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CONTINUOUS TIME INTEGRATION FOR CHANGING TYPE SYSTEMS*

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Abstract. We consider variational time integration using continuous Galerkin-Petrov methods applied to evolutionary systems of changing type. We prove optimal-order convergence of the error in a cGP-like norm and conclude the paper with some numerical examples and conclusions.

Key words. evolutionary equations, changing type system, continuous Galerkin-Petrov, space-time approach

AMS subject classifications. 65J08, 65J10, 65M12, 65M60

1. Introduction.

Let us start with an example where the type of a partial differential equation changes over the spacial domain and the problem is equipped with homogeneous Dirichlet boundary conditions. For this purpose, let $n \in \{1,2,3\}$ be the spatial dimension and $\Omega \subset \mathbb{R}^n$ be a bounded set partitioned into measurable, disjoint sets $\Omega_{\rm ell}, \Omega_{\rm par}$, and $\Omega_{\rm hyp}$. In $\Omega_{\rm hyp}$, a hyperbolic wave equation for $U = (U_1, U_2)$ is given,

$$\partial_t U_1 + \operatorname{div}(U_2) = F_1, \qquad \partial_t U_2 + \operatorname{grad}(U_1) = F_2 \qquad \text{in } \Omega_{\rm hyp},$$

with some force term $F = (F_1, F_2)$. We will come to the boundary conditions for the spatial operators in a moment. In $\Omega_{\rm par}$, a parabolic heat equation is given,

$$\partial_t U_1 + \operatorname{div}(U_2) = F_1, \qquad U_2 + \operatorname{grad}(U_1) = F_2 \qquad \text{in } \Omega_{\rm par},$$

and in $\Omega_{\rm ell}$, an elliptic reaction-diffusion equations completes the setting,

$$U_1 + \operatorname{div}(U_2) = F_1, \qquad U_2 + \operatorname{grad}(U_1) = F_2 \qquad \text{in } \Omega_{\rm ell}.$$

Each of the above equations can also be expressed in their derived second-order formulation for U_1, namely $(\partial_t^2 - \Delta)U_1 = \partial_t F_1 - \operatorname{div} F_2$ for the wave equation, $(\partial_t - \Delta)U_1 = F_1 - \operatorname{div} F_2$ for the heat equation, and $(1 - \Delta)U_1 = F_1 - \operatorname{div} F_2$ for the reaction-diffusion equation.

Denoting by χ_D the characteristic function of a domain $D \subset \Omega$ and defining the linear operators

$$M_0 = \begin{bmatrix} \chi_{\Omega_{\rm hyp}\cup\Omega_{\rm par}} & 0 \\ 0 & \chi_{\Omega_{\rm hyp}} \end{bmatrix}, \quad M_1 = \begin{bmatrix} \chi_{\Omega_{\rm ell}} & 0 \\ 0 & \chi_{\Omega_{\rm par}\cup\Omega_{\rm ell}} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad}^\circ & 0 \end{bmatrix},$$

where $^\circ$ indicates homogeneous Dirichlet boundary conditions with respect to Ω, we can write the above equations in a condensed way as

$$(\partial_t M_0 + M_1 + A)U = F. \tag{1.1a}$$

By defining A as above, we have included the boundary conditions at $\partial\Omega$ into A. All that is left is an initial condition at $t = 0$ as we are only interested in $t \geq 0$:

$$M_0 U(0^+) = M_0 U_0. \tag{1.1b}$$

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Now we are faced with the question, under which conditions the above problem (1.1) has a unique solution.

In the following we assume that U_0 is in $D(A)$. Besides this, we may derive a condition on the operators from a much more general theory. Most of the classical linear partial differential equations arising in mathematical physics can be written in a common operator form. It has been shown in [7] that this is the form of an evolutionary problem, given by (1.1), where ∂_t stands for the derivative with respect to time, $M_0 : \mathbf{H} \to \mathbf{H}$, $M_1 : \mathbf{H} \to \mathbf{H}$ are bounded linear selfadjoint operators on some Hilbert space $\mathbf{H}$, $A : D(A) \subset \mathbf{H} \to \mathbf{H}$ is an unbounded skew-selfadjoint operator on $\mathbf{H}$, and F is a given source term.

We are interested in a unique solution U of the above equation. For this purpose let $\rho > 0$, and define the weighted L^2-function space

$$H_\rho(\mathbb{R}; \mathbf{H}) := \left\{ f : \mathbb{R} \to \mathbf{H} \,:\, f \text{ meas.}, \int_{\mathbb{R}} \|f(t)\|^2_{\mathbf{H}} \exp(-2\rho t)\,\mathrm{d}t < \infty \right\}.$$

The space $H_\rho(\mathbb{R}; \mathbf{H})$ is a Hilbert space endowed with the natural inner product given by

$$\langle f, g\rangle_\rho := \int_{\mathbb{R}} \langle f(t), g(t)\rangle \exp(-2\rho t)\,\mathrm{d}t,$$

for all $f, g \in H_\rho(\mathbb{R}; \mathbf{H})$, where $\langle f(t), g(t)\rangle$ is the inner product of $\mathbf{H}$ and $\|\cdot\|_{\mathbf{H}}$ its associated norm. We obtain a norm by setting $\|f\|^2_\rho := \langle f, f\rangle_\rho$. The associated weighted H^k-function spaces are denoted by $H^k_\rho(\mathbb{R}; \mathbf{H})$, for $k \in \mathbb{N}$. Now from [7, Theorem (solution theory)], the following result can be concluded: If there exists a $\rho_0 > 0$ and a $\gamma > 0$ such that for all $\rho \geq \rho_0$ and $x \in \mathbf{H}$,

$$\langle (\rho M_0 + M_1)x, x\rangle \geq \gamma \langle x, x\rangle = \gamma \|x\|^2_{\mathbf{H}}, \tag{1.2}$$

then there exists a unique solution $U \in H_\rho(\mathbb{R}, \mathbf{H})$ for all right-hand sides $F \in H_\rho(\mathbb{R}, \mathbf{H})$. Furthermore, by the above condition $\langle M_0 x, x\rangle \geq 0$, it follows that there exists a root $M_0^{1/2}$ of M_0. Note that the theory presented in [7] deals with vanishing initial conditions at $t \to -\infty$.

COROLLARY 1.1. *Under condition* (1.2) *and if*

$$F|_{\mathbb{R}_{\geq 0}} \textit{ is continuous and } F(t) = 0,\ t < 0, \tag{1.3a}$$

$$U_0 \in \mathit{dom}(A), \tag{1.3b}$$

and

$$(M_1 + A)U_0 = F(0^+), \tag{1.3c}$$

then problem (1.1) *has a unique solution* U *with*

$$U(0^+) = U_0.$$

Proof. Equation (1.1) stated as a problem on $\mathbb{R}$ reads as

$$(\partial_t M_0 + M_1 + A)U = F + \delta_0 M_0 U_0,$$

where the initial condition $M_0 U(0^+) = M_0 U_0$ is included via the delta distribution δ_0 at $t = 0$ on the right-hand side.

Let H_0 denote the Heaviside function with the jump at $t = 0$. We obtain for $U - H_0U_0$ the evolutionary problem

$$(\partial_t M_0 + M_1 + A)(U - H_0U_0) = F - (M_1 + A)H_0U_0 =: \tilde{F}.$$

By (1.3) we have $\tilde{F}(t) = 0$, $t < 0$, $\tilde{F}(0) = 0$, and $\tilde{F}$ is continuous. Now [10] yields that the problem for $U - H_0U_0$ has a unique solution in $H^1_\rho(\mathbb{R}, \mathbf{H})$. Thus, U is the unique solution of (1.1) and $U(0^+) = U_0$. ∎

In the following we assume conditions (1.2) and (1.3) to be fulfilled. Then U_0 are initial data on the whole domain Ω, explicitly also in the elliptic and parabolic regime. But due to the compatibility condition (1.3) they cannot be chosen independently of F.

In [6], this class of changing type problems was investigated numerically using a discontinuous Galerkin approach for the discretisation in time. Here we want to apply a continuous approach, namely the continuous Galerkin-Petrov method [1, 2, 3, 4, 8, 11].

Note that, like in [6], we deal in this paper with problems that have a changing type over the given domain and could be rewritten into second-order form as shown above. But then transmission conditions would have to be stated, which are automatically included in the first-order formulation. This is a very useful feature of the general approach, and it allows us to combine models from different parts of physics into one well-posed problem. We want to emphasise that the time discretisation presented and analysed in this paper holds for all problems of the above general class of first-order problems, only the spatial discretisation has to be adapted to the operator A.

For our problem with the operator A, the Hilbert space $\mathbf{H}$ and $D(A)$ can now be specified to

$$\mathbf{H} = L^2(\Omega) \otimes (L^2(\Omega))^n \qquad \text{and} \qquad D(A) = H^1_0(\Omega) \otimes H_{\mathrm{div}}(\Omega).$$

REMARK 1.2. The solution theory requires A to be skew-selfadjoint, which in turn restricts the choice of boundary data. Some simple choices are homogeneous Dirichlet boundary conditions in the first component, encoded by grad° in the above operator A, homogeneous Neumann boundary conditions in the second component, encoded by div°, or periodic boundary conditions in both components, encoded by $\mathrm{grad}^\#$ and $\mathrm{div}^\#$. Inhomogeneous conditions can always be transformed into homogeneous ones by a substitution altering the right-hand side of the problem.

The paper is organised as follows. The precise formulation of the considered method is stated in Section 2, while Section 3 deals with the existence of discrete solutions. In Section 4 we present error estimates, and Section 5 provides some numerical examples. Finally, Section 6 contains concluding remarks.

2. Numerical method.

The discrete variational form of equation (1.1) uses a decomposition of $[0, T]$ into M disjoint intervals $I_m = (t_{m-1}, t_m]$ of length $\tau_m = t_m - t_{m-1}$, for $m \in \{1, \ldots, M\}$. Furthermore, let Ω be discretised into Ω_h by a regular simplicial mesh that resolves the sets Ω_{ell}, Ω_{par}, and Ω_{hyp}, i.e., each of these subdomains is a union of mesh cells, and let h be the maximal diameter of the cells of Ω_h. Furthermore, let r, $k \geq 1$ denote polynomial degrees. Then the piecewise polynomial function spaces for the trial and test functions, respectively, are given by

$$\mathcal{U}_h^\tau := \{u \in H^1_\rho([0, T], \mathbf{H}) : u\big|_{I_m} \in \mathcal{P}_r(I_m, V_1 \otimes V_2), m \in \{1, \ldots, M\}\},$$
$$\mathcal{V}_h^\tau := \{v \in H_\rho([0, T], \mathbf{H}) : v\big|_{I_m} \in \mathcal{P}_{r-1}(I_m, V_1 \otimes V_2), m \in \{1, \ldots, M\}\},$$

where the spatial spaces are

$$V_1 := \left\{ v \in H_0^1(\Omega) : \, v|_\sigma \in \mathcal{P}_k(\sigma) \, \forall \sigma \in \Omega_h \right\},$$
$$V_2 := \{ w \in H_{\mathrm{div}}(\Omega) : \, w|_\sigma \in RT_{k-1}(\sigma) \, \forall \sigma \in \Omega_h \},$$

and therefore

$$V_1 \otimes V_2 \subset D(A) \subset \mathbf{H}.$$

Here $\mathcal{P}_k(\sigma)$ is the space of polynomials of degree up to k on the cell σ of Ω_h, and $RT_{k-1}(\sigma)$ is the Raviart-Thomas-space defined by

$$RT_{k-1}(\sigma) = (\mathcal{P}_{k-1}(\sigma))^n + \mathbf{x}\mathcal{P}_{k-1}(\sigma) \subset \mathcal{P}_k(\sigma)^n.$$

Note that we retain the regularity in space of the trial functions also for the test functions in order to define a Galerkin method in space. Furthermore, if the mesh consists of quadrilateral or hexahedral cells, then in the above definitions and statements, the polynomial space $\mathcal{P}_k(\sigma)$ can be replaced by a mapped $\mathcal{Q}_k$-space including all polynomials of total degree k over a reference element mapped onto σ. If the mesh is a combination of both types of cells, then a combination of spaces also works with a suitable mapping ensuring continuity.

Let us localise in addition the scalar product in $H_\rho(\mathbb{R}, \mathbf{H})$ to the time intervals I_m by

$$\langle f, g \rangle_{\rho,m} := \int_{I_m} \langle f(t), g(t) \rangle \exp(-2\rho t) \, \mathrm{d}t,$$

which induces the norm $\|f\|_{\rho,m}^2 := \langle f, f \rangle_{\rho,m}$. Then the variational formulation using the continuous Galerkin-Petrov method reads: Find $U_h^\tau \in \mathcal{U}_h^\tau$ such that for all $V_h^\tau \in \mathcal{V}_h^\tau$ and $m \in \{1, \ldots, M\}$,

$$B_m(U_h^\tau, V_h^\tau) := \langle (\partial_t M_0 + M_1 + A) U_h^\tau, V_h^\tau \rangle_{\rho,m} = \langle F, V_h^\tau \rangle_{\rho,m}, \tag{2.1a}$$

where

$$U_h^\tau(0) = \mathcal{I} U_0 \tag{2.1b}$$

is the initial value. Here $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$ denotes the spatial interpolation operator, where $\mathcal{I}_1 : H_\rho([0,T], H^1(\Omega)) \to H_\rho([0,T], V_1)$ is locally the Scott-Zhang interpolant on each cell σ (see [9] for a precise definition), and $\mathcal{I}_2 : H_\rho((0,t), H(\mathrm{div}, \Omega) \cap (L^s(\Omega))^n) \to H_\rho([0,T], V_2)$, with $s > 2$, is the standard interpolator defined via moments; see [5]. Note that it is appropriate to include the full initial conditions into the discrete problem; see Corollary 1.1.

3. Existence of a discrete solution.

Let us start by defining Π_h^τ as the orthogonal L^2-projection with respect to $\langle \cdot, \cdot \rangle_\rho$ into the test space $\mathcal{V}_h^\tau$, i.e.,

$$\langle U - \Pi_h^\tau U, W_h^\tau \rangle_\rho = 0, \qquad \text{for all } W_h^\tau \in \mathcal{V}_h^\tau, \tag{3.1}$$

and R and N as the projectors onto the range and nullspace of M_0, respectively, and

$$|||U_h^\tau|||_\rho^2 := \frac{1}{2} \|M_0^{1/2} U_h^\tau(T)\|_{\mathbf{H}}^2 \mathrm{e}^{-2\rho T} + \|N U_h^\tau(0)\|_{\mathbf{H}}^2 + \gamma \|\Pi_h^\tau U_h^\tau\|_\rho^2.$$

LEMMA 3.1. *The seminorm $|||U_h^\tau|||_\rho$ is a norm on $\mathcal{U}_h^\tau$.*

Proof. With $\mathcal{U}_h^\tau$ being finite, we only have to show that $|||U_h^\tau|||_\rho = 0$ implies $U_h^\tau = 0$. Thus, let us assume that $|||U_h^\tau|||_\rho = 0$. Then it follows immediately that $\Pi_h^\tau U_h^\tau = 0$, and due to continuity, one degree of freedom is left for U_h^τ. On each time interval, U_h^τ is a multiple of a weighted Legendre polynomial that is orthogonal to V_h^τ with respect to $\langle\cdot,\cdot\rangle_\rho$. From $\|NU_h^\tau(0)\|_{\mathbf{H}} = 0$, we conclude

$$NU_h^\tau(0) = 0,$$

and therefore $NU_h^\tau = 0$ because the Legendre polynomial is nonzero at the left boundary. From $\|M_0^{1/2}U_h^\tau(T)\|_{\mathbf{H}} = 0$, we have similarly

$$RU_h^\tau(T) = 0,$$

and therefore $RU_h^\tau = 0$ because the Legendre polynomial is nonzero at the right boundary. With

$$U_h^\tau = RU_h^\tau + NU_h^\tau = 0,$$

we arrive at the assertion. ∎

LEMMA 3.2. *It holds that*

$$\frac{1}{2}\|M_0^{1/2}U_h^\tau(T)\|_{\mathbf{H}}^2 \mathrm{e}^{-2\rho T} + \gamma\|\Pi_h^\tau U_h^\tau\|_\rho^2 \le \sum_{m=1}^M B_m(U_h^\tau, \Pi_h^\tau U_h^\tau) + \frac{1}{2}\|M_0^{1/2}\mathcal{I}U_0\|_{\mathbf{H}}^2.$$

Proof. Let us consider an arbitrary interval I_m. Then it holds that

$$B_m(U_h^\tau, \Pi_h^\tau U_h^\tau) = \langle \partial_t M_0 U_h^\tau, U_h^\tau\rangle_{\rho,m} + \langle M_1 \Pi_h^\tau U_h^\tau, \Pi_h^\tau U_h^\tau\rangle_{\rho,m},$$

where the skew-symmetry of A and the definition of Π_h^τ was used. For the first term we apply integration by parts and obtain due to the exponential weight

$$\langle \partial_t M_0 U_h^\tau, U_h^\tau\rangle_{\rho,m} = \rho\langle M_0 U_h^\tau, U_h^\tau\rangle_{\rho,m} + \frac{1}{2}\|M_0^{1/2}U_h^\tau(t)\|_{\mathbf{H}}^2 \mathrm{e}^{-2\rho t}\Big|_{t_{m-1}}^{t_m}.$$

By the L^2-orthogonality (3.1), it follows that

$$\begin{aligned}\langle M_0 U_h^\tau, U_h^\tau\rangle_{\rho,m} &= \langle M_0(U_h^\tau - \Pi_h^\tau U_h^\tau), U_h^\tau - \Pi_h^\tau U_h^\tau\rangle_{\rho,m} + \langle M_0\Pi_h^\tau U_h^\tau, \Pi_h^\tau U_h^\tau\rangle_{\rho,m}\\ &\ge \langle M_0\Pi_h^\tau U_h^\tau, \Pi_h^\tau U_h^\tau\rangle_{\rho,m},\end{aligned}$$

and therefore

$$\langle \partial_t M_0 U_h^\tau, U_h^\tau\rangle_{\rho,m} \ge \langle \rho M_0\Pi_h^\tau U_h^\tau, \Pi_h^\tau U_h^\tau\rangle_{\rho,m} + \frac{1}{2}\|M_0^{1/2}U_h^\tau(t)\|_{\mathbf{H}}^2 \mathrm{e}^{-2\rho t}\Big|_{t_{m-1}}^{t_m}.$$

With the general existence assumption $\rho M_0 + M_1 \ge \gamma$ and $M_0 \ge 0$, we obtain

$$B_m(U_h^\tau, \Pi_h^\tau U_h^\tau) \ge \gamma\|\Pi_h^\tau U_h^\tau\|_{\rho,m}^2 + \frac{1}{2}\|M_0^{1/2}U_h^\tau(t)\|_{\mathbf{H}}^2 \mathrm{e}^{-2\rho t}\Big|_{t_{m-1}}^{t_m}.$$

By summing up over the intervals, the statement follows. ∎

It follows that

$$\begin{aligned}|||U_h^\tau|||_\rho^2 &\le \sum_{m=1}^M B_m(U_h^\tau, \Pi_h^\tau U_h^\tau) + \frac{1}{2}\|M_0^{1/2}\mathcal{I}U_0\|_{\mathbf{H}}^2 + \|NU_h^\tau(0)\|_{\mathbf{H}}^2\\ &= \sum_{m=1}^M \langle f, \Pi_h^\tau U_h^\tau\rangle_{\rho,m} + \frac{1}{2}\|M_0^{1/2}\mathcal{I}U_0\|_{\mathbf{H}}^2 + \|NU_h^\tau(0)\|_{\mathbf{H}}^2\\ &\le \frac{1}{2\gamma}\|f\|_\rho^2 + \frac{1}{2}\,|||U_h^\tau|||_\rho^2 + \frac{1}{2}\|M_0^{1/2}\mathcal{I}U_0\|_{\mathbf{H}}^2 + \frac{1}{2}\|N\mathcal{I}U_0\|_{\mathbf{H}}^2,\end{aligned}$$

and therefore

$$|||U_h^\tau|||_\rho^2 \le \frac{1}{\gamma}\|f\|_\rho^2 + \|M_0^{1/2}\mathcal{I}U_0\|_{\mathbf{H}}^2 + \|N\mathcal{I}U_0\|_{\mathbf{H}}^2. \tag{3.2}$$

This shows uniqueness, existence, and continuous dependence on f and U_0 of the discrete solution U_h^τ.

4. Error-estimation.

Let us start by stating some interpolation error estimates.

Interpolation in time. Let us define $P_r : H_\rho^1([0,T],\mathbf{H}) \to H_\rho^1([0,T],\mathbf{H})$, with $P_r u\big|_{I_m} \in \mathcal{P}_r(I_m,\mathbf{H})$ for all $m \in \{1,\ldots,M\}$, as the interpolation operator fulfilling locally for all m and $v \in H_\rho^1([0,T],\mathbf{H})$,

$$\begin{aligned} (P_r v - v)(t_{m-1}) &= 0, \qquad (P_r v - v)(t_m) = 0,\\ \langle P_r v - v, w\rangle_{\rho,m} &= 0 \qquad \forall w \in \mathcal{P}_{r-2}(I_m,\mathbf{H}). \end{aligned}$$

Although we have weighted norms and scalar products, the standard interpolation error estimate

$$\|P_r v - v\|_\rho \le C\tau^{r+1}\|\partial_t^{r+1} v\|_\rho$$

holds for $v \in H_\rho^{r+1}([0,T],\mathbf{H})$, where here and further on, $C > 0$ denotes a generic constant and $\tau := \max\{\tau_m\}$.

Interpolation in space. As previously stated, we use $\mathcal{I} = (\mathcal{I}_1,\mathcal{I}_2)$ as spatial interpolation operator, where the first component

$$\mathcal{I}_1 : H_\rho([0,T],H^1(\Omega)) \to H_\rho([0,T],V_1)$$

is the Scott-Zhang interpolant and the second component

$$\mathcal{I}_2 : H_\rho((0,t),H(\mathrm{div},\Omega)\cap(L^\sigma(\Omega))^n) \to H_\rho([0,T],V_2),$$

with $\sigma > 2$, is the standard Raviart-Thomas interpolator. Here it holds (see [9]) that for all $v \in H_0^1(\Omega)\cap H^s(\Omega)$,

$$\|v - \mathcal{I}_1 v\|_0 \le Ch^s\|v\|_r, \qquad \|\mathrm{grad}(v - \mathcal{I}_1 v)\|_0 \le Ch^{s-1}\|v\|_s,$$

with $1 \le s \le k+1$, $\|v\|_s$ denotes the $H^s(\Omega)$-norm, and for all $q \in H^s(\Omega)$ such that $\mathrm{div}\, q \in H^s(\Omega)$, we have (see [5])

$$\|q - \mathcal{I}_2 q\|_0 \le Ch^s\|q\|_s, \qquad \|\mathrm{div}(q - \mathcal{I}_2 q)\|_0 \le Ch^s\|\mathrm{div}\, q\|_s,$$

with $1 \le s \le k$.

Error analysis. Note that we have for all $V_h^\tau \in \mathcal{V}_h^\tau$ the Galerkin orthogonality

$$B_m(U - U_h^\tau, V_h^\tau) = 0 \tag{4.1}$$

for the solution $U \in H_\rho^1([0,T],\mathbf{H})$ of (1.1) and $U_h^\tau \in \mathcal{U}_h^\tau$ of (2.1). We now want to estimate the error $U - U_h^\tau$ and decompose it into $U - U_h^\tau = \eta + \xi$, where

$$\eta = \eta_1 + \eta_2, \quad \eta_1 = U - P_r U, \quad \eta_2 = P_r(U - \mathcal{I}U), \quad \xi = P_r\mathcal{I}U - U_h^\tau \in \mathcal{U}_h^\tau.$$

With (2.1b) it follows that

$$\xi(0) = P_r \mathcal{I}U(0) - U_h^\tau(0) = \mathcal{I}U(0) - \mathcal{I}U(0) = 0.$$

LEMMA 4.1. *For any $m \in \{1,\dots,M\}$ and $V_h^\tau \in \mathcal{V}_h^\tau$ it holds that*

$$\langle(\partial_t M_0 + M_1 + A)\xi, V_h^\tau\rangle_{\rho,m} \le (\|(2\rho M_0 + M_1)\eta\|_{\rho,m} + \|A\eta\|_{\rho,m})\,\|V_h^\tau\|_{\rho,m}. \tag{4.2}$$

Proof. Using the Galerkin orthogonality (4.1), we obtain the error identity

$$\langle(\partial_t M_0 + M_1 + A)\xi, V_h^\tau\rangle_{\rho,m} = -\langle(\partial_t M_0 + M_1 + A)\eta, V_h^\tau\rangle_{\rho,m}.$$

Using integration by parts and the properties of P_r, we obtain for all $w \in \mathcal{V}_h^\tau$ and $v \in H_\rho^1([0,T],\mathbf{H})$ that

$$\begin{aligned}\langle\partial_t M_0(v - P_r v), w\rangle_{\rho,m} &= 2\rho\langle M_0(v-P_r v), w\rangle_{\rho,m} - \langle v - P_r v, \partial_t M_0 w\rangle_{\rho,m}\\ &\qquad + \langle v - P_r v, w\rangle\,\mathrm{e}^{-2\rho t}\Big|_{t_{m-1}}^{t_m}\\ &= 2\rho\langle M_0(v - P_r v), w\rangle_{\rho,m}.\end{aligned}$$

Thus, we get the error equation

$$\langle(\partial_t M_0 + M_1 + A)\xi, V_h^\tau\rangle_{\rho,m} = -\langle(2\rho M_0 + M_1 + A)\eta, V_h^\tau\rangle_{\rho,m}, \tag{4.3}$$

from which (4.2) follows by the Cauchy-Schwarz inequality. ∎

From the error equation (4.3) and the stability estimate (3.2), we obtain

$$\gamma\|\Pi_h^\tau\xi\|_\rho^2 + \frac{1}{2}\|M_0^{1/2}\xi(T)\|_{\mathbf{H}}^2\mathrm{e}^{-2\rho T} \le \frac{1}{\gamma}(\|(2\rho M_0 + M_1)\eta\|_\rho^2 + \|A\eta\|_\rho^2) \tag{4.4}$$

by substituting $U_h^\tau := \xi$ and $f := -(2\rho M_0 + M_1 + A)\eta$ and noting that $\xi(0) = 0$.

In order to simplify the representation of the main result, let us define

$$\mathbf{H}^k := H^k(\Omega) \otimes (H^k(\Omega))^n \qquad \text{and} \qquad \|U\|_{\mathbf{H}^k,\rho}^2 := \int_0^T \|U(t)\|_k^2 \exp(-2\rho t)\,\mathrm{d}t.$$

THEOREM 4.2. *Assume that the solution U of (1.1) has the regularity*

$$U \in H_\rho^1([0,T];\mathbf{H}^k) \cap H_\rho^{r+1}([0,T];\mathbf{H})$$

as well as

$$AU \in H_\rho([0,T];\mathbf{H}^k) \cap H_\rho^{r+1}([0,T];\mathbf{H}).$$

Then we have for the error of the numerical solution U_h^τ of (2.1) that

$$\begin{aligned}|||U - U_h^\tau|||_\rho \le C\Big(&\tau^{r+1}\left(\|\partial_t^{r+1}U\|_\rho + \|\partial_t^{r+1}AU\|_\rho\right)\\ &+ h^k\left(\|U\|_{\mathbf{H}^k,\rho} + \|AU\|_{\mathbf{H}^k,\rho} + \|U(T)\|_{\mathbf{H}^k}\mathrm{e}^{-\rho T} + \|NU_0\|_{\mathbf{H}^k}\right)\Big).\end{aligned}$$

Proof. By the decomposition of the norm and the error, we have to estimate

$$\begin{aligned}\|\Pi_h^\tau(U - U_h^\tau)\|_\rho &\le \|\Pi_h^\tau\eta_1\|_\rho + \|\Pi_h^\tau\eta_2\|_\rho + \|\Pi_h^\tau\xi\|_\rho,\\ \|M_0^{1/2}(U - U_h^\tau)(T)\|_H &\le \|M_0^{1/2}\eta_1(T)\|_H + \|M_0^{1/2}\eta_2(T)\|_H + \|M_0^{1/2}\xi(T)\|_H\\ &= \|M_0^{1/2}\eta_2(T)\|_H + \|M_0^{1/2}\xi(T)\|_H,\\ \|N(U - U_h^\tau)(0)\|_H &= \|N(U - \mathcal{I}U)(0)\|_H.\end{aligned}$$

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Using the above interpolation error estimates, we obtain

$$\begin{aligned}\|\Pi_h^\tau \eta_1\|_\rho &= \|\eta_1\|_\rho \le C\tau^{r+1}\|\partial_t^{r+1}U\|_\rho,\\ \|\Pi_h^\tau \eta_2\|_\rho &= \|\eta_2\|_\rho \le C\|U-\mathcal{I}U\|_\rho \le Ch^k\|U\|_{\mathbf{H}^k,\rho},\\ \|M_0^{1/2}\eta_2(T)\|_H &\le Ch^k\|U(T)\|_{\mathbf{H}^k},\\ \|N(U-\mathcal{I}U)(0)\|_H &\le Ch^k\|NU_0\|_{\mathbf{H}^k}.\end{aligned}$$

For the remaining two terms we apply (4.4) and find

$$\|\Pi_h^\tau \xi\|_\rho \le C\left(\tau^{r+1}\|\partial_t^{r+1}U\|_\rho + h^k\|U\|_{\mathbf{H}^k,\rho} + \tau^{r+1}\|\partial_t^{r+1}AU\|_\rho + h^k\|AU\|_{\mathbf{H}^k,\rho}\right)$$

and similarly for $\|M_0^{1/2}\xi(T)\|_H$. Combining these results proves the error estimate. ∎

REMARK 4.3. In Theorem 4.2 we assumed a slightly higher regularity for U than what is actually needed. Instead of assuming $U \in H_\rho^1([0,T],\mathbf{H}^k)$ for the point evaluation at $t=T$, the weaker assumption $U \in W_\rho^{0,\infty}([0,T],\mathbf{H}^k)$ suffices. But in order to prove that claim from conditions for the right-hand side, the easiest way is to prove the above regularity and use Sobolev's embedding theorem.

REMARK 4.4. In this section we presented an error analysis for the fully discrete problem of system of changing type. At the same time the result holds for all operators M_0 and M_1 fulfilling assumption (1.2). The analysis can also easily be adapted to general evolutionary problems having a different spatial operator A by defining suitable discrete spatial function spaces and the corresponding interpolation operators and providing sufficient interpolation error estimates.

THEOREM 4.5. *In the case of $M_0 > 0$, e.g., for a purely hyperbolic problem, we can also give a convergence result in the weighted L^2-type norm $\|\cdot\|_\rho$. Under the same conditions as in Theorem 4.2, we have*

$$\begin{aligned}\|U-U_h^\tau\|_\rho \le C\sqrt{1+T}\Big[&\tau^{r+1}\left(\|\partial_t^{r+1}U\|_\rho + \|\partial_t^{r+1}AU\|_\rho\right)\\ &+ h^k\left(\|U\|_{\mathbf{H}^k,\rho} + \|AU\|_{\mathbf{H}^k,\rho} + \|\partial_t U\|_{\mathbf{H}^k,\rho} + \|NU_0\|_{\mathbf{H}^k}\right)\Big].\end{aligned}$$

Proof. For this result we need

- a local norm equivalence for all $W_h^\tau \in \mathcal{U}_h^\tau$,

$$\|W_h^\tau\|_{\rho,m}^2 \le C_1\left(\gamma\|\Pi_h^\tau W_h^\tau\|_{\rho,m}^2 + \tau_m\|M_0^{1/2}W_h^\tau(t_m)\|_H^2 \mathrm{e}^{-2\rho t_m}\right),$$

  with a constant C_1 independent of τ_m and W_h^τ, that holds true because $\Pi_h^\tau W_h^\tau - W_h^\tau$ is a multiple of a weighted Legendre polynomial of degree r, t_m is not a zero of it, and the scaling with respect to τ_m of the two terms is the same;
- a local estimate of the discrete error ξ with a localisation of the norms to the interval $[0,t_m]$ instead of $[0,T]$,

$$\begin{aligned}&\|\xi\|_{\rho,[0,t_m]}^2 + \frac{1}{2}\|M_0^{1/2}\xi(t_m)\|_H^2 \mathrm{e}^{-2\rho t_m}\\ &\le C\Big[\tau^{2(r+1)}\left(\|\partial_t^{r+1}U\|_{\rho,[0,t_m]}^2 + \|\partial_t^{r+1}AU\|_{\rho,[0,t_m]}^2\right)\\ &\quad + h^{2k}\Big(\|U\|_{\mathbf{H}^k,\rho,[0,t_m]}^2 + \|AU\|_{\mathbf{H}^k,\rho,[0,t_m]}^2 + \|U(t_m)\|_{\mathbf{H}^k}^2 \mathrm{e}^{-2\rho T} + \|NU_0\|_{\mathbf{H}^k}^2\Big)\Big],\end{aligned}$$

  which follows by the same arguments as in Theorem 4.2;

- a Sobolev embedding for $t_n < t_m$ and $U \in H^1_\rho([t_n, t_m], \mathbf{H})$,

  $$\|U(t_m)\|_H^2 \mathrm{e}^{-2\rho t_m} \leq C_{inv} \left(\frac{1}{t_m - t_n} \|U\|^2_{\rho,[t_n,t_m]} + (t_m - t_n)\|\partial_t U\|^2_{\rho,[t_n,t_m]} \right),$$

  with a constant C_{inv} independent of U, t_n, and t_m.

Then it follows that

$$\begin{aligned} \|\xi\|_\rho^2 &\leq C_1 \left(\gamma \|\Pi_h^\tau \xi\|_\rho^2 + \sum_{m=1}^{M} \tau_m \|M_0^{1/2} \xi(t_m)\|_H^2 \mathrm{e}^{-2\rho t_m} \right) \\ &\leq C_1(1+T) \Bigg[\tau^{2(r+1)} \left(\|\partial_t^{r+1} U\|_\rho^2 + \|\partial_t^{r+1} AU\|_\rho^2 \right) \\ &\qquad + h^{2k} \Bigg((1 + \frac{C_{inv}}{T}) \|U\|^2_{\mathbf{H}^k,\rho} + \|AU\|^2_{\mathbf{H}^k,\rho} \\ &\qquad\qquad + C_{inv} \|\partial_t U\|^2_{\mathbf{H}^k,\rho} + \|NU_0\|^2_{\mathbf{H}^k} \Bigg) \Bigg], \end{aligned}$$

where the Sobolev embedding for $\|U(T)\|_{\mathbf{H}^k} \mathrm{e}^{-\rho T}$ uses the whole interval $[0, T]$ and for $\|U(t_m)\|_{\mathbf{H}^k} \mathrm{e}^{-\rho t_m}$ only $[t_{m-1}, t_m]$, as well as $\tau_m \leq 1$. Together with the interpolation error bound

$$\|\eta\|_\rho \leq \|\eta_1\|_\rho + \|\eta_2\|_\rho \leq C \left(\tau^{r+1} \|\partial_t^{r+1} U\|_\rho + h^k \|U\|_{\mathbf{H}^k,\rho} \right),$$

the claim follows. ∎

5. Numerical examples. We consider two examples with unknown solutions. Simulations with known smooth solutions were also performed, and the theoretical convergence orders were observed. The following two examples show a more realistic behaviour in the case of systems of changing type. The fact that both examples have initial values zero is not a restriction. We look into the convergence behaviour also with respect to the weighted L^2-norm $\|\cdot\|_\rho$ in addition to the $|||\cdot|||_\rho$-norm, in order to compare the results with those of the discontinuous Galerkin method from [6]. In the finite discrete setting, both norms are equivalent. All computations were done in the finite-element framework $\mathbb{SOFE}$[1].

5.1. 1+1d example. Let us consider a first example with one spatial dimension and a combination of a hyperbolic and an elliptic region. To be more precise, let $\Omega = [-\pi, \pi]$, $\Omega_{\mathrm{hyp}} = [-\pi, 0]$, and $\Omega_{\mathrm{ell}} = [0, \pi]$. As final time, we set $T = 4\pi$. The problem is stated as

$$\left[\partial_t \begin{bmatrix} \chi_{\Omega_{\mathrm{hyp}}} & 0 \\ 0 & \chi_{\Omega_{\mathrm{hyp}}} \end{bmatrix} + \begin{bmatrix} \chi_{\Omega_{\mathrm{ell}}} & 0 \\ 0 & \chi_{\Omega_{\mathrm{ell}}} \end{bmatrix} + \begin{bmatrix} 0 & \partial_x \\ \mathring{\partial}_x & 0 \end{bmatrix} \right] U = F, \tag{5.1}$$

with homogeneous Dirichlet conditions for the first component of $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, the initial condition $U_0 = 0$, and a right-hand side $F(t, x) = (f(t, x), g(t, x)) \cdot \chi_{\geq 0}(t)$, where $\chi_{\geq 0}(t)$ is the characteristic function of the non-negative time line, and

$$f(t, x) = \frac{1}{5} \sin(3t) + \min\{t, \pi\} \cos(3x), \qquad g(t, x) = \sin(t) \left(1 - \frac{x^2}{\pi^2} \right).$$

Thus, F is continuous on $\mathbb{R}$, and it holds that $F(t) = 0$ for $t \leq 0$. Therefore, the solution theory of [7] gives the existence of a unique solution U that is continuous in time. Figure 5.1

[1] `github.com/SOFE-Developers/SOFE`

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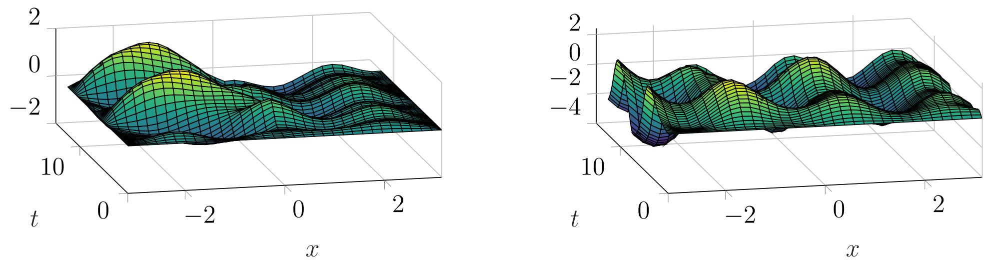

FIG. 5.1. *Solution of the problem* (5.1)*. First component (left) and second component (right).*

TABLE 5.1
Errors and rates for example (5.1)*.*

	cGP-method				dG-method	
$M = 2N$	$\vert\vert\vert U_{ref} - U_h^\tau \vert\vert\vert_\rho$		$\Vert U_{ref} - U_h^\tau\Vert_{\rho,[0,T]}$		$\Vert U_{ref} - U_h^\tau\Vert_{\rho,[0,T]}$	
			$k = 2$, $r = 1$			
256	2.120e-02		8.890e-04		1.808e-04	
512	5.746e-03	1.88	3.136e-04	1.50	7.751e-05	1.22
1024	1.787e-03	1.68	1.380e-04	1.18	3.496e-05	1.15
2048	7.036e-04	1.35	6.739e-05	1.03	1.580e-05	1.15
			$k = 3$, $r = 2$			
256	8.806e-04	1.05	1.187e-04		6.058e-05	
512	4.163e-04	1.08	5.489e-05	1.11	2.642e-05	1.20
1024	1.906e-04	1.13	2.492e-05	1.14	1.137e-05	1.22
2048	8.581e-05	1.15	1.114e-05	1.16	4.669e-06	1.28

displays plots of the components of the solution in the domain. Note that the first component has a kink along $x = 0$, i.e., it is continuous but not differentiable in x. As mesh we use an equidistant mesh of N cells in Ω and M cells in $[0, T]$. In order to calculate the errors, we use a reference solution U_{ref} instead of the unknown solution U. The reference solution is computed on a 4096×2048 mesh with polynomial degrees $k = 4$ and $r = 3$.

Table 5.1 shows the results for different values of M and N and different polynomial degrees k and r. We let $k = r + 1$ as the theory gives for smooth U the convergence order $\min\{k, r + 1\}$ if N and M are proportional. For the continuous Galerkin-Petrov method, we observe only a convergence rate between 1 and 2 in both norms. Increasing the polynomial degree reduces the error, but does not improve the rate much. A reason for this behaviour could be that U is not smooth enough for the error estimates to hold due to jumping coefficients in space and a non-differentiable right-hand side. Unfortunately, the exact solution for this problem and thus its precise regularity are unknown.

For comparison, we also computed approximations with the discontinuous Galerkin method from [6] that uses globally discontinuous piecewise polynomials of degree r in time and the same approximation in space as the method described in this paper. The errors given in the remaining columns show a similar behaviour with convergence rates between 1 and 2. Nevertheless, the errors are smaller for the discontinuous approach.

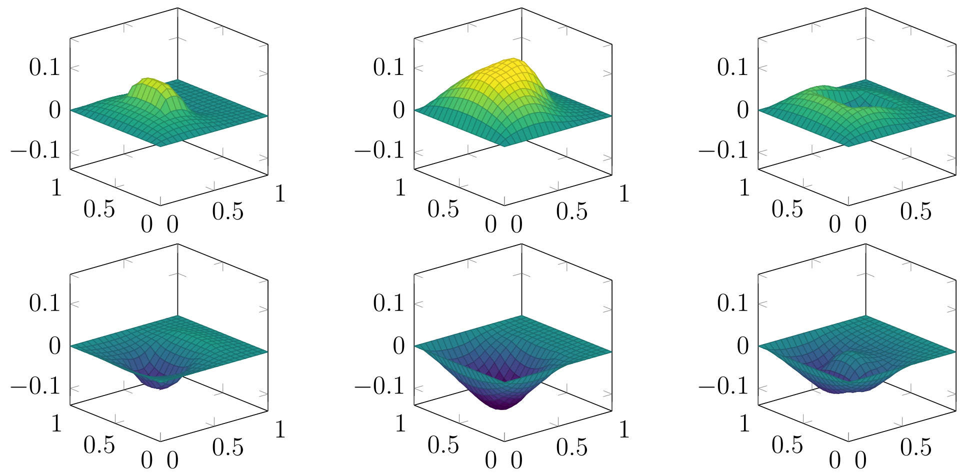

FIG. 5.2. *First component of* U *at times* $t = 5\ell/16$, *for* $\ell \in \{1, \dots, 6\}$, *(top left to bottom right) of the problem* (5.2).

TABLE 5.2
Errors $\|U_{ref} - U_h^\tau\|_{\rho,[0,T]}$ *and rates for example* (5.2).

	cGP-method				dG-method	
$M = 2N$	$\|\|\|U_{ref} - U_h^\tau\|\|\|_\rho$		$\|U_{ref} - U_h^\tau\|_{\rho,[0,T]}$		$\|U_{ref} - U_h^\tau\|_{\rho,[0,T]}$	
			$k = 2, r = 1$			
16	3.989e-02		1.961e-02		7.821e-03	
32	1.972e-02	1.02	9.199e-03	1.09	3.018e-03	1.37
64	9.435e-03	1.06	3.751e-03	1.29	8.813e-04	1.78
96	5.603e-03	1.29	1.324e-03	1.50	2.920e-04	1.59
			$k = 3, r = 2$			
16	1.041e-02		5.499e-03		2.790e-03	
32	3.689e-03	1.50	1.435e-03	1.94	6.385e-04	2.13
64	1.248e-03	1.56	4.430e-04	1.70	2.248e-04	1.51

5.2. 1+2d example. As a second example we consider the last example of [6]. Let $T = 5.2$, $\Omega = (0,1)^2 \subset \mathbb{R}^2$, $\Omega_{\text{hyp}} = \left(\frac{1}{4}, \frac{3}{4}\right)^2$, and $\Omega_{\text{ell}} = \Omega \setminus \bar{\Omega}_{\text{hyp}}$ The problem is given by

$$\left[\partial_t \begin{bmatrix} \chi_{\Omega_{\text{hyp}}} & 0 \\ 0 & \chi_{\Omega_{\text{hyp}}} \end{bmatrix} + \begin{bmatrix} \chi_{\Omega_{\text{ell}}} & 0 \\ 0 & \chi_{\Omega_{\text{ell}}} \end{bmatrix} + \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad}^\circ & 0 \end{bmatrix}\right] U = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{5.2}$$

where

$$f(t, \mathbf{x}) = 2\sin(\pi t) \cdot \chi_{\mathbb{R}_{<1/2} \times \mathbb{R}}(\mathbf{x}).$$

Figure 5.2 displays some snapshots of the first component of the solution $U : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2$, approximated by a numerical simulation. Again we use equidistant meshes with N cells in each dimension of space and M cells in $[0, T]$. As reference solution U_{ref} replacing the unknown exact solution, we use an approximation calculated with

$M = 192$, $N = 96$, $k = 3$, $r = 2$, and $M = 128$, $N = 64$, $k = 4$, $r = 3$, respectively. In Table 5.2, the results are shown. Similarly to the previous example we do not achieve the optimal convergence order for both methods. Here the data and the right-hand side have jumps along interior lines, which reduces the maximum regularity of the solution. Again the discontinuous Galerkin method has smaller errors.

6. Conclusions. The continuous solution of an evolutionary system with continuous right-hand side can be approximated by several methods. Here we investigated the continuous Galerkin-Petrov method that has optimal convergence order for smooth solutions in the $|||\cdot|||_\rho$-norm. The benefit of the continuous method compared to the discontinuous Galerkin method is continuity, which implies a non-dissipative behaviour. In our examples with unknown solutions, which are probably not smooth enough, the discontinuous Galerkin method is slightly better. Furthermore, these examples show that an increase of the polynomial degree in space beyond 2 and in time beyond 1 gives no huge benefit. This is different for smooth solutions—here both methods achieve the high theoretical convergence orders.

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