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DOMAIN DECOMPOSITION ALGORITHMS FOR FIRST-ORDER SYSTEM LEAST SQUARES METHODS*

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Abstract. First-order system least squares methods have been recently proposed and analyzed for second order elliptic equations and systems. They produce symmetric and positive definite discrete systems by using standard finite element spaces which are not required to satisfy the inf-sup condition. In this paper, several domain decomposition algorithms for these first-order least squares methods are studied. Some representative overlapping and substructuring algorithms are considered in their additive and multiplicative variants. The theoretical and numerical results obtained show that the classical convergence bounds (on the iteration operator) for standard Galerkin discretizations are also valid for least squares methods. Therefore, domain decomposition algorithms provide parallel and scalable preconditioners also for least squares discretizations.

Key words. Domain decomposition, first-order system least squares.

AMS subject classifications. 65N30, 65N55.

1. Introduction.

Least squares methods have been proposed in recent years for second-order elliptic problems, Stokes and Navier-Stokes equations; see Chang [12], Bochev and Gunzburger [2], Pehlivanov, Carey, and Lazarov [18], Cai, Lazarov, Manteuffel, and McCormick [7], Cai, Manteuffel, and McCormick [9], Bramble, Lazarov, and Pasciak [3], Bramble and Pasciak [4], Carey, Pehlivanov, and Vassilevski [10], Cai, Manteuffel, and McCormick [8], Bochev, Cai, Manteuffel, and McCormick [1], and the references therein.

Among the possible approaches, we follow here the one introduced in the very recent works of Pehlivanov, Carey, and Lazarov [18] and Cai, Manteuffel, and McCormick [9]. The second-order elliptic problem is rewritten as a first-order system and a least squares functional is introduced. The resulting discrete minimization problem is associated with a bilinear form which is continuous and elliptic on an appropriate space. Therefore, the inf-sup condition is avoided, and standard finite element spaces can be used. The resulting linear system is symmetric, positive definite and has condition number of the same order as standard Galerkin finite element stiffness matrices, $O(1/h^2)$. An interesting alternative approach by Bramble, Lazarov, and Pasciak [3] is based on replacing one of the L^2-terms in the least squares functional by a discrete H^{-1}-norm. We will not consider here such an alternative.

The goal of this paper is to extend to these least squares methods some of the classical domain decomposition algorithms which have been successfully employed for standard Galerkin finite elements and to compare numerically the two approaches for simple model problems. We show that optimal and quasi-optimal convergence bounds follow easily from the standard Galerkin case. Numerical results confirm these bounds and show that domain decomposition algorithms for standard Galerkin and least squares discretizations have comparable convergence rates. Therefore, domain decomposition provides highly parallel and scalable solvers also for first-order system least squares discretizations. An overview of domain decomposition methods can be found in the book by Smith, Bjørstad and Gropp [19] and in the review papers by Chan and Mathew [11], Dryja, Smith, and Widlund [13], Dryja and Widlund [15], Le Tallec [16].

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This paper is organized as follows. In the next section, we briefly review the first-order system least squares methodology and the main results from [9]. In Section 3, we introduce and analyze our domain decomposition algorithms: overlapping additive Schwarz methods (with coupled and uncoupled subspaces; see 3.1), overlapping multiplicative Schwarz methods (3.2), and an iterative substructuring method (3.3). In Section 4, we present numerical results in the plane that confirm the theoretical bounds obtained, and we make a comparison with results for standard Galerkin discretizations.

2. First-Order System Least Squares. We consider the following second-order elliptic problem on a bounded domain $\Omega \subset R^2$ or R^3

$$\begin{cases} -\nabla \cdot (A\nabla p) + Xp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma_D, \\ \mathbf{n} \cdot A\nabla p &= 0 \quad \text{on } \Gamma_N. \end{cases} \tag{2.1}$$

Here A is a symmetric and uniformly positive definite matrix with entries in $L^\infty(\Omega)$, X is a first-order linear differential operator, $\Gamma_D \cup \Gamma_N = \partial\Omega$, and n is the outward unit vector normal to Γ_N.

Defining the new flux variable $\mathbf{u} = -A\nabla p$, the system (2.1) can be rewritten as a first-order system:

$$\begin{cases} \mathbf{u} + A\nabla p &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + Xp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} &= 0 \quad \text{on } \Gamma_N. \end{cases} \tag{2.2}$$

This system can be extended to the equivalent system

$$\begin{cases} \mathbf{u} + A\nabla p &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + Xp &= f \quad \text{in } \Omega, \\ \nabla \times A^{-1}\mathbf{u} &= \mathbf{0} \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} &= 0 \quad \text{on } \Gamma_N, \\ \gamma_\tau(A^{-1}\mathbf{u}) &= 0 \quad \text{on } \Gamma_D, \end{cases} \tag{2.3}$$

where $\nabla\times = curl$ (in two dimensions $\nabla \times \mathbf{u} = 0$ means $\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = 0$) and $\gamma_\tau \mathbf{u} = \mathbf{u} \times \mathbf{n}$ (in two dimensions $\gamma_\tau \mathbf{u} = \mathbf{u} \cdot \tau$).

The following least squares functionals, G_0 for the system (2.2) and G for the augmented system (2.3), were studied in [7] ([18] for the case $X = 0$) and [9] respectively:

$$G_0(\mathbf{v}, q; f) = \|\mathbf{v} + A\nabla q\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathbf{v} + Xq - f\|^2_{L^2(\Omega)}, \tag{2.4}$$

$\forall(\mathbf{v}, q) \in \mathbf{W}_0(div; \Omega) \times V$, and

$$G(\mathbf{v}, q; f) = \|\mathbf{v} + A\nabla q\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathbf{v} + Xq - f\|^2_{L^2(\Omega)} + \|\nabla \times (A^{-1}\mathbf{v})\|^2_{L^2(\Omega)}, \tag{2.5}$$

$\forall(\mathbf{v}, q) \in \mathbf{W} \times V$.
More general functional with scaling parameters in front of each term are possible; see Bochev and Gunzburger [2] and Carey, Pehlivanov and Vassilevski [10]. Here the functional spaces are defined as

$$\mathbf{W}_0(div; \Omega) = \{\mathbf{v} \in \mathbf{H}(div; \Omega) : \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \Gamma_N\},$$

$$\mathbf{W}_0(curlA;\Omega) = \{\mathbf{v} \in \mathbf{H}(curlA;\Omega) : \gamma_\tau(A^{-1}\mathbf{v}) = 0 \text{ on } \Gamma_D\},$$

$$\mathbf{W} = \mathbf{W}_0(div;\Omega) \cap \mathbf{W}_0(curlA;\Omega),$$

$$V = \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_D\}.$$

The least squares minimization problems for (2.2) and (2.3) are respectively:
Find $(\mathbf{u}, p) \in \mathbf{W}_0(div;\Omega) \times V$ such that

$$G_0(\mathbf{u}, p; f) = \inf_{(\mathbf{v},q)\in\mathbf{W}_0(div;\Omega)\times V} G_0(\mathbf{v}, q; f); \tag{2.6}$$

Find $(\mathbf{u}, p) \in \mathbf{W} \times V$ such that

$$G(\mathbf{u}, p; f) = \inf_{(\mathbf{v},q)\in\mathbf{W}\times V} G(\mathbf{v}, q; f). \tag{2.7}$$

Simple calculations show that the associated variational problems are respectively:
Find $(\mathbf{u}, p) \in \mathbf{W}_0(div;\Omega) \times V$ such that

$$a_0(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \quad \forall(\mathbf{v}, q) \in \mathbf{W}_0(div;\Omega) \times V; \tag{2.8}$$

Find $(\mathbf{u}, p) \in \mathbf{W} \times V$ such that

$$a(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \quad \forall(\mathbf{v}, q) \in \mathbf{W} \times V. \tag{2.9}$$

Here the bilinear forms are

$$a_0(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{u} + A\nabla p, \mathbf{v} + A\nabla q)_{L^2} + (\nabla \cdot \mathbf{u} + Xp, \nabla \cdot \mathbf{v} + Xq)_{L^2},$$

$$a(\mathbf{u}, p; \mathbf{v}, q) = a_0(\mathbf{u}, p; \mathbf{v}, q) + (\nabla \times (A^{-1}\mathbf{u}), \nabla \times (A^{-1}\mathbf{v}))_{L^2},$$

and the right-hand side is

$$F(\mathbf{v}, q) = (f, \nabla \cdot \mathbf{v} + Xq)_{L^2}.$$

In [7], it was proved that $a_0(\mathbf{v}, q; \mathbf{v}, q)$ is equivalent to (continuous and elliptic with respect to) the $\mathbf{H}(div;\Omega) \times H^1(\Omega)$-norm on $\mathbf{W}_0(div;\Omega) \times V$, under the assumption (denoted by assumption A0) that a Poincarè-Friedrichs inequality holds for p: there exists a constant C depending only on Ω and the uniform bounds on A such that

$$||p||_{L^2(\Omega)} \leq C||A^{1/2}\nabla p||_{L^2(\Omega)}.$$

For the case $X = 0$, this was proved in [18].

In [9], it was proved that $a(\mathbf{v}, q; \mathbf{v}, q)$ is equivalent to the $[\mathbf{H}(div;\Omega) \cap \mathbf{H}(curlA;\Omega)] \times H^1(\Omega)$ norm on $\mathbf{W} \times V$, under the same assumption A0. Moreover, under three additional technical assumptions denoted by A1, A2, A3, it is proven in [9] that $a(\mathbf{v}, q; \mathbf{v}, q)$ is equivalent to the $H^1(\Omega)^{d+1}$ norm on $\mathbf{W} \times V$ ($d = 2$ or 3). These technical assumptions are made in [9] in order to guarantee the H^2-regularity of the solutions of the elliptic problems considered. We reported them here for completeness and remark that convex polygons and polyhedra satisfy these assumptions (see [9] for a more detailed discussion).

ASSUMPTION A1: *Ω is a bounded, open, connected domain in R^d. Its boundary Γ consists of a finite number of disjoint, simple, closed curves (surfaces) $\Gamma_i,\ i = 0,\dots,L$; Γ_0 is the outer boundary which for $d = 2$ is piecewise $C^{1,1}$ with no reentrant corners and for $d = 3$ is $C^{1,1}$ or a convex polyhedron.*

ASSUMPTION A2: *the boundary is divided into Dirichlet and Neumann parts: $\Gamma = \Gamma_D \cup \Gamma_N$ such that $\Gamma_i \subseteq \Gamma_D$ for $i \in D$ and $\Gamma_i \subseteq \Gamma_N$ for $i \in N$ with $D \cup N = \{1,\dots,L\}$. For $d = 2$, Γ_0 is divided into a finite number of connected pieces: $\Gamma_0 = \cup_{i=1}^M \Gamma_{0,i}$ such that $\Gamma_{0,i} \subseteq \Gamma_D$ for $i \in D_0$ and $\Gamma_{0,i} \subseteq \Gamma_N$ for $i \in N_0$. For $d = 3$, either $\Gamma_0 \subseteq \Gamma_D$ or $\Gamma_0 \subseteq \Gamma_N$.*

ASSUMPTION A3: *The matrix A is $C^{1,1}$ and for $d = 2$ $\mathbf{n}_-^T A \mathbf{n}_+ \le 0$ at each corner $\mathbf{x} \in \Gamma_0$ that separate Γ_D and Γ_N, where $\mathbf{n}_-$ and $\mathbf{n}_+$ are the outward unit normal vectors to Γ_0 at $\mathbf{x}$.*

THEOREM 2.1. *Let $b(\mathbf{u},p;\mathbf{v},q) = (\mathbf{u},\mathbf{v})_{H^1(\Omega)^d} + (p,q)_{H^1(\Omega)}$ be the bilinear form associated with the $H^1(\Omega)^{d+1}$ norm.*

If the assumptions A0-A3 of [9] are verified, then there exist positive constants α and β such that

$$\alpha b(\mathbf{v},q;\mathbf{v},q) \le a(\mathbf{v},q;\mathbf{v},q) \quad \forall (\mathbf{v},q) \in \mathbf{W} \times V,$$

and

$$a(\mathbf{u},p;\mathbf{v},q) \le \beta b(\mathbf{u},p;\mathbf{u},p)^{1/2} b(\mathbf{v},q;\mathbf{v},q)^{1/2},$$

$\forall (\mathbf{u},p),(\mathbf{v},q) \in \mathbf{W} \times V$.

Because of the equivalence of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, from now on we will concentrate on the variational problem (2.9) associated with the augmented system (2.3).

We introduce a triangulation τ_h of Ω and associated finite element subspaces $\mathbf{W}^h \times V^h \subset \mathbf{W} \times V$. We suppose that the domain Ω is first triangulated by a coarse finite element triangulation τ_H consisting of N subdomains Ω_i of diameter H. The fine triangulation τ_h is a refinement of τ_H. For simplicity, we suppose that each subdomain is the image under an affine map of a reference cube. In the general case of curved elements and non-constant matrix A, the boundary condition $\gamma_\tau(A^{-1}\mathbf{u}) = 0 \quad$ on Γ_D cannot be satisfied on the whole boundary Γ_D. If this condition is enforced only at the nodes on Γ_D, we obtain a nonconforming method with $\mathbf{W}^h \not\subset \mathbf{W}$; see Pehlivanov and Carey [17] for a discussion of this approach. In this paper, we confine ourselves to the conforming case, which is obtained for example by using affine elements and a constant matrix A.

We then obtain a finite element discretization of (2.9):

Find $(\mathbf{u}_h,p_h) \in \mathbf{W}^h \times V^h$ such that

$$a(\mathbf{u}_h,p_h;\mathbf{v}_h,q_h) = F(\mathbf{v}_h,q_h) \quad \forall (\mathbf{v}_h,q_h) \in \mathbf{W}^h \times V^h. \tag{2.10}$$

For simplicity, we consider continuous piecewise linear finite elements:

$$\mathbf{W}^h = \{\mathbf{v} \in C^0(\Omega)^d : v_k|_T \in P_1(T),\ \forall T \in \tau_h, \mathbf{v} \in \mathbf{W}\} = W_1^h \times W_2^h \times W_3^h,$$

$$V^h = \{q \in C^0(\Omega) : q|_K \in P_1(K),\ \forall K \in \tau_h, q \in V\},$$

and the subscript h for discrete functions will be dropped in the rest of the paper.

Error estimates and results on the conditioning of the resulting stiffness matrix can be found in [7] (in [18] for the case $X = 0$).

By choosing a basis in $\mathbf{W}^h$ and V_h, the discrete problem (2.10) is turned into a linear system of equation $Ax = b$. We are going to solve such system iteratively by using domain decomposition techniques.

3. Domain Decomposition Algorithms. We will introduce and analyze our domain decomposition algorithms in the Schwarz framework, which has been very successful for standard Galerkin finite elements; see [13], [14], [15], [11], [19]. We illustrate the main ideas on algorithms which are representative of the main classes of domain decomposition (additive, multiplicative, overlapping, iterative substructuring). The same analysis can be applied to the many other algorithms which have been proposed and analyzed for the standard scalar case.

We recall that the domain Ω is the union of N subdomains Ω_i, affine images of a reference cube, which form a coarse finite element triangulation τ_H of Ω. A fine triangulation τ_h is obtained as a refinement of τ_H.

3.1. Overlapping Additive Schwarz Methods. Each subdomain Ω_i is extended to a larger subdomain Ω_i', consisting of all elements of τ_h within a distance δ from Ω_i ($0 < \delta < H$).

Each scalar component of our finite element space $\mathbf{W}^h \times V^h$ is decomposed as in the standard scalar case:

$$W_1^h = \sum_{i=1}^{N} W_{1,i}^h \qquad W_2^h = \sum_{i=1}^{N} W_{2,i}^h \qquad W_3^h = \sum_{i=1}^{N} W_{3,i}^h \qquad V^h = \sum_{i=1}^{N} V_i^h,$$

where

$$W_{k,i}^h = \{u \in W_k^h : \text{support}(u) \subset \Omega_i'\}, \quad k = 1, 2, 3,$$

$$V_i^h = \{u \in V^h : \text{support}(u) \subset \Omega_i'\}.$$

For each scalar component, a global coarse finite element space is associated with the coarse triangulation τ_H:

$$W_{k,0}^h = W_k^H = \{u \in W_k^h : u \text{ is trilinear on each subdomain } \Omega_i\}, \quad k = 1, 2, 3,$$

$$V_0^h = V^H = \{p \in V^h : p \text{ is trilinear on each subdomain } \Omega_i\}.$$

A first additive method is defined by the following decomposition of the discrete space, which maintains the local and coarse coupling between the different scalar components:

$$\mathbf{W}^h \times V^h = \sum_{i=0}^{N} \mathbf{W}_i^h \times V_i^h.$$

The local spaces are

$$\mathbf{W}_i^h \times V_i^h = W_{1,i}^h \times W_{2,i}^h \times W_{3,i}^h \times V_i^h \quad i = 1, 2, \cdots, N,$$

and the coarse space is

$$\mathbf{W}_0^h \times V_0^h = \mathbf{W}^H \times V^H = W_1^H \times W_2^H \times W_3^H \times V^H.$$

We define the local projection operators $P_i : \mathbf{W}^h \times V^h \to \mathbf{W}_i^h \times V_i^h$ by

$$a(P_i(\mathbf{u},p);\mathbf{v},q) = a(\mathbf{u},p;\mathbf{v},q) \quad \forall(\mathbf{v},q) \in \mathbf{W}_i^h \times V_i^h,$$

and the coarse projection operator $P_0 : \mathbf{W}^h \times V^h \to \mathbf{W}_0^h \times V_0^h$ by

$$a(P_0(\mathbf{u},p);\mathbf{v},q) = a(\mathbf{u},p;\mathbf{v},q) \quad \forall(\mathbf{v},q) \in \mathbf{W}_0^h \times V_0^h.$$

It is easy to see that the matrix form of the local projections is $P_i = R_i^T A_i^{-1} R_i A$, where

$$R_i(\mathbf{e}_k) = \begin{cases} 1 & \text{if } \mathbf{e}_k \in \Omega_i' \\ 0 & \text{otherwise} \end{cases}$$

are the restriction matrices selecting only the unknowns in Ω_i' for each component and $A_i = R_i A R_i^T$ are the stiffness matrices of local Dirichlet problems. Analogously, $P_0 = R_H^T A_H^{-1} R_H A$, where R_H^T is the standard piecewise linear interpolation matrix from the coarse grid τ_H to the fine grid τ_h, for each component, and $A_H = R_H A R_H^T$ is the coarse grid discretization of our problem (2.9). Let

$$P_{add1} = \sum_{i=0}^{N} P_i.$$

The original discrete problem is then equivalent to the preconditioned problem

$$P_{add1}(\mathbf{u},p) = \mathbf{g}_{add1},$$

where $\mathbf{g} = \sum_{i=0}^{N} P_i(\mathbf{u},p)$; see Chan and Mathew [11] or Smith, Bjørstad and Gropp [19]. In matrix form, this problem can be written as $M^{-1}Ax = M^{-1}b$, where the preconditioner is $M^{-1} = \sum_{i=1}^{N} R_i^T A_i^{-1} R_i + R_H^T A_H^{-1} R_H$. An optimal convergence bound for this algorithm is given in Theorem 3.1.

A second additive method is obtained by dropping the coupling between the different scalar components of $\mathbf{u}$ and p. Uncoupled local spaces are now defined by

$$\mathbf{W}_{1,i}^h = W_{1,i}^h \times \{0\} \times \{0\} \times \{0\},$$

$$\mathbf{W}_{2,i}^h = \{0\} \times W_{2,i}^h \times \{0\} \times \{0\},$$

$$\mathbf{W}_{3,i}^h = \{0\} \times \{0\} \times W_{3,i}^h \times \{0\},$$

$$\mathbf{V}_i^h = \{0\} \times \{0\} \times \{0\} \times V_i^h,$$

and the coarse spaces by

$$\mathbf{W}_1^H = \mathbf{W}_{1,0}^h = W_{1,0}^h \times \{0\} \times \{0\} \times \{0\},$$

$$\mathbf{W}_2^H = \mathbf{W}_{2,0}^h = \{0\} \times W_{2,0}^h \times \{0\} \times \{0\},$$

$$\mathbf{W}_3^H = \mathbf{W}_{3,0}^h = \{0\} \times \{0\} \times W_{3,0}^h \times \{0\},$$

$$\mathbf{V}^H = \mathbf{V}_0^h = \{0\} \times \{0\} \times \{0\} \times V_0^h.$$

We then have the following decomposition

$$\mathbf{W}^h \times V^h = \sum_{i=1}^N \mathbf{W}_{1,i}^h + \sum_{i=1}^N \mathbf{W}_{2,i}^h + \sum_{i=1}^N \mathbf{W}_{3,i}^h + \sum_{i=1}^N \mathbf{V}_i^h + \mathbf{W}_1^H + \mathbf{W}_2^H + \mathbf{W}_3^H + \mathbf{V}^H$$

$$= \sum_{k=1}^3 \sum_{i=0}^N \mathbf{W}_{k,i}^h + \sum_{i=0}^N \mathbf{V}_i^h.$$

As before, we define projections $P_{k,i} : \mathbf{W}^h \times V^h \to \mathbf{W}_{k,i}^h,\ k = 1,2,3,\ i = 0,1,\cdots,N$ and $P_{4,i} : \mathbf{W}^h \times V^h \to \mathbf{V}_i^h,\ i = 0,1,\cdots,N$, and the additive operator

$$P_{add2} = \sum_{k=1}^3 \sum_{i=0}^N P_{k,i} + \sum_{i=0}^N P_{4,i}.$$

We note that this algorithm can equivalently be defined by the same choice of subspaces as for P_{add1} but using the bilinear form $b(\cdot,\cdot)$ (introduced in Theorem 2.1) instead of $a(\cdot,\cdot)$ in the definition of the projections. In fact this uncoupled preconditioner corresponds to applying four identical copies of a scalar preconditioner to each scalar component. An optimal bound holds also for this algorithm.

THEOREM 3.1. *There exists a positive constant C independent of h, H and δ such that*

$$cond(P) \leq C(1 + \frac{H}{\delta}),$$

where $P = P_{add1}$ or $P = P_{add2}$.

Proof. An upper bound on the spectrum of P is standard, since each point of Ω belongs to a fixed number of extended subdomains independent of N (for example, for $\delta < H/2$ each point belongs to at most four (in 2D) or eight (in 3D) extended subdomains). A lower bound is obtained by classical Schwarz analysis.

For $P = P_{add1}$, since we use exact projections, the lower bound is equivalent to the following partition property (see Dryja and Widlund [15] or Chan and Mathew [11]):
There exists a constant C_0 such that $\forall(\mathbf{u},p) \in \mathbf{W}^h \times V^h$, there exists a decomposition $(\mathbf{u},p) = \sum_{i=0}^N (\mathbf{u}_i, p_i)$, with $(\mathbf{u}_i, p_i) \in \mathbf{W}_i^h \times V_i^h$ such that

$$\sum_{i=0}^N a(\mathbf{u}_i, p_i; \mathbf{u}_i, p_i) \leq C_0^2 a(\mathbf{u}, p; \mathbf{u}, p).$$

By the equivalence of Theorem 2.1, this inequality is equivalent to

$$\sum_{i=0}^N |(\mathbf{u}_i, p_i)|^2_{(H^1)^{d+1}} \leq C_0^2 |(\mathbf{u}, p)|^2_{(H^1)^{d+1}},$$

which is a direct consequence of the scalar result proven by Dryja and Widlund [15]:

$$\sum_{i=0}^N |u_i^k|^2_{H^1} \leq C_0^2 |u^k|^2_{H^1}, \quad \sum_{i=0}^N |p_i|^2_{H^1} \leq C_0^2 |p|^2_{H^1},$$

with $C_0^2 = C(1 + \frac{H}{\delta})$.

For $P = P_{add2}$, since the subspaces are the same but we use inexact projections defined by $b(\cdot,\cdot)$ instead of $a(\cdot,\cdot)$, we need only to show that there exists a constant ω such that $a(\mathbf{u},p;\mathbf{u},p) \leq \omega b(\mathbf{u},p;\mathbf{u},p) \quad \forall(\mathbf{u},p) \in \mathbf{W}_i^h \times V_i^h, \quad i = 0,1,\cdots,N$ (see Dryja and Widlund [14]). This follows immediately from the equivalence of a and b. $\square$

3.2. Overlapping Multiplicative Schwarz Methods. By using the same coupled local and coarse spaces as in the additive algorithm P_{add1}, we can define a multiplicative operator:

$$P_{mult} = I - (I - P_N) \cdots (I - P_1)(I - P_0).$$

The multiplicative algorithm consists in solving the nonsymmetric system

$$P_{mult}(\mathbf{u}, p) = \mathbf{g}_{mult}$$

by an iterative method such as GMRES.

We can also define a symmetrized multiplicative operator

$$P_{mults} = I - (I - P_0) \cdots (I - P_{N-1})(I - P_N)(I - P_{N-1}) \cdots (I - P_0)$$

and a symmetrized algorithm, consisting in solving the symmetric system

$$P_{mults}(\mathbf{u}, p) = \mathbf{g}_{mults}$$

by an iterative method like CG. We have chosen to accelerate this multiplicative version by GMRES or CG because this approach has been proven more efficient and robust than the classical multiplicative Schwarz algorithm; see Cai, Gropp and Keyes [6]. For the symmetrized operator, we have the following optimal bound.

THEOREM 3.2. *There exists a positive constant C independent of h, H and δ such that*

$$cond(P_{mults}) \leq C(1 + \frac{H}{\delta}).$$

The proof is again based on the extension of the scalar result (see Chan and Mathew [11], Smith, Bjørstad and Gropp [19] or the more specific reference Bramble, Pasciak, Wang and Xu [5]) by using the equivalence of Theorem 2.1. Analogously, multiplicative versions of P_{add2} could be built using uncoupled local and coarse spaces.

3.3. An Iterative Substructuring Method. For a complete and detailed analysis of this class of methods, we refer to Dryja, Smith and Widlund [13]. Here we only consider a simple representative of this class, namely the analog of Algorithm 6.2 in [13], which is vertex-based and has a standard coarse space. For simplicity, we only consider the uncoupled additive version.

The standard first step of nonoverlapping methods is the elimination of the variables interior to each subdomain (at least implicitly). We then work with the Schur complement $S = K_{BB} - K_{IB}^T K_{II}^{-1} K_{IB}$ of the stiffness matrix

$$K = \begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^T & K_{BB} \end{pmatrix}.$$

The reduced linear system with S involves only variables on the interface $\Gamma = \cup \partial\Omega_i \setminus \Gamma_D$. When solving with a preconditioned iterative method, we only need the action of S on a given vector and there is no need to explicitly assemble S.

In the Schwarz framework, working with S corresponds to working with the discrete harmonic subspace $\tilde{\mathbf{W}}^h \times \tilde{V}^h$ of the original space $\mathbf{W}^h \times V^h$. Local spaces are associated with the geometric objects (faces F_i, edges E_i and vertices v_i) forming the interface Γ. Each scalar space is decomposed as

$$\tilde{W}_k^h = \sum_{F_i} \tilde{W}_{k,F_i}^h + \sum_{E_i} \tilde{W}_{k,E_i}^h + \sum_{v_i} \tilde{W}_{k,v_i}^h, \quad k = 1, 2, 3,$$

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and

$$\tilde{V}^h = \sum_{F_i} \tilde{V}^h_{F_i} + \sum_{E_i} \tilde{V}^h_{E_i} + \sum_{v_i} \tilde{V}^h_{v_i}.$$

Here, for example, $\tilde{W}^h_{k,F_i} = \{u \in \tilde{W}^h : u = 0 \text{ on } \Gamma_h - F_{i,h}\}$, where Γ_h and $F_{i,h}$ are the set of nodes on Γ and F_i respectively. The other spaces are defined analogously. As for the overlapping case, we then embed these scalar spaces in our product space $\tilde{\mathbf{W}}^h \times \tilde{V}^h$: for example, $\tilde{\mathbf{W}}^h_{1,F_i} = \tilde{W}^h_{1,F_i} \times \{0\} \times \{0\} \times \{0\}$. As a coarse space, we consider the discrete harmonic subspace of the same coarse space used for P_{add2}, i.e. $\tilde{\mathbf{W}}^H_1 + \tilde{\mathbf{W}}^H_2 + \tilde{\mathbf{W}}^H_3 + \tilde{\mathbf{V}}^H$. We obtain the following decomposition

$$\tilde{\mathbf{W}}^h \times \tilde{V}^h = \sum_{k=1}^{3} (\sum_{F_i} \tilde{\mathbf{W}}^h_{k,F_i} + \sum_{E_i} \tilde{\mathbf{W}}^h_{k,E_i} + \sum_{v_i} \tilde{\mathbf{W}}^h_{k,v_i} + \tilde{\mathbf{W}}^H_k)$$

$$+ \sum_{F_i} \tilde{\mathbf{V}}^h_{F_i} + \sum_{E_i} \tilde{\mathbf{V}}^h_{E_i} + \sum_{v_i} \tilde{\mathbf{V}}^h_{v_i} + \tilde{\mathbf{V}}^H.$$

By defining as before projection operators into the subspaces, we form the additive operator

$$P_{is} = \sum_{k=1}^{4} (\sum_{F_i} P_{k,F_i} + \sum_{E_i} P_{k,E_i} + \sum_{v_i} P_{k,v_i} + P_{k,0}),$$

where again for $k = 4$ the projections are into the $\tilde{\mathbf{V}}^h_i$ spaces.

THEOREM 3.3. *There exists a positive constant C independent of h and H such that*

$$cond(P_{is}) \leq C(1 + \log(H/h))^2.$$

As before, the proof is based on the extension of the scalar result (see Dryja, Smith and Widlund [13], Theorem 6.2) by using the equivalence of Theorem 2.1.

4. Numerical Results. In this section, we report the results of numerical experiments which confirm the optimal convergence bounds obtained in the previous sections. We also compare the same domain decomposition methods applied to least squares discretizations and to standard Galerkin discretization of (2.1) with piecewise linear finite elements. We have run numerical experiments for symmetric positive definite problems, for which the domain decomposition theory is completely understood. More general nonsymmetric or indefinite problems, for which the domain decomposition theory is still undergoing important developments, will be the subject of future studies.

All the results have been obtained with Matlab 4.2 running on Sun Sparcstations. The model problem considered is the standard Poisson equation (A=I, X=0) on the unit square, with $p = 0$ on $\Gamma_D = \partial\Omega$ and $\gamma_\tau \mathbf{u} = 0$ on $\partial\Omega$ (i.e. $u_1 = 0$ on $\{y = 0\}$ and $\{y = 1\}$; $u_2 = 0$ on $\{x = 0\}$ and $\{x = 1\}$). The right-hand side f is chosen such that we have $p(x, y) = sin(\pi x)sin(\pi y)$ as exact solution. The region Ω is decomposed into a regular grid of N square subdomains, with N varying from 2×2 to 8×8. The fine grid mesh size h varies from $1/32$ to $1/128$.

The Krylov method used for all the symmetric problems is PCG, while we use GMRES for the nonsymmetric problem with P_{mult}. The initial guess is always zero and the stopping criterion is $\|r_k\|_2/\|r_0\|_2 < 10^{-6}$, where r_k is the residual at step k.

The local and coarse problems involved in the application of the preconditioners are always solved directly. For each method, we report the number of iterations and the Lanczos-based estimates of the condition number and the extreme eigenvalues (except for the multiplicative algorithm, where we report the average convergence factor instead).

TABLE 4.1
P_{add1}: *Overlapping Additive Schwarz with fixed overlap size* $\delta = h$.

N	h^{-1}	$iter.$	$cond(P_{add1})$	λ_{max}	λ_{min}
4	32	16	11.2172	4.0048	0.3570
9	48	19	12.1787	4.0068	0.3290
16	64	20	11.9775	4.0050	0.3343
25	80	20	11.1689	4.0052	0.3586
36	96	21	12.5450	4.0044	0.3192
49	112	20	11.9944	4.0050	0.3339
64	128	21	12.5500	4.0047	0.3191

TABLE 4.2
P_{add1}: *Overlapping Additive Schwarz with fixed number of subdomains* $N = 64$.

δ	h^{-1}	$iter.$	$cond(P_{add1})$	λ_{max}	λ_{min}
h	128	21	12.5500	4.0047	0.3191
$2h$	128	17	7.1316	4.0307	0.5651
$3h$	128	16	5.5769	4.0765	0.7309
$4h$	128	15	4.9540	4.1396	0.8356
$5h$	128	15	4.6460	4.2170	0.9076
$6h$	128	15	4.5125	4.3054	0.9541
$7h$	128	16	4.5859	4.4018	0.9598

4.1. Overlapping additive methods. We have first studied the coupled method P_{add1} with fixed minimal overlap size $\delta = h$. The mesh size h is decreased while the number of subdomains N is increased proportionally, so that the subdomain size $H/h = 16$ is kept constant ($H = 1/\sqrt{N}$). The results are reported in Table 4.1 and clearly show a constant bound for the condition number $cond(P_{add1}) = \lambda_{max}/\lambda_{min}$, for problem sizes from $3007 (N = 4)$ to $48895 (N = 64)$.

In Table 4.2, we fix the mesh size ($h = 1/128$) and the decomposition ($N = 64$), and we vary the overlap size δ from h to $7h$. As in the scalar case, the condition number $cond(P_{add1})$ improves as δ increases, because of λ_{min} being closer to unity. For large overlap, the improvement becomes negligible or negative, because of to the growth of λ_{max}.

The same sets of results for the uncoupled method P_{add2} are reported in Table 4.3 and Table 4.4, respectively. Again, the tables clearly show a constant bound on the condition number $cond(P_{add2})$, which improves as the overlap increases. For this simple model problem, the uncoupled method is only slightly worse than the coupled one, in terms of iteration count (some condition number are almost the same or even better for P_{add2}). We point out that although $A = I$, eliminating diffusive coupling between the flux components, there is still coupling between the flux variables and p, so the strong performance of P_{add2} is encouraging.

4.2. Overlapping multiplicative methods. In Table 4.5, we compare the multiplicative method P_{mult} accelerated with GMRES and the symmetrized multiplicative method P_{mults} accelerated with CG. We consider the two methods with minimal overlap and constant subdomain size. Since P_{mult} is nonsymmetric, we report the average convergence factor $\rho = (r_i/r_0)^{1/i}$ instead of the condition number. For both methods, the number of iterations is clearly bounded by a constant. Even if the symmetrized version is approximately twice as expensive as the standard one, the number of iterations is almost the same for the two methods. Therefore, the symmetrized version is less efficient on this simple problem.

TABLE 4.3
P_{add2} *: Overlapping Additive Schwarz with fixed overlap size* $\delta = h$.

N	h^{-1}	$iter.$	$cond(P_{add2})$	λ_{max}	λ_{min}
4	32	17	10.3521	4.0050	0.3868
9	48	20	12.6290	4.0051	0.3171
16	64	20	11.9811	4.0051	0.3342
25	80	21	11.3821	4.0052	0.3518
36	96	21	12.5458	4.0043	0.3191
49	112	20	11.9997	4.0052	0.3337
64	128	21	12.5261	4.0047	0.3197

TABLE 4.4
P_{add2} *: Overlapping Additive Schwarz with fixed number of subdomains* $N = 64$.

δ	h^{-1}	$iter.$	$cond(P_{add2})$	λ_{max}	λ_{min}
h	128	21	12.5261	4.0047	0.3197
$2h$	128	17	7.1206	4.0315	0.5661
$3h$	128	16	5.5513	4.0777	0.7345
$4h$	128	16	5.3850	4.1442	0.7695
$5h$	128	16	5.4545	4.2233	0.7742
$6h$	128	16	5.5306	4.3158	0.7803
$7h$	128	16	5.6176	4.4297	0.7885

4.3. Iterative substructuring. Table 4.6 shows the results for the iterative substructuring methods P_{is} with fixed subdomain size $H/h = 16$. The condition numbers and iteration counts have a less smooth behavior compared to overlapping methods, but they still show a constant bound. The smaller dimension of the reduced Schur complement being solved here accounts for the smaller condition numbers and iteration counts.

4.4. Comparison with the standard Galerkin formulation. In this subsection, we provide a comparison of the least squares results with the standard Galerkin formulation of the same model problem, using piecewise linear finite elements for the original variable p. More complete results for domain decomposition methods for standard Galerkin formulations of elliptic problems in the plane can be found in Cai, Gropp and Keyes [6]. We start with the overlapping additive Schwarz method P_{add}, which corresponds to P_{add1} or P_{add2} (there is only one version in the scalar case). Tables 4.7 and 4.8 report the results which should be compared with the least squares results of Tables 4.1 and 4.2. Besides the well-known optimality of the method, we remark that for this simple model problem, the least squares condition numbers are almost the same as (and the iterations count only slightly worse than) the corresponding results for standard Galerkin, in spite of the much larger size of the least squares system (roughly three times larger in the plane). This fact is very encouraging for the least squares approach with these domain decomposition preconditioners, since a much larger system with both the original and the flux variables can be solved in almost the same number of iterations.

The situation is analogous for multiplicative methods. The results for both the nonsymmetric and symmetrized multiplicative version reported in Table 4.9 are almost the same as the results in Table 4.5. Again, the much larger least squares system requires the same number of iterations as (or just one more than) the standard Galerkin system.

On the other hand, for the substructuring method P_{is}, the results reported in Table 4.10

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TABLE 4.5
P_{mult} and P_{mults}: Overlapping Multiplicative Schwarz with fixed overlap size $\delta = h$.

		multiplicative (GMRES)		symmetrized multiplicative (CG)			
N	h^{-1}	$iter.$	$\rho = (r_i/r_0)^{1/i}$	$iter.$	$cond(P_{mults})$	λ_{max}	λ_{min}
4	32	8	0.1847	7	1.8576	0.9994	0.5379
9	48	7	0.1433	6	1.7398	0.9999	0.5749
16	64	6	0.1233	6	1.7600	0.9999	0.5681
25	80	6	0.1102	6	1.6810	0.9999	0.5948
36	96	6	0.1021	6	1.6940	0.9999	0.5902
49	112	6	0.0952	6	1.6661	0.9999	0.6001
64	128	5	0.0849	6	1.7308	0.9999	0.6079

TABLE 4.6
P_{is}: Iterative Substructuring .

N	h^{-1}	$iter.$	$cond(P_{is})$	λ_{max}	λ_{min}
4	32	9	3.4035	1.5691	0.4610
9	48	17	7.8812	1.8497	0.2347
16	64	18	7.8543	1.7962	0.2287
25	80	18	8.5822	1.8864	0.2198
36	96	19	9.4115	1.8511	0.1966
49	112	18	8.6646	1.8939	0.2185
64	128	19	9.6532	1.8617	0.1928

show a considerable difference between the two discretizations. The condition numbers and iteration counts of the standard Galerkin system are about one half the corresponding ones for the least squares system.

Even if more results for more realistic problems are needed in order to better compare domain decomposition methods for the two discretizations, this initial comparison seems to indicate that overlapping methods are very efficient for least squares discretizations, particularly in their multiplicative form.

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TABLE 4.7
Standard Galerkin: Overlapping Additive Schwarz with fixed overlap size $\delta = h$.

N	h^{-1}	$iter.$	$cond(P_{add})$	λ_{max}	λ_{min}
4	32	11	9.9403	4.0048	0.4029
9	48	15	12.6274	4.0025	0.3170
16	64	16	12.1403	4.0051	0.3299
25	80	18	12.8580	4.0038	0.3114
36	96	18	12.6448	4.0051	0.3167
49	112	18	12.7554	4.0044	0.3139
64	128	17	12.6752	4.0044	0.3159

TABLE 4.8
Standard Galerkin: Overlapping Additive Schwarz with fixed number of subdomains $N = 64$.

δ	h^{-1}	$iter.$	$cond(P_{add})$	λ_{max}	λ_{min}
h	128	17	12.6752	4.0044	0.3159
$2h$	128	15	7.4472	4.0301	0.5412
$3h$	128	13	5.8085	4.0757	0.7017
$4h$	128	13	5.1185	4.1408	0.8090
$5h$	128	13	4.8248	4.2320	0.8771
$6h$	128	13	4.7267	4.3393	0.9180
$7h$	128	13	4.7085	4.4452	0.9441

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TABLE 4.9
Standard Galerkin: Overlapping Multiplicative Schwarz with fixed overlap size $\delta = h$.

		multiplicative (GMRES)		symmetrized multiplicative (CG)			
N	h^{-1}	$iter.$	$\rho = (r_i/r_0)^{1/i}$	$iter.$	$cond(P_{mults})$	λ_{max}	λ_{min}
4	32	7	0.1283	7	1.9154	0.9999	0.5221
9	48	6	0.1317	7	1.8269	0.9999	0.5474
16	64	6	0.1219	6	1.7212	0.9999	0.5809
25	80	6	0.1161	6	1.6851	0.9999	0.5934
36	96	5	0.0979	6	1.6586	0.9999	0.6029
49	112	5	0.0895	6	1.6355	0.9999	0.6114
64	128	5	0.0829	5	1.4554	0.9999	0.6870

TABLE 4.10
Standard Galerkin: Iterative Substructuring.

N	h^{-1}	$iter.$	$cond(P_{is})$	λ_{max}	λ_{min}
4	32	7	3.2874	1.5758	0.4793
9	48	9	3.5248	1.6525	0.4688
16	64	9	3.6773	1.7336	0.4714
25	80	10	3.7989	1.7854	0.4699
36	96	10	3.8530	1.8122	0.4703
49	112	10	3.8863	1.8287	0.4705
64	128	10	3.9091	1.8399	0.4707

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