < 0$. Then the polynomial $\tilde{g}(x)$ gives rise to the entire function $\sum_{k=0}^{\infty}\frac{\tilde{g}(k)(-1)^k x^k}{k!} = g(x)e^{-x}$, where $g(x)$ is a polynomial. We next approximate the entire function $g(x)e^{-x}$ by means of the polynomials $q_n(x) = g(x)\left[\left(1-\frac{x}{2n}\right)^{2n}+\epsilon_n\right]$, where $\epsilon_n > 0$ and $\lim_{n\to\infty}\epsilon_n = 0$ (see Remark 2.9(1)). We note, in particular, that $q_n(x)$ has exactly the same real zeros as $g(x)$ has. Moreover, as $n\to\infty$, $q_n(x)\to g(x)e^{-x}$ uniformly on compact subsets of $\mathbb{C}$. If we set $\Lambda = \{\frac{1}{h(k)}\}_{k=0}^{\infty}$, then by Proposition 3.8, the function

$$\Lambda[g(x)e^{-x}] = F(x,a,b) = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!(k^2+ak+b)\phi(k)}$$

has infinitely many sign changes in the interval $(0,\infty)$. Also, as $n\to\infty$, $f_n(x) := \Lambda[q_n(x)]$ converges to $F(x,a,b)$ uniformly on compact subsets of $\mathbb{C}$. Thus, for all sufficiently large n, each of the approximating polynomials $f_n(x)$ has more real zeros than $g(x)$ has. Since T is a CZDS, $Z_c([T[f_n(x)]) \le Z_c(f_n(x))$, and since $\deg q_n = \deg f_n$ consequently, for all n sufficiently large, the polynomial $T[f_n(x)] = T[\Lambda[q_n(x)]] = q_n(x)$ has more real zeros than $g(x)$ has. This is the desired contradiction. □

THEOREM 3.10. *Suppose that $\phi(x) \in \mathcal{L}-\mathcal{P}I$, $\phi(x) > 0$ if $x > 0$. Let $p(x)$ be a real polynomial with no nonreal zeros in the left half-plane $\Re z < 0$. Suppose that $p(0)\phi(0) = 1$ and set $h(x) = p(x)\phi(x)$. Then $T = \{h(k)\}_{k=0}^{\infty}$ is a CZDS if and only if $p(x)$ has only real negative zeros.*

Proof. If $p(x)$ has only real negative zeros, the theorem follows from Laguerre's theorem. Conversely, suppose T is a CZDS. We may assume, without loss of generality, that all the zeros of $p(x)$ lie in the right half-plane; indeed, by the assumption, the zeros of $p(x)$ in the left half-plane are all real and these may be incorporated into $\phi(x)$. The case where $\phi(x)$ is of the form $ce^{\beta x}$ is covered by Proposition 3.1. Otherwise, Theorem 3.9 implies that $p(x)$ has only real zeros and, by Theorem 3.6, $p(x)$ can have only real negative zeros. □

The corresponding problem when the nonreal zeros of $p(x)$ lie in the left half-plane is still open. Note that the technique employed here was to show that the existence of nonreal zeros in the right half-plane implied that a certain reciprocal sequence was not a λ-sequence. This is no longer true if the nonreal zeros lie in the left half-plane. There are also specific examples which show that one does not get a CZDS in any generality. To wit, take $\{(k^2+k+1)\cosh(\sqrt{k+m})\}_{k=0}^{\infty}$ and apply it to $(x+1)^6(x^2+x/2+1/5)$. The resulting polynomial will have four nonreal zeros if $m \ge 4$.

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