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MULTIGRID METHOD FOR $H(\mathrm{DIV})$ IN THREE DIMENSIONS*

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Abstract. We are concerned with the design and analysis of a multigrid algorithm for $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic linear variational problems. The discretization is based on $\boldsymbol{H}(\mathrm{div};\Omega)$–conforming Raviart–Thomas elements. A thorough examination of the relevant bilinear form reveals that a separate treatment of vector fields in the kernel of the divergence operator and its complement is paramount. We exploit the representation of discrete solenoidal vector fields as $\mathbf{curl}$s of finite element functions in so-called Nédélec spaces. It turns out that a combined nodal multilevel decomposition of both the Raviart–Thomas and Nédélec finite element spaces provides the foundation for a viable multigrid method. Its Gauß–Seidel smoother involves an extra stage where solenoidal error components are tackled. By means of elaborate duality techniques we can show the asymptotic optimality in the case of uniform refinement. Numerical experiments confirm that the typical multigrid efficiency is actually achieved for model problems.

Key words. multigrid, Raviart–Thomas finite elements, Nédélec's finite elements, multilevel, mixed finite elements.

AMS subject classifications. 65N55, 65N30.

1. Introduction. The Hilbert–space $\boldsymbol{H}(\mathrm{div};\Omega)$ is the space of square integrable vector fields with a square integrable divergence, defined on a domain Ω. The inner product is given by the bilinear form

$$a(\boldsymbol{v},\boldsymbol{j}) := (\boldsymbol{v},\boldsymbol{j})_{\boldsymbol{L}^2(\Omega)} + (\mathrm{div}\,\boldsymbol{v},\mathrm{div}\,\boldsymbol{j})_{L^2(\Omega)} \ ,\ \boldsymbol{v},\boldsymbol{j}\in\boldsymbol{H}(\mathrm{div};\Omega)\ .$$

In this paper Ω is supposed to be a bounded subset of $\mathbb{R}^3$ with polyhedral boundary $\partial\Omega$. Moreover, Ω and $\partial\Omega$ should be simply connected.

The significance of this space is due to the fact that it provides an appropriate description for vector-valued quantities whose flux through surfaces is of physical relevance. Consequently, the space $\boldsymbol{H}(\mathrm{div};\Omega)$ looms large in many mathematical models, when they are cast into variational form.

Suitable (Dirichlet–)boundary conditions can be imposed by prescribing the normal flux $\langle\boldsymbol{v},\boldsymbol{n}\rangle$ of a vectorfield $\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div};\Omega)$ on parts of the boundary. For the space with homogeneous boundary conditions throughout we adopt the notation $\boldsymbol{H}_0(\mathrm{div};\Omega)$. Yet the technical difficulties arising from imposing boundary conditions have not been totally overcome. For this reason we have to confine ourselves to free boundary values throughout this presentation.

In this paper the focus is on the variational problem: For $\boldsymbol{f}\in\boldsymbol{H}(\mathrm{div};\Omega)'$, seek $\boldsymbol{j}\in\boldsymbol{H}(\mathrm{div};\Omega)$ such that

$$a(\boldsymbol{j},\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{q}) \quad \forall\boldsymbol{q}\in\boldsymbol{H}(\mathrm{div};\Omega)\ . \tag{1.1}$$

As a concise operator notation we adopt $A\boldsymbol{j}=\boldsymbol{f}$. This equation obviously has a unique solution. The same applies to the discrete problem $A_h\boldsymbol{j}_h=\boldsymbol{f}_h$ that arises from restricting (1.1) to a conforming finite element subspace of $\boldsymbol{H}(\mathrm{div};\Omega)$. The present paper studies an algorithm that yields a fast iterative solver for the large linear system of equations the discrete problem boils down to. This is not merely a mathematical challenge, but matches an urgent demand for such a solver in several areas.

To begin with, variational problems posed over $\boldsymbol{H}(\mathrm{div};\Omega)$ naturally occur in the context of *mixed methods* for second order elliptic boundary value problems (see [10]). One option is to tackle the resulting saddle point problem by means of a preconditioned minimal residual

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algorithm. As pointed out in [3], §7 and [23], Sect. 3.4, a step in a powerful preconditioning scheme involves the approximate solution of (1.1).

Other ways to treat the mixed saddle point problems also zero in on variational problems similar to (1.1). Among them the penalty method (see [20]) and augmented Lagrangian techniques (see [36]) are prominent. In these cases we are generally faced with a bilinear form like

$$a_r(\boldsymbol{v}, \boldsymbol{j}) := (\boldsymbol{v}, \boldsymbol{j})_{\boldsymbol{L}^2(\Omega)} + r \cdot (\operatorname{div} \boldsymbol{v}, \operatorname{div} \boldsymbol{j})_{L^2(\Omega)} \tag{1.2}$$

over $\boldsymbol{H}(\mathrm{div}; \Omega)$, where $r > 0$ is a parameter which is usually chosen to be fairly large. This raises the issue of how the convergence of the multigrid method is affected by increasing the value of r. Fortunately it turns out to be robust with respect to large r as was shown in [23]. However, for the sake of lucidity, the investigations in this paper will not take into account r.

Furthermore, the variational problem (1.1) is also the key to efficient preconditioners for *first order system least squares* (FOSLS) formulations of second order elliptic boundary values problems. In [12] and [32] a close connection between the $\boldsymbol{H}(\mathrm{div}; \Omega)$–norm and the least squares functional has been established. These results revealed that a fast solver for (1.1) can be extremely useful for the treatment of the FOSLS systems of equations. For a more detailed discussion the reader is referred to §7 of [3].

Eventually, apart from second order problems, (1.1) emerges in the numerical treatment of the incompressible Navier–Stokes equations, as well. The so-called sequential regularization method (cf. [26]) requires the solution of a discrete equation of the form (1.1) in each timestep.

Our ultimate goal is to devise an efficient multigrid method for this discrete problem. In this context the notion of "efficient" implies two essential requirements:

1. A single step of the iteration should require a computational effort proportional to the number of unknowns.
2. The rate of convergence must be well below 1 and must not deteriorate on very fine finite element meshes

The first criterion is naturally met by a multigrid algorithm that relies on purely local operations. To confirm that the second is satisfied is much harder; to this end we rely on the modern algebraic theory of multilevel methods as outlined in [7, 21, 38]. Its essential message is that we only need to specify a multilevel decomposition of the finite element space used to approximate $\boldsymbol{H}(\mathrm{div}; \Omega)$. Then the multigrid algorithm can be recovered as a simple multiplicative Schwarz scheme based on this very decomposition. In addition, two fundamental estimates can completely describe the stability of the decomposition with respect to the energy norm $\|\cdot\|_A$ induced by the bilinear form $a(\cdot, \cdot)$ (which coincides with the natural norm on $\boldsymbol{H}(\mathrm{div}; \Omega)$). The constants occurring in these estimates provide rather comprehensive information on the convergence properties of the multigrid V–cycle iteration. The bulk of this paper will be devoted to determining on what the size of these constants does not depend.

The importance of $\boldsymbol{H}(\mathrm{div}; \Omega)$–related problems has prompted vigorous research into efficient multilevel schemes. An early attempt was the construction of a hierarchical basis in the paper [11] by Cai, Goldstein, and Pasciak. In the 2D case, it has been shown by Hoppe and Wohlmuth [25] that this scheme leads to a slightly suboptimal growth $O(L^2)$ of the condition number of the preconditioned system, where L is the total number of refinement levels. Surprisingly enough, this behaviour carries over to three dimensions.

An alternative multilevel splitting of a $\boldsymbol{H}(\mathrm{div}; \Omega)$–conforming finite element space was proposed by Vassilevski and Wang in [37]. In two dimensions this approach actually achieves uniformly bounded convergence rates independent of the number of levels involved, as has been proved in [24]. Both domain decomposition methods and multigrid schemes for (1.1) have been introduced by Arnold, Falk, and Winther [3, 4]. In 2D they managed to show that the convergence rates of their methods remain neatly bounded independently of the depth of refinement.

Mixed saddle point problems have also been tackled directly with multilevel methods. The schemes presented in [17, 18, 27] are based on the insight that the saddle point problem can be converted into a symmetric, positive definite problem in the subspace of divergence-free vectorfield. Theoretical optimality of the multilevel methods could be established in two dimensions. The same could be shown in [13] for a domain decomposition method in three dimensions which also employs the prior reduction to a solenoidal problem.

The method to be developed in the current paper owes much to the ideas of Vassilevski and Wang [37] and Arnold, Falk, and Winther [3], as far as the central role of *Helmholtz decompositions* is concerned. The term Helmholtz decomposition designates an L^2–orthogonal splitting of a function space into the kernel of a differential operator (div or $\mathbf{curl}$) and its complement. Obviously the kernel of the divergence operator has a decisive impact on the properties of the bilinear form $a(\cdot,\cdot)$. By using the Helmholtz decomposition of $\boldsymbol{H}(\mathrm{div};\Omega)$, this can be taken into account.

The principles guiding the design of the multigrid algorithm presented in this paper are basically the same in any dimension. Yet the algorithmic details and the technical devices employed in the proofs in three dimensions significantly differ from those used by the authors mentioned above in the 2D case. Additional complications are due to the different nature of *"vector potential spaces"* in 2D and 3D. Vector potentials provide a representation of solenoidal vector fields. In 2D those can be obtained as rotated gradients of H^1–functions, whereas in $\mathbb{R}^3$ the $\mathbf{curl}$–operator and the Hilbert space $\boldsymbol{H}(\mathbf{curl};\Omega)$ have to be used (see [20], Ch. I). Clearly, the $\mathbf{curl}$ operator is much more difficult to handle than the gradient. This offers an explanation why rigorous results for the 3D case were long missing.

The plan of the paper is a follows: In the next section we provide a brief description of the finite element spaces used in the construction of the multigrid algorithm. Those are the $\boldsymbol{H}(\mathrm{div};\Omega)$–conforming Raviart–Thomas spaces and $\boldsymbol{H}(\mathbf{curl};\Omega)$–conforming Nédélec spaces. We also summarize their relevant properties and discuss the close relationship between them.

In the third section the multilevel decomposition of the Raviart–Thomas spaces is specified. Prior to that, we try to give a sound motivation of the construction by scrutinising the properties of the bilinear form $a(\cdot,\cdot)$. Finally we recall the basic estimates that guarantee an optimal convergence of the multigrid iteration based on the decomposition.

The fourth section examines one of the crucial concept in the design and analysis of the multigrid method, namely *Helmholtz–decompositions*. In the discrete setting we are forced to introduce different kinds of these decompositions and then have to establish several auxiliary estimates linking them.

The fifth section is devoted to proving the central estimate related to the stability of the decomposition with respect to the energy norm. We show uniform stability (w.r.t. the depth of refinement) by means of duality techniques applied to both $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$–conforming finite element spaces.

The sixth section provides the second estimate, a strengthened Cauchy–Schwarz inequality, for the multilevel decomposition. The proof is purely local and adapts techniques invented for standard $H^1(\Omega)$–conforming problems.

In the next to last section we discuss the implementation of the scheme in a standard multigrid fashion and explain a few algorithmic details.

In the last section we report on numerical experiments which bolster the claim that the multigrid method developed in this paper actually provides a competitive iterative solver for discrete $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic variational problems.

2. Finite element spaces. Let $\mathcal{T}_h := \{T_i\}_i$ denote a quasiuniform simplicial or hexaedral triangulation of Ω with meshwidth $h := \max\{\mathrm{diam}\,T_i\}$. We demand that the elements are uniformly shape–regular in the sense of[14]. Based on this mesh we introduce several conforming finite element spaces:

$\mathcal{S}_d(\mathcal{T}_h) \subset H^1(\Omega)$ stands for the space of continuous finite element functions, piecewise polynomial of degree $d \in \mathbb{N}$. $\boldsymbol{\mathcal{ND}}_d(\mathcal{T}_h) \subset \boldsymbol{H}(\mathbf{curl};\Omega)$ designates the so-called Nédélec finite element space of order $d \in \mathbb{N}$ introduced in [29]. We write $\boldsymbol{\mathcal{RT}}_d(\mathcal{T}_h) \subset \boldsymbol{H}(\mathrm{div};\Omega)$ for the Raviart–Thomas finite element space of order $d \in \mathbb{N}_0$ (see [10, 29, 33]). Finally, the space of discontinuous functions, that are piecewise polynomial of degree $d \in \mathbb{N}_0$, is denoted by $\mathcal{Q}_d(\mathcal{T}_h) \subset L^2(\Omega)$. Supplemented by a subscript 0 the same notations cover the spaces equipped with homogeneous boundary conditions (in the sense of an appropriate trace operator). In addition, $\mathcal{Q}_{d,0}(\mathcal{T}_h)$ contains only functions with zero mean value. We hope the reader will not mind our policy to stick with somewhat bulky notations rather than run the risk of ambiguity and confusion.

All finite element spaces are equipped with sets $\Xi(\mathcal{X}_d, \mathcal{T}_h)$, $\mathcal{X} = \mathcal{S}$, $\boldsymbol{\mathcal{ND}}$, $\boldsymbol{\mathcal{RT}}$, $\mathcal{Q}$, of *global degrees of freedom* (d.o.f.) which ensure conformity. They can be defined in a canonical fashion so that they remain invariant under the respective canonical transformations of finite element functions. Consequently, all finite element spaces form affine families in the sense of [14]. We refer to [29] for a comprehensive exposition. Besides, we impose a p–hierarchical arrangement on the sets of degrees of freedom by requiring that $\Xi(\mathcal{X}_{d-1}, \mathcal{T}_h)$ is contained in $\Xi(\mathcal{X}_d, \mathcal{T}_h)$, and all functionals from $\Xi(\mathcal{X}_d, \mathcal{T}_h)/\Xi(\mathcal{X}_{d-1}, \mathcal{T}_h)$ have to vanish on $\mathcal{X}_{d-1}$.

Based on the degrees of freedom, sets of *canonical nodal basis functions* can be introduced as bidual bases for $\Xi(\mathcal{X}_d, \mathcal{T}_h)$. They are locally supported and form an L^2*–frame*: We can find generic constants $\underline{C}, \overline{C} > 0$, independent of the meshwidth h and only depending on d and the shape regularity of $\mathcal{T}_h$, such that

$$
\begin{array}{lllllll}
\underline{C}\|\boldsymbol{\xi}_h\|^2_{\boldsymbol{L}^2(\Omega)} & \le & \sum_\kappa \kappa(\boldsymbol{\xi}_h)^2\|\boldsymbol{\psi}_\kappa\|^2_{\boldsymbol{L}^2(\Omega)} & \le & \overline{C}\,\|\boldsymbol{\xi}_h\|^2_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_d(\mathcal{T}_h) \\
\underline{C}\|\boldsymbol{v}_h\|^2_{\boldsymbol{L}^2(\Omega)} & \le & \sum_\kappa \kappa(\boldsymbol{v}_h)^2\|\boldsymbol{j}_\kappa\|^2_{\boldsymbol{L}^2(\Omega)} & \le & \overline{C}\,\|\boldsymbol{v}_h\|^2_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_d(\mathcal{T}_h)\ ,
\end{array}
\tag{2.1}
$$

where κ runs through all degrees of freedom of the respective finite element space and $\boldsymbol{\psi}_\kappa$ stands for the canonical basis function of $\boldsymbol{\mathcal{ND}}_d(\mathcal{T}_h)$ belonging to the d.o.f. $\kappa \in \Xi(\boldsymbol{\mathcal{ND}}_d, \mathcal{T}_h)$, $\boldsymbol{j}_\kappa$ for the basis function in $\boldsymbol{\mathcal{RT}}_d(\mathcal{T}_h)$ associated with $\kappa \in \Xi(\boldsymbol{\mathcal{RT}}_d, \mathcal{T}_h)$. Moreover, following a popular convention, a capital C will be used as a generic constant. Its value can vary between different occurrences, but we will always specify what it must not depend on.

Now, given the degrees of freedom, for sufficiently smooth functions the nodal projections (nodal interpolation operators) $\Pi^{\mathcal{X}_d}_{\mathcal{T}_h}$, $\mathcal{X} = \mathcal{S}, \boldsymbol{\mathcal{ND}}, \boldsymbol{\mathcal{RT}}, \mathcal{Q}$ are well defined. The nodal interpolation operators are exceptional in that they satisfy the following *commuting diagram property* [10, 15, 19] (for $d \in \mathbb{N}_0$)

$$
\begin{array}{ccccccc}
C^\infty(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\mathrm{div}} & C^\infty(\Omega) \\
\Big\downarrow \Pi^{\mathcal{S}_{d+1}}_{\mathcal{T}_h} & & \Big\downarrow \boldsymbol{\Pi}^{\boldsymbol{\mathcal{ND}}_{d+1}}_{\mathcal{T}_h} & & \Big\downarrow \boldsymbol{\Pi}^{\boldsymbol{\mathcal{RT}}_d}_{\mathcal{T}_h} & & \Big\downarrow \Pi^{\mathcal{Q}_d}_{\mathcal{T}_h} \\
\mathcal{S}_{d+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{\mathcal{ND}}_{d+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{RT}}_d(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{Q}_d(\mathcal{T}_h)\ ,
\end{array}
$$

which links nodal projectors and differential operators. The commuting diagram property is the key to the proof of the following *representation theorem*, which shows that essential algebraic properties of the function spaces are preserved in the discrete setting:

THEOREM 2.1 (Discrete potentials). *The following sequences of vector spaces are exact for any $d > 0$:*

$$\{\text{const.}\} \longrightarrow \mathcal{S}_d(\mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{ND}_d(\mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{RT}_{d-1}(\mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathcal{Q}_{d-1}(\mathcal{T}_h) \longrightarrow \{0\}$$
$$\{0\} \xrightarrow{Id} \mathcal{S}_{d,0}(\mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{ND}_{d,0}(\mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{RT}_{d-1,0}(\mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathcal{Q}_{d-1,0}(\mathcal{T}_h) \longrightarrow \{0\}$$

Proof. See [23], Theorem 2.36. □

Another consequence of the commuting diagram property is that p–hierarchical surpluses are preserved when the appropriate differential operator is applied. For Nédélec spaces this reads:

$$\mathbf{curl}\left(\mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{ND}_{d+1}} - \mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{ND}_d}\right)\mathcal{ND}_{d+1}(\mathcal{T}_h) \subset \left(\mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{RT}_d} - \mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{RT}_{d-1}}\right)\mathcal{RT}_d(\mathcal{T}_h)\ . \tag{2.2}$$

An inconvenient trait of the nodal projectors has to be stressed: Except in the case of $\mathcal{Q}_k$, they cannot be extended to the respective continuous function spaces. A slightly enhanced smoothness of the argument function is required, which drastically complicates the use of these projectors. Nevertheless, we cannot dispense with them; no other projectors are known that satisfy the commuting diagram property (compare Remark 3.1 in [19]).

To cope with the $\mathcal{ND}_d$–projectors' need for smooth arguments, we have to resort to the following approximation property in fractional Sobolev spaces: From a variant of the Bramble–Hilbert lemma ([16], Theorem 6.1) we get for $d \geq 2$

$$\left\|\boldsymbol{\xi} - \mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{ND}_d}\boldsymbol{\xi}\right\|_{\boldsymbol{L}^2(\Omega)} \leq C\,h^s \|\boldsymbol{\xi}\|_{\boldsymbol{H}^s(\Omega)} \quad \forall \boldsymbol{\xi} \in \boldsymbol{H}^s(\Omega), 1 < s \leq 2\ , \tag{2.3}$$

with $C > 0$ only depending on s, d and the shape–regularity of $\mathcal{T}_h$. For Raviart-Thomas spaces we can settle for a simpler approximation property (see [10, 29]):

$$\left\|\boldsymbol{v} - \mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{RT}_d}\boldsymbol{v}\right\|_{\boldsymbol{L}^2(\Omega)} \leq C\,h\,|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \tag{2.4}$$

Other important estimates can be obtained via the commuting diagram property (see [29])

$$\begin{array}{rcll} \left\|\mathbf{curl}\left(\boldsymbol{\xi} - \mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{ND}_d}\boldsymbol{\xi}\right)\right\|_{\boldsymbol{L}^2(\Omega)} & \leq & C\,h\,|\mathbf{curl}\,\boldsymbol{\xi}|_{\boldsymbol{H}^1(\Omega)} & \forall \boldsymbol{\xi}; \mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{H}^1(\Omega) \\ \left\|\mathrm{div}\left(\boldsymbol{v} - \mathbf{\Pi}_{\mathcal{T}_h}^{\mathcal{RT}_d}\boldsymbol{v}\right)\right\|_{L^2(\Omega)} & \leq & C\,h\,|\mathrm{div}\,\boldsymbol{v}|_{H^1(\Omega)} & \forall \boldsymbol{v}; \mathrm{div}\,\boldsymbol{v} \in H^1(\Omega) \end{array} \tag{2.5}$$

with $C > 0$ independent of h.

To steer clear of problems arising from irregularly shaped domains it turns out to be convenient that the following discrete extension theorem holds (see [1]):

THEOREM 2.2 (Discrete extension theorem for $\mathcal{RT}_0$). *Let $\widetilde{\Omega} \subset \mathbb{R}^3$ be a large polyhedron which contains Ω in its interior. Further, $\widetilde{\Omega}$ must allow us to extend the mesh $\mathcal{T}_h$ on Ω to a triangulation $\widetilde{\mathcal{T}}_h$ of $\widetilde{\Omega}$ without a loss of shape regularity or quasiuniformity. Then there are linear continuous extension operators mapping vector fields in $\mathcal{RT}_d(\mathcal{T}_h)$ to $\mathcal{RT}_{d,0}(\widetilde{\mathcal{T}}_h)$, whose norms do not depend on the meshwidth h.*

Proof. See the proof of Thm. 2.46 in [23] □

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3. Multilevel decomposition. The performance of standard multilevel schemes for linear discrete variational problems crucially hinges on the "ellipticity" of the bilinear form. Crudely speaking, ellipticity implies that the eigenvalue belonging to an eigenfunction of the associated operator depends only on the "frequency" of the eigenfunction and becomes greater with higher frequency.

Obviously the bilinear form $a(\cdot,\cdot)$ lacks outright ellipticity. If restricted to the kernel $\mathcal{N}(\mathrm{div})$ of the divergence operator, it agrees with the L^2-inner product. In other words, in the subspace $\mathcal{N}(\mathrm{div})$ no amplification of highly oscillatory functions occurs. Conversely, we may expect a proper elliptic character of $a(\cdot,\cdot)$ on the L^2–orthogonal complement $\mathcal{N}(\mathrm{div})^\perp$, where the $(\mathrm{div}\,\cdot,\mathrm{div}\,\cdot)_{L^2(\Omega)}$–part prevails. By and large, it is precisely the two components of the Helmholtz decomposition of $\boldsymbol{H}(\mathrm{div};\Omega)$ that require a different treatment, reflecting the different character of the problem (1.1) on these components.

To elucidate this further, let us temporarily switch to the entire space $\mathbb{R}^3$. Straightforward calculations in the frequency domain bear out the ellipticity on $\mathcal{N}(\mathrm{div})^\perp$:

$$a(\boldsymbol{v},\boldsymbol{j}) = (\boldsymbol{v},\boldsymbol{j})_{\boldsymbol{L}^2(\mathbb{R}^3)} + (\nabla\boldsymbol{v},\nabla\boldsymbol{j})_{\boldsymbol{L}^2(\mathbb{R}^3)} \quad \forall \boldsymbol{v},\boldsymbol{j}\in\boldsymbol{H}(\mathrm{div};\mathbb{R}^3)\cap\mathcal{N}(\mathrm{div})^\perp .$$

This means that when restricted to $\mathcal{N}(\mathrm{div})^\perp$, the differential operator $\mathbf{grad}\,\mathrm{div}$ associated with A agrees with the Laplacian plus a zero order term. Putting it crudely, we have

$$A \approx Id + \Delta \quad \text{on}\ \mathcal{N}(\mathrm{div})^\perp\ . \tag{3.1}$$

To deal with $\mathcal{N}(\mathrm{div})$ we make use of the representation theorem $\mathcal{N}(\mathrm{div}) = \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl};\Omega)$ (Thm. I.3.4 in [20]), which holds due to our special assumptions on Ω. It furnishes a lifting to a second order operator in potential space. Thus we can formulate the equivalence

$$a(\cdot,\cdot)_{|\mathcal{N}(\mathrm{div})} \Longleftrightarrow (\mathbf{curl}\,\cdot,\mathbf{curl}\,\cdot)_{\boldsymbol{L}^2(\Omega)}$$

with the right hand side being restricted to a suitable subspace of $\boldsymbol{H}(\mathbf{curl};\Omega)$. Consequently the bilinear form $(\boldsymbol{\xi},\boldsymbol{\eta}) \mapsto (\mathbf{curl}\,\boldsymbol{\xi},\mathbf{curl}\,\boldsymbol{\eta})_{\boldsymbol{L}^2(\Omega)}$ becomes our next target. In contrast to the 2D case, we confront a large nontrivial kernel $\mathcal{N}(\mathbf{curl})$. As before, we use a Helmholtz decomposition to switch to the L^2–orthogonal complement $\mathcal{N}(\mathbf{curl})^\perp$ and find that for $\Omega = \mathbb{R}^3$

$$(\mathbf{curl}\,\boldsymbol{\xi},\mathbf{curl}\,\boldsymbol{\eta})_{\boldsymbol{L}^2(\mathbb{R}^3)} = (\nabla\boldsymbol{\xi},\nabla\boldsymbol{\eta})_{\boldsymbol{L}^2(\mathbb{R}^3)} \quad \text{for}\ \boldsymbol{\xi},\boldsymbol{\eta}\in\boldsymbol{H}(\mathbf{curl};\mathbb{R}^3)\cap\mathcal{N}(\mathbf{curl})^\perp\ .$$

In a terse manner we can write

$$\mathbf{curl}^*\circ A\circ\mathbf{curl} = \Delta \quad \text{on}\ \mathcal{N}(\mathbf{curl})^\perp\ . \tag{3.2}$$

This time we do not have to worry about $\mathcal{N}(\mathbf{curl})$, since no zero order term is present in potential space. The gist of these considerations is that we can arrive at neat second order elliptic problems by treating the two components of the Helmholtz decomposition separately. It is well known how multilevel methods for such problems should look like (see [21, 30]): they should be based on a *nodal multilevel decomposition* of the finite element space encompassing all basis functions on several levels of refinement. This gives rise, for instance, to the standard V–cycle for the Laplacian discretized in $\mathcal{S}_1$ (see [21]), which doubtlessly gives superb efficiency.

Therefore, (3.1) and (3.2) suggest that we should give similar nodal multilevel decompositions of discrete spaces corresponding to $\mathcal{N}(\mathrm{div})^\perp$ and $\mathcal{N}(\mathbf{curl})^\perp$ a try. As the discussion in the next section will reveal, no convenient finite element bases are available for any reasonable choice of these spaces. However, keep in mind that we are only interested in an approximate inverse of A_h, which is provided by one sweep of the multigrid iteration. So it is acceptable to put up with a splitting that only approximates the exact Helmholtz decomposition. A hint is offered by the estimates

$$\begin{aligned} \|\boldsymbol{\psi}_\kappa\|_{\boldsymbol{L}^2(\Omega)} &\leq C\,h\|\mathbf{curl}\,\boldsymbol{\psi}_\kappa\|_{\boldsymbol{L}^2(\Omega)} \quad \forall\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_1,\mathcal{T}_h) \\ \|\boldsymbol{j}_\kappa\|_{\boldsymbol{L}^2(\Omega)} &\leq C\,h\|\mathrm{div}\,\boldsymbol{j}_\kappa\|_{L^2(\Omega)} \quad \forall\kappa\in\Xi(\boldsymbol{\mathcal{RT}}_0,\mathcal{T}_h)\;, \end{aligned} \tag{3.3}$$

which hold with constants independent of h. They imply that the nodal basis functions in either space come "close" to being orthogonal to the kernels of the differential operators. Moreover, (3.3) indicates that the basis functions on fine grids actually have an oscillatory character, giving evidence that a nodal multilevel decomposition makes sense.

To fix the setting, we assume that we have a nested sequence of quasiuniform triangulations $\mathcal{T}_l$, $l = 0,\dots,L$, of Ω, created by regular refinement of an initial mesh $\mathcal{T}_0$ as, for instance, described in [5] for simplicial meshes. Then the meshwidths h_l, $l = 0,\dots,L$, can be expected to decrease in geometric progression, usually $h_l = 2^{-l}h_0$. Moreover, we will treat only the lowest order case $\boldsymbol{\mathcal{RT}}_0$ and $\boldsymbol{\mathcal{ND}}_1$ in the sequel. Nevertheless, we emphasise that the approach can be extended to higher order finite elements in a straightforward fashion.

The concrete multilevel decomposition into mainly one-dimensional subspaces then reads

$$\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_L) = \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_0) + \sum_{l=1}^{L}\sum_{\kappa\in\Xi(\boldsymbol{\mathcal{RT}}_0,\mathcal{T}_l)}\mathrm{Span}\,\{\boldsymbol{j}_\kappa\} + \sum_{l=1}^{L}\sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_1,\mathcal{T}_l)}\mathrm{Span}\,\{\mathbf{curl}\,\boldsymbol{\psi}_\kappa\}\;. \tag{3.4}$$

In a multiplicative Schwarz framework, (3.4) immediately gives rise to a multigrid V–cycle. The discussion of the details of the algorithm will be postponed to Sect. 6.

However convincing the above heuristics, we have to provide a rigorous underpinning for the claim that this decomposition is a sound basis for a fast multigrid method. We have to show that (3.4) guarantees a sufficient decoupling of its components in terms of energy, no matter how big L might be. According to modern multilevel theory [34, 38, 40], this property can be gauged by means of two estimates: Formally writing $\{\mathcal{V}_j\}_j$ for the set of subspaces in (3.4), the first, which we chose to label a *stability estimate*, can be stated as

$$\inf\{\sum_j\|\boldsymbol{v}_j\|_A^2;\ \sum_j\boldsymbol{v}_j=\boldsymbol{v},\ \boldsymbol{v}_j\in\mathcal{V}_j\}\leq C_{\mathrm{stab}}\|\boldsymbol{v}\|_A^2 \quad \forall\boldsymbol{v}\in\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_L)\;, \tag{3.5}$$

where $\|\cdot\|_A$ stands for the "energy–norm" induced by the bilinear form $a(\cdot,\cdot)$.

The second is a *strengthened Cauchy–Schwarz inequality* of the form

$$a(\boldsymbol{v}_j,\boldsymbol{v}_k)\leq C_{\mathrm{orth}}\,q^{|k-j|}\|\boldsymbol{v}_j\|_A\|\boldsymbol{v}_k\|_A \quad \forall\boldsymbol{v}_j\in\mathcal{V}_j,\ \boldsymbol{v}_k\in\mathcal{V}_k\;, \tag{3.6}$$

where $0\leq q<1$. It makes a statement about the *quasi-orthogonality* of the subspaces. From [38], Thm. 4.4, and [40], Thm. 5.1, we have

THEOREM 3.1. *Provided that (3.5) and (3.6) hold, the convergence rate ρ_A of the multigrid V–cycle in the energy norm $\|\cdot\|_A$ is bounded above by*

$$\rho_A\leq 1-\frac{1}{C_{\mathrm{stab}}(1+\rho_E)^2}\quad \textit{with}\quad \rho_E:=C_{\mathrm{orth}}\frac{1+q}{1-q}\;.$$

It is now our main objective to prove that the constants in (3.5) and (3.6) do not depend on L, as should be expected from a decent multigrid method.

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4. Helmholtz decompositions. The considerations that led us to the multilevel decomposition centred around Helmholtz decompositions of vector fields. They are indispensable for theoretical investigations, but in the finite element setting their usefulness is tainted by the elusive character of some components.

The natural discrete Helmholtz decomposition of a vector field $\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{d,0}(\mathcal{T}_h)$ is given by

$$\boldsymbol{v}_h = \boldsymbol{v}_h^+ + \boldsymbol{v}_h^0 \,, \tag{4.1}$$

where

$$\boldsymbol{v}_h^0 \in \boldsymbol{\mathcal{RT}}^0_{d,0}(\mathcal{T}_h) := \{\boldsymbol{j}_h \in \boldsymbol{\mathcal{RT}}_{d,0}(\mathcal{T}_h) :\ \mathrm{div}\,\boldsymbol{j}_h = 0\}$$

and

$$\boldsymbol{v}_h^+ \in \boldsymbol{\mathcal{RT}}^+_{d,0}(\mathcal{T}_h) := \{\boldsymbol{j}_h \in \boldsymbol{\mathcal{RT}}_{d,0}(\mathcal{T}_h) : \left(\boldsymbol{j}_h, \boldsymbol{q}_h^0\right)_{\boldsymbol{L}^2(\Omega)} = 0 \ \forall \boldsymbol{q}_h^0 \in \boldsymbol{\mathcal{RT}}^0_{d,0}(\mathcal{T}_h)\} \,.$$

Analogously, we have for $\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_d(\mathcal{T}_h)$:

$$\boldsymbol{\xi}_h = \boldsymbol{\xi}_h^+ + \boldsymbol{\xi}_h^0 \,, \tag{4.2}$$

with

$$\boldsymbol{\xi}_h^0 \in \boldsymbol{\mathcal{ND}}^0_{d,0}(\mathcal{T}_h) := \{\boldsymbol{\eta}_h \in \boldsymbol{\mathcal{ND}}_{d,0}(\mathcal{T}_h) :\ \mathbf{curl}\,\boldsymbol{\eta}_h = 0\}$$

and

$$\boldsymbol{\xi}_h^+ \in \boldsymbol{\mathcal{ND}}^+_{d,0}(\mathcal{T}_h) := \{\boldsymbol{\eta}_h \in \boldsymbol{\mathcal{ND}}_{d,0}(\mathcal{T}_h) :\ \left(\boldsymbol{\eta}_h, \boldsymbol{\nu}_h^0\right)_{L^2(\Omega)} = 0 \ \forall \boldsymbol{\nu}_h^0 \in \boldsymbol{\mathcal{ND}}^0_{d,0}(\mathcal{T}_h)\} \,.$$

The spaces $\boldsymbol{\mathcal{RT}}^+_{d,0}(\mathcal{T}_h)$ and $\boldsymbol{\mathcal{ND}}^+_{d,0}(\mathcal{T}_h)$ seem to be just the right environments for investigations into the stability of the multilevel decomposition. At second glance, this hope turns out to be premature, since these spaces are not nested, i.e.

$$\begin{array}{rcl} \boldsymbol{\mathcal{RT}}^+_{d,0}(\mathcal{T}_{j-1}) & \not\subset & \boldsymbol{\mathcal{RT}}^+_{d,0}(\mathcal{T}_j) \\ \boldsymbol{\mathcal{ND}}^+_{d,0}(\mathcal{T}_{j-1}) & \not\subset & \boldsymbol{\mathcal{ND}}^+_{d,0}(\mathcal{T}_j) \,, \end{array}$$

nor are they contained in the corresponding continuous function spaces $\mathcal{N}(\mathrm{div})^\perp$ and $\mathcal{N}(\mathbf{curl})^\perp$. In a sense, they display all awkward properties of nonconforming finite element spaces. Many successful attempts have been made to tackle nonconforming schemes with multigrid [9, 31]. What renders these techniques futile in this case is the lack of a localised basis. After all, the "+-spaces" are not generic finite element spaces!

On the other hand we can regard the finite element functions $\boldsymbol{v}_h$ and $\boldsymbol{\xi}_h$ as generic members of the continuous function spaces. As such, they have alternative Helmholtz decompositions:

$$\boldsymbol{v}_h = \boldsymbol{v}_h^\perp + \boldsymbol{v}_h^* \tag{4.3}$$

where $\boldsymbol{v}_h^* \in \mathcal{N}(\mathrm{div})$ and $\boldsymbol{v}_h^\perp \in \mathcal{N}(\mathrm{div})^\perp$, and

$$\boldsymbol{\xi}_h = \boldsymbol{\xi}_h^\perp + \boldsymbol{\xi}_h^* \tag{4.4}$$

with $\boldsymbol{\xi}^* \in \mathcal{N}(\mathbf{curl})$ and $\boldsymbol{\xi}_h^\perp \in \mathcal{N}(\mathbf{curl})^\perp$. We write $\boldsymbol{\mathcal{RT}}_{d,0}^\perp(\mathcal{T}_h)$ and $\boldsymbol{\mathcal{ND}}_{d,0}^\perp(\mathcal{T}_h)$ for the finite dimensional spaces of all possible $\boldsymbol{v}_h^\perp$ and $\boldsymbol{\xi}_h^\perp$, respectively. It is easy to see that now all components of the Helmholtz decompositions are perfectly nested, in particular

$$\begin{array}{rcl} \boldsymbol{\mathcal{RT}}_{d,0}^\perp(\mathcal{T}_{j-1}) & \subset & \boldsymbol{\mathcal{RT}}_{d,0}^\perp(\mathcal{T}_j) \\ \boldsymbol{\mathcal{ND}}_{d,0}^\perp(\mathcal{T}_{j-1}) & \subset & \boldsymbol{\mathcal{ND}}_{d,0}^\perp(\mathcal{T}_j) \; . \end{array}$$

Yet, functions from theses spaces are no longer piecewise polynomial, but at least their images under the differential operators are. To see this, note that $\operatorname{div} \boldsymbol{v}_h^\perp = \operatorname{div} \boldsymbol{v}_h$ and $\mathbf{curl}\, \boldsymbol{\xi}_h^\perp = \mathbf{curl}\, \boldsymbol{\xi}_h$. This permits us to establish fundamental estimates in the next section. However, since the multilevel decomposition (3.4) is ultimately set in the original finite element spaces, we have to bridge the gap between both types of Helmholtz decompositions.

To this end we have to rely on the following regularity assumptions:

$$\left.\begin{array}{rcll} \operatorname{div} \boldsymbol{j} & \in & L^2(\Omega) & \\ \mathbf{curl}\, \boldsymbol{j} & = & 0 & \text{in } \Omega \\ \langle \boldsymbol{j}, \boldsymbol{n} \rangle & = & 0 & \text{on } \partial\Omega \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \boldsymbol{j} \in \boldsymbol{H}^1(\Omega) \\ \|\boldsymbol{j}\|_{H^1(\Omega)} \leq C \|\operatorname{div} \boldsymbol{j}\|_{L^2(\Omega)} \end{array}\right. , \tag{4.5}$$

and for some $0 < \epsilon \leq 1$

$$\left.\begin{array}{rcll} \mathbf{curl}\, \boldsymbol{\xi} & \in & H^\epsilon(\Omega) & \\ \operatorname{div} \boldsymbol{\xi} & = & 0 & \text{in } \Omega \\ \boldsymbol{\xi} \times \boldsymbol{n} & = & 0 & \text{on } \partial\Omega \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \boldsymbol{\xi} \in H^{1+\epsilon}(\Omega) \\ \|\boldsymbol{\xi}\|_{H^{1+\epsilon}(\Omega)} \leq C \|\mathbf{curl}\, \boldsymbol{\xi}\|_{H^\epsilon(\Omega)} \end{array}\right. . \tag{4.6}$$

LEMMA 4.1. *Provided that the regularity assumption (4.5) holds, we can estimate the difference between the non-solenoidal components of both Helmholtz decompositions (4.1) and (4.3) for Raviart–Thomas vector fields by*

$$\left\|\boldsymbol{v}_h^+ - \boldsymbol{v}_h^\perp\right\|_{\boldsymbol{L}^2(\Omega)} \leq C\, h \|\operatorname{div} \boldsymbol{v}_h\|_{L^2(\Omega)}$$

with $C > 0$ independent of $\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{0,0}(\mathcal{T}_h)$ and the meshwidth h.

Proof. Thanks to the regularity assumption (4.5) we immediately have $\boldsymbol{v}_h^\perp \in \boldsymbol{H}^1(\Omega)$. Furthermore by (2.4) we get the approximation estimate

$$\left\|\boldsymbol{v}_h^\perp - \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{RT}}_0} \boldsymbol{v}_h^\perp\right\|_{\boldsymbol{L}^2(\Omega)} \leq Ch \left\|\boldsymbol{v}_j^\perp\right\|_{H^1(\Omega)} \leq Ch \left\|\operatorname{div} \boldsymbol{v}_h^\perp\right\|_{L^2(\Omega)} .$$

From the commuting diagram property of the nodal interpolation operator we conclude

$$\operatorname{div}(\boldsymbol{v}_h^+ - \boldsymbol{v}_h^\perp) = 0 \quad \Rightarrow \quad \operatorname{div}\left(\boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{RT}}_0}(\boldsymbol{v}_h^+ - \boldsymbol{v}_h^\perp)\right) = 0 \, .$$

This means $\boldsymbol{z}_h := \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{RT}}_0}(\boldsymbol{v}_h^+ - \boldsymbol{v}_h^\perp) \in \boldsymbol{\mathcal{RT}}_{0,0}^0(\mathcal{T}_h)$ so that by its definition,

$$\left(\boldsymbol{v}_h^+, \boldsymbol{z}_h\right)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \text{and} \quad \left(\boldsymbol{v}_h^\perp, \boldsymbol{z}_h\right)_{\boldsymbol{L}^2(\Omega)} = 0 \, .$$

This together with a straightforward application of the Cauchy–Schwarz inequality finishes the proof:

$$\begin{aligned} \left\|\boldsymbol{v}_h^+ - \boldsymbol{v}_h^\perp\right\|_{\boldsymbol{L}^2(\Omega)}^2 &= \left(\boldsymbol{v}_h^+ - \boldsymbol{v}_h^\perp, (\boldsymbol{v}_h^\perp - \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{RT}}_0} \boldsymbol{v}_h^\perp) + (\boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{RT}}_0}(\boldsymbol{v}_h^\perp - \boldsymbol{v}_h^+)\right)_{L^2(\Omega)} \\ &\leq \left\|\boldsymbol{v}_h^+ - \boldsymbol{v}_h^\perp\right\|_{\boldsymbol{L}^2(\Omega)} Ch \|\operatorname{div} \boldsymbol{v}_h\|_{L^2(\Omega)}. \end{aligned}$$

□

LEMMA 4.2. *Assuming (4.6), we get the following estimate for the components of the Helmholtz decompositions (4.2) and (4.4) of a vector field in 2nd order Nédélec space with* $C > 0$ *independent of the meshwidth* h*:*

$$\left\|\boldsymbol{\xi}_h^+ - \boldsymbol{\xi}_h^\perp\right\|_{\boldsymbol{L}^2(\Omega)} \leq Ch\|\mathbf{curl}\,\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_{2,0}(\mathcal{T}_h).$$

Proof. The proof is similar to that of the previous lemma, slightly compounded by the tighter smoothness requirements of the interpolation operators in Nédélec space.

We start with the trivial observation that $\mathbf{curl}\,\boldsymbol{\xi}_h^\perp = \mathbf{curl}\,\boldsymbol{\xi}_h$ is piecewise polynomial. Now, recall the important fact that any piecewise polynomial function $f \in L^2(\Omega)$ belongs to $H^\varepsilon(\Omega)$ for all $0 \leq \varepsilon < 1/2$ and fulfils the inverse estimate

$$\|f\|_{H^\varepsilon(\Omega)} \leq C(\varepsilon) h_l^{-\varepsilon} \|f\|_{L^2(\Omega)} \tag{4.7}$$

with $C(\varepsilon)$ independent of f (cf. the appendix of [8]).

We conclude that $\mathbf{curl}\,\boldsymbol{\xi}_h^\perp \in \boldsymbol{H}^\varepsilon(\Omega)$ for some $\varepsilon \in]0;1/2[$. According to (4.6), this means that $\boldsymbol{\xi}_h^\perp \in \boldsymbol{H}^{1+\varepsilon}(\Omega)$ and

$$\left\|\boldsymbol{\xi}_h^\perp\right\|_{\boldsymbol{H}^{\varepsilon+1}(\Omega)} \leq C(\varepsilon)\left\|\mathbf{curl}\,\boldsymbol{\xi}_h^\perp\right\|_{\boldsymbol{H}^\varepsilon(\Omega)},$$

where we made tacit use of $\operatorname{div}\boldsymbol{\xi}_h^\perp = 0$. This makes sure that the nodal interpolation operator $\boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_2}$ is well defined for $\boldsymbol{\xi}_h^\perp$.

The commuting diagram property again guarantees that the interpolant $\boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_2}(\boldsymbol{\xi}_h^+ - \boldsymbol{\xi}_h^\perp)$ is $\mathbf{curl}$-free. Since $\boldsymbol{\xi}_h^+$ and $\boldsymbol{\xi}_h^\perp$ are both L^2–orthogonal to $\boldsymbol{\mathcal{ND}}_{2,0}^0(\mathcal{T}_h)$ we get

$$\left(\boldsymbol{\xi}_h^+ - \boldsymbol{\xi}_h^\perp, \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_2}(\boldsymbol{\xi}_h^+ - \boldsymbol{\xi}_h^\perp)\right)_{\boldsymbol{L}^2(\Omega)} = 0\,.$$

Using the approximation property (2.3) and the inverse estimate (4.7) we confirm

$$\left\|\boldsymbol{\xi}_h^\perp - \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_2}\boldsymbol{\xi}_h^\perp\right\|_{\boldsymbol{L}^2(\Omega)} \leq Ch\|\mathbf{curl}\,\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\Omega)}\,.$$

The final steps of the proof are almost the same as in the previous proof, so that we can skip them here. □

5. Proof of stability. In this section we are going to prove that inequality (3.5) holds for the splitting (3.4), uniformly in the depth L of refinement. Owing to the discrete extension theorem Thm. 2.2, it suffices to establish the stability of the multilevel decomposition for convex domains only: Since Ω is bounded we can find a convex domain $\widetilde{\Omega}$ such that Ω, equipped with the coarse mesh $\mathcal{T}_0$, and $\widetilde{\Omega}$ satisfy the assumptions of Thm. 2.2. $\widetilde{\mathcal{T}}_0$ denotes the extended mesh on $\widetilde{\Omega}$. Its regular refinement yields a nested sequence $\{\widetilde{\mathcal{T}}_j\}_{j=0}^L$ of triangulations which match the original meshes on Ω.

Then Thm. 2.2 tells us that for any $\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_L)$ there is a $\widetilde{\boldsymbol{v}}_h \in \boldsymbol{\mathcal{RT}}_{0,0}(\widetilde{\mathcal{T}}_L)$ defined on all of $\widetilde{\Omega}$ such that $\|\widetilde{\boldsymbol{v}}_h\|_A \leq C\|\boldsymbol{v}_h\|_A$. The constant $C > 0$ depends only on the domains Ω, $\widetilde{\Omega}$ and the shape regularity of $\widetilde{\mathcal{T}}_0$.

Provided that the estimate (3.5) is true for $\widetilde{\Omega}$ with a constant independent of L, we first pick a certain splitting of $\widetilde{\boldsymbol{v}}_h$ that satisfies (3.5). Sheer plain restriction of the individual terms of the decomposition to Ω will then provide a specimen of a decomposition of $\boldsymbol{v}_h$ for which

(3.5) is fulfilled in $\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_L)$. The constant C_{stab} remains the same. Thus the problem can be reduced to the case of a convex domain Ω.

What accounts for the particular appeal of a convex domain is the availability of powerful regularity results: Firstly, for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ with $\mathbf{curl}\,\boldsymbol{f} = 0$ in weak sense we have

$$
(5.1)\qquad \left.\begin{array}{rcll} -\mathbf{grad}\,\mathrm{div}\,\boldsymbol{j} + \boldsymbol{j} & = & \boldsymbol{f} & \text{in } \Omega \\ \mathbf{curl}\,\boldsymbol{j} & = & 0 & \text{in } \Omega \\ \langle \boldsymbol{j}, \boldsymbol{n} \rangle & = & 0 & \text{on } \partial\Omega \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \boldsymbol{j} \in \boldsymbol{H}^1(\Omega) \ \wedge \ \mathrm{div}\,\boldsymbol{j} \in H^1(\Omega) \\ \|\boldsymbol{j}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \\ \|\mathrm{div}\,\boldsymbol{j}\|_{H^1(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \ . \end{array}\right.
$$

Secondly, we have for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\mathrm{div}\,\boldsymbol{f} = 0$ weakly

$$
(5.2)\qquad \left.\begin{array}{rcll} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\eta} & = & \boldsymbol{f} & \text{in } \Omega \\ \mathrm{div}\,\boldsymbol{\eta} & = & 0 & \text{in } \Omega \\ \boldsymbol{\eta} \times \boldsymbol{n} & = & 0 & \text{on } \partial\Omega \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega) \ \wedge \ \mathbf{curl}\,\boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega) \\ \|\boldsymbol{\eta}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \\ \|\mathbf{curl}\,\boldsymbol{\eta}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \ . \end{array}\right.
$$

In addition, we point out that the regularity assumptions (4.5) and (4.6) can be verified for a convex domains as well [2, 35]. This is due to their close relationship with the regularity of Dirichlet and Neumann problems for the Laplacian [2].

To begin with, we pick an arbitrary $\boldsymbol{j}_L \in \boldsymbol{\mathcal{RT}}_{0,0}(\mathcal{T}_L)$. Our aim is to find a concrete decomposition according to (3.4) that complies with (3.5) and permits us to fix a C_{stab} for all L. The construction is pursued in the spirit of the work of Arnold, Falk and Winther [3] and involves an $a(\cdot,\cdot)$-orthogonal splitting followed by a levelwise discrete Helmholtz decomposition.

Writing $P_l : \boldsymbol{H}_0(\mathrm{div};\Omega) \mapsto \boldsymbol{\mathcal{RT}}_{0,0}(\mathcal{T}_l)$, $l = 0, \ldots, L$ for the $a(\cdot,\cdot)$-orthogonal projection onto the finite element spaces on different levels, and setting $P_{-1} := 0$, the first stage of the decomposition reads

$$
(5.3)\qquad \boldsymbol{j}_L = \sum_{l=0}^{L} (P_l - P_{l-1})\,\boldsymbol{j}_L =: \sum_{l=0}^{L} \boldsymbol{v}_l \ .
$$

The next stage involves discrete Helmholtz decompositions according to (4.1) on each level:

$$
(5.4)\qquad \boldsymbol{v}_l = \boldsymbol{v}_l^0 + \boldsymbol{v}_l^+ \ ,
$$

with $\mathrm{div}\,\boldsymbol{v}_l^0 = 0$ and $\boldsymbol{v}_l^+ \in \boldsymbol{\mathcal{RT}}_{0,0}^+(\mathcal{T}_l)$. It is important to note that $\boldsymbol{v}_l^0$ and $\boldsymbol{v}_l^+$ are $a(\cdot,\cdot)$-orthogonal too. Now, the crucial step consists of showing that the vector fields $\boldsymbol{v}_l^0$ and $\boldsymbol{v}_l^+$ can be chopped up into multiples of basis functions without a drastic increase in the overall energy. This is only possible for oscillatory functions. The following two lemmata, whose proof will be postponed a short while, validate this property for the components of the current decomposition.

LEMMA 5.1. *Using the notations from above we have*

$$
\left\|\boldsymbol{v}_l^+\right\|_{\boldsymbol{L}^2(\Omega)} \leq C\, h_l \|\boldsymbol{v}_l\|_A \ ,
$$

with $C > 0$ independent of $\boldsymbol{j}_L$ and l.

LEMMA 5.2. *There is a constant $C > 0$, independent of $\boldsymbol{j}_L$ and l, such that for $\boldsymbol{v}_l^0$ we can always find an $\boldsymbol{\eta}_l \in \boldsymbol{\mathcal{ND}}_{1,0}(\mathcal{T}_l)$ with $\mathbf{curl}\,\boldsymbol{\eta}_l = \boldsymbol{v}_l^0$ and*

$$
\|\boldsymbol{\eta}_l\|_{\boldsymbol{L}^2(\Omega)} \leq C h_l \|\mathbf{curl}\,\boldsymbol{\eta}_l\|_{\boldsymbol{L}^2(\Omega)} \ .
$$

Based on these auxiliary estimates we are able to prove the main theorem

THEOREM 5.3. *If we represent both $\boldsymbol{\eta}_l$ from Lemma 5.2 and $\boldsymbol{v}_l^+$ as a sum of components belonging to the one-dimensional subspaces that the splitting (3.4) is based on, i.e.,*

$$\begin{aligned} \boldsymbol{\eta}_l &= \sum_\kappa \boldsymbol{\eta}_{\kappa,l} \,, \quad \boldsymbol{\eta}_{\kappa,l} \in \mathrm{Span}\,\{\boldsymbol{\psi}_\kappa\}\,,\ \kappa \in \Xi(\boldsymbol{\mathcal{ND}}_{1,0}, \mathcal{T}_l) \\ \boldsymbol{v}_l^+ &= \sum_\kappa \boldsymbol{v}_{\kappa,l} \,, \quad \boldsymbol{v}_{\kappa,l} \in \mathrm{Span}\,\{\boldsymbol{j}_\kappa\}\,,\ \kappa \in \Xi(\boldsymbol{\mathcal{RT}}_{0,0}, \mathcal{T}_l)\,, \end{aligned}$$

then we get, with $C > 0$ independent of $\boldsymbol{j}_L$ and L,

$$\|\boldsymbol{v}_0\|_A^2 + \sum_{l=1}^{L}\sum_\kappa \|\boldsymbol{v}_{\kappa,l}\|_A^2 + \sum_{l=1}^{L}\sum_\kappa \left\|\mathbf{curl}\,\boldsymbol{\eta}_{\kappa,l}\right\|_A^2 \leq C\|\boldsymbol{j}_L\|_A^2\,.$$

Proof. Employing the inverse estimates (3.3) we immediately get

$$\begin{aligned} \|\boldsymbol{v}_{\kappa,l}\|_A^2 &= \|\boldsymbol{v}_{\kappa,l}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathrm{div}\,\boldsymbol{v}_{\kappa,l}\|_{L^2(\Omega)}^2 &\leq (1 + Ch_l^{-2})\|\boldsymbol{v}_{\kappa,l}\|_{\boldsymbol{L}^2(\Omega)}^2 \\ \left\|\mathbf{curl}\,\boldsymbol{\eta}_{\kappa,l}\right\|_A^2 &= \left\|\mathbf{curl}\,\boldsymbol{\eta}_{\kappa,l}\right\|_{\boldsymbol{L}^2(\Omega)}^2 &\leq Ch_l^{-2}\left\|\boldsymbol{\eta}_{\kappa,l}\right\|_{\boldsymbol{L}^2(\Omega)}^2\,, \end{aligned} \tag{5.5}$$

with constants independent of the functions and the level l.

Thanks to the L^2–stability of the nodal bases (cf. (2.1)), we can estimate

$$\begin{aligned} \sum_\kappa \|\boldsymbol{v}_{\kappa,l}\|_{\boldsymbol{L}^2(\Omega)}^2 &\leq C\left\|\boldsymbol{v}_l^+\right\|_{\boldsymbol{L}^2(\Omega)}^2 \\ \sum_\kappa \left\|\boldsymbol{\eta}_{\kappa,l}\right\|_{\boldsymbol{L}^2(\Omega)}^2 &\leq C\|\boldsymbol{\eta}_l\|_{\boldsymbol{L}^2(\Omega)}^2\,. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) and exploiting the L^2–orthogonality of (5.4) and the $a(\cdot,\cdot)$-orthogonality of (5.3) we can finish the proof:

$$\begin{aligned} &\|\boldsymbol{v}_0\|_A^2 + \sum_{l=1}^{L}\sum_\kappa \|\boldsymbol{v}_{\kappa,l}\|_A^2 + \sum_{l=1}^{L}\sum_\kappa \left\|\mathbf{curl}\,\boldsymbol{\eta}_{\kappa,l}\right\|_A^2 \leq \\ &\quad \leq \|\boldsymbol{v}_0\|_A^2 + \sum_{l=1}^{L}\left\{(1 + Ch_l^{-2})\left\|\boldsymbol{v}_l^+\right\|_{\boldsymbol{L}^2(\Omega)}^2 + Ch_l^{-2}\|\boldsymbol{\eta}_l\|_{\boldsymbol{L}^2(\Omega)}^2\right\} \leq \\ &\qquad \leq \|\boldsymbol{v}_0\|_A^2 + C\sum_{l=1}^{L}\left\{\left\|\boldsymbol{v}_l^+\right\|_A^2 + \left\|\boldsymbol{v}_l^0\right\|_A^2\right\} \leq C\|\boldsymbol{j}_L\|_A^2. \end{aligned}$$

The final step could be accomplished by virtue of Lemmata (5.1) and (5.2). ☐

The proofs of Lemmata 5.1 and 5.2 make heavy use of duality techniques. They adapt ideas that were first employed in multilevel theory for problems in H^1 (see e.g. [41]).

Proof. (Of Lemma 5.1) Since duality techniques are mainly suited to nested sequences of spaces, we first focus on the continuous Helmholtz decomposition (4.3) of $\boldsymbol{v}_l$. Then determine $\boldsymbol{z} \in \mathcal{N}(\mathrm{div})^\perp$ as the unique solution of

$$a(\boldsymbol{z}, \boldsymbol{q}) = \left(\boldsymbol{v}_l^\perp, \boldsymbol{q}\right)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{q} \in \mathcal{N}(\mathrm{div})^\perp\,.$$

Now, we can conclude from regularity assumption (5.1) that

$$\boldsymbol{z} \in \boldsymbol{H}^1(\Omega) \quad \text{and} \quad \|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} \leq C\left\|\boldsymbol{v}_j^\perp\right\|_{\boldsymbol{L}^2(\Omega)} \quad \text{and} \quad \|\mathrm{div}\,\boldsymbol{z}\|_{H^1(\Omega)} \leq C\left\|\boldsymbol{v}_l^\perp\right\|_{\boldsymbol{L}^2(\Omega)}\,.$$

Now, recall that $\boldsymbol{z}$ is $a(\cdot,\cdot)$–orthogonal to any divergence free vectorfield. Because of $\operatorname{div}(\boldsymbol{v}_l - \boldsymbol{v}_l^{\perp}) = 0$ this leads to

$$a(\boldsymbol{z}, \boldsymbol{v}_l^{\perp}) = a(\boldsymbol{z}, \boldsymbol{v}_l) = a(\boldsymbol{z} - \boldsymbol{q}_{l-1}, \boldsymbol{v}_l) \quad \forall \boldsymbol{q}_{l-1} \in \boldsymbol{\mathcal{RT}}_{0,0}(\mathcal{T}_{l-1}) ,$$

where we also made use of the $a(\cdot,\cdot)$–orthogonality of $\boldsymbol{v}_l$ and any finite element function on a coarser mesh which is implied by (5.3). Moreover, the smoothness of $\boldsymbol{z}$ permits us to apply the canonical projection operators onto the finite element spaces. So we arrive at

$$\left\|\boldsymbol{v}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} = a(\boldsymbol{z}, \boldsymbol{v}_l^{\perp}) \leq \left\|\boldsymbol{v}_l\right\|_A \cdot \left\| z - \boldsymbol{\Pi}_{\mathcal{T}_{l-1}}^{\boldsymbol{\mathcal{RT}}_0} \boldsymbol{z} \right\|_A .$$

We employ the approximation estimates (2.3) and (2.5) to get

$$\begin{aligned} \left\| \boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}_{l-1}}^{\boldsymbol{\mathcal{RT}}_0} \boldsymbol{z} \right\|_{\boldsymbol{L}^2(\Omega)} &\leq Ch_{l-1} \left\|\boldsymbol{z}\right\|_{\boldsymbol{H}^1(\Omega)} \leq Ch_l \left\|\boldsymbol{v}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} \\ \left\| \operatorname{div}(\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}_{l-1}}^{\boldsymbol{\mathcal{RT}}_0} \boldsymbol{z}) \right\|_{L^2(\Omega)} &\leq Ch_{l-1} \left\|\operatorname{div} \boldsymbol{z}\right\|_{H^1(\Omega)} \leq Ch_l \left\|\boldsymbol{v}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} , \end{aligned}$$

from which we infer

$$\left\|\boldsymbol{v}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} \leq Ch_l \left\|\boldsymbol{v}_l\right\|_A .$$

An application of Lemma 4.1 completes the proof.□

Proof. (Of Lemma 5.2) The idea much resembles that of the previous proof. Yet in order to use Lemma 4.2 we have to switch to higher order finite element spaces intermittently. In the beginning we exploit the $a(\cdot,\cdot)$–orthogonality of (5.3):

$$a(\boldsymbol{v}_l^0, \boldsymbol{q}_{l-1}^0) = a(\boldsymbol{v}_l, \boldsymbol{q}_{l-1}^0) = 0 \quad \forall \boldsymbol{q}_{l-1}^0 \in \boldsymbol{\mathcal{RT}}_{0,0}^0(\mathcal{T}_{l-1}) \tag{5.7}$$

Then we pick the unique $\boldsymbol{\zeta}_l^{\perp} \in \boldsymbol{\mathcal{ND}}_{2,0}^{\perp}(\mathcal{T}_l)$ with $\mathbf{curl}\, \boldsymbol{\zeta}_l^{\perp} = \boldsymbol{v}_l^0$ and get $\boldsymbol{\omega} \in \mathcal{N}(\mathbf{curl})^{\perp}$ from the variational equation

$$(\mathbf{curl}\, \boldsymbol{\omega}, \mathbf{curl}\, \boldsymbol{\xi})_{\boldsymbol{L}^2(\Omega)} = \left(\boldsymbol{\zeta}_l^{\perp}, \boldsymbol{\xi}\right)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{\xi} \in \mathcal{N}(\mathbf{curl})^{\perp} .$$

Under the regularity assumption (5.2) we have

$$\boldsymbol{\omega} \in \boldsymbol{H}^1(\Omega) \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{\omega} \in \boldsymbol{H}^1(\Omega) \quad \text{and} \quad \left\|\mathbf{curl}\, \boldsymbol{\omega}\right\|_{\boldsymbol{H}^1(\Omega)} \leq C \left\|\boldsymbol{\zeta}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} .$$

Next, we conclude from the commuting diagram property that $\mathbf{curl}\, \boldsymbol{\Pi}_{\mathcal{T}_{l-1}}^{\boldsymbol{\mathcal{ND}}_1} \boldsymbol{\omega} \in \boldsymbol{\mathcal{RT}}_{0,0}^0(\mathcal{T}_{l-1})$. Then, using the approximation estimate (2.5) and taking into account (5.7) we get

$$\begin{aligned} \left\|\boldsymbol{\zeta}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} &= \left(\mathbf{curl}\, \boldsymbol{\omega}, \mathbf{curl}\, \boldsymbol{\zeta}_l^{\perp}\right)_{\boldsymbol{L}^2(\Omega)} \\ &\leq \left\|\mathbf{curl}\, \boldsymbol{\zeta}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} \cdot \left\|\mathbf{curl}(\boldsymbol{\omega} - \boldsymbol{\Pi}_{\mathcal{T}_{l-1}}^{\boldsymbol{\mathcal{ND}}_1} \boldsymbol{\omega})\right\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq Ch_{l-1} \left\|\mathbf{curl}\, \boldsymbol{\zeta}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} \left\|\mathbf{curl}\, \boldsymbol{\omega}\right\|_{\boldsymbol{H}^1(\Omega)} \\ &\leq Ch_l \left\|\boldsymbol{\zeta}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} \left\|\mathbf{curl}\, \boldsymbol{\zeta}_l^{\perp}\right\|_{\boldsymbol{L}^2(\Omega)} \end{aligned}$$

If $\boldsymbol{\zeta}_l^{+} \in \boldsymbol{\mathcal{ND}}_{2,0}^{+}(\mathcal{T}_l)$ is characterised by $\mathbf{curl}\, \boldsymbol{\zeta}_l^{+} = \boldsymbol{v}_l^0$, then Lemma 4.2 provides us with the estimate

$$\left\|\boldsymbol{\zeta}_l^{+}\right\|_{\boldsymbol{L}^2(\Omega)} \leq Ch_l \left\|\mathbf{curl}\, \boldsymbol{\zeta}_l^{+}\right\|_{\boldsymbol{L}^2(\Omega)} .$$

Still, we have to return to the lowest order spaces. To this end we carry out a p–hierarchical splitting of $\boldsymbol{\zeta}_l^+$

$$\boldsymbol{\zeta}_l^+ = \boldsymbol{\eta}_l + \widehat{\boldsymbol{\zeta}}_l \ ,$$

where $\boldsymbol{\eta}_l \in \boldsymbol{\mathcal{ND}}_{1,0}(\mathcal{T}_l)$ and $\widehat{\boldsymbol{\zeta}}_l$ stands for the p–hierarchical surplus in $\boldsymbol{\mathcal{ND}}_{2,0}(\mathcal{T}_l)$. As $\mathbf{curl}\,\boldsymbol{\zeta}_l^+$ belongs to the lowest order Raviart–Thomas space and the p–hierarchical splitting constitutes a direct sum, we learn from (2.2) that $\mathbf{curl}\,\widehat{\boldsymbol{\zeta}}_l = 0$. Additionally, the L^2–stability (2.1) of the nodal bases of Nédélec spaces ensures

$$\|\boldsymbol{\eta}_l\|_{L^2(\Omega)} \leq C \|\boldsymbol{\zeta}_l^+\|_{L^2(\Omega)} \ .$$

In sum, $\boldsymbol{\eta}_l$ is the function in lowest order potential space with the desired properties.□

6. Quasi-orthogonality. To establish the strengthened Cauchy–Schwarz inequality (3.6) we resort to tricks that have been conceived e.g. in [6, 38, 39] for the standard H^1–conforming case. For applications of these techniques in the 2D case, which is hardly different from 3D, we refer to [4, 24].

To begin with, we put the basis functions of the finite element spaces into a small number of classes, such that the supports of two basis functions in the same class do not overlap. $\boldsymbol{\mathcal{RT}}_0$–degrees of freedom are attached to faces, whereas $\boldsymbol{\mathcal{ND}}_1$–degrees of freedom are associated with edges (c.f. [29]). Hence, we may as well start with partitioning the faces/edges of the meshes $\mathcal{T}_l$ into disjoint sets $\mathsf{F}^i_{[l]}$ and $\mathsf{E}^i_{[l]}$. A finite number N_F and N_E, respectively, of sets will do on any level of refinement due to the uniform shape regularity of the meshes. We introduce the notations $\boldsymbol{\mathcal{D}}^i_l$, $i = 1, \dots, N_\mathsf{F}$ and $\boldsymbol{\mathcal{U}}^i_l$, $i = 1, \dots, N_\mathsf{E}$, for the subspaces of $\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l)$ and $\boldsymbol{\mathcal{ND}}_1(\mathcal{T}_l)$, respectively, spanned by the basis functions in class number i. Note that the basis functions in one class are mutually orthogonal. This is the reason why we might well replace the one-dimensional subspaces in (3.4) by the $\boldsymbol{\mathcal{D}}^i_l$ and $\boldsymbol{\mathcal{U}}^i_l$ without the slightest impact on C_{stab} and C_{orth}.

We separately deal with solenoidal and non-solenoidal vector fields in the finer mesh. The first results concerns a local estimate in the divergence free case:

LEMMA 6.1. *Pick an arbitrary triangle $\tilde{T} \in \mathcal{T}_m$ ($m \in \{0, \dots, L-1\}$) and a $L \geq k > m$. For any $\boldsymbol{\xi}^i_k \in \boldsymbol{\mathcal{U}}^i_k$, ($1 \leq i \leq N_\mathsf{F}$) and $\boldsymbol{z}_m \in \mathcal{RT}_0(\mathcal{T}_m)$ there holds*

$$a_{|\tilde{T}}(\mathbf{curl}\,\boldsymbol{\xi}^i_k, \boldsymbol{z}_m) \leq C \sqrt{\frac{h_k}{h_m}} \left\|\mathbf{curl}\,\boldsymbol{\xi}^i_k\right\|_{\boldsymbol{L}^2(\tilde{T})} \|\boldsymbol{z}_m\|_{\boldsymbol{L}^2(\tilde{T})}.$$

Proof. Consider the basis representation

$$\boldsymbol{\xi}^i_k = \sum_e \kappa_e(\boldsymbol{\xi}^i_k)\boldsymbol{\psi}_e \ ,$$

where the sum covers all edges in $\mathsf{E}^i_{[l]}$ that lie in the closure of $\tilde{T}$. The the degrees of freedom and nodal basis functions are tagged with the associated edge. The classical idea from [6] is to isolate an internal part of $\boldsymbol{\xi}^i_k$ and a boundary part:

$$\boldsymbol{\xi}^i_k = \sum_{e \subset \partial T, e \in \mathsf{E}^i_{[l]}} \kappa_e(\boldsymbol{\xi}^i_k)\boldsymbol{\psi}_e + \sum_{e \subset T, e \in \mathsf{E}^i_{[l]}} \kappa_e(\boldsymbol{\xi}^i_k)\boldsymbol{\psi}_e =: \boldsymbol{\xi}^i_{k,bd} + \boldsymbol{\xi}^i_{k,int}. \tag{6.1}$$

This approach is motivated by an interesting feature of the $\mathcal{RT}_0$ flux ansatz: We have $\mathbf{curl}(\boldsymbol{a}+\beta\boldsymbol{x})=0$. This permits us to drop the internal part of $\boldsymbol{\xi}_k^i$, because of

$$\int_{\tilde{T}} \left\langle \mathbf{curl}\,\boldsymbol{\xi}_{k,int}^i, \boldsymbol{z}_m \right\rangle d\boldsymbol{x} = \underbrace{\int_{\tilde{T}} \left\langle \boldsymbol{\xi}_{k,int}^i, \mathbf{curl}\,\boldsymbol{z}_m \right\rangle d\boldsymbol{x}}_{=0,\text{ as }\mathbf{curl}\,\boldsymbol{z}_m=0} - \underbrace{\int_{\partial\tilde{T}} \left\langle \boldsymbol{z}^m, \boldsymbol{\xi}_{k,int}^i \times \boldsymbol{n} \right\rangle d\boldsymbol{x}}_{=0,\text{ as }\boldsymbol{\xi}_{k,int}^i\times\boldsymbol{n}=0\text{ on }\partial\tilde{T}} = 0\ . \tag{6.2}$$

How we benefit from this is illustrated by

$$\int_{\tilde{T}} \left\langle \mathbf{curl}\,\boldsymbol{\xi}_k^i, \boldsymbol{z}_m \right\rangle d\boldsymbol{x} = \int_{\tilde{T}} \left\langle \mathbf{curl}\,\boldsymbol{\xi}_{k,bd}^i, \boldsymbol{z}_m \right\rangle d\boldsymbol{x} = \int_{\Gamma} \left\langle \mathbf{curl}\,\boldsymbol{\xi}_{k,bd}^i, \boldsymbol{z}_m \right\rangle d\boldsymbol{x}\ ,$$

where

$$\Gamma := \bigcup \{\operatorname{supp}\boldsymbol{\psi}_e; e\subset\partial T\}$$

is a sort of narrow fringe along the boundary of $\tilde{T}$. In summary, integration can be confined to a small part of $\tilde{T}$. As a preliminary result we have by the Cauchy–Schwarz inequality

$$\int_{\tilde{T}} \left\langle \mathbf{curl}\,\boldsymbol{\xi}_{i,bd}^k, \boldsymbol{z}^m \right\rangle d\boldsymbol{x} \leq \left\| \mathbf{curl}\,\boldsymbol{\xi}_{i,bd}^k \right\|_{\boldsymbol{L}^2(\Gamma)} \|\boldsymbol{z}^m\|_{L^2(\Gamma)}\ . \tag{6.3}$$

The first factor can be easily bounded, thanks to the orthogonality of $\mathbf{curl}\,\boldsymbol{\xi}_{k,int}^i$ and $\mathbf{curl}\,\boldsymbol{\xi}_{k,bd}^i$ (Remember that the supports of these functions are disjoint by construction):

$$\left\|\mathbf{curl}\,\boldsymbol{\xi}_{k,bd}^i\right\|_{\boldsymbol{L}^2(\Gamma)} \leq \left\|\mathbf{curl}\,\boldsymbol{\xi}_k^i\right\|_{\boldsymbol{L}^2(\tilde{T})}.$$

A bound for the second factor in (6.3) is determined making use of the fact that the area of Γ is only a fraction of $|\tilde{T}|$, if k and m differ widely. More precisely, we have the bound

$$|\Gamma| \leq 2\cdot\frac{h_k}{h_m}\,|\tilde{T}|\ .$$

$\boldsymbol{z}_m$ is linear over $\widetilde{T}$ in each component. Tedious but elementary computations (cf. [24]) then show

$$\|\boldsymbol{z}_m\|_{\boldsymbol{L}^2(\Gamma)} \leq C\sqrt{\frac{h_k}{h_m}}\,\|\boldsymbol{z}_m\|_{\boldsymbol{L}^2(\widetilde{T})}\ .$$

Plugging this into (6.3) completes the proof. □

Next we establish a local precursor of the strengthened Cauchy–Schwarz inequality for components with nonzero divergence:

LEMMA 6.2. *Let $\tilde{T}$ be an arbitrary element of $\mathcal{T}_m$ ($m\in\{0,\dots,L-1\}$) and $L\geq k>m$. Then we have for any $\boldsymbol{z}_m\in\mathcal{RT}_0(\mathcal{T}_m)$ and $\boldsymbol{q}_k\in\boldsymbol{\mathcal{D}}_k^i$, $i=1,\dots,N_{\mathcal{RT}}$ that*

$$a_{|\tilde{T}}(\boldsymbol{q}_k,\boldsymbol{z}_m) \leq C\left\{ h_k\|\operatorname{div}\boldsymbol{q}_k\|_{L^2(\tilde{T})}\|\boldsymbol{z}_m\|_{\boldsymbol{L}^2(\tilde{T})} + \sqrt{\frac{h_k}{h_m}}\|\operatorname{div}\boldsymbol{q}_k\|_{L^2(\tilde{T})}\|\operatorname{div}\boldsymbol{z}_m\|_{L^2(\tilde{T})} \right\}$$

with both constants independent of $\boldsymbol{z}_m$, $\boldsymbol{q}_k$, i, and k.

Proof. The first part of the proof is rather similar to the proof of the previous lemma. Again we can dismiss an interior part of $\boldsymbol{q}_k$ and restrict integration to a tiny zone along $\partial\widetilde{T}$. The outcome accounts for the second term in the estimate. We will skip the details.

To take care of the L^2–inner product we initially rely on the Cauchy–Schwarz inequality

$$(\boldsymbol{q}_k, \boldsymbol{z}_m)_{\boldsymbol{L}^2(\tilde{T})} \leq \left\|\boldsymbol{q}^k\right\|_{\boldsymbol{L}^2(\tilde{T})} \cdot \|\boldsymbol{z}^m\|_{L^2(\tilde{T})} . \tag{6.4}$$

The mesh $\mathcal{T}_k$, restricted to $\tilde{T}$, forms a valid triangulation of $\tilde{T}$. It is evident that the same inequalities that apply to a single basis function also hold for $\boldsymbol{q}_k$, as in particular in (3.3), which furnishes the estimate

$$\left\|\boldsymbol{q}^k\right\|_{\boldsymbol{L}^2(\tilde{T})} \leq Ch_k \left\|\operatorname{div} \boldsymbol{q}^k\right\|_{L^2(\tilde{T})} . \tag{6.5}$$

Now, we obtain the assertion from merely joining the estimates (6.4) and (6.5). ☐

We point out that the strengthened Cauchy–Schwarz inequality in the form (3.6) refers to the lumped subspaces $\boldsymbol{\mathcal{D}}_l^i$ and $\boldsymbol{\mathcal{U}}_l^i$ rather than the spans of individual basis functions. For both types of lumped subspaces on level l we introduce the token $\mathcal{Y}_l^i$:

THEOREM 6.3 (Strengthened Cauchy–Schwarz inequality). *For the decomposition (3.4) there holds a strengthened Cauchy–Schwarz inequality of the form*

$$a(\boldsymbol{y}_k^i, \boldsymbol{y}_m^j) \leq C \min\left\{\sqrt{\frac{h_m}{h_k}}, \sqrt{\frac{h_k}{h_m}}\right\}, \left\|\boldsymbol{y}_k^i\right\|_A \left\|\boldsymbol{y}_m^j\right\|_A,$$

for all $\boldsymbol{y}_k^i \in \boldsymbol{\mathcal{Y}}_k^i, \boldsymbol{y}_m^j \in \boldsymbol{\mathcal{Y}}_m^j$, j, i suitable indices. The positive constant C does neither depend on $\boldsymbol{y}_k^i, \boldsymbol{y}_m^j$ nor on the number L of refinement levels.

Proof. The proof boils down to applying the Lemmata 6.1 and 6.2 on elements of the coarser mesh and then getting to a global result by means of an ordinary Cauchy–Schwarz inequality. For details the reader is referred to [24]. ☐

7. Algorithm. Let us take a closer look at the actual implementation of the scheme to convince ourselves that one cycle of the iteration really comes as cheaply as contended earlier. It is by now obvious that the multiplicative Schwarz method, based upon (3.4) and used as linear iteration, can be cast in the form of a multigrid V–cycle. This has been a revolutionary insight and a thorough discussion can be found in [21, 34, 38]. The principal idea is to avoid visiting the finest grid after each correction in the direction of a coarse grid function. Instead the exchange of information between different levels of refinement is effected by evaluating transfer operators (restriction, prolongation) only once for each level. In Fig. 7.1 the general recursive structure of the algorithm is depicted, to convey that it can be implemented in a perfect multigrid fashion. Symbols with small arrows on top designate coefficient vectors with respect to the canonical bases of the finite element spaces.

The operators $P_{l-1}^l : \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_{l-1}) \mapsto \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l)$ and $R_l^{l-1} : \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l) \mapsto \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_{l-1})$ designate the canonical intergrid transfers, prolongation and restriction, in the Raviart–Thomas spaces, induced by the natural embedding of these spaces (see [22]). They are transposes of each other and lend themselves to a purely local evaluation.

The only special thing about the method is the design of the smoother $S_l(\cdot,\cdot)$, whose steps are described in Fig. 7.2. It might be dubbed a "hybrid" Gauß–Seidel smoother, since smoothing sweeps both in the Raviart–Thomas space and the Nédélec space of vector potentials are carried out. In Fig. 7.2, C_l stands for the linear operator (i.e., a matrix) related to the bilinear form $(\boldsymbol{\xi}_l, \boldsymbol{\eta}_l) \mapsto (\mathbf{curl}\,\boldsymbol{\xi}_l, \mathbf{curl}\,\boldsymbol{\eta}_l)_{\boldsymbol{L}^2(\Omega)}$ in $\boldsymbol{\mathcal{ND}}_1(\mathcal{T}_l)$. The Gauß–Seidel relaxation

Initial guess: $\vec{\boldsymbol{v}}_L$, right hand side $\vec{\boldsymbol{s}}_L$

```
MGVC(int k, v_l ∈ RT_0(T_l), s_l ∈ RT_0(T_l))
  {
    if (l==0) v_0 ← A_0^{-1} s_0
    else
      {
        v_l ← S_l(v_l, s_l)      [Presmoothing]
        q_{k-1} ← 0
        MGVC(l-1, q_{l-1}, R_l^{l-1}(s_l - A_l v_l))
        v_l ← v_l + P_{l-1}^l q_{l-1}
        v_l ← S_l(v_l, s_l)      [Postsmoothing]
      }
  }
```

FIG. 7.1. *Recursive implementation of multigrid V(1,1)–cycle for discrete problem (1.1).*

```
S_l(v_l ∈ RT_0(T_l), s_l ∈ RT_0(T_l))
  {
    Gauß-Seidel sweep on A_l v_l = s_l
    r_l ← s_l - A_l v_l
    ρ_l ← T_l^* r_l
    ξ_l ← 0
    Gauß-Seidel sweep on C_l ξ_l = ρ_l
    return v_l + T_l ξ_l
  }
```

FIG. 7.2. *Evaluation of the hybrid smoother $S_l(\boldsymbol{v}_l, \boldsymbol{s}_l)$.*

of any linear system is invariably supposed to be based on the canonical bases of the finite element spaces.

The significance of the smoother warrants a discussion in greater detail: The evaluation of $S_l(\vec{\boldsymbol{v}}_l, \vec{\boldsymbol{s}}_l)$ boils down to a multiplicative Schwarz method based on the decomposition

$$\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l) = \sum_{\kappa' \in \Xi(\boldsymbol{\mathcal{RT}}_0, \mathcal{T}_l)} \operatorname{Span}\{\boldsymbol{j}_{\kappa'}\} + \sum_{\kappa \in \Xi(\boldsymbol{\mathcal{ND}}_1, \mathcal{T}_l)} \operatorname{Span}\{\mathbf{curl}\,\boldsymbol{\psi}_\kappa\} \ . \tag{7.1}$$

This is just the part of (3.4) associated with level l. The targeted matrix A_l is the stiffness matrix belonging to the bilinear form $a(\cdot,\cdot)$ and the nodal basis $\{\boldsymbol{j}_\kappa\}$, $\kappa \in \Xi(\boldsymbol{\mathcal{RT}}_0, \mathcal{T}_l)$. Thus we end up with a straightforward Gauß–Seidel sweep as the part of the multiplicative Schwarz algorithm related to the first sum in (7.1).

As for the second part of (7.1), a local correction $\boldsymbol{\gamma}_l \in \operatorname{Span}\{\boldsymbol{\psi}_\kappa\}$, $\kappa \in \Xi(\boldsymbol{\mathcal{ND}}_1, \mathcal{T}_l)$, in potential space of the intermediate solution $\boldsymbol{v}_l$ is obtained from

$$a(\mathbf{curl}\,\boldsymbol{\gamma}_l, \mathbf{curl}\,\boldsymbol{\psi}_\kappa) = (\boldsymbol{s} - A_l \boldsymbol{v}_l, \mathbf{curl}\,\boldsymbol{\psi}_\kappa)_{\boldsymbol{L}^2(\Omega)} \ .$$

Actually, this is a scalar equation, whose right hand side is calculated by evaluating the residual, which is a linear form on $\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l)$, for the argument vector $\mathbf{curl}\,\boldsymbol{\psi}_\kappa$. If the residual corresponds to the coefficient vector $\vec{\boldsymbol{r}} := (\boldsymbol{r}_{\kappa'})$ with respect to the canonical dual basis of $\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l)'$, then we get

$$(\boldsymbol{s} - A_l \boldsymbol{v}_l, \mathbf{curl}\,\boldsymbol{\psi}_\kappa)_{\boldsymbol{L}^2(\Omega)} = \sum_{\kappa' \in \Xi(\boldsymbol{\mathcal{RT}}_0, \mathcal{T}_l)} \omega_{\kappa'} \cdot \boldsymbol{r}_{\kappa'} \ . \tag{7.2}$$

The weights $\omega_{\kappa'}$ agree with the coefficients of $\mathbf{curl}\,\boldsymbol{\psi}_\kappa$ in the basis of $\boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l)$. Due to the local nature of the basis functions in Nédélec space, only a few $\omega_{\kappa'}$ are different from zero, namely, those belonging to faces contained in the support of $\boldsymbol{\psi}_\kappa$. Moreover, Stokes' theorem reveals that the non–vanishing weights are either $+1$ or -1, depending on the orientation of the faces. Thus (7.2) can be implemented by summing up the weighted nodal values located at faces adjacent to the edge κ is associated with. This takes a fixed number of operations for each edge.

For the sake of efficiency it makes sense to rearrange the steps in which the computation of the correction in potential space is carried out; the residual can be formed first, then it should be transferred into the dual space of $\boldsymbol{\mathcal{ND}}_1(\mathcal{T}_l)$ all at once: this amounts to the collective execution of the summing–up operation outlined above and can be characterised as the transpose of the transfer operator $T_l : \boldsymbol{\mathcal{ND}}_1(\mathcal{T}_l) \mapsto \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l)$ induced by the embedding $\mathbf{curl}\,\boldsymbol{\mathcal{ND}}_1(\mathcal{T}_l) \subset \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_l)$.

The bottom line is that everything about this algorithm is about as inexpensive as with any other smart multigrid method.

We wish to emphasise that the algorithm can be easily extended to an adaptive setting. In the sense of "local multigrid" (see [28]) only those degrees of freedom should be relaxed belonging to a basis function that does not occur on any coarser level. This rule applies to both parts of the hybrid smoother. Then optimal computational complexity can be maintained despite local refinement.

8. Numerical experiments. Now we have to address the second criterion for efficiency, the rate of convergence. We have provided a rigorous proof that it does not get infinitely poor on very fine meshes. Yet the estimates are riddled with ominous constants, whose actual size is not known, but which have a strong influence on the rate of convergence (c.f. Thm. 3.1). Only numerical experiments can provide some clues.

The *first experiment* was carried out on a cube $\Omega :=]0;1[^3$ with assumed homogeneous Dirichlet boundary conditions imposed on all of $\partial\Omega$. The coarsest grid $\mathcal{T}_0$ comprised eight equal cubes, which were successively regularly refined to create $\mathcal{T}_1, \ldots, \mathcal{T}_L$. Based on $\mathcal{T}_0, \ldots, \mathcal{T}_L$, a multigrid V(1,1)–cycle as outlined in the previous section was applied to the discretized problem (1.1) with $\boldsymbol{f} = 0$. A random initial guess was provided and the rate of convergence has been determined from the reduction of the error in the final three of ten multigrid iteration sweeps.

In this first experiment we investigated the dependence of the convergence rate of the depth of refinement and the choice of the parameter r for the more general bilinear form (1.2). The results are recorded in Tab. 8.1. We observe the uniform boundedness of the convergence rates as predicted by the theory and, in addition, the robustness of the method with respect to the choice of the parameter r.

L	2	3	4	5	6
$r = 0$	0.09	0.09	0.09	0.09	0.1
$r = 0.01$	0.11	0.12	0.15	0.16	0.16
$r = 0.05$	0.13	0.15	0.16	0.16	0.16
$r = 0.25$	0.16	0.16	0.16	0.16	0.16
$r = 1.25$	0.17	0.17	0.17	0.17	0.16
$r = 6.25$	0.17	0.17	0.17	0.17	0.17

TABLE 8.1
Convergence rates for multigrid V(1,1)–cycle obtained in numerical experiment 1.

The second experiment relied on almost the same setting as the first, except for the domain, which was a three dimensional "L-shaped" domain in this case: $\Omega :=]0;1[^3/[0;\frac{1}{2}]^3$. This is a domain for which the regularity assumptions of Sect. 4 and 5 are definitely not fulfilled.

The outcome of the second experiment is documented in Tab. 8.2. Despite the lack of regularity the multigrid convergence is hardly affected. This suggests that regularity can be dispensed with, ultimately.

L	2	3	4	5	6
$r=1$	0.16	0.16	0.17	0.17	0.18

TABLE 8.2
Multigrid convergence rates for experiment 2.

The *third experiment* dealt with the setting mainly treated in this paper, namely the case of free boundary values. Again, we resort to $\Omega :=]0;1[^3$. All the other circumstances agree with those of experiment 1.

The results are given in Tab. 8.3. As expected, the convergence of the multigrid cycle hardly slows down as the mesh is more and more refined.

L	2	3	4	5	6
$r=1$	0.14	0.17	0.18	0.19	0.19

TABLE 8.3
Multigrid convergence rates for $\Omega :=]0;1[^3$, free boundary values.

9. Conclusion. We have presented a new multigrid method to tackle $\boldsymbol{H}(\mathrm{div};\Omega)$–elliptic problems discretized by means of Raviart–Thomas elements. Under several restrictive assumptions, free boundary values on the entire boundary and uniform refinement. we proved the asymptotic optimality of the method. The restrictions are mainly to blame on technical obstacles encountered in the proof; the algorithm is well suited to cope with local refinement and Dirichlet boundary conditions imposed on parts of $\partial\Omega$. In our opinion it has a considerable potential that still awaits to be harnessed for the various applications addressed in the introduction.

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