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AN ANALYSIS OF THE POLE PLACEMENT PROBLEM II. THE MULTI-INPUT CASE*

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Abstract. For the solution of the multi-input pole placement problem we derive explicit formulas for the subspace from which the feedback gain matrix can be chosen and for the feedback gain as well as the eigenvector matrix of the closed-loop system. We discuss which Jordan structures can be assigned and also when diagonalizability can be achieved. Based on these formulas we study the conditioning of the pole-placement problem in terms of perturbations in the data and show how the conditioning depends on the condition number of the closed loop eigenvector matrix, the norm of the feedback matrix and the distance to uncontrollability.

Key words. pole placement, condition number, perturbation theory, Jordan form, explicit formulas, Cauchy matrix, Vandermonde matrix, stabilization, feedback gain, distance to uncontrollability.

AMS subject classifications. 65F15, 65F35, 65G05, 93B05, 93B55.

1. Introduction. In this paper we continue the analysis of the conditioning of the pole placement problem in [22] with the multi-input case. We study multi-input time-invariant linear systems

$$\dot{x} = dx(t)/dt = Ax(t) + Bu(t),\ x(0) = x_0, \tag{1.1}$$

with $A \in \mathcal{C}^{n\times n}$, $B \in \mathcal{C}^{n\times m}$. For such systems we analyse the following problem:

PROBLEM 1. *Multi-input pole placement (MIPP):* Given a set of n complex numbers $\mathcal{P} = \{\lambda_1, \ldots, \lambda_n\} \subset \mathcal{C}$, find a matrix $F \in \mathcal{C}^{m\times n}$, such that the set of eigenvalues of $A - BF$ is equal to $\mathcal{P}$. (Here we assume in the real case that the set $\mathcal{P}$ is closed under complex conjugation.) It is well-known [14, 37] that a *feedback gain* matrix F that solves this problem for all possible sets $\mathcal{P} \subset \mathcal{C}$ exists if and only if (A, B) is *controllable*, i.e.,

$$\operatorname{rank}[A - \lambda I_n, B] = n,\ \forall \lambda \in \mathcal{C} \tag{1.2}$$

or

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n. \tag{1.3}$$

Due to its wide range of applications there is a vast literature on this problem. Extensions of Ackermann's explicit formula [1] for the single-input case were given in [33, 32]. Also many numerical algorithms were developed for this problem, see [27, 36, 15, 24, 25] For some of these methods numerical backward stability has been established, see e.g. [15, 25, 24, 5, 6, 3]. Nonetheless it is observed very often that the numerical results (even from numerically stable methods or explicit formulas) are very inaccurate. This observation led to the conjecture in [12] (supported by intensive numerical testing) that the pole placement problem becomes inherently ill-conditioned when the system size is increasing. This conjecture has been heavily debated, since some of the perturbation results derived in recent years do not seem to support this conjecture [2, 17, 29, 18].

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The reason for the discrepancy in opinions about the conditioning of the pole assigment problem is that one has to distinguish between two aspects of the pole placement problem, the computation of the feedback F and the computation of the closed loop matrix $A - BF$ or its spectrum, respectively. Both can be viewed as *result* of the pole placement problem but they exhibit different pertubation results. A striking example for the difference is given in [22] for the single-input case, where the exact feedback was used but the poles of the computed closed loop system were nowhere near to the desired poles. In our opinion the most important goal of pole placement is that the poles of the closed loop system obtained with the computed feedback are close to the desired ones. If the desired poles of the exact closed loop system are very sensitive to perturbations then this ultimate goal cannot be guaranteed. And this may happen even if the computation of F is reliable or even exact.

A new analysis that covers all the aspects of the problem is therefore necessary and it was given for the single-input case in [22]. In this paper we will continue this analysis for the multi-input case. We will derive explicit formulas for the feedback matrix F. These formulas are different from the formulas derived in [33, 32] and display all the freedom in the solution, which is clearly for $m > 1$ not uniquely determined from the data $A, B, \mathcal{P}$. To remove the non-uniqueness several directions can be taken. The most common approach is to try to minimize the norm of the feedback matrix F under all possible feedbacks F that achieve the desired pole assigment, see [24, 27, 36, 25, 16]. Another approach is to optimize the robustness of the closed-loop system [15].

In this paper we study the whole solution set, i.e., the set of feedbacks that place the poles and describe it analytically. We also derive explicit formulas for the closed-loop eigenvector matrix. Based on these formulas we will then give perturbation bounds which are multi-input versions of the bounds for the single-input problem in [22] and display the problems that can arise when choosing one or the other method for making the feedback unique.

Throughout the paper we will assume that (A, B) is controllable and that $\operatorname{rank} B = m$. We will use the superscript H to represent the conjugate transpose. All used norms are spectral norms.

2. The null space of $[A - \lambda I, B]$. We begin our analysis with a characterization of the nullspace of $[A - \lambda I, B]$ for a given $\lambda \in \mathcal{C}$. Since (A, B) is controllable, from (1.2) we have that $\operatorname{rank}[A - \lambda I, B] = n$, $\forall \lambda \in \mathcal{C}$. So the dimension of the null space is m.

Let $\begin{bmatrix} U_\lambda \\ -V_\lambda \end{bmatrix}$, with $U_\lambda \in \mathcal{C}^{n \times m}, V_\lambda \in \mathcal{C}^{m \times m}$, be such that its columns span the null space $\mathcal{N}_\lambda$ of $[A - \lambda I, B]$, i.e.,

$$\begin{bmatrix} A - \lambda I_n & B \end{bmatrix} \begin{bmatrix} U_\lambda \\ -V_\lambda \end{bmatrix} = 0, \tag{2.1}$$

or

$$(A - \lambda I_n) U_\lambda = B V_\lambda. \tag{2.2}$$

Before we can characterize this nullspace, we have to introduce some notation and recall some well-known facts from linear systems theory.

The basis for most of the results concerning the analysis and also the numerical solution of the control problem under consideration are canonical and condensed forms. The most useful form in the context of numerical methods is the staircase orthogonal form [34, 35].

LEMMA 2.1. *[35] Let $A \in \mathcal{C}^{n \times n}$, $B \in \mathcal{C}^{n \times m}$, (A, B) controllable and* $\operatorname{rank}(B) = m$.

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Then there exists a unitary matrix $Q \in \mathcal{C}^{n\times n}$ such that

$$
(2.3)\quad Q^HAQ = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_s \end{array}
\begin{array}{c}
\begin{array}{ccccc} n_1 & n_2 & \dots & \dots & n_s \end{array} \\
\begin{bmatrix}
A_{1,1} & A_{1,2} & \dots & \dots & A_{1,s} \\
A_{2,1} & A_{2,2} & \dots & \dots & A_{2,s} \\
0 & A_{3,2} & \ddots & \dots & A_{3,s} \\
 & & \ddots & \ddots & \vdots \\
 & & & A_{s,s-1} & A_{s,s}
\end{bmatrix}
\end{array}, \quad
Q^HB = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_s \end{array}
\begin{array}{c}
n_1 = m \\
\begin{bmatrix} B_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\end{array},
$$

with B_1, $A_{1,1}, \dots, A_{s,s}$ square, B_1 nonsingular, and the matrices $A_{i,i-1} \in \mathcal{C}^{n_i \times n_{i-1}}$, $i = 2, \dots, s$, all have full row rank. ($n_1 \geq n_2 \geq \dots \geq n_s$). The indices n_i play an important role in the following constructions and we will also need the following indices derived from the n_i. Set

$$
(2.4)\qquad d_i := n_i - n_{i+1},\ i = 1, \dots, s-1,\ d_s := n_s,
$$

and

$$
(2.5)\qquad \pi_i := d_1 + \dots + d_i = m - n_{i+1},\ i = 1, \dots, s-1, \quad \pi_s = m.
$$

An immediate consequence of the staircase form is that the indices n_i, d_i, π_i are invariant under adding multiples of the identity to A, i.e., these indices are the same for the pairs (A, B) and $(A - \lambda I, B)$. This follows, since the subdiagonal blocks in the staircase form, which determine these invariants, are the same if we add a shift to the diagonal.

If we allow nonunitary transformations we can get a more condensed form, similar to the Luenberger canonical form [21], which follows directly from the staircase orthogonal form.

LEMMA 2.2. *[21] Let $A \in \mathcal{C}^{n\times n}$, $B \in \mathcal{C}^{n\times m}$, (A, B) controllable and $\operatorname{rank}(B) = m$. Then there exist nonsingular matrices $S \in \mathcal{C}^{n\times n}, T \in \mathcal{C}^{m\times m}$ such that*

$$
\begin{aligned}
\hat{A} &:= S^{-1}AS \\
&= \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_{s-1} \\ n_s \end{array}
\begin{array}{c}
\begin{array}{cccccccc} d_1 & n_2 & d_2 & n_3 & \dots & d_{s-1} & n_s & d_s \end{array} \\
\begin{bmatrix}
\hat{A}_{1,1} & 0 & \hat{A}_{1,2} & 0 & \dots & \hat{A}_{1,s-1} & 0 & \hat{A}_{1,s} \\
0 & I_{n_2} & \hat{A}_{2,2} & 0 & \dots & \hat{A}_{2,s-1} & 0 & \hat{A}_{2,s} \\
 & & 0 & I_{n_3} & \dots & \hat{A}_{3,s-1} & 0 & \hat{A}_{3,s} \\
 & & & & \ddots & \vdots & \vdots & \vdots \\
 & & & & & \hat{A}_{s-1,s-1} & 0 & \hat{A}_{s-1,s} \\
 & & & & & 0 & I_{n_s} & \hat{A}_{s,s}
\end{bmatrix}
\end{array},
\end{aligned}
$$

$$
(2.6)\qquad \hat{B} := S^{-1}BT = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix},
$$

where the indices n_i and d_i are related as in (2.4).

Let us further introduce the Krylov matrices

$$
(2.7)\qquad K_k := [B, AB, \dots, A^{k-1}B],\ \hat{K}_k := [\hat{B}, \hat{A}\hat{B}, \dots, \hat{A}^{k-1}\hat{B}],
$$

and the block matrices

$$
(2.8)\qquad \hat{X}_k := \begin{bmatrix} \hat{X}_{1,1} & \dots & \hat{X}_{1,k} \\ & \ddots & \vdots \\ & & \hat{X}_{k,k} \end{bmatrix} \in \mathcal{C}^{km\times \pi_k},
$$

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$$X_k := \begin{bmatrix} X_{1,1} & \dots & X_{1,k} \\ & \ddots & \vdots \\ & & X_{k,k} \end{bmatrix} := \mathrm{diag}(T,\dots,T)\hat{X}_k \in \mathcal{C}^{km\times\pi_k}, \tag{2.9}$$

$$\hat{R}_k := [\hat{A}_{1,1}, \hat{A}_{1,2}, \dots, \hat{A}_{1,k}], \quad R_k := T\hat{R}_k \in \mathcal{C}^{m\times\pi_k}, \tag{2.10}$$

where

$$\hat{X}_{i,i} := \begin{array}{c} \\ \pi_{i-1} \\ d_i \\ n_{i+1} \end{array} \begin{array}{c} d_i \\ \begin{bmatrix} 0 \\ I_{d_i} \\ 0 \end{bmatrix} \end{array}, \; i = 1,\dots,k,$$

$$\hat{X}_{i,j} := \begin{array}{c} \\ \pi_i \\ n_{i+1} \end{array} \begin{array}{c} d_j \\ \begin{bmatrix} 0 \\ -\hat{A}_{i+1,j} \end{bmatrix} \end{array}, \; i = 1,\dots,k-1, \; j = i+1,\dots,k,$$

$$X_{i,j} := T\hat{X}_{i,j}, \; i = 1,\dots,k, \; j = i,\dots,k.$$

Let us also abbreviate $X := X_s$, $R := R_s$, $K := K_s$. Then we can characterize the nullspace of $[A, B]$ as follows.

LEMMA 2.3. *Let X_k, $\hat{X}_k$, R_k, $\hat{R}_k$, K_k, $\hat{K}_k$ be as introduced in (2.7)–(2.10). Then*

$$AK_kX_k = BR_k, \; \hat{A}\hat{K}_k\hat{X}_k = \hat{B}\hat{R}_k, \; k = 1,\dots,s \tag{2.11}$$

and the columns of

$$\begin{bmatrix} U_0 \\ -V_0 \end{bmatrix} = \begin{bmatrix} KX \\ -R \end{bmatrix}$$

span the nullspace $\mathcal{N}_0$ of $[A, B]$.

Proof. The proof follows directly from the fact that $AK_kX_k = S(\hat{A}\hat{K}_k\hat{X}_k)$, $BR_k = S(\hat{B}\hat{R}_k)$ and the special structure of the block columns in $\hat{K}_k$, i.e., for $1 \le l \le s$,

$$\hat{A}^{l-1}\hat{B} = \begin{array}{c} \\ n_1 \\ \vdots \\ n_{l-1} \\ n_l \\ n_{l+1} \\ \vdots \\ n_s \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & \dots & d_{l-2} & d_{l-1} & n_l \end{array} \\ \begin{bmatrix} * & \dots & * & \hat{A}_{1,l-1} & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ * & \dots & * & \hat{A}_{l-1,l-1} & 0 \\ 0 & \dots & 0 & 0 & I_{n_l} \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{2.12}$$

by just multiplying out both sides of the equations in (2.11). Note that it follows directly from the controllability assumption and the staircase form (2.3) that the full nullspace is obtained for $k = s$, since then the dimension of the space spanned by the columns of

$$\begin{bmatrix} KX \\ -R \end{bmatrix}$$

is $m = n_1$, which, as noted before, is the dimension of the nullspace of $[A, B]$. $\square$

We need some further notation. Let

$$\Theta_{i,j} := \sum_{l=i}^{j} A^{l-i} B X_{l,j},\ \hat{\Theta}_{i,j} := \sum_{l=i}^{j} \hat{A}^{l-i} \hat{B} \hat{X}_{l,j},\ i = 1, \ldots, s,\ j = i, \ldots, s, \tag{2.13}$$

and set

$$\begin{aligned} W_i &:= [\Theta_{i,i}, \ldots, \Theta_{i,s}] \in \mathcal{C}^{n \times n_i},\ i = 1, \ldots, s, \\ W &:= [W_1, W_2, \ldots, W_s] \in \mathcal{C}^{n \times n}, \\ Y_i &:= [X_{i,i}, \ldots, X_{i,s}] \in \mathcal{C}^{m \times n_i},\ i = 1, \ldots, s, \\ Y &:= [Y_1, Y_2, \ldots, Y_s] \in \mathcal{C}^{m \times n}. \end{aligned} \tag{2.14}$$

Furthermore define

$$\mathcal{I}_{i,j} := \begin{array}{c c} & \begin{array}{cc} n_j - n_i & n_i \end{array} \\ n_i & \left[\begin{array}{cc} 0 & I_{n_i} \end{array} \right] \end{array},\ i \geq j, \tag{2.15}$$

and

$$N := \begin{bmatrix} 0 & & & & \\ \mathcal{I}_{2,1} & 0 & & & \\ 0 & \mathcal{I}_{3,2} & \ddots & & \\ \vdots & & \ddots & 0 & \\ 0 & \ldots & 0 & \mathcal{I}_{s,s-1} & 0 \end{bmatrix},\ \tilde{N} = \begin{bmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & \ddots & I_m \\ & & & 0 \end{bmatrix}.$$

LEMMA 2.4. *The matrices W, W_1 defined in (2.14) have the following properties.*

i)

$$W_1 = KX, \quad W = K\tilde{X}, \quad \tilde{X} = [X, \tilde{N}X, \ldots, \tilde{N}^{s-1}X]. \tag{2.16}$$

ii)

$$W = AWN + BY. \tag{2.17}$$

iii) W is nonsingular.

Proof.

i) follows directly from the definition of W_1 and W.

ii) Using the form of W, N we have

$$\begin{aligned} AWN &= A[W_1, W_2, \ldots, W_s]N \\ &= A[0, W_2; \ldots; 0, W_{s-1}; 0] \\ &= [0, A\Theta_{2,2}, \ldots, A\Theta_{2,s}; \ldots; 0, A\Theta_{s,s}; 0] \\ &= [0, \Theta_{1,2}, \ldots, \Theta_{1,s}; \ldots; 0, \Theta_{s-1,s}; 0] \\ &\qquad -B[0, X_{1,2}, \ldots, X_{1,s}; \ldots; 0, X_{s-1,s}; 0] \\ &= W - BY. \end{aligned}$$

iii) We have

$$W = S(S^{-1}W)$$

$$= S[\hat{\Theta}_{1,1}, \ldots, \hat{\Theta}_{1,s}; \ldots; \hat{\Theta}_{s-1,s-1}, \hat{\Theta}_{s-1,s}, \hat{\Theta}_{s,s}]$$
$$= S \begin{bmatrix} I_{n_1} & * & \ldots & * \\ & I_{n_2} & \ldots & \vdots \\ & & \ddots & * \\ & & & I_{n_s} \end{bmatrix} P$$

for an appropriate permutation matrix P, which follows directly from the definition of $\hat{\Theta}_{i,j}$ in (2.13). Thus, W is nonsingular. □

REMARK 1. If $m = 1$ (the single-input case), then $X = [a_1, \ldots, a_{n-1}, 1]^T$, $R = -a_0$, and by (1.3) $K = [B, \ldots, A^{n-1}B]$ is nonsingular. Since $AKX = BR$, we find that $a_0, \ldots, a_{n-1}$ are the coefficients of the (monic) characteristic polynomial of A, i.e.,

$$\xi(\lambda) := \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k = \det(\lambda I_n - A).$$

Using $\mathrm{adj}(\lambda I_n - A) := \sum_{k=0}^{n-1} A_k \lambda^k$, where $\mathrm{adj}(A)$ represents the adjoint matrix of the square matrix A, it is not difficult to verify that $W_1 = A_0 B$ and $W = [A_0 B, \ldots A_{n-1} B]$.

We are now able to give a simple characterization of the nullspace of $[A - \lambda I, B]$ for an arbitrary λ.

THEOREM 2.5. *Let* $E_{\lambda,k} := (I - \lambda N)^{-1} \begin{bmatrix} I_{\pi_k} \\ 0 \end{bmatrix}$. *Then the columns of*

$$\begin{bmatrix} U_{\lambda,k} \\ -V_{\lambda,k} \end{bmatrix} := \begin{bmatrix} W E_{\lambda,k} \\ -(R_k - \lambda Y E_{\lambda,k}) \end{bmatrix}, \ k = 1, 2, \ldots, s, \tag{2.18}$$

span the subspaces $\mathcal{N}_{\lambda,k}$ *of dimension* π_k *of the nullspace of* $[A - \lambda I, B]$*. In particular, for* $k = s$ *we obtain the whole nullspace* $\mathcal{N}_\lambda$ *spanned by the columns of*

$$\begin{bmatrix} U_{\lambda} \\ -V_{\lambda} \end{bmatrix} := \begin{bmatrix} W E_{\lambda,s} \\ -(R - \lambda Y E_{\lambda,s}) \end{bmatrix}, \tag{2.19}$$

which has dimension $\pi_s = m$*. Hence, we have* $(A - \lambda I)U_\lambda = BV_\lambda$*.*

Proof. By (2.17) we have

$$(A - \lambda I)W = AW - \lambda W = AW - \lambda AWN - \lambda BY = AW(I - \lambda N) - \lambda BY.$$

Since $I - \lambda N$ is nonsingular, we get

$$(A - \lambda I)W(I - \lambda N)^{-1} = AW - \lambda BY(I - \lambda N)^{-1}$$

and then by multiplying with $\begin{bmatrix} I_{\pi_k} \\ 0 \end{bmatrix}$ from the right we obtain

$$(A - \lambda I)W E_{\lambda,k} = AW \begin{bmatrix} I_{\pi_k} \\ 0 \end{bmatrix} - \lambda BY E_{\lambda,k}.$$

By Lemma 2.3 and (2.16) we have that $AW \begin{bmatrix} I_{\pi_k} \\ 0 \end{bmatrix} = BR_k$ and hence the result follows. The dimension of $\mathcal{N}_{\lambda,k}$ is directly determined from the fact that

$$\mathrm{rank}\, U_{\lambda,k} = \mathrm{rank}\, W E_{\lambda,k} = \mathrm{rank}\, E_{\lambda,k} = \pi_k.$$

□

In this section we have derived explicit formulas for matrices whose columns span the right nullspace of $[A - \lambda I, B]$. These formulas will be used in the following section to derive explicit expressions for F and also the closed loop eigenvector matrix.

3. Formulas for F and the closed loop eigenvector matrix . In this section we derive explicit expressions for the feedback matrix F and the closed loop eigenvector matrix. Other explicit formulas for the feedback matrix F are given in [33, 32]. They are different from our formulas in that they do not display the whole solution set and also do not give the closed loop Jordan canonical form.

Set

$$\mathcal{U}_{\lambda,k} := \operatorname{range} U_{\lambda,k}, \quad \mathcal{V}_{\lambda,k} := \operatorname{range} V_{\lambda,k}, \quad k = 1, \ldots, s, \tag{3.1}$$

where $U_{\lambda,k}, V_{\lambda,k}$ are defined in (2.18). In particular we set $\mathcal{U}_\lambda := \operatorname{range} U_\lambda$ ($U_\lambda = U_{\lambda,s}$), $\mathcal{V}_\lambda := \operatorname{range} V_\lambda$ ($V_\lambda = V_{\lambda,s}$).

Let (λ, g) be an eigenpair of $A - BF$, i.e.,

$$(A - BF)g = \lambda g \text{ or } (A - \lambda I)g = BFg =: Bz.$$

Using the representation of the nullspace of $[A - \lambda I, B]$ in (2.19) there is a vector $\phi \in \mathcal{C}^m$ such that $g = U_\lambda \phi$, $z = V_\lambda \phi$. Clearly $\mathcal{U}_\lambda$ is just the space containing all possible eigenvectors of $A - BF$ associated with λ.

Let us first consider a single Jordan block $J_p = \lambda I + N_p$, where

$$N_p := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ & 0 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}_{p \times p}.$$

LEMMA 3.1. *Suppose that $A - BF$ has a Jordan block of size $p \times p$ associated with λ and the corresponding chain of principle vectors is $g_1, \ldots, g_p$, i.e.,*

$$(A - BF)[g_1, \ldots, g_p] = [g_1, \ldots, g_p]J_p. \tag{3.2}$$

Let $G_p := [g_1, \ldots, g_p]$, $Z_p =: FG_p =: [z_1, \ldots, z_p]$. Then there exist matrices $\Phi_p = [\phi_1, \ldots, \phi_p] \in \mathcal{C}^{m \times p}$ and $\Gamma_p \in \mathcal{C}^{n \times p}$ such that

$$G_p = W\Gamma_p, \ Z_p = R\Phi_p - Y\Gamma_p J_p, \tag{3.3}$$

where

$$\Gamma_p = \begin{bmatrix} \Phi_p \\ \mathcal{I}_{2,1}\Phi_p J_p \\ \vdots \\ \mathcal{I}_{s,1}\Phi_s J_p^{s-1} \end{bmatrix} \tag{3.4}$$

satisfies $\operatorname{rank} \Gamma_p = p$. (Here the matrices $\mathcal{I}_{i,1}$ are as defined in (2.15).)

Proof. By adding $-\lambda WN$ on both sides of (2.17) we obtain

$$W(I - \lambda N) = (A - \lambda I)WN + BY.$$

Hence we have that

$$W = (A - \lambda I)WN(I - \lambda N)^{-1} + BY(I - \lambda N)^{-1}. \tag{3.5}$$

Let $E = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$ then via induction we prove that there exist vectors $\phi_j \in \mathcal{C}^m$ such that the following expressions hold for g_k, z_k.

$$g_k = W \sum_{j=1}^{k} N^{j-1}(I - \lambda N)^{-j} E\phi_{k+1-j}, \tag{3.6}$$

$$z_k = V_\lambda \phi_k - Y \sum_{j=2}^{k} N^{j-2}(I - \lambda N)^{-j} E\phi_{k+1-j}, \tag{3.7}$$

for $k = 1, 2, \ldots, p$.

For $k = 1$ we have from (3.2) that g_1 is an eigenvector of $A - BF$. So there exists a $\phi_1 \in \mathcal{C}^m$ such that

$$g_1 = WE_{\lambda,s}\phi_1 = W(I - \lambda N)^{-1} E\phi_1,\ z_1 = V_\lambda \phi_1. \tag{3.8}$$

Suppose now that (3.6) and (3.7) hold for k, we will show that they also hold for $k+1$. By (3.2), $(A - \lambda I)g_{k+1} = Bz_{k+1} + g_k$. By (3.6), (3.5) it follows that

$$\begin{aligned} g_k = (A - \lambda I)W \sum_{j=1}^{k} N^{j}(I - \lambda N)^{-(j+1)} E\phi_{k+1-j} \\ + BY \sum_{j=1}^{k} N^{j-1}(I - \lambda N)^{-(j+1)} E\phi_{k+1-j}. \end{aligned}$$

Then there exists $\phi_{k+1} \in \mathcal{C}^m$, (note that $N^k = 0$ for $k \geq s$,) such that

$$\begin{aligned} g_{k+1} &= W\{(I - \lambda N)^{-1} E\phi_{k+1} + \sum_{j=1}^{k} N^{j}(I - \lambda N)^{-(j+1)} E\phi_{k+1-j}\} \\ &= W \sum_{j=1}^{k+1} N^{j-1}(I - \lambda N)^{-j} E\phi_{k+2-j} \end{aligned}$$

and

$$\begin{aligned} z_{k+1} &= V_\lambda \phi_{k+1} - Y \sum_{j=1}^{k} N^{j-1}(I - \lambda N)^{-(j+1)} E\phi_{k+1-j} \\ &= V_\lambda \phi_{k+1} - Y \sum_{j=2}^{k+1} N^{j-2}(I - \lambda N)^{-j} E\phi_{k+2-j}. \end{aligned}$$

Now with (3.6) and (3.7) we obtain

$$G_p = W \sum_{j=1}^{p} N^{j-1}(I - \lambda N)^{-j} E\Phi_p N_p^{j-1} =: W\Gamma_p,$$

$$Z_p = V_\lambda \Phi_p - Y \sum_{j=2}^{p} N^{j-2}(I - \lambda N)^{-j} E\Phi_p N_p^{j-1}.$$

Using the formula

$$N^{j-1}(I-\lambda N)^{-j} = \sum_{k=j}^{s} \binom{k-1}{j-1} \lambda^{k-j} N^{k-1},$$

we obtain

$$\begin{aligned}
\Gamma_p &= \sum_{j=1}^{s} (\sum_{k=j}^{s} \binom{k-1}{j-1} \lambda^{k-j} N^{k-1}) E \Phi_p N_p^{j-1} \\
&= \sum_{j=1}^{s} (\sum_{k=j}^{s} \binom{k-1}{j-1} \lambda^{k-j} \begin{bmatrix} 0 \\ \mathcal{I}_{k,1}\Phi_p \\ 0 \end{bmatrix}) N_p^{j-1} \\
&= \sum_{k=1}^{s} \begin{bmatrix} 0 \\ \mathcal{I}_{k,1}\Phi_p \\ 0 \end{bmatrix} (\sum_{j=1}^{k} \binom{k-1}{j-1} \lambda^{k-j} N_p^{j-1}) \\
&= \sum_{k=1}^{s} \begin{bmatrix} 0 \\ \mathcal{I}_{k,1}\Phi_p(\lambda I_p + N_p)^{k-1} \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_p \\ \mathcal{I}_{2,1}\Phi_p J_p \\ \vdots \\ \mathcal{I}_{s,1}\Phi_p J_p^{s-1} \end{bmatrix}.
\end{aligned}$$

Since

$$\sum_{j=2}^{p} N^{j-2}(I-\lambda N)^{-j} E \Phi_p N_p^{j-1} = (I-\lambda N)^{-1} \Gamma_p N_p,$$

we get $Z_p = V_\lambda \Phi_p - Y(I-\lambda N)^{-1}\Gamma_p N_p$, and then with $V_\lambda = R - \lambda Y(I-\lambda N)^{-1}E$ we obtain

$$Z_p = R\Phi_p - Y(I-\lambda N)^{-1} \begin{bmatrix} \Phi_p J_p \\ \mathcal{I}_{2,1}\Phi_p J_p N_p \\ \vdots \\ \mathcal{I}_{s,1}\Phi_p J_p^{s-1} N_p \end{bmatrix}.$$

It is then easy to check that $Z_p = R\Phi_p - Y\Gamma_p J_p$ by using the explicit formula for the inverse of $(I-\lambda N)^{-1}$ and by calculating the blocks from top to bottom. Then $\operatorname{rank} \Gamma_p = p$ follows from $\operatorname{rank} W = n$ and $\operatorname{rank} G_p = p$. □

After having obtained the formula for each different Jordan block, we have the following theorem for a general Jordan matrix.

THEOREM 3.2. *Let*

$$J = \operatorname{diag}(J_{1,1}, \ldots, J_{1,r_1}, \ldots, J_{q,1}, \ldots, J_{q,r_q}), \tag{3.9}$$

where $J_{ij} = \lambda_i I_{p_{ij}} + N_{p_{ij}}$. There exists an F so that J is the Jordan canonical form of $A - BF$ if and only if there exists a matrix $\Phi \in \mathcal{C}^{m \times n}$ so that

$$\Gamma := \begin{bmatrix} \Phi \\ \mathcal{I}_{2,1}\Phi J \\ \vdots \\ \mathcal{I}_{s,1}\Phi J^{s-1} \end{bmatrix} \tag{3.10}$$

is nonsingular. If such a nonsingular Γ *exists, then with* $G := W\Gamma$ *and* $Z := R\Phi - Y\Gamma J$*, we have that* $F = ZG^{-1}$ *is a feedback gain that assigns the desired eigenstructure and moreover* $A - BF = GJG^{-1}$.

Proof. The necessity follows directly from Lemma 3.1. For sufficiency, using (2.16), (2.11) and (2.17), we have

$$\begin{aligned} AW\Gamma &= AW_1\Phi + A[W_2, \ldots, W_s] \begin{bmatrix} \mathcal{I}_{2,1}\Phi J \\ \vdots \\ \mathcal{I}_{s,1}\Phi J^{s-1} \end{bmatrix} \\ &= AW_1\Phi + A[0, W_2; \ldots; 0, W_s; 0]\Gamma J \\ &= BR\Phi + AWN\Gamma J \\ &= BR\Phi + W\Gamma J - BY\Gamma J = BZ + W\Gamma J. \end{aligned}$$

Since Γ and W are nonsingular, we get

$$A - BZ(W\Gamma)^{-1} = W\Gamma J(W\Gamma)^{-1}$$

and thus $F = Z(W\Gamma)^{-1}$ is a feedback matrix which completes the task. $\square$

REMARK 2.

Note that $Z := [R, -Y] \begin{bmatrix} \Phi \\ \Gamma J \end{bmatrix} =: [R, -Y]\Psi$, and one can easily verify that $\Psi\Gamma^{-1}$ has a condensed form as the Luenberger like form (2.6). This fact indicates the relationship of the formula (3.10) to the formulas used in the well known assignment methods via canonical forms [37]. The following results follow from the special structure of Γ.

COROLLARY 3.3. *Consider the pole placement problem of Theorem 3.2, with* J *given as in (3.9). A necessary condition for the existence of* F *with* J *as the Jordan canonical form of* $A - BF$ *is that* Φ *is chosen so that* (J^H, Φ^H) *is controllable. A sufficient condition is that there exists* $\Psi \in \mathcal{C}^{m\times n}$ *so that* (J^H, Ψ^H) *is controllable and has the same indices* n_k *as* (A, B).

Proof. The necessary condition is obvious. For the sufficient condition observe that we can write $\tilde{B} = BT$ with T as in (2.6). Then $W = \tilde{W}H$, where

$$\begin{aligned} \tilde{W} &= [\tilde{B}, A\tilde{B}\mathcal{I}_{2,1}^H, \ldots, A^{s-1}\tilde{B}\mathcal{I}_{s,1}^H], \\ H &= \operatorname{diag}(\mathcal{I}_{2,1}, \ldots, \mathcal{I}_{s,1})[\hat{X}, \tilde{N}\hat{X}, \ldots, \tilde{N}^{s-1}\hat{X}]. \end{aligned}$$

Thus $\tilde{W}$ has a dual structure to Γ. Therefore $\Phi = \Psi\tilde{T}$ can be used to determine a feedback gain F, where $\tilde{T} \in \mathcal{C}^{m\times m}$ is nonsingular and is determined by computing the condensed from (2.6) for (J^H, Ψ^H). $\square$

Theorem 3.2 also leads to a characterization of the set of feedbacks that assign a desired Jordan structure.

COROLLARY 3.4. *The set of all feedbacks* F *that assign the Jordan structure in (3.9) is given by*

$$\{F = ZG^{-1} = (R\Phi - Y\Gamma J)(W\Gamma)^{-1} |\, \det\Gamma \neq 0,\ \Gamma \text{ as in (3.10)}\}. \tag{3.11}$$

REMARK 3. Note that we do not have to choose a matrix J in Jordan form in Theorem 3.2. In fact J can be chosen arbitrarily, since for an arbitrary nonsingular Q,

$$\Gamma Q = \begin{bmatrix} \Phi Q \\ \mathcal{I}_{2,1}\Phi Q(Q^{-1}JQ) \\ \vdots \\ \mathcal{I}_{s,1}\Phi Q(Q^{-1}JQ)^{s-1} \end{bmatrix} = \begin{bmatrix} \hat{\Phi} \\ \mathcal{I}_{2,1}\hat{\Phi}\hat{J} \\ \vdots \\ \mathcal{I}_{s,1}\hat{\Phi}\hat{J}^{s-1} \end{bmatrix},$$

where $\hat{\Phi} = \Phi Q$, $\hat{J} = Q^{-1}JQ$. In particular for a real problem we can choose J in real canonical form and also choose a real Φ.

REMARK 4. In the single-input case, i.e., $m = 1$, the Jordan form must be nondegenerate, see [22]. Hence for J in (3.9), we need $r_1 = \ldots = r_q = 1$. Let $\Phi = [\phi_1, \ldots, \phi_q]$ and $\phi_k = [\phi_{k,1}, \ldots, \phi_{k,p_k}] \in \mathcal{C}^{1\times p_k}$, let $\xi(\lambda) = \det(\lambda I_n - A)$, $\Xi(\lambda) = \text{adj}(\lambda I_n - A)$, as in Remark 1. Then we can easily verify that

$$G = W\Gamma = [G_1, \ldots, G_q]\,\mathrm{diag}(\hat{\Phi}_1, \ldots, \hat{\Phi}_q),$$

$$Z = -[Z_1, \ldots, Z_q]\,\mathrm{diag}(\hat{\Phi}_1, \ldots, \hat{\Phi}_q),$$

where

$$G_k = [\Xi(\lambda_k)B, \Xi^{(1)}(\lambda_k)B, \ldots, \Xi^{(p_k-1)}(\lambda_k)B], \tag{3.12}$$

$$Z_k = [\xi(\lambda_k), \xi^{(1)}(\lambda_k), \ldots, \xi^{(p_k-1)}(\lambda_k)], \tag{3.13}$$

$$\hat{\Phi}_k = \sum_{j=0}^{p_k-1} \phi_{k,j+1} N_{p_k}^j.$$

Here $\xi^{(k)}$ and $\Xi^{(k)}$ represent the k-th derivatives with respect to λ. Obviously we need $\hat{\Phi}_k$ nonsingular for $1 \le k \le q$, so in this case the formulas reduce to

$$G := [G_1, \ldots, G_q], \quad , F = -[Z_1, \ldots, Z_q]G^{-1},$$

with G_k, Z_k defined in (3.12) and (3.13).

Note that this is another variation of the formulas for the single-input case, see [22]. By using the properties of $\xi(\lambda)$ and $\Xi(\lambda)$, it is easy to rederive the formulas in [22] when $\lambda(A) \cap \mathcal{P} = \emptyset$.

Though it is well known that for an arbitrary pole set $\mathcal{P}$, if (A, B) is controllable then there always exists an F that assigns the elements of $\mathcal{P}$ as eigenvalues, it is not true that we can assign an arbitrary Jordan structure in $A-BF$ when there are multiple poles. This already follows from the single-input case. See also [22, 7, 30, 31, 4, 9]. We see from Theorem 3.2 that in order to have a desired Jordan structure, the existence of a nonsingular matrix Γ as in (3.10) is needed.

We will now discuss when we can obtain a diagonalizable $A - BF$. Note that in order to have a robust closed loop system, it is absolutely essential that the closed loop system is diagonalizable and has no multiple eigenvalues, since it is well known from the perturbation theory for eigenvalues [11, 28] that otherwise small perturbations may lead to large perturbations in the closed loop eigenvalues.

In the following we study necessary and sufficient conditions for the existence of a feedback that assigns for a given controllable matrix pair (A, B) and poles $\lambda_1, \ldots, \lambda_q$ with multiplicities $r_1, \ldots, r_q$ and a diagonal Jordan canonical form of the closed loop system

$$A - BF = G\,\mathrm{diag}(\lambda_1 I_{r_1}, \ldots, \lambda_q I_{r_q})G^{-1} =: G\Lambda G^{-1}. \tag{3.14}$$

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This problem has already been solved in [26, 19] using the theory of invariant polynomials. It is also discussed in [15], where necessary conditions are given even if (A, B) is uncontrollable.

Here we will give a different characterization in terms of the results of Theorem 3.2 and the multiplicites $r_1, \ldots, r_q$. In the proof we will also show a way to explicitely construct the eigenvector matrix G and the feedback gain F, provided they exist.

Notice that multiplication with $E_{\lambda,s}$ defined in Theorem 2.5 sets up a one to one mapping between $\mathcal{C}^m$ and the eigenspace of $A - BF$ associated with a pole λ. By (2.18) a vector

$$\phi := \begin{bmatrix} \hat{\phi} \\ 0 \end{bmatrix} \in \mathcal{C}^m, \hat{\phi} \in \mathcal{C}^{\pi_k}$$

uniquely determines an eigenvector as $g = W(I - \lambda N)^{-1} E\phi \in \mathcal{U}_{\lambda,k}$.

LEMMA 3.5. *Let (A, B) be controllable. Given arbitrary poles $\lambda_1, \ldots, \lambda_k$, and an integer l with $1 \le l \le s$. For each pole λ_i choose an arbitrary vector $g_i \in \mathcal{U}_{\lambda_i,l}$, where $\mathcal{U}_{\lambda_i,l}$ defined in (3.1) is the subspace of the nullspace of $[A - \lambda_i I, B]$. If $k > \sum_{i=1}^{l} d_i i$, then the vectors $g_1, \ldots, g_k$ must be linear dependent.*

Proof. Since $g_i \in \mathcal{U}_{\lambda_i,l}$, there exists a corresponding $\phi_i = \begin{bmatrix} \hat{\phi}_i \\ 0 \end{bmatrix}$, with $\hat{\phi}_i \in \mathcal{C}^{\pi_l}$ such that $g_i = U_{\lambda_i,s}\phi_i$. Let $\Phi_k := [\phi_1, \ldots, \phi_k]$, $\Lambda_k := \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$ and $\Gamma_k = \begin{bmatrix} \Phi_k \\ \mathcal{I}_{2,1}\Phi_k\Lambda_k \\ \vdots \\ \mathcal{I}_{s,1}\Phi_k\Lambda_k^{s-1} \end{bmatrix}$. By Lemma 3.1, $G_k = [g_1, \ldots, g_k] = W\Gamma_k$ and, since W is invertible, $\operatorname{rank} G_k = \operatorname{rank}\Gamma_k$. Applying an appropriate row permutation, Γ_k can be transformed to $\begin{bmatrix} \hat{\Gamma}_k \\ 0 \end{bmatrix}$, with $\hat{\Gamma}_k = \begin{bmatrix} \hat{\Phi}_{k,1} \\ \hat{\Phi}_{k,2}\Lambda_k \\ \vdots \\ \hat{\Phi}_{k,l}\Lambda_k^{l-1} \end{bmatrix}$ and where $\hat{\Phi}_{k,1} = [\hat{\phi}_1, \ldots, \hat{\phi}_k]$, $\hat{\Phi}_{k,i}$ is the bottom $(\pi_l - \pi_{i-1}) \times k$ submatrix of $\hat{\Phi}_{k,1}$. Because the number of rows of $\hat{\Gamma}_k$ is $\sum_{i=1}^{l}(\pi_l - \pi_{i-1}) = \sum_{i=1}^{l} d_i i$,

$$\operatorname{rank} G_k = \operatorname{rank}\Gamma_k = \operatorname{rank}\hat{\Gamma}_k \le \sum_{i=1}^{l} d_i i.$$

So $k > \sum_{i=1}^{l} d_i i$ implies that $g_1, \ldots, g_k$ are linear dependent. □

THEOREM 3.6. *Let (A, B) be controllable. Given poles $\lambda_1, \ldots, \lambda_q$ with multiplicities $r_1, \ldots, r_q$ satisfying $r_1 \ge r_2 \ge \cdots \ge r_q$. Then there exists a feedback matrix F so that $\Lambda(A - BF) = \{\lambda_1, \ldots, \lambda_q\}$ and $A - BF$ is diagonalizable if and only if*

$$\sum_{i=1}^{k} r_i \le \sum_{i=1}^{k} n_i, \ k = 1, \ldots, q. \tag{3.15}$$

Proof. To prove the necessity, suppose that a feedback matrix F and a nonsingular G exist, such that (3.14) holds. Partition $G := [G_1, \ldots, G_q]$, where $G_i \in \mathcal{C}^{n \times r_i}$ with $\operatorname{range} G_i \subseteq \mathcal{U}_{\lambda_i}$. We will prove (3.15) by induction.

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If $k = 1$, then from Theorem 2.5 we have that $\dim \mathcal{U}_{\lambda_1} = m = n_1$. Since $\operatorname{range} G_1 \subseteq \mathcal{U}_{\lambda_1}$, $\operatorname{rank} G_1 \leq n_1$. On the other hand, G nonsingular implies that $\operatorname{rank} G_1 = r_1$ and therefore $r_1 \leq n_1$.

Now suppose that (3.15) holds for k. If (3.15) would not hold for $k+1$ then by applying the induction hypothesis, we obtain $r_1 \geq \ldots \geq r_{k+1} > n_{k+1}$. Since G_i is of full column rank and by Theorem 2.5, $n_{k+1} = m - \pi_k = \dim \mathcal{U}_{\lambda_i} - \dim \mathcal{U}_{\lambda_i,k}$, it follows that $l_i := \dim(\operatorname{range} G_i \cap \mathcal{U}_{\lambda_i,k}) \geq r_i - n_{k+1}$, $i = 1, \ldots, k+1$. Let $g_{i,1}, \ldots, g_{i,l_i}$ be a basis of $\operatorname{range} G_i \cap \mathcal{U}_{\lambda_i,k}$. As

$$\sum_{i=1}^{k+1} l_i \geq \sum_{i=1}^{k+1} (r_i - n_{k+1}) > \sum_{i=1}^{k} (n_i - n_{k+1}) = \sum_{i=1}^{k} d_i i,$$

by Lemma 3.5, $g_{1,1}, \ldots, g_{1,l_1}, \ldots, g_{k+1,1}, \ldots, g_{k+1,l_{k+1}}$ are linear dependent. In other words, there exists a nonzero vector ν such that $[G_1, \ldots, G_{k+1}]\nu = 0$. Hence G is singular, which is a contradiction.

To prove sufficiency, using Theorem 3.2, we construct a matrix $\Phi \in \mathcal{C}^{m \times n}$ so that

$$\Gamma = \begin{bmatrix} \Phi \\ \mathcal{I}_{2,1}\Phi\Psi \\ \vdots \\ \mathcal{I}_{s,1}\Phi\Psi^{s-1} \end{bmatrix}$$

is nonsingular, where Ψ is diagonal and has the form $P\Lambda P^T$ with Λ is as in (3.14) and P a permutation matrix. Let

$$\Phi := \begin{array}{c} \\ d_1 \\ d_2 \\ \vdots \\ d_s \end{array} \!\!\begin{array}{c} \begin{array}{cccc} d_1 & 2d_2 & \ldots & sd_s \end{array} \\ \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} & \ldots & \Phi_{1,s} \\ & \Phi_{2,2} & \ldots & \Phi_{2,s} \\ & & \ddots & \vdots \\ & & & \Phi_{s,s} \end{bmatrix} \end{array}, \text{ with } \Phi_{i,i} = \begin{bmatrix} \phi_{1,1}^{(i)} & \ldots & \phi_{1,d_i}^{(i)} \\ & \ddots & \vdots \\ & & \phi_{d_i,d_i}^{(i)} \end{bmatrix}$$

and $\phi_{j,j}^{(i)} = \begin{bmatrix} \omega_1^{(i,j)}, \ldots, \omega_i^{(i,j)} \end{bmatrix} \in \mathcal{C}^{1 \times i}$ with $\omega_l^{(i,j)} \neq 0$ for all $i = 1, \ldots, s$, $j = 1, \ldots, d_i$, $l = 1, \ldots, i$. Partition Ψ accordingly as

$$\begin{array}{c} \\ d_1 \\ 2d_2 \\ \vdots \\ sd_s \end{array} \!\!\begin{array}{c} \begin{array}{cccc} d_1 & 2d_2 & \ldots & sd_s \end{array} \\ \begin{bmatrix} \Psi_1 & & & \\ & \Psi_2 & & \\ & & \ddots & \\ & & & \Psi_s \end{bmatrix} \end{array}, \text{ with } \Psi_i = \begin{bmatrix} \psi_{i,1} & & \\ & \ddots & \\ & & \psi_{i,d_i} \end{bmatrix}$$

and $\psi_{i,j} = \operatorname{diag}(\nu_1^{(i,j)}, \ldots, \nu_i^{(i,j)})$.

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Then we obtain

$$\Gamma = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} & \dots & \dots & \Phi_{1,s} \\ & \Phi_{2,2} & & & \Phi_{2,s} \\ & & \ddots & & \vdots \\ & & & \ddots & \Phi_{s,s} \\ & \Phi_{2,2}\Psi_2 & \dots & \dots & \Phi_{2,s}\Psi_s \\ & & \ddots & & \vdots \\ & & & & \Phi_{s,s}\Psi_s \\ & & & & \vdots \\ & & & \Phi_{s-1,s-1}\Psi_{s-1}^{s-2} & \Phi_{s-1,s}\Psi_s^{s-2} \\ & & & 0 & \Phi_{s,s}\Psi_s^{s-2} \\ & & & & \Phi_{s,s}\Psi_s^{s-1} \end{bmatrix}.$$

It follows from the form of $\Phi_{i,i}$ that, by applying a row permutation, Γ can be transformed to the form

$$\hat{\Gamma} = \begin{bmatrix} \hat{\Gamma}_1 & * & \dots & * \\ & \hat{\Gamma}_2 & & \vdots \\ & & \ddots & \vdots \\ & & & \hat{\Gamma}_s \end{bmatrix}, \quad \text{with } \hat{\Gamma}_i = \begin{bmatrix} \hat{\Gamma}_{1,1}^{(i)} & * & \dots & * \\ & \hat{\Gamma}_{2,2}^{(i)} & & \vdots \\ & & \ddots & \vdots \\ & & & \hat{\Gamma}_{d_i,d_i}^{(i)} \end{bmatrix}$$

and

$$\hat{\Gamma}_{j,j}^{(i)} = \begin{bmatrix} 1 & \dots & 1 \\ \nu_1^{(i,j)} & \dots & \nu_i^{(i,j)} \\ \vdots & & \vdots \\ (\nu_1^{(i,j)})^{i-1} & \dots & (\nu_i^{(i,j)})^{i-1} \end{bmatrix} \operatorname{diag}(\omega_1^{(i,j)}, \dots, \omega_i^{(i,j)}).$$

Since $\hat{\Gamma}$ is block upper triangular and since each $\hat{\Gamma}_{j,j}^{(i)}$ is a product of a nonsingular diagonal matrix and a Vandermonde matrix, which is nonsingular if $\nu_1^{(i,j)}, \dots, \nu_i^{(i,j)}$ are distinct, it follows that the matrix $\hat{\Gamma}$, or equivalently Γ, is nonsingular. So it remains to show that the $\nu_j^{(i,j)}$ can be chosen from the eigenvalues so that all the occuring Vandermonde matrices are nonsingular. It is easy to see that condition (3.15) guarantees this choice. □

4. Perturbation Theory.

In this section we consider how the feedback gain and the actual poles of the closed loop system change under small perturbations to the system matrices and the given poles. It is clear from the perturbation theory for the eigenvalue problem [28] that we need a diagonalizable closed loop system with distinct poles if we want that the closed loop system is insensitive to perturbations. The following result, which is a generalization of the perturbation result of Sun [29], also holds in the case of multiple poles if diagonalizable closed loop systems exist for some choice of feedback.

THEOREM 4.1. *Given a controllable matrix pair (A, B), and a set of poles $\mathcal{P} = \{\lambda_1, \dots, \lambda_n\}$. Consider a perturbed system $(\hat{A}, \hat{B})$ which is also controllable and a perturbed set of poles $\hat{\mathcal{P}} = \{\hat{\lambda}_1, \dots, \hat{\lambda}_n\}$. Set $\hat{A} - A =: \delta A$, $\hat{B} - B =: \delta B$ and $\hat{\lambda}_k - \lambda_k =: \delta\lambda_k$, $k = 1, \dots, n$. Suppose that both the pole placement problems with $A, B, \mathcal{P}$ and $\hat{A}, \hat{B}, \hat{\mathcal{P}}$ have solutions with a diagonalizable closed loop matrix. Set*

$$\epsilon := \|[\delta A, \delta B]\|. \tag{4.1}$$

and suppose that

$$\max_i \frac{\epsilon + |\delta\lambda_i|}{\sigma_n([A-\lambda_i I, B])} < \frac{3}{4}, \tag{4.2}$$

Then there exists a feedback gain $\hat{F} := F + \delta F$ *of* $(\hat{A}, \hat{B})$ *such that*

$$\|\delta F\| < \frac{5\sqrt{n}}{4}\kappa\sqrt{1+\|\hat{F}\|^2}\max_i\{\frac{\sqrt{1+(\|B^\dagger(A-\lambda_i I)\|)^2}\,(\epsilon+|\delta\lambda_i|)}{\sigma_n([A-\lambda_i I, B])}\}, \tag{4.3}$$

$\lambda(\hat{A}-\hat{B}\hat{F}) = \hat{\mathcal{P}}$ *and* $\hat{A}-\hat{B}\hat{F}$ *is diagonalizable.*

Moreover, for each eigenvalue μ_i *of the closed loop matrix* $A - B\hat{F}$*, (i.e., the perturbed feedback is used for the unperturbed system), there is a corresponding* $\lambda_i \in \mathcal{P}$ *such that*

$$|\mu_i - \lambda_i| < |\delta\lambda_i| + \epsilon\hat{\kappa}\sqrt{1+\|\hat{F}\|^2}. \tag{4.4}$$

Here κ*,* $\hat{\kappa}$ *are the scaled spectral condition numbers of* $A-BF$ *and* $\hat{A}-\hat{B}\hat{F}$*, respectively (cf. [8]),* $\sigma_n(A)$ *is the smallest singular value of* A*, and* $B^\dagger$ *is the Moore-Penrose pseudoinverse of* B.

Proof. Suppose that $A-BF = G\,\mathrm{diag}(\lambda_1,\ldots,\lambda_n)G^{-1}$. Let $G := [g_1,\ldots,g_n]$, $\|g_i\| = 1$, $i = 1,\ldots,n$. Let $Z = FG := [z_1,\ldots,z_n]$ and $w_i = \begin{bmatrix} g_i \\ -z_i \end{bmatrix}$, then

$$[A-\lambda_i I, B]w_i = 0. \tag{4.5}$$

Consider a singular value decomposition

$$[A-\lambda_i I, B] = U\begin{bmatrix} \Sigma & 0 \end{bmatrix}V^H, \text{ with } \quad \Sigma = \mathrm{diag}(\sigma_1,\ldots,\sigma_n),\ \sigma_1 \geq \ldots \geq \sigma_n. \tag{4.6}$$

The controllability of (A,B) implies that $\sigma_n \neq 0$. Set $V = [V_1, V_2]$ with $V_2 \in \mathcal{C}^{n\times m}$, then $\mathrm{range}\,V_2 = \mathrm{kernel}[A-\lambda_i I, B]$. So there exists a nonzero $\alpha \in \mathcal{C}^m$ such that $w_i = V_2\alpha$. Since we require $\|g_i\| = 1$, from (4.5) it follows that $\|z_i\| \leq \|B^\dagger(A-\lambda_i I)\|$ and thus

$$\|\alpha\| = \sqrt{\|g_i\|^2 + \|z_i\|^2} \leq \sqrt{1+\|B^\dagger(A-\lambda_i I)\|^2}. \tag{4.7}$$

Similarly for the perturbed problem there exists a matrix $\hat{V}_2 \in \mathcal{C}^{n\times m}$ with $\hat{V}_2^H\hat{V}_2 = I_m$, such that $[\hat{A}-\hat{\lambda}_i I, \hat{B}]\hat{V}_2 = 0$, or $[A-\lambda_i I, B]\hat{V}_2 = -[\delta A - \delta\lambda_i I, \delta B]\hat{V}_2$. Using (4.6) we have that $V_1^H\hat{V}_2 = -\Sigma^{-1}U^H[\delta A-\delta\lambda_i I, \delta B]\hat{V}_2$. Hence

$$\|V_1^H\hat{V}_2\| \leq (\epsilon + |\delta\lambda_i|)/\sigma_n =: \tau < \frac{3}{4}.$$

Performing a singular value decomposition $V_2^H\hat{V}_2 = Y_1\Sigma_2Y_2^H$, we obtain from $I_m = \hat{V}_2^H\hat{V}_2 = (V_1^H\hat{V}_2)^H(V_1^H\hat{V}_2) + (V_2^H\hat{V}_2)^H(V_2^H\hat{V}_2)$, that $\|I_m - \Sigma_2^2\| = \|V_1^H\hat{V}_2\|^2 \leq \tau^2$. Hence $\|I_m - \Sigma_2\| \leq \tau^2$.

Let $\hat{w}_i$ be chosen analogous to w_i but for the perturbed problem and assume that $\hat{w}_i = \hat{V}_2\hat{\alpha}$, with $\hat{\alpha} = Y_2Y_1^H\alpha$. Note that it may happen that for this choice of $\hat{\alpha}$ the related $\hat{G}$ is singular. We can overcome this difficulty as follows. By our hypothesis, a nonsingular $\tilde{G}$ always exists for the perturbed problem. Consider the matrix $\hat{G}(t) = \hat{G} + t\tilde{G}$. Since $\det\hat{G}(t) \neq 0$ for sufficiently large t and $\det\hat{G}(t)$ is a polynomial in t, it has at most n roots. So we can choose a nonsingular $\hat{G}(t)$ with $|t| > 0$ arbitrary small. This is equivalent to chosing a $\tilde{w}_i$ for each i,

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which tends to $\hat{w}_i$. Moreover, in this sense, the determined $\hat{F}$ makes $\hat{A}-\hat{B}\hat{F}$ diagonalizable. By (4.2), (4.7) we obtain

$$\begin{aligned}\|w_i-\hat{w}_i\| &= \|(V_2Y_1-\hat{V}_2Y_2)Y_1^H\alpha\| = \|V^H(V_2Y_1-\hat{V}_2Y_2)Y_1^H\alpha\| \\ &= \sqrt{\|V_1^H\hat{V}_2Y_2Y_1^H\alpha\|^2+\|(I_m-Y_1^HV_2^H\hat{V}_2Y_2)Y_1^H\alpha\|^2} \\ &\leq \|\alpha\|\sqrt{\|V_1^H\hat{V}_2\|^2+\|I_m-\Sigma_2\|^2} \leq \|\alpha\|\sqrt{\tau^2+\tau^4} \leq \frac{5}{4}\tau\|\alpha\|.\end{aligned}$$

Let $\hat{G}$, $\hat{Z}$ be constructed analogous to G, Z, then $F=ZG^{-1}$, $\hat{F}=F+\delta F=\hat{Z}\hat{G}^{-1}$. Therefore

$$\begin{aligned}\delta F &= \hat{Z}\hat{G}^{-1}-ZG^{-1} = (\hat{Z}-Z)G^{-1}+\hat{Z}(\hat{G}^{-1}-G^{-1}) \\ &= -[\hat{F},I]\begin{bmatrix}\hat{G}-G \\ -(\hat{Z}-Z)\end{bmatrix}G^{-1} \\ &= -[\hat{F},I]\begin{bmatrix}\hat{w}_1-w_1 & \ldots & \hat{w}_n-w_n\end{bmatrix}G^{-1}.\end{aligned}$$

By (4.7) and $\kappa:=\|G^{-1}\|\|G\|\geq\|G^{-1}\|$, we have

$$\begin{aligned}\|\delta F\| &\leq \sqrt{n}\kappa\sqrt{1+\|\hat{F}\|^2}\max_i\|w_i-\hat{w}_i\| \\ &\leq \frac{5\sqrt{n}}{4}\kappa\sqrt{1+\|\hat{F}\|^2}\max_i\{\frac{\sqrt{1+\|B^\dagger(A-\lambda_iI)\|^2}(\epsilon+|\delta\lambda_i|)}{\sigma_n([A-\lambda_iI,B])}\},\end{aligned}$$

which implies (4.3).

For (4.4), rewrite $A-B\hat{F}$ as $\hat{A}-\hat{B}\hat{F}-(\delta A-\delta B\hat{F})$. Since $\lambda(\hat{A}-\hat{B}\hat{F})=\hat{\mathcal{P}}$, by applying the Bauer-Fike Theorem, e.g. [11, pp. 342], for each eigenvalue μ_i of $A-B\hat{F}$ there exists a corresponding $\hat{\lambda}_i$, so that $|\mu_i-\hat{\lambda}_i|\leq\hat{\kappa}\|\delta A-\delta B\hat{F}\|\leq\epsilon\hat{\kappa}\sqrt{1+\|\hat{F}\|^2}$. Using $\hat{\lambda}_i=\lambda_i+\delta\lambda_i$, we obtain (4.4). □

Note that under additional mild restrictions on the perturbed matrices and poles we obtain a similar upper bound for $\|\delta F\|$ with $\|\hat{F}\|$ replaced by $\|F\|$. Such a bound for the single-input case was given in [22]. We prefer the given bound from the computational point of view, since $\hat{F}$ is the quantity that is computed.

In the given upper bounds the norm of F and the spectral condition number κ are related.

COROLLARY 4.2. *Under the hypotheses of Thereom 4.1 we have*

$$\|F\|\leq\sqrt{n}\kappa\max_i\|B^\dagger(A-\lambda_iI)\|. \tag{4.8}$$

Proof. Using (4.5) we obtain that $z_i=B^\dagger(A-\lambda_iI)g_i$. Since $\|g_i\|=1$, $\|Z\|\leq\sqrt{n}\max_i\|B^\dagger(A-\lambda_iI)\|$ and then $F=ZG^{-1}$ yields (4.8). □

Theorem 4.1 only gives upper bounds for the perturbations. This is the usual situation in most perturbation results. To complete the perturbation theory it would be nice to show that these bounds are tight and actually close to the exact perturbations. We will demonstrate the tightness of the bounds via a numerical example below. The main factor that contributes to the sensitivity of the feedback gain F and the poles of the the closed-loop system $A-B\hat{F}$ obtained with the perturbed feedback $\hat{F}$, is $\mathcal{S}:=\kappa\sqrt{1+\|F\|^2}$. In the bound for F there is an additional factor $d:=1/\min_i\sigma_n[A-\lambda_iI,B]$. This latter factor is closely related to the distance to uncontrollability

$$d_u(A,B)=\min_{\lambda\in\mathcal{C}}\sigma_n[A-\lambda I,B], \tag{4.9}$$

[10]. It is obvious if $d_u(A,B)$ is small then d can be very large and the problem to compute F is likely to be ill-conditioned. If $d_u(A,B)$ is large, then clearly d is small and then this factor plays a minor role in the perturbation bounds. The other dominating factor $\mathcal{S}$ is more difficult to analyze. In the single-input case it was discussed in [22, 23] how this factor is influenced by the choice of poles. It was observed that $\mathcal{S}$ is essentially given by the condition number of the Cauchy matrix $C = [\frac{1}{\nu_i - \lambda_j}]$, where the ν_i are the eigenvalues of A and the λ_i are the desired poles. Unfortunately this condition number is usually very large, in particular if the system dimension n is large. In [23] it was also discussed how the poles λ_j can be chosen to minimize $\mathcal{S}$. The situation in the multi-input case is much more complicated. Notice that (4.8) implies that $\mathcal{S}$ essentially behaves like κ^2. Furthermore we see from (2.16) and Theorem 3.2, with a diagonal matrix J, that

$$\begin{aligned} G &= K\tilde{X}\Gamma \\ &= [B, AB, \ldots, A^{s-1}B]\tilde{X}\,\mathrm{diag}(I, \mathcal{I}_{2,1}, \ldots \mathcal{I}_{s,1}) \begin{bmatrix} \Phi \\ \Phi J \\ \vdots \\ \Phi J^{s-1} \end{bmatrix}, \end{aligned}$$

where $K, \tilde{X}$ are as in (2.16). If as a special case

$$A = \mathrm{diag}(\nu_1, \ldots, \nu_n), \qquad J = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

then there exists a permutation matrix P such that

$$[B, AB, \ldots, A^{s-1}B]P = [\mathrm{diag}(b_{11}, \ldots, b_{n1})V_A, \ldots, \mathrm{diag}(b_{1m}, \ldots, b_{nm})V_A],$$

with the Vandermonde-like matrix

$$V_A = \begin{bmatrix} 1 & \nu_1 & \ldots & \nu_1^{s-1} \\ \vdots & \vdots & & \vdots \\ 1 & \nu_n & \ldots & \nu_n^{s-1} \end{bmatrix}.$$

We also obtain analogously that there exists a premutation matrix $\tilde{P}$, such that

$$P^T \begin{bmatrix} \Phi \\ \Phi J \\ \vdots \\ \Phi J^{s-1} \end{bmatrix} = [\mathrm{diag}(\phi_{11}, \ldots, \phi_{1n})V_J, \ldots, \mathrm{diag}(\phi_{m1}, \ldots, \phi_{mn})V_J]^T$$

with a Vandermonde-like matrix V_J formed from the λ_j. It is well known that such Vandermonde matrices are usually very ill-conditioned (see [13, Chapter 21] and the references therein), in particular if s is large.

There may be some fortunate circumstances by which the ill conditioning of the Vandermonde factors is cancelled out by the middle term or when forming the product, but in general this cannot be expected.

From the relationship $\frac{n}{m} \leq s \leq n - m + 1$, which follows from the staircase from, we see that for large n in order to have a small s and thus a reasonable conditioning of the Vandermonde matrices, we need that also m is large.

We see from this rough analysis that $\mathcal{S}$ depends critically on the choice of poles and we can expect that $\mathcal{S}$ is large if s is large. Thus, we can conclude that if s is large then the pole

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assignment problem will in general be ill-conditioned. It is not difficult, however, to contrive examples with good conditioning, by taking $\tilde{A}$ with well-conditioned eigenvector matrix (say a normal matrix) and then, choosing B and F of small norm, forming $A = \tilde{A} + BF$, see [2, 3]. But in general we can expect neither $\|F\|$ nor $\mathcal{S}$ to be small.

Another way to analyze the conditioning of G is obtained from

$$AG - BZ = GJ, \qquad Z = FG.$$

If again A and J are diagonal, and if $\nu_i \neq \lambda_j$, $i, j = 1, \ldots, n$ and $G = [g_{ij}]_{n \times n}$, then $g_{ij} = e_i^H BZe_j/(\nu_i - \lambda_j)$, where e_i is the ith column of I_m. So G is a generalized Cauchy matrix which usually has a large condition number.

Let us demonstrate the above analysis via an example.

EXAMPLE 1. Let $A = \mathrm{diag}(1, \ldots, 20)$, $\mathcal{P} = \{-1, \ldots, -20\}$ let B be formed from the first m columns of a random 20×20 orthogonal matrix.

The following results were obtained on a pentium-s PC with machine precision $eps = 2.22 \times 10^{-16}$, under Matlab Version 4.2. The MATLAB pole placement code of Miminis and Paige [24] was used to compute the feedback gain. We ran m from 1 to 20 and in each case we computed 20 times with 20 random updated matrices B. In Table 1 we list the geometric means (over the 20 experiments) of $\hat{\kappa}$, $\hat{F}$, bound, err, where bound=$eps\|[A, B]\|\hat{\kappa}\sqrt{1 + \|\hat{F}\|^2}$, and err=$\max_{1 \le i \le 20} |\mu_i - \lambda_i|$, with λ_i and the real parts of μ_i arranged in increased order. In the second column we list the average value for s taken over the 20 random tests for each m. (This value of s is actually the least possible value or generic value.) Note that for all 400 tests the values of $\min_i \sigma_n([A - \lambda_i I, B])$ varied from 2.0 to 2.24.

m	s	$\hat{\kappa}$	$\hat{F}$	Bound	Err
1	20	3.5×10^9	1.1×10^{14}	1.7×10^9	7.3×10^4
2	10	1.8×10^{11}	5.0×10^9	3.9×10^6	2.7×10^2
3	7	2.1×10^{10}	2.4×10^{10}	2.2×10^6	1.4×10^2
4	5	7.4×10^{11}	5.8×10^7	1.9×10^5	2.4×10^1
5	4	1.2×10^{14}	1.3×10^5	7.3×10^4	1.0×10^1
6	4	2.1×10^{14}	2.6×10^4	2.5×10^4	5.8
7	3	1.7×10^{14}	4.2×10^4	3.1×10^4	2.0
8	3	1.7×10^{14}	1.1×10^4	8.6×10^3	7.8×10^{-1}
9	3	2.4×10^{14}	9.0×10^3	9.8×10^3	6.6×10^{-1}
10	2	2.1×10^{14}	2.6×10^3	2.9×10^3	3.8×10^{-1}
11	2	1.8×10^{13}	7.9×10^2	6.5×10^1	1.0×10^{-4}
12	2	9.2×10^{12}	5.0×10^2	2.0×10^1	3.6×10^{-3}
13	2	5.7×10^{11}	4.5×10^2	1.1	1.5×10^{-4}
14	2	2.1×10^{11}	3.2×10^2	3.0×10^{-1}	6.7×10^{-5}
15	2	3.4×10^{10}	2.8×10^2	4.2×10^{-2}	1.3×10^{-5}
16	2	5.9×10^8	2.6×10^2	6.7×10^{-4}	3.0×10^{-7}
17	2	3.1×10^7	2.2×10^2	3.0×10^{-5}	1.6×10^{-8}
18	2	1.6×10^5	2.0×10^2	1.4×10^{-7}	1.0×10^{-10}
19	2	7.0×10^2	1.9×10^2	5.9×10^{-10}	9.9×10^{-13}
20	1	1.0	3.5×10^1	1.5×10^{-13}	2.6×10^{-14}

Table 1

In this example, if we consider the poles of the closed loop matrices, it makes sense to interpret the results of the numerical method only for $m \geq 8$, since only then the error is less

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than 1 . When m becomes larger, then the computed poles become more accurate and the bounds become tighter. Usually the bounds tend to overestimate the error by $1-4$ orders of magnitude, but they really reveal the relation of the conditioning of the closed loop matrices and the scale of the input. Notice that the bounds are valid regardless of the methods that is used. For each numerical method it may use some micro structures of the matrices. So the accuray of the computed results (feedback matrices or the poles of the closed loop matrices) may be improved.

5. Analysis of pole placement strategies. We see from the perturbation analysis that serious numerical difficulties may arise in the pole assignment problem. First of all, if s is large, then we can expect that the problem is ill-conditioned, regardless which strategy is used to resolve the freedom in F. But even if s is small, then not every strategy to fix the freedom in F will lead to a robust closed loop system. Clearly a minimization of $\|F\|$ as approached in [4, 16] or a local minimization as done in [36] will improve the upper bound in (4.3). Corollary 4.2, however, indicates that the minimization of κ as described in [15] may be a better choice, since it automatically also minimizes an upper bound for $\|F\|$. Note, however, that the computational cost is an order of magnitude larger, than in the methods that minimize $\|F\|$.

Certainly an optimization of $\mathcal{S} := \kappa\sqrt{1+\|F\|^2}$ would be even more promising, since then a smaller upper bound is optimized. Using the explicit characterization of F, it is actually possible to write down an explicit optimization problem for $\mathcal{S}$ in the form

$$\min_{\Phi} \mathcal{S} = \min_{\Phi}\{\|W\Gamma(\Phi)\|\|\Gamma(\Phi)^{-1}W^{-1}\|\sqrt{1+\|Z(\Phi)\Gamma(\Phi)^{-1}W^{-1}\|}\}, \tag{5.1}$$

where $Z(\Phi)$, $\Gamma(\Phi)$ are as in Theorem 3.2. For mn not too large, we can approach this minimization problem with standard optimization software and actually in practice one probably usually does not need the global minimum, but just one, where $\mathcal{S}$ is small enough to guarantee a small bound (4.3), which then can be actually computed and used as condition estimator.

6. Future research. Pole placement is often used as a substitute problem for the solution of another problem, like stabilization or damped stabilization, see, e.g., [12]. If this is the case, then also the poles $\{\lambda_1,\ldots,\lambda_n\}$ are free to vary in a given set $\Omega\subset\mathcal{C}$. For the single-input case, where we have no freedom in F, this problem was discussed in [23].

But for the substitute problem, $\mathcal{S}$ might not be the right measure to optimize, since one usually also wants that the poles are robustly bounded away from the boundary of Ω, e.g., are robustly stable. Then also the distance to the complement of Ω should be included in the measure. This topic is currently under investigation.

The analysis that we have given can also be used to study pole assignment via output feedback, i.e., the problem of determining a feedback $F\in\mathcal{C}^{m\times p}$, such that $A-BFC$ has a desired set of poles, where $C\in\mathcal{C}^{p\times n}$ describes an output equation of the form

$$y = Cx. \tag{6.1}$$

It is evident from Theorem 3.2 that a solution to the output feedback problem exists if and only if there exists a matrix $\Phi\in\mathcal{C}^{m\times n}$ so that Γ as in (3.10) is nonsingular and

$$FC = ZG^{-1} \tag{6.2}$$

with Z, G as in Theorem 3.2. This condition is equivalent to

$$\mathrm{range}(Z\Gamma^{-1})^H \subseteq \mathrm{range}(CW)^H. \tag{6.3}$$

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If (A^H, C^H) is also controllable, anagalous to Theorem 3.2, by considering the problem $A^H - C^H(BF)^H = G^{-H}\Gamma^H G^H$, there are $Z_c := R_c\Psi - Y_c\Gamma_c$ and $G^{-H} = W_c\Gamma_c$, so that

$$BF = (Z_c G^H)^H = W_c^{-H}\Gamma_c^{-H} Z_c^H. \tag{6.4}$$

Here W_c, R_c, Y_c and Γ_c are similar as in the state feedback case, but for (A^H, C^H), $\Psi \in \mathcal{C}^{p\times n}$ still has to be chosen. In this case the output feedback problem is solvable if and only if there exist Φ and Ψ so that

$$W\Gamma = W_c^{-H}\Gamma_c^{-H}, \text{ or } \Gamma\Gamma_c^H = (W_c^H W)^{-1}. \tag{6.5}$$

Note that in this case (6.2) and (6.4) are automatically satisfied from the structures of W and W_c. It is currently under investigation to obtain more explicit formulas for this problem.

Another important variant of the pole assignment problem is when not only poles but also some eigenvectors of the closed loop system are given, see [20].

7. Conclusion. We have continued the analysis of the pole placement problem in [22] for the multi-input case and we have derived explicit formulas for the feedback matrix and the closed loop eigenvector matrix as well as new perturbation results.

We observe a similar behaviour as in the single-input case, and we come to a similar conclusion, that we can expect the pole placement problem to be ill-conditioned if s the number of blocks in the staircase form of the matrix is large. This is definitely the case when n is large and m is small.

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