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ERROR BOUNDS FOR KRONROD EXTENSION OF GENERALIZATIONS OF MICCHELLI-RIVLIN QUADRATURE FORMULA FOR ANALYTIC FUNCTIONS*

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Dedicated to Walter Gautschi on the occasion of his 90th birthday

Abstract. We consider the Kronrod extension of generalizations of the Micchelli-Rivlin quadrature formula for the Fourier-Chebyshev coefficients with the highest algebraic degree of precision. For analytic functions, the remainder term of these quadrature formulas can be represented as a contour integral with a complex kernel. We study the kernel on elliptic contours with foci at the points ∓ 1 and the sum of semi-axes $\rho > 1$ for the mentioned quadrature formulas. We derive L^∞-error bounds and L^1-error bounds for these quadrature formulas. Finally, we obtain explicit bounds by expanding the remainder term. Numerical examples that compare these error bounds are included.

Key words. Kronrod extension of generalizations of the Micchelli-Rivlin quadrature formula, Chebyshev weight function of the first kind, error bound, remainder term for analytic functions, contour integral representation

AMS subject classifications. 65D32, 65D30, 41A55

1. Introduction.

The Gaussian quadrature formula with multiple nodes

$$\int_{-1}^{1} f(t)T_n(t)\,\frac{dt}{\sqrt{1-t^2}} = \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} A_{i,\nu} f^{(i)}(\tau_\nu) + E_{n,s}(f) \tag{1.1}$$

for calculating the Fourier-Chebyshev coefficients of an analytic function f, where $n, s \in \mathbb{N}$, with respect to the Chebyshev weight function of the first kind $\omega(t) = 1/\sqrt{1-t^2}$, was introduced in [1, p. 383] and then examined in more detail in [11]. Here T_n is the Chebyshev polynomial of the first kind of degree n, and the nodes τ_ν are its zeros. The quadrature rule has algebraic degree of precision $n(2s+1)-1$. The special case $s = 1$ of (1.1) represents the well-known Micchelli-Rivlin quadrature formula; see [7]. For more details on the theory of Gaussian quadrature formulas with simple and multiple nodes for calculating the Fourier-Chebyshev coefficients, see [1, 10, 11].

The authors of [11] considered the Kronrod extension of (1.1) in the form

$$\int_{-1}^{1} f(t)T_n(t)\,\frac{dt}{\sqrt{1-t^2}} = \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{j=1}^{n+1} C_j f(\hat{\tau}_j) + R_{n,s}(f), \tag{1.2}$$

which has algebraic degree of precision $2sn + 2n + 1$. The nodes τ_ν are the same as in (1.1), and the $\hat{\tau}_j$ are the zeros of the monic polynomial

$$F_{n+1}(t) = \frac{1}{2^n}(T_{n+1}(t) - T_{n-1}(t)) = \frac{1}{2^{n-1}}(t^2-1)U_{n-1}(t),$$

where U_{n-1} is the Chebyshev polynomial of the second kind of degree $n-1$. A nice and detailed survey of Kronrod rules in the last fifty years is provided by Notaris [12].

Error bounds for the Micchelli-Rivlin quadrature formula, and then for (1.1), for functions being analytic on confocal ellipses that contain the interval $[-1, 1]$ in the interior, have been

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considered in [13] and [15], respectively. In this paper, our aim is to do the same for the quadrature formula (1.2).

2. The remainder term of the Kronrod extension of the generalization of Micchelli-Rivlin quadrature formula for analytic functions. Let f be an analytic function on a domain D which contains the interval $[-1,1]$ in its interior, and let Γ be a simple closed curve in D surrounding $[-1,1]$. Assume that we know the values of the function f and its derivatives $f^{(i)}$, $i=1,2,\ldots,2s-1$, at the nodes $x_1,x_2,\ldots,x_n$ in the interval $[-1,1]$ and that we also know the values of the function f at the nodes $y_1,y_2,\ldots,y_{n+1}$ in the interval $[-1,1]$ satisfying

$$y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n < y_{n+1}.$$

Let

$$t_{2\nu} = x_\nu,\ \nu = 1,2,\ldots,n, \quad t_{2\nu-1} = y_\nu, \qquad \nu = 1,2,\ldots,n+1.$$

Using a result by Gončarov [4], the error in the Hermite interpolation of the function f can be written in the form

$$r_{n,s}(f;t) = f(t) - \sum_{\nu=1}^{2n+1}\sum_{i=0}^{2s_\nu-1} \ell_{i,\nu}(t) f^{(i)}(t_\nu) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)\Omega_{n,s}(t)}{(z-t)\Omega_{n,s}(z)}\,dz, \tag{2.1}$$

where the $\ell_{i,\nu}$ are the basis functions for Hermite interpolation, $s_\nu = s$ if $t_\nu \in \{x_1,\ldots,x_n\}$, $s_\nu = 1/2$ if $t_\nu \in \{y_1,\ldots,y_{n+1}\}$, and

$$\Omega_{n,s}(z) = \prod_{\nu=1}^{2n+1}(z-t_\nu)^{2s_\nu} = \prod_{\nu=1}^{n}(z-x_\nu)^{2s}\cdot\prod_{\nu=1}^{n+1}(z-y_\nu).$$

Let x_ν be the zeros of the Chebyshev polynomial T_n, i.e., $x_\nu = \tau_\nu$ for $\nu = 1,\ldots,n$. For $\nu = 1,\ldots,n-1$, let η_ν be the zeros of the Chebyshev polynomial U_{n-1}. Set $y_1 = \hat\tau_1 = -1$, $y_{\nu+1} = \hat\tau_{\nu+1} = \eta_\nu$, for $\nu = 1,2,\ldots,n-1$, and $y_{n+1} = \hat\tau_{n+1} = 1$. Thus, $y_\nu = \hat\tau_\nu$ are the zeros of the polynomial $(t^2-1)U_{n-1}$, which are the corresponding nodes in (1.2); see [11]. In this case, by multiplying (2.1) by $\omega(t)T_n(t)$, where $\omega(t) = 1/\sqrt{1-t^2}$, and integrating with respect to t over $(-1,1)$, we get a contour integral representation of the remainder term in (1.2):

$$R_{n,s}(f) = \frac{1}{2\pi i}\oint_\Gamma K_{n,s}(z) f(z)\,dz, \tag{2.2}$$

where the kernel is given by

$$K_{n,s}(z) = \frac{\rho_{n,s}(z)}{(1-z^2)T_n^{2s}(z)\,U_{n-1}(z)} \tag{2.3}$$

and

$$\rho_{n,s}(z) = \int_{-1}^{1} \frac{\omega(t)}{z-t}\,(1-t^2)T_n^{2s+1}(t)\,U_{n-1}(t)\,dt. \tag{2.4}$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., it holds that $|K_{n,s}(\bar z)| = |K_{n,s}(z)|$. Also, note that, due to the symmetry of the Jacobi Polynomials $T_n(z)$

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and $U_n(z)$, we have $|K_{n,s}(-\bar{z})| = |K_{n,s}(z)|$. Hence, the modulus of the kernel is symmetric with respect to both axes.

Applying Hölder's inequality to (2.2) yields

$$|R_{n,s}(f)| \leq \frac{1}{2\pi}\|K_{n,s}\|_r\|f\|_{r'}, \tag{2.5}$$

where $1 \leq r \leq +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r = \begin{cases} \left(\oint_\Gamma |f(z)|^r |dz|\right)^{1/r}, & 1 \leq r < +\infty, \\ \max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

In the case $r = +\infty$ and $r' = 1$, the estimate (2.5) reduces to

$$|R_{n,s}(f)| \leq \frac{1}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\oint_\Gamma |f(z)||dz|\right),$$

which leads to the error bound

$$|R_{n,s}(f)| \leq \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{2.6}$$

where $\ell(\Gamma)$ is the length of the contour Γ. We refer to it as the L^∞-error bound.

On the other hand, for $r = 1$ and $r' = +\infty$, the estimate (2.5) reduces to

$$|R_{n,s}(f)| \leq \frac{1}{2\pi}\left(\oint_\Gamma |K_{n,s}(z)||dz|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{2.7}$$

which is stronger than the estimate (2.6). We refer to (2.7) as the L^1-error bound.

In this paper, we take the contour Γ to be an ellipse $\mathcal{E}_\rho$ with foci at the points ± 1 and the sum of its semi-axes $\rho > 1$, i.e.,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid \; z = \frac{1}{2}\left(u + u^{-1}\right),\; 0 \leq \theta \leq 2\pi \right\}, \quad u = \rho\, e^{i\theta}.$$

The choice of the family of ellipses $\mathcal{E}_\rho$ as basic contours of integration is natural when dealing with analytic functions in a neighborhood of $[-1, 1]$ since they are the level curves of the Green's function for the region $\mathbb{C} \setminus [-1, 1]$ with a pole at infinity. When $\rho \to 1^+$, the ellipse $\mathcal{E}_\rho$ shrinks to the interval $[-1, 1]$, and when $\rho \to \infty$, the interior of $\mathcal{E}_\rho$ approaches the whole complex plane (which is useful when we deal with entire integrands such as those in Section 6).

In the sequel we present three types of bounds:

- In Section 3, we derive L^∞-error bounds by means of contour integration techniques, applying essentially a method introduced by Gautschi and Varga in [3]. Here, one has to calculate
$$\max_{z\in\mathcal{E}_\rho} |K_{n,s}(z)|.$$
- In Section 4, L^1-error bounds are derived, which are stronger than the L^∞-error bounds and require an estimate for
$$\frac{1}{2\pi}\oint_{\mathcal{E}_\rho} |K_{n,s}(z)||dz|.$$

- In Section 5, we derive the bounds resulting from an expansion of the remainder term $R_{n,s}(f)$ in the form
$$R_{n,s}(f) = \sum_{k=0}^{\infty} \alpha_{(2s+3)n+k} \varepsilon_{n,k}^{(s)},$$
following the method introduced by Hunter in [6] and then estimating $\alpha_{(2s+3)n+k}$ using the result of Elliott in [2].

Finally, numerical examples illustrating these three estimates are given in Section 6.

In the case of standard Gaussian quadrature formulas (with simple or multiple nodes), L^∞-error bounds are considered by Gautschi and Varga [3], Schira [18], Milovanović and Spalević [8], Pejčev and Spalević [14], and others. For the mentioned quadrature formulas, error bounds analogous to those in Sections 4 and 5, are considered by Hunter [6], Milovanović and Spalević [9], and others. Here we also mention the general method of estimating the error in Gauss-Turán quadrature formulas for functions analytic inside ellipses proposed by Spalević [16].

3. L^∞-error bounds. In order to find an explicit formula for the kernel (2.3), we determine the integral (2.4). Substituting $t = \cos\theta$ into (2.4) yields

$$\rho_{n,s}(z) = \int_0^\pi \frac{\sin^2\theta}{z-\cos\theta} (\cos n\theta)^{2s+1} \frac{\sin n\theta}{\sin\theta} d\theta.$$

We use formula 1.320.7 in [5] for $(\cos n\theta)^{2s+1}$ to obtain

$$\rho_{n,s}(z) = \frac{1}{2^{2s}} \int_0^\pi \frac{\sum_{k=0}^{s} \binom{2s+1}{k} \cos(2s-2k+1)n\theta \sin n\theta \sin\theta}{z-\cos\theta} d\theta,$$

which is transformed into

$$\rho_{n,s}(z) = \frac{1}{2^{2s+2}} \int_0^\pi \frac{\sum_{k=-1}^{s-1} \left(\binom{2s+1}{k+1} - \binom{2s+1}{k}\right) \cos((2s-2k)n-1)\theta}{z-\cos\theta} d\theta$$
$$- \frac{1}{2^{2s+2}} \int_0^\pi \frac{\sum_{k=-1}^{s-1} \left(\binom{2s+1}{k+1} - \binom{2s+1}{k}\right) \cos((2s-2k)n+1)\theta}{z-\cos\theta} d\theta,$$

where $\binom{2s+1}{k} := 0$ for $k < 0$. Using formula 3.613.1 in [5] (see also [3, p. 1176]), we obtain

$$\rho_{n,s}(z) = \frac{\pi}{2^{2s+2}} \frac{\sum_{k=-1}^{s-1} \left(\binom{2s+1}{k+1} - \binom{2s+1}{k}\right) \left(v^{(2s-2k)n-1} - v^{(2s-2k)n+1}\right)}{\sqrt{z^2-1}},$$

where $v = z - \sqrt{z^2-1}$.

Substituting $z = \frac{1}{2}\left(u + u^{-1}\right)$ into the above expression (where $u = 1/v$) yields

$$\rho_{n,s}(z) = \frac{\pi}{2^{2s+1}} \sum_{k=-1}^{s-1} \left(\binom{2s+1}{k+1} - \binom{2s+1}{k} \right) u^{-(2s-2k)n}. \tag{3.1}$$

Finally, since $T_n(z) = (u^n + u^{-n})/2$ and $U_{n-1}(z) = (u^n - u^{-n})/(u - u^{-1})$ (see, e.g., [3]), from (2.3) and (3.1), we get

$$K_{n,s}(z) = 2\pi \frac{\sum_{k=0}^{s} \left(\binom{2s+1}{s-k-1} - \binom{2s+1}{s-k}\right) u^{-2kn-2n}}{(u-u^{-1})(u^n+u^{-n})^{2s}(u^n-u^{-n})}. \tag{3.2}$$

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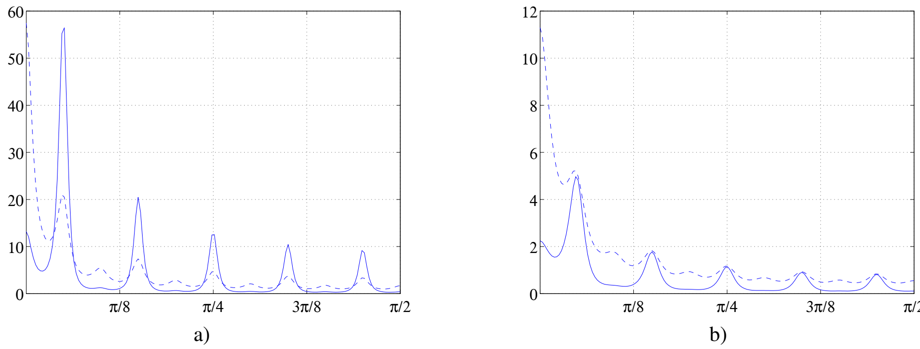

FIG. 3.1. *The functions* $\theta \mapsto |K_{10,s}(z)|$ *for* $s = 1$ *(dashed line) and* $s = 2$ *(solid line) for a)* $\rho = 1.03$*, and b)* $\rho = 1.05$*.*

Let $u = \rho e^{i\theta}$ and $C_{s,k} = \binom{2s+1}{s-k-1} - \binom{2s+1}{s-k}$. In order to find the modulus of the kernel, set

$$\begin{aligned} a &= (\rho^{2sn+2n})^2 \left| \sum_{k=0}^{s} C_{s,k} u^{-2kn-2n} \right|^2 \\ &= \left(\sum_{k=0}^{s} C_{s,k} \rho^{2sn-2kn} \cos(2kn+2n)\theta \right)^2 + \left(\sum_{k=0}^{s} C_{s,k} \rho^{2sn-2kn} \sin(2kn+2n)\theta \right)^2, \\ b &= \rho^2 |u - u^{-1}|^2 = \rho^4 - 2\rho^2 \cos 2\theta + 1, \\ c &= \rho^{2n} |u^n - u^{-n}|^2 = \rho^{4n} - 2\rho^{2n} \cos 2n\theta + 1, \\ d &= \rho^{2n} |u^n + u^{-n}|^2 = \rho^{4n} + 2\rho^{2n} \cos 2n\theta + 1. \end{aligned}$$

Then

$$|K_{n,s}(z)|^2 = \frac{4\pi^2}{\rho^{2n-2}} \frac{a}{bcd^{2s}}.$$

The graphs of the functions $\theta \mapsto |K_{n,s}(z)|$ for certain values of n, s, and ρ are displayed in Figure 3.1. Since the modulus of the kernel is symmetric with respect to both axes, it suffices to consider the interval $[0, \pi/2]$. We can now state the following result.

THEOREM 3.1. *For each* $n \in \mathbb{N}$*,* $n > 1$*, and each* $s \in \mathbb{N}$*, there exists* $\rho_0 = \rho_0(n, s)$ *such that*

$$\max_{z \in \mathcal{E}_\rho} |K_{n,s}(z)| = \left| K_{n,s}\left(\frac{1}{2}(\rho + \rho^{-1}) \right) \right|,$$

for each $\rho > \rho_0$.

Proof. We have to show that the inequality $\dfrac{a}{bcd^{2s}} \leqslant \dfrac{A}{BCD^{2s}}$ holds, i.e., that for each ρ greater than some $\rho_0 = \rho_0(n, s)$, it holds that $I = aBCD^{2s} - Abcd^{2s} \leqslant 0$, where A, B, C, D denote the values of a, b, c, d for $\theta = 0$.

The expression $I = I(\rho)$ is a polynomial of degree $12sn + 4n + 2$ with leading coefficient $2C_{s,0}^2(\cos 2\theta - 1)$, which is negative for $\theta \in (0, \pi/2]$. Therefore $I(\rho)$ is negative for sufficiently large ρ. ∎

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For $n=1$ the corresponding result is slightly different.

THEOREM 3.2. *Let* $n=1$.

a) For $s=1$*, there exists* ρ_0 *such that*

$$\max_{z\in\mathcal{E}_\rho}|K_{1,1}(z)| = \left|K_{1,1}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|,$$

for each $\rho>\rho_0$.

b) For $s>1$*, there exists* $\rho_0=\rho_0(s)$ *such that*

$$\max_{z\in\mathcal{E}_\rho}|K_{1,s}(z)| = \left|K_{1,s}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right|,$$

for each $\rho>\rho_0$.

Proof. For $n=1$ we have

$$a = C_{s,0}^2\rho^{4s} + 2C_{s,0}C_{s,1}\rho^{4s-2}\cos 2\theta + \cdots,$$
$$b = c = \rho^4 - 2\rho^2\cos 2\theta + 1, \qquad d = \rho^4 + 2\rho^2\cos 2\theta + 1.$$

Let A,B,C,D be the values of a,b,c,d for $\theta=0$, i.e.,

$$A = C_{s,0}^2\rho^{4s} + 2C_{s,0}C_{s,1}\rho^{4s-2} + \cdots,$$
$$B = C = \rho^4 - 2\rho^2 + 1, \qquad D = \rho^4 + 2\rho^2 + 1.$$

Then we have

$$aBCD^{2s} = C_{s,0}^2\rho^{12s+8} + (C_{s,0}^2(4s-4) + 2C_{s,0}C_{s,1}\cos 2\theta)\rho^{12s+6} + \cdots$$

and

$$Abcd^{2s} = C_{s,0}^2\rho^{12s+8} + (C_{s,0}^2(4s-4)\cos 2\theta + 2C_{s,0}C_{s,1})\rho^{12s+6} + \cdots,$$

where $C_{s,0} = -\frac{2}{s}\binom{2s+1}{s-1}$, $C_{s,1} = -\frac{4}{s+3}\binom{2s+1}{s-1}$, and $C_{s,1}/C_{s,0} = \frac{2s}{s+3}$. The coefficient in the expression $aBCD^{2s} - Abcd^{2s}$ at ρ^{12s+6} equals

$$(C_{s,0}^2(4s-4) - 2C_{s,0}C_{s,1})(1-\cos 2\theta) = 4C_{s,0}^2\frac{s^2+s-3}{s+3}(1-\cos 2\theta),$$

which is negative for each $\theta\in(0,\pi/2]$ and $s=1$.

Now we consider the values $\bar{A},\bar{B},\bar{C},\bar{D}$ of a,b,c,d for $\theta=\pi/2$, i.e.,

$$\bar{A} = C_{s,0}^2\rho^{4s} - 2C_{s,0}C_{s,1}\rho^{4s-2} + \cdots,$$
$$\bar{B} = \bar{C} = \rho^4 + 2\rho^2 + 1, \qquad \bar{D} = \rho^4 - 2\rho^2 + 1.$$

The leading coefficient of the expression $I = a\bar{B}\bar{C}\bar{D}^{2s} - \bar{A}bcd^{2s}$, similarly to the previous case, is

$$-4C_{s,0}^2\frac{s^2+s-3}{s+3}(1+\cos 2\theta),$$

which is negative for each $\theta\in[0,\pi/2)$ and $s>1$. $\square$

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4. L^1-error bounds. In order to derive an L^1-error bound, we use (2.7) and study the expression

$$L_{n,s}(\mathcal{E}_\rho) = \frac{1}{2\pi} \oint_{\mathcal{E}_\rho} |K_{n,s}(z)||dz|, \tag{4.1}$$

where the kernel $K_{n,s}(z)$ is given by (3.2).

Let $z = \frac{1}{2}(u + u^{-1})$, where $u = \rho e^{i\theta}$. Denoting $a_j = \frac{1}{2}(\rho^j + \rho^{-j})$, for $j \in \mathbb{N}$, we obtain

$$|u^n \pm u^{-n}|^2 = 2(a_{2n} \pm \cos 2n\theta).$$

Substituting (3.2), the previous equations, and $|dz| = 2^{-\frac{1}{2}}\sqrt{a_2 - \cos 2\theta}\,d\theta$ (see [6]) into (4.1), we get

$$L_{n,s}(\mathcal{E}_\rho) = \frac{1}{2^{s+\frac{1}{2}}} \int_0^\pi \frac{\left|\sum_{k=0}^s \left(\binom{2s+1}{k-1} - \binom{2s+1}{k}\right) u^{-2sn+2kn-2n}\right|}{\sqrt{a_{2n} - \cos 2n\theta}(a_{2n} + \cos 2n\theta)^s} d\theta. \tag{4.2}$$

Let $D_{s,k} = \binom{2s+1}{k-1} - \binom{2s+1}{k}$. The squared modulus of the sum in (4.2) equals (cf. [9])

$$\begin{aligned}
\left|\sum_{k=0}^s D_{s,k} u^{-2sn+2kn-2n}\right|^2 &= \left|u^{-2sn-2n} \sum_{k=0}^s D_{s,k} u^{2kn}\right|^2 \\
&= \rho^{-4sn-4n} \left(\sum_{l=0}^s D_{s,l} \rho^{2ln} e^{2iln\theta}\right) \left(\sum_{j=0}^s D_{s,j} \rho^{2jn} e^{-2ijn\theta}\right) \\
&= \rho^{-4sn-4n} \sum_{l,j=0}^s D_{s,l} D_{s,j} \rho^{2(l+j)n} e^{2i(l-j)n\theta} \\
&= \rho^{-4sn-4n} \sum_{k=0}^s A_k \cos 2kn\theta,
\end{aligned}$$

where

$$A_k = \sum_{\substack{|l-j|=k \\ l,j=0,\ldots,s}} D_{s,l} D_{s,j} \rho^{2(l+j)n},$$

i.e.,

$$A_0 = \sum_{j=0}^s D_{s,j}^2 \rho^{4jn}, \qquad A_k = 2\sum_{j=0}^{s-k} D_{s,j} D_{s,j+k} \rho^{4jn+2kn}, \quad k = 1, 2, \ldots, s. \tag{4.3}$$

Hence, from (4.2) we get

$$L_{n,s}(\mathcal{E}_\rho) = \frac{1}{2^{s+\frac{1}{2}} \rho^{2sn+2n}} \int_0^\pi \sqrt{\frac{\sum_{k=0}^s A_k \cos 2kn\theta}{(a_{2n} - \cos 2n\theta)(a_{2n} + \cos 2n\theta)^{2s}}} d\theta.$$

Since the integrand is periodic, this expression reduces to

$$L_{n,s}(\mathcal{E}_\rho) = \frac{1}{2^{s+\frac{1}{2}} \rho^{2sn+2n}} \int_0^\pi \sqrt{\frac{\sum_{k=0}^s A_k \cos k\theta}{(a_{2n} - \cos \theta)(a_{2n} + \cos \theta)^{2s}}} d\theta. \tag{4.4}$$

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Using [5, eq. 3.616.7], one finds

$$\int_0^{\pi} \frac{\cos k\theta}{(a_{2n}+\cos\theta)^{2s}}d\theta = \frac{2^{2s}\pi(-1)^k\rho^{2(2s-k)n}}{(\rho^{4n}-1)^{4s-1}}\sum_{i=0}^{2s-1}E_{s,i}^k(\rho^{4n}-1)^i, \tag{4.5}$$

where

$$E_{s,i}^k = \binom{2s+k-1}{i}\binom{4s-i-2}{2s-1}.$$

We are now ready to prove the main result.

THEOREM 4.1. *For the expression $L_{n,s}(\mathcal{E}_\rho)$ given by (4.1), it holds that*

$$L_{n,s}(\mathcal{E}_\rho) \leqslant \frac{\pi\sqrt{Q_s(\rho^{4n})}}{\rho^n(\rho^{4n}-1)^{2s}}, \tag{4.6}$$

where

$$Q_s(\rho^{4n}) = 2\sum_{k=0}^{s}{}'(-1)^k\left(\sum_{j=0}^{s-k}D_{s,j}D_{s,j+k}\rho^{4jn}\right)\left(\sum_{i=0}^{2s-1}E_{s,i}^k(\rho^{4n}-1)^i\right), \tag{4.7}$$

$$D_{s,j} = \binom{2s+1}{j-1}-\binom{2s+1}{j}, \qquad E_{s,i}^k = \binom{2s+k-1}{i}\binom{4s-i-2}{2s-1}.$$

Proof. Upon applying Cauchy's inequality to (4.4), we obtain

$$L_{n,s}(\mathcal{E}_\rho) \leqslant \frac{1}{2^{s+\frac{1}{2}}\rho^{2sn+2n}}\sqrt{\int_0^{\pi}\frac{d\theta}{a_{2n}-\cos\theta}}\sqrt{\int_0^{\pi}\frac{\sum_{k=0}^{s}A_k\cos k\theta}{(a_{2n}+\cos\theta)^{2s}}d\theta}, \tag{4.8}$$

where the coefficients A_k, for $k=0,1,\ldots,s$, are defined by (4.3). One finds

$$\int_0^{\pi}\frac{d\theta}{a_{2n}-\cos\theta} = \frac{\pi}{\sqrt{a_{2n}^2-1}} = \frac{2\pi\rho^{2n}}{\rho^{4n}-1},$$

and, by (4.3) and (4.5),

$$\begin{aligned}
&\int_0^{\pi}\frac{\sum_{k=0}^{s}A_k\cos k\theta}{(a_{2n}+\cos\theta)^{2s}}d\theta \\
&\quad = \sum_{k=0}^{s}{}'\left(2\sum_{j=0}^{s-k}D_{s,j}D_{s,j+k}\rho^{4jn+2kn}\right)\left(\frac{2^{2s}\pi(-1)^k\rho^{2(2s-k)n}}{(\rho^{4n}-1)^{4s-1}}\sum_{i=0}^{2s-1}E_{s,i}^k(\rho^{4n}-1)^i\right) \\
&\quad = \frac{2^{2s}\pi\rho^{4ns}}{(\rho^{4n}-1)^{4s-1}}Q_s(\rho^{4n}),
\end{aligned}$$

where $Q_s(\rho^{4n})$ is given by (4.7). Thus, (4.8) reduces to (4.6). ∎

REMARK 4.2. Let $x=\rho^{4n}$. The first few polynomials $Q_s(x)$ in (4.7) are

$$\begin{aligned}
Q_1 &= 1-3x+4x^2, \\
Q_2 &= 1-7x+22x^2-42x^3+81x^4+25x^5, \\
Q_3 &= 1-11x+56x^2-176x^3+385x^4-431x^5+3536x^6+2744x^7+196x^8.
\end{aligned}$$

Note that $\deg Q_s = 3s-1$.

5. Error bounds based on an expansion of the remainder term. If f is an analytic function in the interior of $\mathcal{E}_\rho$, then it can be expanded as

$$f(z) = {\sum_{k=0}^{\infty}}' \alpha_k T_k(z), \tag{5.1}$$

where

$$\alpha_k = \frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2} f(t) T_k(t) dt.$$

This series converges for all z in the interior of $\mathcal{E}_\rho$. The prime symbol on the summation sign means that the first term of the sum is to be halved.

For the expansion of the kernel we rewrite (3.2) as

$$K_{n,s}(z) = 2\pi \frac{\sum_{k=0}^{s} \left(\binom{2s+1}{s-k-1} - \binom{2s+1}{s-k} \right) u^{-2kn-2n}}{u(1-u^{-2}) u^{2sn} (1+u^{-2n})^{2s} u^n (1-u^{-2n})}. \tag{5.2}$$

Thus, for $|u^{-2n}| < 1$, we have

$$(1+u^{-2n})^{-2s} = \sum_{j=0}^{\infty} u^{-2jn} \binom{-2s}{j} = \sum_{j=0}^{\infty} (-1)^j \binom{2s-1+j}{2s-1} u^{-2jn}, \tag{5.3}$$

whereas the other term is expanded as

$$\frac{1}{(1-u^{-2})(1-u^{-2n})} = \sum_{i=0}^{\infty} u^{-2i} \sum_{j=0}^{\infty} u^{-2jn} = \sum_{i=0}^{\infty} \left(1 + \left[\frac{i}{n} \right] \right) u^{-2i}. \tag{5.4}$$

Here $[x]$ denotes the integer part of a real number x. From (5.2), (5.3), and (5.4), we get

$$K_{n,s}(z) = 2\pi \sum_{k=0}^{s} \sum_{i,j=0}^{\infty} C_{s,k} (-1)^j \binom{2s-1+j}{2s-1} \left(1 + \left[\frac{i}{n} \right] \right) u^p,$$

where

$$C_{s,k} = \binom{2s+1}{s-k-1} - \binom{2s+1}{s-k}$$

and $p = -(2s+3)n - 2((j+k)n+i) - 1$. The coefficient of $u^{-(2s+3)n-2(an+b)-1}$ with $0 \leqslant a$ and $0 \leqslant b \leqslant n-1$ in the above expression equals

$$\omega_{n,2an+2b}^{(s)} = 2\pi \sum_{k=0}^{s} \sum_{j=0}^{a-k} C_{s,k} (-1)^j \binom{2s-1+j}{2s-1} (a-k-j+1), \tag{5.5}$$

so we obtain

$$K_{n,s}(z) = \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} \omega_{n,2jn+2i}^{(s)} u^{-(2s+3)n-2jn-2i-1}. \tag{5.6}$$

THEOREM 5.1. *The remainder term $R_{n,s}(f)$ can be represented in the form*

$$R_{n,s}(f) = \sum_{k=0}^{\infty} \alpha_{(2s+3)n+k} \varepsilon_{n,k}^{(s)},$$

where the coefficients $\varepsilon_{n,k}^{(s)}$ are independent of f. Furthermore, $\varepsilon_{n,2j+1}^{(s)} = 0$ for $j = 0, 1, \ldots$

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Proof. Substituting (5.1) and (5.6) in (2.2) gives

$$R_{n,s}(f)=\frac{1}{2\pi i}\oint_{\mathcal{E}_\rho}\left(\sum_{k=0}^{\infty}{}'\alpha_k T_k(z)\sum_{j=0}^{\infty}\sum_{i=0}^{n-1}\omega_{n,2jn+2i}^{(s)}u^{-(2s+3)n-2jn-2i-1}\right)dz.$$

According to [6, Lemma 5], this reduces to

$$R_{n,s}(f)=\sum_{j=0}^{\infty}\sum_{i=0}^{n-1}\alpha_{(2s+3)n+2jn+2i}\varepsilon_{n,2jn+2i}^{(s)}$$

with

$$\varepsilon_{n,0}^{(s)}=\frac{1}{4}\omega_{n,0}^{(s)},\qquad \varepsilon_{n,1}^{(s)}=\frac{1}{4}\omega_{n,1}^{(s)},\qquad \varepsilon_{n,k}^{(s)}=\frac{1}{4}\left(\omega_{n,k}^{(s)}-\omega_{n,k-2}^{(s)}\right),\quad k=2,3,\dots \tag{5.7}$$

Obviously, $\varepsilon_{n,2j+1}^{(s)}=0$, $j=0,1,\dots$ Finally, from (5.5) and (5.7) we find

$$\varepsilon_{n,k}^{(s)}=\begin{cases}\dfrac{\pi}{2}\displaystyle\sum_{l=0}^{s}\sum_{i=0}^{j-l}\left(\binom{2s+1}{s-l-1}-\binom{2s+1}{s-l}\right)(-1)^i\binom{2s-1+i}{2s-1}, & k=2jn,\\ 0, & k\neq 2jn,\end{cases} \tag{5.8}$$

for $j=0,1,\dots$ ☐

Since

$$|R_{n,s}(f)|\leqslant\sum_{k=0}^{\infty}|\alpha_{(2s+3)n+k}||\varepsilon_{n,k}^{(s)}|, \tag{5.9}$$

in order to obtain explicit error estimates for general s, we need to know the sign of $\varepsilon_{n,k}^{(s)}$. This requires some computation. We first note that $C_{s,l}=0$ for $l>s$ and

$$\sum_{l=0}^{p}C_{s,l}=\binom{2s+1}{s-p-1}-\binom{2s+1}{s}.$$

Thus, the sum in (5.8) can be rewritten as

$$\varepsilon_{n,k}^{(s)}=\frac{\pi}{2}\sum_{i=0}^{j}\sum_{l=0}^{j-i}C_{s,l}(-1)^i\binom{2s-1+i}{2s-1}=(-1)^{j+1}\frac{\pi}{2}(S_1-S_2), \tag{5.10}$$

where

$$S_1=\binom{2s+1}{s}\sum_{i=0}^{j}(-1)^{j-i}\binom{2s-1+i}{2s-1}$$

and

$$S_2=\sum_{i=j-s+1}^{j}(-1)^{j-i}\binom{2s-1+i}{2s-1}\binom{2s+1}{s-j+i-1}.$$

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The sum S_2 can be calculated using [15, Lemma 3], yielding

$$S_2 = \frac{2js + 2s^2 + 3s + 1}{(j+s+1)(j+s+2)} \binom{2s}{s-1} \binom{2s+j}{j}. \tag{5.11}$$

However, we could not determine the sum S_1 explicitly. Since we only need the sign of $S_1 - S_2$, it will be enough to find sufficiently accurate bounds for S_1. An upper bound for S_1 is provided by the following statement.

LEMMA 5.2. *For each $j \in \mathbb{N}_0$ and $s \in \mathbb{N},\ s > 1$, we have*

$$\frac{1}{\binom{2s+1}{s}} S_1 = \sum_{i=0}^{j} (-1)^{j-i} \binom{2s-1+i}{2s-1} \leqslant \frac{2s-1+j}{2s-1+2j} \binom{2s-1+j}{2s-1}. \tag{5.12}$$

Proof. We use induction on j. For $j = 0$ and $j = 1$ the inequality (5.12) trivially holds, as it is equivalent to

$$1 \leqslant 1 \quad \text{and} \quad -1 + 2s \leqslant \frac{2s}{2s+1} 2s,$$

respectively.

Suppose that (5.12) holds for some $j \in \mathbb{N}_0$. We shall prove that it also holds for $j+2$, i.e.,

$$\sum_{i=0}^{j+2} (-1)^{j-i} \binom{2s-1+i}{2s-1} \leqslant \frac{2s+1+j}{2s+3+2j} \binom{2s+1+j}{2s-1}.$$

Since

$$\begin{aligned}\sum_{i=0}^{j+2} (-1)^{j-i} \binom{2s+1+i}{2s-1} &\leqslant \frac{2s-1+j}{2s-1} \binom{2s-1+j}{2s-1} - \binom{2s+j}{2s-1} + \binom{2s+1+j}{2s-1} \\ &= \binom{2s+1+j}{2s-1} \left(\frac{2s-1+j}{2s-1+2j} \frac{(j+1)(j+2)}{(2s+j)(2s+1+j)} - \frac{j+2}{2s+1+j} + 1 \right),\end{aligned}$$

it suffices to verify that

$$\frac{2s-1+j}{2s-1+2j} \frac{(j+1)(j+2)}{(2s+j)(2s+1+j)} - \frac{j+2}{2s+1+j} + 1 \leqslant \frac{2s+1+j}{2s+3+2j},$$

but this can be simplified to

$$\frac{(j+2)(4(s-2)s+3)}{(j+2s)(j+2s+1)(2j+2s-1)(2j+2s+3)} \geqslant 0,$$

which is evident for $s > 1$. ▯

We shall use this result to derive a lower bound for S_1.

LEMMA 5.3. *For each $j \in \mathbb{N}_0$ and $s \in \mathbb{N},\ s > 1$, we have*

$$S_1 \geqslant \frac{(2s-3+j)(2s-1+j)+j}{(2s+2j-3)(2s+j-1)} \binom{2s-1+j}{2s-1} \binom{2s+1}{s}. \tag{5.13}$$

Proof. Lemma 5.2 for $j-1$ instead of j gives

$$\sum_{i=0}^{j-1} (-1)^{j-1-i} \binom{2s-1+i}{2s-1} \leqslant \frac{2s-2+j}{2s-3+2j} \binom{2s-2+j}{2s-1},$$

which implies that

$$S_1 = \binom{2s+1}{s}\left(\binom{2s-1+j}{2s-1} - \sum_{i=0}^{j-1}(-1)^{j-1-i}\binom{2s-1+i}{2s-1}\right)$$
$$\geqslant \binom{2s+1}{s}\left(\binom{2s-1+j}{2s-1} - \frac{2s-2+j}{2s-3+2j}\binom{2s-2+j}{2s-1}\right).$$

This is equivalent to the desired inequality. □

Now we can compare S_1 with S_2. Since it holds that $\binom{2s}{s-1} = \frac{s}{2s+1}\binom{2s+1}{s}$ and $\binom{2s+j}{j} = \frac{2s+j}{2s}\binom{2s-1+j}{j}$, equation (5.11) is equivalent to

$$\frac{S_2}{\binom{2s-1+j}{2s-1}\binom{2s+1}{s}} = \frac{2js+2s^2+3s+1}{(j+s+1)(j+s+2)}\frac{2s+j}{2(2s+1)}. \tag{5.14}$$

From (5.13) and (5.14) we get

$$\frac{S_1 - S_2}{\binom{2s-1+j}{2s-1}\binom{2s+1}{s}} \geqslant L,$$

where

$$L = \frac{(2s-3+j)(2s-1+j)+j}{(2s+2j-3)(2s+j-1)} - \frac{2js+2s^2+3s+1}{(j+s+1)(j+s+2)}\frac{2s+j}{2(2s+1)}.$$

The previous expression is positive if and only if

$$\begin{aligned} 0 < I &= ((2s-3+j)(2s-1+j)+j)(j+s+1)(j+s+2)(4s+2) \\ &\quad -(2js+2s^2+3s+1)(2s+j)(2s+2j-3)(2s+j-1) \\ &= 2j^4 + 2j^3(-1+8s) + j^2(-3-15s+46s^2) + j(3-17s-46s^2+64s^3) \\ &\quad +4(3+s-14s^2-4s^3+8s^4). \end{aligned}$$

For $s > 1$, all coefficients of I as a polynomial in j are positive, and hence, $I > 0$ whenever $j \in \mathbb{N}_0$, so $S_1 - S_2 > 0$, which together with (5.10) implies

$$\operatorname{sgn}(\varepsilon_{n,2jn}^{(s)}) = (-1)^{j+1} \tag{5.15}$$

for $s \in \mathbb{N},\ s > 1$, and $j \in \mathbb{N}_0$. Moreover, if $s = 1$, it follows from (5.8) that

$$\varepsilon_{n,2jn}^{(1)} = -\frac{\pi}{8}((-1)^j(2j+5)+3), \qquad j = 0, 1, \ldots,$$

and $\varepsilon_{n,2jn}^{(1)} = 0$ otherwise, so (5.15) remains valid in this case.

Thus we have proved the following result.

LEMMA 5.4. *For $\varepsilon_{n,2jn}^{(s)}$ defined by (5.8) we have*

$$\operatorname{sgn}(\varepsilon_{n,2jn}^{(s)}) = (-1)^{j+1}, \tag{5.16}$$

for $s \in \mathbb{N}$ and $j \in \mathbb{N}_0$.

Now, from (5.9) we shall derive an explicit bound for $|R_{n,s}(f)|$.

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In general, the Chebyshev coefficients α_k in (5.1) are unknown. However, Elliott [2] described a number of ways to estimate or bound them. In particular, under our assumptions,

$$|\alpha_k| \leqslant \frac{2}{\rho^k}\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right). \tag{5.17}$$

Substituting (5.8) and (5.17) into (5.9) gives

$$|R_{n,s}(f)| \leqslant \pi \max_{z\in\mathcal{E}_\rho}|f(z)|F(\rho), \tag{5.18}$$

where

$$F(\rho) = \frac{1}{\rho^{(2s+3)n}}\sum_{j=0}^{\infty}|\varepsilon_{n,2jn}^{(s)}|\rho^{-2jn}, \tag{5.19}$$

and $\varepsilon_{n,2jn}^{(s)}$ are defined by (5.8). Although the $\varepsilon_{n,2jn}^{(s)}$ are sums themselves, it turns out that $F(\rho)$ can be simplified to a single finite sum.

LEMMA 5.5. *For $F(\rho)$ given by (5.19) with $\rho > 1$, it holds that*

$$F(\rho) = \frac{\sum_{l=0}^{s}(-1)^{l+1}\left(\binom{2s+1}{s-l-1}-\binom{2s+1}{s-l}\right)\rho^{2(s-l)n}}{\rho^n(\rho^{2n}-1)^{2s}(\rho^{2n}+1)}. \tag{5.20}$$

Proof. From (5.3) we find

$$\begin{aligned}\frac{1}{(\rho^{2n}-1)^{2s}(\rho^{2n}+1)} &= \frac{1}{\rho^{2n(2s+1)}}\sum_{i=0}^{\infty}\binom{2s-1+i}{2s-1}\rho^{-2in}\sum_{l=0}^{\infty}(-1)^l\rho^{-2ln}\\ &= \frac{1}{\rho^{2n(2s+1)}}\sum_{m=0}^{\infty}\sum_{i=0}^{m}(-1)^{m-i}\binom{2s-1+i}{2s-1}\rho^{-2mn}.\end{aligned} \tag{5.21}$$

Let $C_{s,l} = \binom{2s+1}{s-l-1}-\binom{2s+1}{s-l}$. From (5.21) and (5.20) we obtain

$$\begin{aligned}F(\rho) &= \frac{1}{\rho^{(2s+3)n}}\sum_{l=0}^{s}\sum_{m=0}^{\infty}\sum_{i=0}^{m}(-1)^{l+m+i+1}C_{s,l}\binom{2s-1+i}{2s-1}\rho^{-2(m+l)n}\\ &= \frac{1}{\rho^{(2s+3)n}}\sum_{j=0}^{\infty}\sum_{l=0}^{s}\sum_{i=0}^{j-l}(-1)^{j+i+1}C_{s,l}\binom{2s-1+i}{2s-1}\rho^{-2jn},\end{aligned}$$

which reduces to (5.19), using (5.8) and (5.16). ◻

Now we can formulate the main result.

THEOREM 5.6. *For $s \in \mathbb{N}$, the estimate (5.18) can be expressed in the form*

$$|R_{n,s}(f)| \leqslant \pi \max_{z\in\mathcal{E}_\rho}|f(z)|\frac{\sum_{l=0}^{s}(-1)^{l+1}\left(\binom{2s+1}{s-l-1}-\binom{2s+1}{s-l}\right)\rho^{2(s-l)n}}{\rho^n(\rho^{2n}-1)^{2s}(\rho^{2n}+1)}. \tag{5.22}$$

6. Numerical examples. Let us now compute the integral

$$\int_{-1}^{1} T_n(t)f(t)\,\frac{dt}{\sqrt{1-t^2}}$$

by using the quadrature formula (1.2) for two entire functions.

Denoting the bounds in Sections 3, 4, and 5 by $|R^{(i)}_{n,s}(f)| \leqslant r_i(f)$, for $i = 1,2,3$, respectively, we find

$$r_1(f) = \inf_{\rho_0<\rho<+\infty} \left(B_1 \cdot \max_{z\in\mathcal{E}_\rho} |f(z)| \right), \tag{6.1}$$

$$r_i(f) = \inf_{1<\rho<+\infty} \left(B_i \cdot \max_{z\in\mathcal{E}_\rho} |f(z)| \right), \qquad i = 2,3, \tag{6.2}$$

where ρ_0 is defined in Theorem 3.1 and the B_i, for $i = 1,2,3$, are defined below. Numerical experiments show that for all n and all s, the corresponding values of $\rho_0(n,s)$ are very close to 1 (in most cases they are less than 1.1).

The length of the ellipse in (2.6) can be estimated by (cf. [17, Eq. (2.2)])

$$l(\mathcal{E}_\rho) \leqslant 2\pi a_1 \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6} \right),$$

where $a_1 = \frac{1}{2}(\rho+\rho^{-1})$. Therefore from (2.6), Theorem 3.1 and (6.1) for $n > 1$, we get

$$B_1 = a_1 \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6} \right) \left| K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1}) \right) \right|.$$

From (2.7), (4.6), and (6.2) we get

$$B_2 = \frac{\pi\sqrt{Q_s(x)}}{\rho^n(\rho^{4n}-1)^{2s}},$$

where Q_s are defined by (4.7). Finally, from (5.22) and (6.2) we obtain

$$B_3 = \pi \frac{\sum_{l=0}^{s}(-1)^{l+1}\left(\binom{2s+1}{s-l-1} - \binom{2s+1}{s-l} \right)\rho^{2(s-l)n}}{\rho^n(\rho^{2n}-1)^{2s}(\rho^{2n}+1)}.$$

We have calculated these bounds for some values of n, s, and ω. "Error" is the actual (sharp) error and I_ω is the exact value of the integral. All computations reported in this paper were carried out in MATLAB with high-precision arithmetic. The computations were carried out with 150 significant decimal digits.

EXAMPLE 1. Let

$$f(z) = f_0(z) = e^{\omega z^2}, \quad \omega > 0.$$

Since the function f_0 is entire, the obtained estimates hold for $\mathcal{E}_\rho$, $\rho > 1$. One finds

$$\max_{z\in\mathcal{E}_\rho} |e^{\omega z^2}| = e^{\omega a_1^2}, \qquad a_1 = \frac{1}{2}(\rho+\rho^{-1}).$$

The results are reported in Table 6.1.

EXAMPLE 2. Let

$$f(z) = f_1(z) = e^{\cos(\omega z)}, \quad \omega > 0.$$

Similarly to the previous example, the obtained estimates hold for $\mathcal{E}_\rho$, $\rho > 1$. We have

$$\max_{z\in\mathcal{E}_\rho} |e^{\cos(\omega z)}| = e^{\cosh(\omega b_1)}, \qquad b_1 = \frac{1}{2}(\rho-\rho^{-1}).$$

The results are reported in Table 6.2.

One can notice that for the results in these examples, all three estimates are within the same range. These results are comparable to those obtained for the quadrature formula with multiple nodes for the Fourier-Chebyshev coefficients; see [15].

TABLE 6.1
The values of the derived bounds $r_1(f_0), r_2(f_0), r_3(f_0)$, for some values of n, s, ω.

n, s, ω	$r_1(f_0)$	$r_2(f_0)$	$r_3(f_0)$	Error	I_ω
$8, 1, 1$	$4.43(-29)$	$4.37(-29)$	$4.37(-29)$	$3.88(-30)$	$8.53...(-4)$
$8, 2, 1$	$1.59(-44)$	$1.57(-44)$	$1.57(-44)$	$1.18(-45)$	$8.53...(-4)$
$8, 1, 5$	$3.54(-14)$	$3.32(-14)$	$3.32(-14)$	$2.94(-15)$	$5.28...(+0)$
$8, 2, 5$	$4.78(-24)$	$4.56(-24)$	$4.56(-24)$	$3.42(-25)$	$5.28...(+0)$
$8, 1, 10$	$6.12(-7)$	$5.35(-7)$	$5.35(-7)$	$4.69(-8)$	$2.38...(+3)$
$8, 2, 10$	$1.94(-14)$	$1.76(-14)$	$1.76(-14)$	$1.32(-15)$	$2.38...(+3)$
$8, 1, 15$	$3.91(-2)$	$3.17(-2)$	$3.17(-2)$	$2.73(-3)$	$4.99...(+5)$
$8, 2, 15$	$2.79(-8)$	$2.41(-8)$	$2.41(-8)$	$1.78(-9)$	$4.99...(+5)$
$10, 1, 1$	$7.55(-39)$	$7.48(-39)$	$7.48(-39)$	$5.95(-40)$	$4.25...(-5)$
$10, 2, 1$	$3.18(-59)$	$3.16(-59)$	$3.16(-59)$	$2.13(-60)$	$4.25...(-5)$
$10, 1, 5$	$1.84(-20)$	$1.75(-20)$	$1.75(-20)$	$1.39(-21)$	$1.25...(+0)$
$10, 2, 5$	$7.36(-34)$	$7.10(-34)$	$7.10(-34)$	$4.77(-35)$	$1.25...(+0)$
$10, 1, 10$	$9.58(-12)$	$8.61(-12)$	$8.61(-12)$	$6.78(-13)$	$1.00...(+3)$
$10, 2, 10$	$3.66(-22)$	$3.39(-22)$	$3.39(-22)$	$2.27(-23)$	$1.00...(+3)$
$10, 1, 15$	$4.25(-6)$	$3.60(-6)$	$3.60(-6)$	$2.81(-7)$	$2.74...(+5)$
$10, 2, 15$	$8.43(-15)$	$7.51(-15)$	$7.51(-15)$	$5.00(-16)$	$2.74...(+5)$
$14, 1, 1$	$1.27(-59)$	$1.26(-59)$	$1.26(-59)$	$8.51(-61)$	$6.32...(-8)$
$14, 2, 1$	$2.38(-90)$	$2.36(-90)$	$2.36(-90)$	$1.35(-91)$	$6.32...(-8)$
$14, 1, 5$	$2.94(-34)$	$2.84(-34)$	$2.84(-34)$	$1.91(-35)$	$4.39...(-2)$
$14, 2, 5$	$3.28(-55)$	$3.20(-55)$	$3.20(-55)$	$1.82(-56)$	$4.39...(-2)$
$14, 1, 10$	$1.46(-22)$	$1.36(-22)$	$1.36(-22)$	$9.09(-24)$	$1.19...(+2)$
$14, 2, 10$	$2.55(-39)$	$2.41(-39)$	$2.41(-39)$	$1.37(-40)$	$1.19...(+2)$
$14, 1, 15$	$3.37(-15)$	$3.01(-15)$	$3.01(-15)$	$2.00(-16)$	$5.92...(+4)$
$14, 2, 15$	$1.59(-29)$	$1.47(-29)$	$1.47(-29)$	$8.29(-31)$	$5.92...(+4)$

7. Concluding remarks. Three kinds of effective error bounds of the quadrature formulas with multiple nodes that generalize the well-known Micchelli-Rivlin quadrature formula, when the integrand is an analytic function in the regions containing the confocal ellipses, were considered recently in [13, 15]. In this paper, we continue with the analogous analysis for their Kronrod extensions and obtain effective error bounds of them, which is confirmed by the given numerical examples.

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TABLE 6.2
The values of the derived bounds $r_1(f_1), r_2(f_1), r_3(f_1)$, for some values of n, s, ω.

n, s, ω	$r_1(f_1)$	$r_2(f_1)$	$r_3(f_1)$	Error	I_ω
$8, 1, 0.5$	$5.54(-39)$	$5.50(-39)$	$5.50(-39)$	$3.34(-40)$	$1.08...(-6)$
$8, 2, 0.5$	$1.21(-56)$	$1.21(-56)$	$1.21(-56)$	$6.03(-58)$	$1.08...(-6)$
$8, 1, 1$	$3.07(-27)$	$2.99(-27)$	$2.99(-27)$	$1.80(-28)$	$1.98...(-4)$
$8, 2, 1$	$3.81(-40)$	$3.73(-40)$	$3.73(-40)$	$1.85(-41)$	$1.98...(-4)$
$8, 1, 5$	$1.89(-5)$	$1.16(-5)$	$1.16(-5)$	$5.81(-7)$	$3.11...(-1)$
$8, 2, 5$	$8.23(-9)$	$5.39(-9)$	$5.39(-9)$	$2.25(-10)$	$3.11...(-1)$
$8, 1, 10$	$3.14(-1)$	$9.48(-2)$	$9.65(-2)$	$4.06(-3)$	$1.07...(+0)$
$8, 2, 10$	$1.93(-2)$	$6.80(-3)$	$7.10(-3)$	$1.98(-4)$	$1.07...(+0)$
$10, 1, 0.5$	$3.30(-50)$	$3.29(-50)$	$3.29(-50)$	$1.75(-51)$	$-1.29...(-8)$
$10, 2, 0.5$	$8.16(-73)$	$8.12(-73)$	$8.12(-73)$	$3.56(-74)$	$-1.29...(-8)$
$10, 1, 1$	$1.71(-35)$	$1.67(-35)$	$1.67(-35)$	$8.85(-37)$	$-9.13...(-6)$
$10, 2, 1$	$3.73(-52)$	$3.66(-52)$	$3.66(-52)$	$1.60(-53)$	$-9.13...(-6)$
$10, 1, 5$	$9.30(-8)$	$5.96(-8)$	$5.96(-8)$	$2.65(-9)$	$-1.72...(-1)$
$10, 2, 5$	$2.35(-12)$	$1.60(-12)$	$1.60(-12)$	$5.93(-14)$	$-1.72...(-1)$
$10, 1, 10$	$3.36(-2)$	$1.09(-2)$	$1.09(-2)$	$3.95(-4)$	$-8.80...(-1)$
$10, 2, 10$	$5.72(-4)$	$2.09(-4)$	$2.12(-4)$	$6.01(-6)$	$-8.80...(-1)$
$14, 1, 0.5$	$3.26(-73)$	$3.25(-73)$	$3.25(-73)$	$1.42(-74)$	$-1.37...(-12)$
$14, 2, 0.5$	$6.75(-106)$	$6.73(-106)$	$6.73(-106)$	$2.43(-107)$	$-1.37...(-12)$
$14, 1, 1$	$1.49(-52)$	$1.47(-52)$	$1.47(-52)$	$6.39(-54)$	$-1.47...(-8)$
$14, 2, 1$	$6.70(-77)$	$6.59(-77)$	$6.59(-77)$	$3.68(-78)$	$-1.47...(-8)$
$14, 1, 5$	$9.42(-13)$	$6.38(-13)$	$6.38(-13)$	$2.37(-14)$	$-4.23...(-2)$
$14, 2, 5$	$5.79(-20)$	$4.10(-20)$	$4.10(-20)$	$2.72(-21)$	$-4.23...(-2)$
$14, 1, 10$	$2.32(-4)$	$8.36(-5)$	$8.36(-5)$	$2.45(-6)$	$9.03...(-2)$
$14, 2, 10$	$2.39(-7)$	$9.50(-8)$	$9.51(-8)$	$2.30(-9)$	$9.03...(-2)$

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