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ASYMPTOTIC STABILITY OF A 9-POINT MULTIGRID ALGORITHM FOR CONVECTION-DIFFUSION EQUATIONS*

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Abstract. We consider the solution of the convection-diffusion equation in two dimensions by a compact high-order 9-point discretization formula combined with multigrid algorithm. We prove the ϵ-asymptotic stability of the coarse-grid operators. Two strategies are examined. A method to compute the asymptotic convergence is described and applied to the multigrid algorithm.

Key words. multigrid method, high-order discretization, asymptotic stability, convection-diffusion equation.

AMS subject classifications. 65F10, 65N06, 65N22, 65N55, 76D07.

1. Introduction.

Consider the convection-diffusion equation

$$L_\epsilon u \equiv -\epsilon \triangle u + p(x,y)u_x + q(x,y)u_y \quad = \quad f(x,y), \qquad (x,y) \in \Omega, \tag{1.1}$$

where $p(x,y)$ and $q(x,y)$ are functions of x and y and $\epsilon > 0$, and Ω is a convex domain.

We are mainly interested in the case $\epsilon \ll 1$. In applications, many different multigrid methods for solving convection-dominated problems are used. In general, the "standard" multigrid approach used for a diffusion problem deteriorates when applied to convection-dominated problems. For these types of problems, artificial viscosity are in general introduced to obtain convergence.

de Zeeuw and van Asselt [8] introduced some strategies for choosing the artificial viscosity on the coarse grids in the multigrid algorithm (MGA). They show analytically and through numerical experiments that, for a proper choice of the artificial viscosity term, the convergence of MGA is obtained.

Gupta, Manohar and Stephenson [1] proposed a 4^{th} order compact 9-point finite difference scheme (NPF). Gupta et al. [2], Zhang [10] and Kouatchou [6] used NPF combined with MGA (NPF-MGA) to solve (1.1). Their numerical experiments show that the method converges for any values of the cell Reynolds numbers.

The focus of this article is to present a proof of the stability of NPF-MGA. The analysis is based on the constant coefficient case. This paper is organized as follows. In Section 2 we present NPF and prove the ϵ-asymptotic stability of NPF-MGA for two coarse grid strategies. A method to approximate the asymptotic convergence rate is described in Section 3 and in Section 4 we briefly show how to implement the coarse grid strategies introduced. In Section 5 numerical experiments are presented. Finally, some conclusions are formulated in Section 6.

2. Stability Analysis.

In this section we derive theoretical results for the constant coefficient case by local mode analysis neglecting the boundaries. We first present the 9-point discretization scheme for (1.1) with constant functions p and q.

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2.1. The Finite Difference Scheme. We consider the following 9-point compact stencil:

$$
\begin{aligned}
&\begin{bmatrix} \alpha_6 & \alpha_2 & \alpha_5 \\ \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_4 & \alpha_8 \end{bmatrix} \\
&= \frac{1}{h^2} \begin{bmatrix} -\epsilon - \frac{h}{2}(p-q) + \frac{h^2}{4\epsilon}pq & -4\epsilon + 2hq - \frac{h^2}{2\epsilon}q^2 & -\epsilon + \frac{h}{2}(p+q) - \frac{h^2}{4\epsilon}pq \\ -4\epsilon - 2hp - \frac{h^2}{2\epsilon}p^2 & 20\epsilon + \frac{h^2}{\epsilon}(p^2+q^2) & -4\epsilon + 2hp - \frac{h^2}{2\epsilon}p^2 \\ -\epsilon - \frac{h}{2}(p+q) - \frac{h^2}{4\epsilon}pq & -4\epsilon - 2hq - \frac{h^2}{2\epsilon}q^2 & -\epsilon + \frac{h}{2}(p-q) + \frac{h^2}{4\epsilon}pq \end{bmatrix},
\end{aligned} \tag{2.1}
$$

where h is the uniform meshwidth in the x and y directions. This fourth-order scheme was introduced by Gupta et al. [1]. The scheme was shown in [2, 6, 10] to be stable and to give accurate solutions.

2.2. Some Definitions. We consider the MGA with $l+1$ levels, $0, \cdots, l$, and uniform square meshes on each level with meshwidths h_0 and $h_k = h_{k-1}/2$ for $k = 1, \cdots, l$. Let $\{L_\epsilon^{k,l}\}_{k=0,\cdots,l}$ be a sequence of discretizations of L_ϵ. For the constant-coefficient equation we let $\hat{L}_\epsilon(\omega)$, $\omega \in R^2$, be the symbol (or the characteristic form) of the continuous operator L_ϵ. Let $\hat{L}_\epsilon^{k,l}(\omega)$, $\omega \in T_k \equiv [-\frac{\pi}{h_k}, \frac{\pi}{h_k}] \times [-\frac{\pi}{h_k}, \frac{\pi}{h_k}]$, be the symbol of the discrete operator $L_\epsilon^{k,l}$.

When a symbol is small, the corresponding operator is unstable in the sense that small changes in the right hand side cause great changes in the solution. Depending on the boundary conditions the continuous problem can be well posed. Therefore we allow the symbol of the discrete operator to be small only for those frequencies for which the symbol of the continuous operator is small. This idea is formalized in the following definitions.

DEFINITION 2.2.1. *The ϵ-asymptotic stability degree of L_ϵ with respect to the mode $e^{\underline{i}\omega x}$ is the quantity* $\lim_{\epsilon\to 0} |\hat{L}_\epsilon|$.

DEFINITION 2.2.2. *The δ-domain of L_ϵ is the set of all $\omega \in R^2$ for which* $\lim_{\epsilon\to 0} |\hat{L}_\epsilon| > \delta > 0$.

DEFINITION 2.2.3. *The ϵ-asymptotic stability degree of $L_\epsilon^{k,l}$ with respect to the mode $e^{\underline{i}\omega x}$ is the quantity* $\lim_{\epsilon\to 0} |\hat{L}_\epsilon^{k,l}|$.

DEFINITION 2.2.4. *The δ-domain of $L_\epsilon^{k,l}$ is the set of all $\omega \in T_k$ for which* $\lim_{\epsilon\to 0} |\hat{L}_\epsilon^{k,l}| > \delta > 0$.

DEFINITION 2.2.5. *A strategy for coarse-grid operators is a set $\{L_\epsilon^0, L_\epsilon^1, \cdots, L_\epsilon^l, \cdots\}$ with $L_\epsilon^l \equiv \{L_\epsilon^{0,l}, \cdots, L_\epsilon^{l,l}\}$*

DEFINITION 2.2.6. *Let S be a strategy for coarse-grid operators, then S is ϵ-asymptotically stable with respect to L_ϵ if for every $\delta_0 > 0$, there exists a $\delta_1 > 0$ such that for all $0 \le k \le l$, we have the δ_1-domain of $L_\epsilon^{k,l} \supset \delta_0$-domain of $L_\epsilon \cap T_k$.*

REMARK 2.2.1. *To define a strategy (as in Definition 2.2.5) is equivalent to determine in the multigrid algorithm the coarse grid operators on all the grid levels.*

In order to avoid residual transfers in the MGA that are useless due to oscillating solutions, we require that a strategy is ϵ-asymptotically stable with respect to L_ϵ. We need also a relaxation method for which the smoothing factors on all grids are less than 1. We then expect rapid convergence of the MGA.

If the smoother is not very good (smoothing number closed to 1 for example), the restriction operator may transfer on the coarse grid bad "components" of the residual (components related to high frequency modes that are supposed to be removed on the fine grid). These components will appear in the right hand side of the coarse grid equation. If we do not have the ϵ-asymptotic stability property the coarse grid equation may transfer to the fine grid erroneous solution that will badly affect the convergence of MGA.

Another approach would be to admit an ϵ-asymptotically unstable strategy and require that the relaxation method is such that bad components in the residuals are sufficiently smoothed.

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This poses very strong demand upon the relaxation method. But no smoother can be made good enough to compensate for the poor coarse grid correction if the strategy is not ϵ-asymptotically stable and the number of grid levels is increasing. If the coarse-grid operators are not ϵ-asymptotically stable then, at a (very) coarse grid, a small perturbation of the right-hand side of the linear system to be solved may produce a huge perturbation of the solution, an effect that is not present for the fine grid problem (nor for the continuous problem). In the limit-case, roughly speaking, the kernel of the matrix-operator is not empty, in contrast with the empty kernel of the matrix-operator at the finest grid.

2.3. Stability of NPF-MGA. In this section, we introduce two coarse grid strategies for NPF-MGA and prove that they are ϵ-asymptotically stable with respect to L_ϵ. We start with some definitions.

DEFINITION 2.3.1. *Let $L_{\epsilon,k,l}$ be the NPF discretization of L_ϵ with meshwidth h_k. Define the following strategies:*
Strategy $\mathcal{S}_1$:

$$L_\epsilon^{k,l} = L_{\epsilon,k,l},\ k = 0, \cdots, l. \tag{2.2}$$

Strategy $\mathcal{S}_2$:

$$L_\epsilon^{l,l} = L_{\epsilon,l,l},\ L_\epsilon^{k,l} \equiv R_{k,k+1} L_\epsilon^{k+1,l} P_{k+1,k},\ k = l-1, \cdots, 0, \tag{2.3}$$

where $R_{k,k+1}$ and $P_{k+1,k}$ are the restriction and the prolongation operators, repsectively.

Strategy $\mathcal{S}_1$ falls into the category of multigrid algorithms called discretization coarse grid approximation (DCA) whereas strategy $\mathcal{S}_2$ falls into the Galerkin coarse grid approximation (GCA). In [7, p. 82] a comparison of DCA and GCA is presented.

When $R_{k,k+1}$ and $P_{k+1,k}$ are fixed (independent of the discretization) and are transposes of each other, $\mathcal{S}_2$ is the standard Galerkin coarse grid approximation. When they are matrix-dependent and transposes of each other, $\mathcal{S}_2$ corresponds to the matrix-dependent prolongation and restriction method [9]. This method performs better than the standard Galerkin coarse grid approximation.

In $\mathcal{S}_1$, many choices are possible for the grid transfer operators. Gupta et al. [2] found that the choice of full-weighting (as restriction) and bilinear prolongation operator may lead to divergence of the NPF-MGA when ϵ vanishes. They propose instead the scaled injection operator (as restriction) and bilinear prolongation. Their numerical experiments suggest that we regain convergence with this approach. In [4], we compare the performances of $\mathcal{S}_1$ (with scaled injection) and $\mathcal{S}_2$ (with matrix-dependent prolongation and restriction) using NPF-MGA. We found that both methods converge for any ϵ and give comparable results. Since numerical experiments suggest that $\mathcal{S}_2$ converges whenever $\mathcal{S}_1$ does, we state without proof the following conjecture:

CONJECTURE 2.3.1. *If $\mathcal{S}_1$ is ϵ-asymptotically stable then $\mathcal{S}_2$ is also ϵ-asymptotically stable.*

We will now focus our attention on $\mathcal{S}_1$ only.

DEFINITION 2.3.2. *The characteristic form of $L_\epsilon^{k,l}$ is given by*

$$\begin{aligned} \hat{L}_\epsilon^{k,l}(\omega) \;=\;& \alpha_0 + (\alpha_1+\alpha_3)\cos\omega_1 h_k + (\alpha_2+\alpha_4)\cos\omega_2 h_k + \\ & (\alpha_5+\alpha_7)\cos(\omega_1+\omega_2)h_k + (\alpha_6+\alpha_8)\cos(\omega_1-\omega_2)h_k \\ +\;& \underline{i}[(\alpha_1-\alpha_3)\sin\omega_1 h_k + (\alpha_2-\alpha_4)\sin\omega_2 h_k + \\ & (\alpha_5-\alpha_7)\sin(\omega_1+\omega_2)h_k + (\alpha_6-\alpha_8)\sin(\omega_2-\omega_1)h_k] \end{aligned} \tag{2.4}$$

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and the characteristic form of L_ϵ reads

$$\hat{L}_\epsilon(\omega) = \epsilon(\omega_1^2 + \omega_2^2) + \underline{i}(p\omega_1 + q\omega_2). \tag{2.5}$$

Before giving a proof of the asymptotic stability, we show the following result:

LEMMA 2.1. *Let θ_1 and θ_2 be any real numbers in $[-\pi, \pi]$ and let a, b be any numbers such that $a^2 + b^2 = 1$. Assume that θ_1, θ_2, a, b satisfy the relation*

$$|a\theta_1 + b\theta_2| > 0.$$

Then we have

$$G_{a,b}(\theta_1, \theta_2) = 1 + ab \sin\theta_1 \sin\theta_2 - a^2 \cos\theta_1 - b^2 \cos\theta_2 > 0.$$

Proof. Assume $b = 0$. Then $a = 1$ and $G_{1,0}(\theta_1, \theta_2) = 1 - \cos\theta_1$. But $|a\theta_1 + b\theta_2| = |\theta_1| > 0$, and therefore $G_{1,0}(\theta_1, \theta_2) > 0$. The proof for $a = 0$ and $b = 1$ is similar/.

We now consider the case where both $a \neq 0$ and $b \neq 0$. Let us find the maximum or minimum of $G_{a,b}(\theta_1, \theta_2)$.

$$\begin{array}{rclcl} \frac{\partial G_{a,b}}{\partial \theta_1} & = & ab\cos\theta_1 \sin\theta_2 + a^2 \sin\theta_1 & = & 0, \\ \frac{\partial G_{a,b}}{\partial \theta_2} & = & ab\cos\theta_2 \sin\theta_1 + b^2 \sin\theta_2 & = & 0. \end{array}$$

After simplifications we get

$$\begin{array}{rcl} b\cos\theta_1 \sin\theta_2 + a\sin\theta_1 & = & 0, \\ (1 - \cos\theta_1 \cos\theta_2)\sin\theta_2 & = & 0. \end{array}$$

The above equalities are true if and only if $\theta_1 = 0,\ \pm\pi$ and $\theta_2 = 0,\ \pm\pi$. It is easy to check that the minimum of $G_{a,b}$ is at $(0,0)$ ($G_{a,b}(0,0) = 0$). But $(0,0)$ can not be considered for $|a\theta_1 + b\theta_2| > 0$. Therefore $G_{a,b}(\theta_1, \theta_2) > 0$. □

THEOREM 2.2. *The strategy $\mathcal{S}_1$ is ϵ-asymptotically stable.*

Proof. We have to prove that for all $\delta_0 > 0$, there exists $\delta_1 > 0$ such that $k, l, 0 \le k \le l$ $\delta_0 - domain\ of\ L_\epsilon \cap T_k \subset \delta_1 - domain\ of\ L_\epsilon^{k,l}$. In fact we will show that, given an arbitrary δ_0, δ_1 can be any positive number. Without loss of generality we can introduce the normalization $p^2 + q^2 = 1$.

For any $\delta_0 > 0$, $\omega = (\omega_1, \omega_2) \in \delta_0 - domain\ of L_\epsilon \cap T_k$ implies (by Def. 2.2.2 and Eq. (2.5))

$$0 < \delta_0 h_k < |p\omega_1 h_k + q\omega_2 h_k|.$$

It follows from $\omega \in T_k \equiv [-\frac{\pi}{h_k}, \frac{\pi}{h_k}] \times [-\frac{\pi}{h_k}, \frac{\pi}{h_k}]$ then $h_k\omega_1,\ h_k\omega_2$ are in $[-\pi, \pi]$. Using (2.4) we have

$$\begin{array}{rcl} \lim\limits_{\epsilon\to 0} |\hat{L}_\epsilon^{k,l}(\omega)| & > & \lim\limits_{\epsilon\to 0} |\Re\{\hat{L}_\epsilon^{k,l}(\omega)\}| \\ & = & \lim\limits_{\epsilon\to 0} \left| \begin{array}{c} \alpha_0 + (\alpha_1 + \alpha_3)\cos\omega_1 h_k + (\alpha_2 + \alpha_4)\cos\omega_2 h_k \\ +(\alpha_5 + \alpha_7)\cos(\omega_1 + \omega_2)h_k + (\alpha_6 + \alpha_8)\cos(\omega_1 - \omega_2)h_k \end{array} \right| \\ & = & \frac{1}{h_k^2} \lim\limits_{\epsilon\to 0} \left| \begin{array}{c} -(20\epsilon + \frac{p^2 h_k^2}{\epsilon} + \frac{q^2 h_k^2}{\epsilon}) + 4(2\epsilon + \frac{p^2 h_k^2}{4\epsilon})\cos\omega_1 h_k + 4(2\epsilon + \frac{q^2 h_k^2}{4\epsilon})\cos\omega_2 h_k \\ +2(\epsilon + \frac{pq h_k^2}{4\epsilon})\cos(\omega_1 + \omega_2)h_k + 2(\epsilon - \frac{pq h_k^2}{4\epsilon})\cos(\omega_1 - \omega_2)h_k \end{array} \right| \end{array}$$

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$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left|\begin{array}{c} -p^2-q^2+p^2\cos\omega_1 h_k+q^2\cos\omega_2 h_k \\ +\frac{pq}{2}\cos\left(\omega_1+\omega_2\right)h_k-\frac{pq}{2}\cos\left(\omega_1-\omega_2\right)h_k \end{array}\right| \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}|1-p^2\cos\omega_1 h_k-q^2\cos\omega_2 h_k+pq\sin\omega_1 h_k\sin\omega_2 h_k| \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}G_{p,q}(\omega_1 h_k,\omega_2 h_k) \quad (G_{p,q}>0 \quad by \;\; the \;\; Lemma) \\
&= \infty
\end{aligned}
$$

This shows that, for any $\omega \in \delta_0 - domain\ of\ L_\epsilon \cap T_k$, $\lim_{\epsilon\to 0}|\hat{L}_\epsilon^{k,l}(\omega)| = \infty > \delta_1$ (for any $\delta_1 > 0$) for all k such that $0 \le k \le l$. Therefore $\omega \in \delta_0 - domain\ of\ L_\epsilon^{k,l}$. □

Using Conjecture 2.3.1, we can state the Corollary:

COROLLARY 2.3. *The strategy $\mathcal{S}_2$ is ϵ-asymptotically stable.*

3. Numerical Approximation of the Asymptotic Rate of Convergence.

We describe in this section the method used to determine the asymptotic rate of convergence of the multi-grid algorithm (MGA). Let

$$A_h u_h = f_h. \tag{3.1}$$

be a discretization of (1.1). The MG algorithm used to solve (3.1) can be described as

$$
\begin{array}{l} u_h^0 \;\; given \;\; start \;\; approximation, \\ u_h^{i+1} = M_h u_h^i + B_h^{-1} f_h, \quad i = 0, 1, \ldots \end{array} \tag{3.2}
$$

with amplification matrix $M_h = I_h - B_h^{-1}A_h$. I_h is the identity matrix, and B_h^{-1} is an approximate inverse of A_h, determined by coarse-grid smoothing operators, prolongation and restriction. We suppose A_h and B_h to be nonsingular. For the error $e_h^i = u_h - u_h^i$, $i = 0, 1, \ldots$, the following relation holds:

$$e_h^{i+1} = M_h e_h^i. \tag{3.3}$$

The convergence behavior of MGA is determined by the spectral radius of M_h, $\rho(M_h)$.

DEFINITION 3.0.3. *The asymptotic convergence rate of the MGA (3.2) is* $-\log_{10}\rho(M_h)$.

An approximation of $\rho(M_h)$ given by

$$\rho_{m,k} = \left(\frac{||M_h^{m+k}e_h^0||_2}{||M_h^m e_h^0||_2}\right)^{1/k} = \left(\frac{||e_h^{m+k}||_2}{||e_h^m||_2}\right)^{1/k}. \tag{3.4}$$

for m and k large; see [[8]]. As the true solution must be known in advance to compute the error, we will instead use the formula

$$\tilde{\rho}_{m,k} = \left(\frac{||r_h^{m+k}||_2}{||r_h^m||_2}\right)^{1/k} \tag{3.5}$$

to estimate the convergence of the MGA. Here r_h^i is the residual. As explained in [3], $\rho_{m,k}$ is preferable to approximate the asymptotic convergence rate because it approaches more smoothly than $\tilde{\rho}_{m,k}$ the asymptotic rate.

4. Implementation of the two Strategies. The proof of the asymptotic stability property does not involve the smoother (used for relaxation), the grid transfer operators and does not show how fast NPF-MGA converges. The asymptotic stability property is not sufficient to guarantee the convergence of MGA, because the convergence depends also on the robustness of the smoother and the appropriate choice of the restriction ($\mathcal{R}$) and prolongation ($\mathcal{P}$) operators.

For strategy $\mathcal{S}_1$, the scaled injection as restriction operator ($\mathcal{R}$) and the bi-linear interpolation as prolongation operator ($\mathcal{P}$) is the choice that displays the best convergence rate or even that guarantee convergence [2, 6, 10] for any value of ϵ and the coefficients. The multigrid implementation of $\mathcal{S}_1$ is found in [2, 5].

The scaled injection can be described as follows

$$\mathcal{R} = \beta\mathcal{I},$$

where β is a positive constant and $\mathcal{I}$ is the injection operator. Here on all the grid levels, the fine grid residuals are directly injected to the corresponding coarse grid points weighted by the constant β. One of the main challenge of this technique is to find for a particular problem the optimal β, i.e., the one that gives the best convergence rate.

For strategy $\mathcal{S}_2$, the matrix-dependent prolongation and restriction technique is the best choice possible. de Zeeuw provides detailed informations on how to implement this startegy for a general 9-point scheme [9]. This approach is based on an automatic adaptation of prolongation and restriction operators to the discrete problem to be solved.

In [4] we compare $\mathcal{S}_1$ and $\mathcal{S}_2$ when p and q are constant. The numerical experiments show that $\mathcal{S}_1$ (with the scaled injection operator for $\beta = 2$) and $\mathcal{S}_2$ (with the matrix-dependent approach) have comparable results but $\mathcal{S}_2$ is more robust.

In the next section, we present numerical approximations of the asymptotic rate of convergence of $\mathcal{S}_1$ and $\mathcal{S}_2$ on a constant coefficient problem. These approximations are calculated for different values of ϵ and different convection directions.

5. Numerical Results. In this section we give the results of numerical expriments applied to both strategies $\mathcal{S}_1$ and $\mathcal{S}_2$. We consider a constant coefficient problem with Dirichlet boundary conditions.

$$-\epsilon\triangle u + \cos\alpha u_x + \sin\alpha u_y = f(x,y),\ (x,y) \in \Omega = [-0.5, 0.5] \times [-0.5, 0.5], \tag{5.1}$$

where the exact solution is $u(x,y) = 2x(x-1)(\cos(2\pi y) - 1)$.
For our computations, we choose $m = 10$ and $k = 20$ and consider two smoothers: red-black Gauss-Seidel (RBGS) and a horizontal line follows by a vertical line Gauss-Seidel (XY-LINE). In the XY-LINE smoother, we first update simultaneously the unknowns corresponding to horizontal lines (x-direction) in the grid and we carry out the same process for unknowns corresponding to vertical lines (y-direction).

First we present in Table 5.1, the smoothing number of the XY-LINE relaxation for different values of ϵ, α and mesh size $h = 1/N$. We observe that the smoothing number is always less than 1. It is important to note that it is impossible to compute the smoothing number for RBGS (when our 9-point formula is used) since red points cannot be decoupled from black points.

A W-cycle is applied to both strategies $\mathcal{S}_1$ and $\mathcal{S}_2$. In $\mathcal{S}_2$ we employ matrix-dependent prolongations and restrictions technique. In $\mathcal{S}_1$, we utilize $\beta = 2$ as the weight (injection factor) in the scaled injection.

For $\mathcal{S}_1$ and $\mathcal{S}_2$ we calculate the asymptotic rates as function of α, the mesh size h when $\epsilon = 10^{-6}$. The results are summarized in Table 5.2 and Table 5.3. We observe that the

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	α								
ϵ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{5\pi}{7}$	$\frac{11\pi}{9}$	$\frac{9\pi}{5}$	N
10^{-6}	0	.46	.31	.46	0	.37	.36	.40	8
10^{-7}	0	.46	.31	.46	0	.37	.36	.40	
10^{-6}	0	.46	.31	.46	0	.37	.36	.40	16
10^{-7}	0	.46	.31	.46	0	.37	.36	.40	
10^{-6}	0	.46	.31	.46	0	.37	.36	.40	32
10^{-7}	0	.46	.31	.46	0	.37	.36	.40	

TABLE 5.1
Smoothing numbers for the horizontal line follows by the vertical line Gauss-Seidel relaxation for different values of ϵ, α and mesh size $h = 1/N$.

asymptotic rates are positive for any α and h. In addition $\mathcal{S}_2$ seems to display the best result in particular for $h = 1/16, 1/32$.

	α						
	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{7}$	$\frac{11\pi}{9}$	$\frac{9\pi}{5}$	N
RBGS	.48	.49	.49	.48	.49	.49	8
XY-LINE	.43	.44	.36	.41	.42	.41	
RBGS	.30	.30	.30	.29	.30	.30	16
XY-LINE	.34	.31	.37	.34	.33	.33	
RBGS	.19	.20	.19	.19	.19	.19	32
XY-LINE	.17	.16	.18	.17	.16	.16	

TABLE 5.2
Strategy $\mathcal{S}_1$, $\epsilon = 10^{-6}$. Approximation of the asymptotic convergence rate for different values of α and mesh size $h = 1/N$.

	α						
	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{7}$	$\frac{11\pi}{9}$	$\frac{9\pi}{5}$	N
RBGS	.42	.53	.48	.52	.53	.53	8
XY-LINE	.21	.26	.20	.28	.27	.26	
RBGS	.31	.32	.30	.37	.32	.33	16
XY-LINE	.41	.42	.41	.41	.43	.43	
RBGS	.20	.19	.17	.17	.18	.19	32
XY-LINE	.36	.32	.35	.31	.31	.31	

TABLE 5.3
Strategy $\mathcal{S}_2$, $\epsilon = 10^{-6}$. Approximation of the asymptotic convergence rate for different values of α and mesh size $h = 1/N$.

For $\mathcal{S}_1$, our experiments were done for $\beta = 2$. As stated in [6, 10], β is a function (increasing and bounded) of the cell Reynolds number. In particular, for a given ϵ and h (also the $\sup$ over the domain Ω of the coefficients in Eq. (1.1)), there exists a value of β that yields the best asymptotic rate. Unfortunately the optimal β is problem dependent and depends on the smoother. At least we can state that, for a given cell Reynolds number, an estimate of the optimal injection factor β can be seen by

- finding the best ω in the SOR method, or
- finding the appropriate artificial viscosity coefficient to be introduced in Eq. (1.1).

Given the cell Reynolds number, to obtain the optimal β can be seen as

- finding the best ω in the SOR method,
- or finding the appropriate artificial viscosity coefficient to be introduced in Eq. (1.1) when ϵ is small.

For the case $\alpha = \frac{\pi}{3}$, $h = \frac{1}{32}$ and $\epsilon = 10^{-6}$, we plot for strategy $\mathcal{S}_1$ the asymptotic rate as a function of the the injection factor β when RBGS and XY-LINE are employed. The graph is presented in Figure 5.1. The best rates are obtained for $\beta \approx 3.0$ for RBGS and for $\beta \approx 6.0$ for XY-LINE. These rates (found for the optimal β) are better than the corresponding ones obtained with $\mathcal{S}_2$.

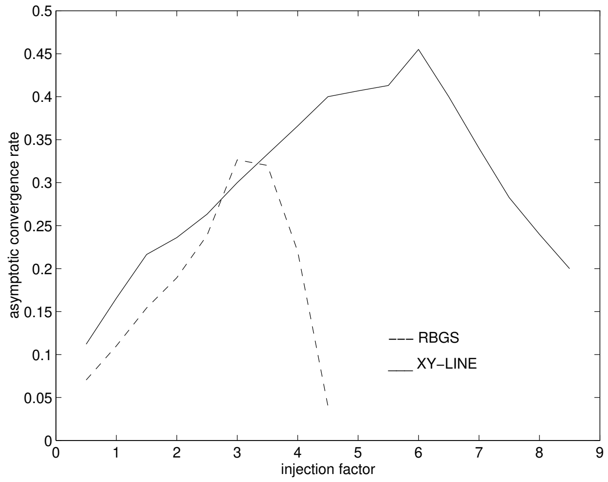

FIG. 5.1. Strategy $\mathcal{S}_1$: asymptotic rate as function of the injection factor β when $h = 1/32$, $\alpha = \frac{\pi}{3}$.

REMARK 5.0.1. *When ϵ is small, for the constant coefficient problem presented here, the optimal β was found to be* 3.0 *for RBGS and* 6.0 *for $XY - LINE$. The picture looks different for variable coefficient problems. In fact, the optimal β is in general a value between* 1 *and* 2 *for RBGS and a value between* 4 *and* 5 *for XY-LINE [5].*

REMARK 5.0.2. *The introduction of artificial viscosity affects the accuracy of the solution but this is not the case for the injection factor β. All the β for which the convergence of the multigrid algorithm is obtained, give the same accuracy. In addition, the accuracy obtained from the scaled injection operator is the same as that of any other restriction operator (half-injection, full-weighting, half-weighting) [2, 6].*

6. Conclusion. In order to solve the convection-diffusion equation in two dimension by a multigrid algorithm (MGA), we propose a 9-point compact formula (NPF). Two strategies for coarse-grid operators were considered and shown to be ϵ-asymptotically stable. Numerical experiments confirmed this stability property.

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