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ON A POSTERIORI ERROR ESTIMATORS IN THE FINITE ELEMENT METHOD ON ANISOTROPIC MESHES*

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Abstract. On anisotropic finite element meshes, the standard residual based error indicator is derived and it is proved that it is not efficient if the aspect ratio deteriorates. For a nonlocal error indicator it is proved that it is reliable and efficient independent of the aspect ratio. This is also confirmed by some numerical calculations.

Key words. finite elements, a posteriori estimators, anisotropic meshes.

AMS subject classifications. 65N30, 65N15.

1. Formulation of the problem. Let $\Omega \subset I\!\!R^2$ be a bounded polygonal domain. Consider Poisson's equation

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

For approximating problem (1.1) we use the standard conforming finite element method on an anisotropic triangulation $\Pi = \{\Lambda\}$ which is defined by the following conditions:

(1.2)
a) The intersection of two triangles is void or coincides with a common side or vertex.
b) The interior angles of the triangles are bounded from above by $\alpha < \pi$.
c) Let $U(\Lambda)$ denote the union of the triangles adjacent to Λ. It is assumed that each $U(\Lambda)$ can be rotated such that it can be represented as the image of an isotropic reference configuration $\hat{U}(\hat{\Lambda})$ of size O(1) under the mapping $x_i = h_i \hat{x}_i$.

The last condition ensures that the direction of the anisotropic mesh does not change too rapidly. Since condition (1.2b) guarantees that in the anisotropic case two of the sides of Λ are long and nearly perpendicular to the small side, there exists a local orthogonal coordinate system $(e_1, e_2) = (e_1(\Lambda), e_2(\Lambda))$ where e_1 can be chosen to be the direction of one of the larger sides. Similarly, the long and short local step sizes are denoted by $(h_1, h_2) = (h_1(\Lambda), h_2(\Lambda))$. The sets of long and small sides are denoted by Γ_l and Γ_s, respectively.

For $k = 1, 2$ we define the standard finite element spaces consisting of continuous and piecewise linear or quadratic shape functions,

$$S_k = \left\{ v \in C(\overline{\Omega}) : v|_\Lambda \in I\!\!P_k \text{ for all } \Lambda \in \Pi \text{ and } v|_{\partial\Omega} = 0 \right\}.$$

Denoting by $I_k : C^0(\overline{\Omega}) \cap H_0^{1,2}(\Omega) \to S_k$ the standard interpolation operators we obtain from Theorem 2 in [1] the estimates

$$||D_1(u - I_k u)||_{2;\Lambda} \le c h_1^{-1} \sum_{|\alpha|=k+1} h^\alpha ||D^\alpha u||_{2;\Lambda}, \tag{1.3}$$

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$$||D_2(u - I_k u)||_{2;\Lambda} \le c \sum_{|\alpha|=k} h^\alpha ||D^\alpha D u||_{2;\Lambda}, \tag{1.4}$$

where $D_i = \frac{\partial}{\partial e_i}$ and $h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2}$.

The finite element approximation $P_k u \in S_k$ is defined by

$$(DP_k u, Dv) = (f, v) \quad \forall v \in S_k. \tag{1.5}$$

We are interested in a posteriori estimates for the error $e = u - P_1 u$ by local error indicators η_Λ satisfying

$$m_1 ||De||^2 - T(h,f) \le \sum_\Lambda \eta_\Lambda^2 \le m_2 ||De||^2 + T(h,f), \tag{1.6}$$

where $T(h,f)$ is usually a small term depending on f and converging to 0 for $h \to 0$.

For earlier work on a posteriori error estimators on isotropic meshes we refer to Babuška and Rheinboldt [2] and to the survey Verfürth [10]. Of special interest are the residual based indicator of Verfürth [9] and the indicators of Bank and Weiser [4]. The crucial point of anisotropic a posteriori estimating is the fact that all classical estimators deteriorate if the aspect ratio $a(\Lambda) = h_1(\Lambda)/h_2(\Lambda)$ tends to infinity. Siebert [8] solves this problem by locally balancing the directional errors avoiding anisotropic overrefinement. On the other hand, overrefinement occurs in elliptic systems where one equation is singularly perturbed and the others are not.

The outline of the paper is as follows. In section 2, we show that the standard error estimator based on the residual does not satisfy (1.6) with constants m_1, m_2 independent of the aspect ratio a. Section 3 is devoted to the study of a nonlocal error estimator inspired by the third indicator of Bank and Weiser [4]. Despite the fact that the estimator is nonlocal, it is proved that it can be computed economically on isotropic and anisotropic meshes. On anisotropic meshes, the estimator shows a significant propagation of local errors along the small mesh direction e_2, which clearly indicates that local a posteriori error estimation is impossible as long as the standard norm $||D \cdot ||$ is used. Some numerical computations demonstrate that the nonlocal indicator behaves exactly as predicted by the theory.

2. An error estimator based on local residuals. By $R_1 : H_0^{1,2} \to S_1$ we denote the local approximation operator constructed by Scott and Zhang [7] which satisfies, on an isotropic mesh with mesh parameter $h = 1$,

$$||v - R_1 v||_{\hat\Lambda}^2 \le c ||Dv||_{U(\hat\Lambda)}^2, \tag{2.1}$$

$$||v - R_1 v||_{\hat\Gamma}^2 \le c ||Dv||_{U(\hat\Gamma)}^2, \tag{2.2}$$

where $U(\hat\Gamma)$ consists of the union of the triangles adjacent to $\hat\Gamma$. In view of condition (1.2) c) we can transform (2.1) , (2.2) by $x_i = h_i \hat{x}_i$ and obtain

$$||v - R_1 v||_\Lambda^2 \le c \left\{ h_1^2 ||D_1 v||_{U(\Lambda)}^2 + h_2^2 ||D_2 v||_{U(\Lambda)}^2 \right\}, \tag{2.3}$$

$$||v - R_1 v||_\Gamma^2 \le c h_2^{-1} \left\{ h_1^2 ||D_1 v||_{U(\Gamma)}^2 + h_2^2 ||D_2 v||_{U(\Gamma)}^2 \right\}, \quad \Gamma \in \Gamma_l, \tag{2.4}$$

$$||v - R_1 v||_\Gamma^2 \le c h_1^{-1} \left\{ h_1^2 ||D_1 v||_{U(\Gamma)}^2 + h_2^2 ||D_2 v||_{U(\Gamma)}^2 \right\}, \quad \Gamma \in \Gamma_s. \tag{2.5}$$

Using the orthogonality relation $(De, Dv) = 0$ for all $v \in S_1$ and integration by parts, it follows that

$$\begin{aligned} ||De||^2 &= (De, D(e - R_1 e)) \\ &= \sum_\Lambda \left\{ \int_\Lambda (-\Delta u)(e - R_1 e)\, dx + \int_{\partial\Lambda} D_n e(e - R_1 e)\, d\sigma \right\} \\ &= \sum_\Lambda \int_\Lambda f(e - R_1 e)\, dx + \sum_\Gamma \int_\Gamma [D_n P_1 u]_J (e - R_1 e)\, d\sigma, \end{aligned}$$

where $[D_n v]_J$ denotes the "jump" of the normal derivative $D_n v$ across Γ. The right hand side can be bounded by Cauchy's inequality, and (2.3) - (2.5) which gives

$$\begin{aligned} ||De||^2 \le{}& c \sum_\Lambda ||f||_\Lambda \left\{ h_1^2 ||D_1 e||^2_{U(\Lambda)} + h_2^2 ||D_2 e||^2_{U(\Lambda)} \right\}^{1/2} \\ &+ c \sum_{\Gamma_l} ||[D_n P_1 u]_J||_\Gamma \left\{ h_2^{-1} h_1^2 ||D_1 e||^2_{U(\Gamma)} + h_2 ||D_2 e||^2_{U(\Gamma)} \right\}^{1/2} \\ &+ c \sum_{\Gamma_s} ||[D_n P_1 u]_J||_\Gamma \left\{ h_1 ||D_1 e||^2_{U(\Gamma)} + h_1^{-1} h_2^2 ||D_2 e||^2_{U(\Gamma)} \right\}^{1/2}. \end{aligned}$$

In view of the fact that $h_1 \ge h_2$ we have found the a posteriori bound

$$||De||^2 \le c \sum_\Lambda h_1^2 ||f||^2_\Lambda + c \sum_{\Gamma_l} h_2^{-1} h_1^2 ||[D_n P_1 u]_J||^2_\Gamma + c \sum_{\Gamma_s} h_1 ||[D_n P_1 u]_J||^2_\Gamma \tag{2.6}$$

with local mesh sizes $h_i(\Lambda)$. Denoting by $\Gamma(\Lambda)$ the set of the sides of Λ we define the local estimator η_Λ by

$$\eta^2_\Lambda = \sum_{\Gamma_l \cap \Gamma(\Lambda)} h_2^{-1} h_1^2 ||[D_n P_1 u]_J||^2_\Gamma + \sum_{\Gamma_s \cap \Gamma(\Lambda)} h_1 ||[D_n P_1 u]_J||^2_\Gamma. \tag{2.7}$$

Though the right hand side definitely deteriorates for $h_2 << h_1$, one can argue that the corresponding jump $[D_n P_1 u]_J$ becomes smaller in this case. But in the sequel, we will prove that $\sum_\Lambda \eta^2$ leads to an arbitrarily large overestimation of the true error $||De||^2$ if the aspect ratio tends to infinity.

As an example, we consider the orthogonal subdivision of the unit square with mesh sizes $\hat h_1, \hat h_2, \hat h_1 = a \hat h_2, a > 1$. In order to transform this mesh to an isotropic mesh we use $x_2 = a \hat x_2, x_1 = \hat x_1$ and get the operator

$$Lu = -D^2_{11} u - a^2 D^2_{22} u$$

on the rectangle $[0,1] \times (0,a)$. Denoting by S^b_1 the space of continuous and piecewise bilinear functions satisfying a 0–boundary condition the finite element method is defined by

$$(D_1 P_1 u, D_1 v) + a^2 (D_2 P_1 u, D_2 v) = (f, v) \quad \forall v \in S^b_1. \tag{2.8}$$

Let Γ_i be the set of edges with direction $e_i, i = 1, 2$. By a similar analysis as before we obtain for the error $e = u - P_1 u$,

$$||D_1 e||^2 + a^2 ||D_2 e||^2 = \sum_\Lambda \int_\Lambda Lu(e - R_1 e)\, dx + \sum_{\Gamma_2} \int_\Gamma [D_1 P_1 u]_J (e - R_1 e)\, dx_2$$

$$
\begin{aligned}
&+a^2 \sum_{\Gamma_1} \int_\Gamma [D_2 P_1 u]_J (e - Re)\, dx_1 \\
&\le ch \sum_\Lambda ||f||_\Lambda ||De||_{U(\Lambda)} + ch^{1/2} \sum_{\Gamma_2} ||[D_1 P_1 u]_J||_\Gamma ||De||_{U(\Gamma)} \\
&+ca^2 h^{1/2} \sum_{\Gamma_1} ||[D_2 P_1 u]_J||_\Gamma ||De||_{U(\Gamma)}.
\end{aligned}
$$

By Young's inequality it follows that $(a \ge 1)$

$$||D_1 e||^2 + a^2 ||D_2 e||^2 \le ch^2 ||f||_\Omega^2 + ch \sum_{\Gamma_2} ||[D_1 P_1 u]_J||_\Gamma^2 + ca^4 h \sum_{\Gamma_1} ||[D_2 P_1 u]_J||_\Gamma^2.$$

The local error indicator is now defined by

$$\eta_\Lambda^2 = a^4 h \sum_{\Gamma_1 \cap \Gamma(\Lambda)} ||[D_2 P_1 u]_J||_\Gamma^2 + h \sum_{\Gamma_2 \cap \Gamma(\Lambda)} ||[D_1 P_1 u]_J||_\Gamma^2. \tag{2.9}$$

We remark that we obtain the same error indicator by simply transforming (2.6) using $x_1 = \hat{x}_1, x_2 = a\hat{x}_2, h = \hat{h}_1 = a\hat{h}_2$,

$$
\begin{aligned}
||D\hat{e}||^2 &\to a^{-1} \left(||D_1 e||^2 + a^2 ||D_2 e||^2 \right), \\
\hat{h}_1^2 ||\hat{f}||^2 &\to a^{-1} h^2 ||f||^2, \\
\sum_{\Gamma_l} \hat{h}_2^{-1} \hat{h}_1^2 ||[D_n P_1 \hat{u}]_J||_{\hat{\Gamma}}^2 &\to \sum_{\Gamma_1} a^3 h ||[D_2 P_1 u]_J||_\Gamma^2, \\
\sum_{\Gamma_s} \hat{h}_1 ||[D_n P_1 \hat{u}]_J||_{\hat{\Gamma}}^2 &\to \sum_{\Gamma_2} a^{-1} h ||[D_1 P_1 u]_J||_\Gamma^2.
\end{aligned}
$$

LEMMA 2.1. *For the finite element approximation $P_1 u$ in (2.8) we have the error estimate*

$$||D_1 e||^2 + a^2 ||D_2 e||^2 \le ca^2 h^2 ||D^2 u||^2.$$

Proof. From the interpolation estimates (1.3), (1.4) it follows that

$$
\begin{aligned}
||D_1 e||^2 + a^2 ||D_2 e||^2 &= (D_1 e, D_1(u - I_1 u)) + a^2 (D_2 e, D_2(u - I_1 u)) \\
&\le \frac{1}{2} ||D_1 e||^2 + \frac{a^2}{2} ||D_2 e||^2 + ca^2 h^2 \|D^2 u\|^2.
\end{aligned}
$$

□

Let D_2^+ be the forward finite difference operator approximating D_2, i.e.

$$D_2^+ v(x) = \frac{1}{h} (v(x + he_2) - v(x)).$$

LEMMA 2.2. *For the finite element approximation $P_1 u$ in (2.8) we have the estimate*

$$||D_2^+ D_1 e||_{\Omega_0}^2 + a^2 ||D_2^+ D_2 e||_{\Omega_0}^2 \le ca^2 h ||u||_{3,2}^2$$

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for every $\Omega_0 \subset\subset \Omega$, $\epsilon > 0$, *and* $0 < h \leq h_0(\Omega_0)$.

Proof. Since the subdivision is uniform we have

$$(D_2^+ D_1 e, D_1 v) + a^2 (D_2^+ D_2 e, D_2 v) = 0 \tag{2.10}$$

for all $v \in S_1^b$ with dist (supp $(v), \partial\Omega) \geq 2h$. Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ be domains that are sufficiently far away from $\partial\Omega$, let τ be a cut–off function with respect to $\{\Omega_0, \Omega_1\}$, i.e. $\tau \in C_0^\infty(\Omega_1)$ with $\tau = 1$ in Ω_0. For τ we have the estimates $|D^k \tau| \leq c$, $k = 1, 2$, with a constant c depending on Ω_0, Ω_1. Using (2.10) we obtain that

$$\begin{aligned}\int_{\Omega_0} \left\{ |D_2^+ D_1 e|^2 + a^2 |D_2^+ D_2 e|^2 \right\} dx &\leq \int_\Omega \tau \left\{ |D_2^+ D_1 e|^2 + a^2 |D_2^+ D_2 e|^2 \right\} dx \\ &= \big(D_2^+ D_1 e, D_1 (D_2^+ e\tau - v)\big) \\ &\quad + a^2 \left(D_2^+ D_2 e, D_2 (D_2^+ e\tau - v)\right) \\ &\quad - (D_2^+ D_1 e, D_2^+ e D_1 \tau) - a^2 (D_2^+ D_2 e, D_2^+ e D_2 \tau).\end{aligned}$$

We choose $v = I_1(D_2^+ e\tau)$ and obtain from (1.3), (1.4) that

$$\begin{aligned}||D_2^+ D_1 e||_{\Omega_0}^2 + a^2 ||D_2^+ D_2 e||_{\Omega_0}^2 &\leq ch ||D_2^+ D_1 e||_{\Omega_1} ||D^2 (D_2^+ e\tau)||_{\Omega_1} \\ &\quad + c a^2 h ||D_2^+ D_2 e||_{\Omega_1} ||D^2 (D_2^+ e\tau)||_{\Omega_1} &(2.11)\\ &\quad + c ||D_2^+ D_1 e||_{\Omega_1} ||D_2^+ e||_{\Omega_1} &(2.12)\\ &\quad + a^2 ||D_2^+ D_2 e||_{\Omega_1} ||D_2^+ e||_{\Omega_1}.\end{aligned}$$

Let Ω_2 be a slightly larger domain than Ω_1, such that

$$||D_2^+ e||_{\Omega_1} \leq ||D_2 e||_{\Omega_2}$$

(see [6] p.161). By Lemma 2.1, this term can then be bounded by

$$||D_2^+ e||_{\Omega_1} \leq ch ||u||_{2,2}. \tag{2.13}$$

Moreover, we have the simple inequality

$$||D^2 (D_2^+ e\tau)||_{\Omega_1} \leq ||D^2 D_2^+ u||_{\Omega_1} + c ||D D_2^+ e||_{\Omega_1} + c ||D_2^+ e||_{\Omega_1}.$$

Applying the interpolation estimates (1.3), (1.4) and the usual inverse inequality to the second term on the right hand side of the last expression, we obtain

$$\begin{aligned}||D D_2^+ e||_{\Omega_1} &\leq ||D D_2^+ (u - I_1 u)||_{\Omega_1} + ||D D_2^+ (I_1 u - P_1 u)||_{\Omega_1} \\ &\leq ch ||u||_{2,2} + ch^{-1} ||D_2^+ (I_1 u - u)||_{\Omega_1} + ch^{-1} ||D_2^+ (u - P_1 u)||_{\Omega_1} &(2.14)\\ &\leq c ||u||_{2,2},\end{aligned}$$

from which it follows that

$$||D^2 (D_2^+ e\tau)||_{\Omega_1} \leq c ||u||_{3,2}.$$

Inserting the last inequality and (2.13), (2.14) into (2.11), we obtain

$$\begin{aligned}||D_2^+ D_1 e||_{\Omega_0}^2 + a^2 ||D_2^+ D_2 e||_{\Omega_0}^2 &\leq ch ||D_2^+ D_1 e||_{\Omega_1} ||u||_{3,2} &(2.15)\\ &\quad + c a^2 h ||D_2^+ D_2 e||_{\Omega_1} ||u||_{3,2}.\end{aligned}$$

In view of the property that $||D_j D_2^+ e||_{\Omega_1} = ||D_2^+ D_j e||_{\Omega_1}$ for $j = 1, 2$ and (2.14) we have

$$||D_2^+ D_j e||_{\Omega_1} \le c||u||_{2,2} \le c||u||_{3,2}$$

which leads to

$$||D_2^+ D_1 e||_{\Omega_0}^2 + a^2 ||D_2^+ D_2 e||_{\Omega_0}^2 \le ch||u||_{3,2}^2 + ca^2 h||u||_{3,2}^2 \le ca^2 h||u||_{3,2}^2.$$

□

Remark: Note that one can get a better estimation than stated in the Lemma by iterating (2.15) for a sequence of domains $\Omega_0 \subset\subset \Omega_1 \subset\subset \dots \Omega_k \subset\subset \dots$. Arguing in this way we would get the inequality

$$||D_2^+ D_1 e||_{\Omega_0}^2 + a^2 ||D_2^+ D_2 e||_{\Omega_0}^2 \le ca^2 h^{2-\epsilon} ||u||_{3,2}^2$$

for all $\epsilon > 0$.

Now let Λ be a square with upper neighboring square $\tilde{\Lambda}$ and common side Γ. For $v \in S_0^b$ we have

$$\int_\Lambda |D_2^+ D_2 v|^2 \, dx_1 dx_2 = \frac{1}{h^2} \int_\Lambda |D_2 v(x + he_2) - D_2 v(x)|^2 \, dx_1 dx_2.$$

In view of the fact that $D_2 v$ depends only on the variable x_1 we get

$$||D_2^+ D_2 v||_\Lambda^2 = \frac{1}{h} \int_\Gamma [D_2 v]_J^2 \, dx_1.$$

Let $\Omega_0 \subset\subset \Omega$ be a fixed domain. Denoting the set of sides of Ω_0 in direction e_1 by Γ_0 we obtain from Lemma 2.2 that

$$\begin{aligned} ha^4 \sum_{\Gamma_0} ||[D_2 P_1 u]_J||_\Gamma^2 &\ge a^4 h^2 ||D_2^+ D_2 P_1 u||_{\Omega_0}^2 \\ &\ge \frac{1}{2} a^4 h^2 ||D_2^+ D_2 u||_{\Omega_0}^2 - a^4 h^2 ||D_2^+ D_2 e||_{\Omega_0}^2 \\ &\ge \frac{1}{2} a^4 h^2 ||D_2^+ D_2 u||_{\Omega_0}^2 - ca^4 h^3 ||u||_{3,2;\Omega}^2. \end{aligned} \tag{2.16}$$

Choosing a smooth function u which behaves like $\sin x_2$ in Ω_0 such that

$$||D_2^+ D_2 u||_{\Omega_0} \ge c_1, \qquad ||u||_{3,2;\Omega} \le c_2,$$

with constants $c_1, c_2 > 0$, then (2.16) shows, that for h sufficiently small

$$ha^4 \sum_{\Gamma_0} ||[D_2 P_1 u]_J||_\Gamma^2 \ge \frac{1}{4} a^4 h^2 c_1^2.$$

From Lemma 2.1 we conclude that the error estimator gives an overestimation with factor a^2 for functions of this type.

3. A nonlocal error indicator. In this section we return to Poisson's equation (1.1) and its finite element approximation (1.5). Recall that $I_k, k = 1, 2$, are the standard interpolation operators into the spaces S_k, and define the space

$$S^0 = \{v \in S_2 : \; I_1 v = 0\},$$

which means that the elements of S^0 vanish in the nodal points of the triangulation.

The nonlocal version of the third error indicator of Bank–Weiser [4] is given by $\overset{o}{e} \in S^0$ such that

$$(D\overset{o}{e}, Dv) = (De, Dv) \quad \forall v \in S^0. \tag{3.1}$$

In view of

$$(De, Dv) = (f, v) - (DP_1 u, Dv)$$

the right-hand side of (3.1) is known if $P_1 u$ is known.

Let us compare $\overset{o}{e}$ with the original third indicator of [4] which also gives some insight into error propagation on anisotropic meshes. Let $\tilde{S}$ be the discontinuous version of S^0, i.e. $\tilde{S}$ consists of all piecewise quadratic functions vanishing in the nodal points of the triangulation. The indicator $\tilde{e} \in \tilde{S}$ is then defined by

$$(D\tilde{e}, Dv)_\Lambda = F_\Lambda(v) \quad \forall v \in \tilde{S}, \tag{3.2}$$

where

$$F_\Lambda(v) = (f, v)_\Lambda + \frac{1}{2} \sum_{\Gamma \in \Gamma(\Lambda)} \int_\Gamma [D_n P_1 u]_J [v]_A \, d\sigma,$$

and where $[v]_A$ is the average of v on the neighboring triangles of Γ. Summing (3.2) over Λ and using integration by parts yield

$$\sum_\Lambda (D\tilde{e}, Dv)_\Lambda = \sum_\Lambda (De, Dv)_\Lambda - \sum_\Gamma \int_\Gamma [D_n e]_J [v]_A \quad \forall v \in \tilde{S}.$$

Comparing this with (3.1) shows that $\overset{o}{e}$ is the continuous and nonlocal counterpart of $\tilde{e}$. For the actual computation of $\tilde{e}$, a 3×3 linear system has to be solved on each triangle Λ in contrast to the large system required for the computation of $\overset{o}{e}$. On the other hand, a complicated computation using the symbolic program "mathematica" shows that the system corresponding to (3.1) is strictly diagonally dominant if the largest interior angle is bounded by $\alpha < \frac{\pi}{2}$. Let $0 < \beta \leq \alpha < \frac{\pi}{2}$. Since the triangles with interior angles between β and α are compactly parametrized we obtain that the system in (3.1) is uniformly strictly diagonally dominant in this class of triangles and can efficiently be solved by the simple Gauß–Seidel method. Furthermore, we conclude that local errors decrease exponentially on such meshes. This is the reason why local error estimation is possible. For isotropic triangles with angles bounded by $\alpha < \pi$ the reasoning is similar. Since local error estimation is also possible in this case, we conclude that the system in (3.1) must be strictly dominant "in the mean" and can again be solved by the Gauß–Seidel method.

Now we turn to the anisotropic case and consider the orthogonal mesh with parameters $h_2 << h_1$ shown in Fig. 1. The entries of the matrix in (3.1) can be represented by stencils. For the midpoints of the larger sides we obtain

$$S_l = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix} + O\left(\frac{h_2}{h_1}\right),$$

whereas the entries of the stencil for the shorter sides are of the type

$$S_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O\left(\frac{h_2}{h_1}\right).$$

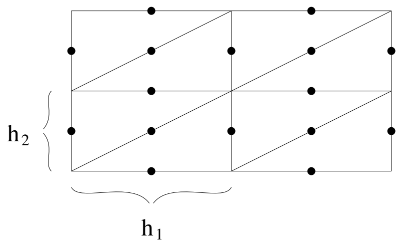

FIG. 3.1. *An anisotropic mesh.*

Thus, local errors propagate in the direction of the small sides with a stencil approximating $-h_2^2 D_{yy}^2$. Since the indicator $\overset{o}{e}$ is reliable and efficient in this case, we believe that local error estimation is inherently inaccurate on anisotropic meshes.

On the anisotropic mesh shown in Fig. 1, the indicator $\overset{o}{e}$ can efficiently be determined with the aid of a Block–Gauß–Seidel method since there is nearly no coupling normal to the smaller sides. On general anisotropic meshes, we use a mesh–dependent Block–Gauß–Seidel method where points coupled by smaller sides are updated simultaneously.

Since the local error indicator is equivalent to the residual based indicator on anisotropic meshes, the counterexample given in section 2 can also be applied. It remains to prove that the nonlocal indicator is reliable and efficient. We start with a preparatory result.

LEMMA 3.1. *There is a constant c_0 depending only on the angle α in condition (1.2) b), but not on the shape of the triangle Λ such that*

$$||DI_1 v||_\Lambda \le c_0 ||Dv||_\Lambda \quad \forall v \in S_2.$$

Proof. On isotropic triangles, this estimate can simply be proved by using a reference element and transforming the corresponding estimate to Λ. In the anisotropic case, our proof requires some notations, but is simpler and shows that $c_0 \sim 1$ for moderate α. Let P_1, P_2, P_3 be the nodes of Λ numbered counterclockwise. The edge opposite to P_i is denoted by e_i with midpoint a_i. The derivative in direction e_i is denoted by D_i. For $w \in I\!P_2(\Lambda)$ we have

$$D_i w(a_i) = \frac{1}{|e_i|}\left(w(P_{i+1}) - w(P_{i-1})\right), \tag{3.3}$$

$$\int_\Lambda w(x)\,dx = \frac{\mu(\Lambda)}{3}\sum_{i=1}^{3} w(a_i). \tag{3.4}$$

Assuming that (e_1, e_2) is a pair of a larger and a smaller side we obtain

$$||DI_1 v||_\Lambda^2 \le c\left\{||D_1 I_1 v||_\Lambda^2 + ||D_2 I_1 v||_\Lambda^2\right\},$$

where c depends on the angle α in condition (1.2) b) . Using (3.3), (3.4) and the fact that $D_i I_1 v$ is constant in Λ, we obtain

$$||DI_1 v||_\Lambda^2 \le c\mu(\Lambda)\left\{|D_1 I_1 v(a_1)|^2 + |D_2 I_1 v(a_2)|^2\right\} \le c||Dv||_\Lambda^2.$$

□

Since the spaces S^0, S_1 are locally three dimensional, the strengthened Cauchy–inequality

$$(Dv, Dw) \leq \gamma ||Dv|| \, ||Dw|| \quad \forall v \in S_1, \, \forall w \in S^0 \tag{3.5}$$

holds.

Furthermore, a well-known saturation assumption is required, namely

$$||D(u - P_2 u)|| \leq \beta(h) ||De|| \tag{3.6}$$

with $\beta(h) \to 0$ for $h \to 0$. This condition is satisfied on arbitrary anisotropic meshes independent of the aspect ratio if the solution u is sufficiently regular or the mesh is appropriately refined (see (1.3), (1.4)).

THEOREM 3.2. *With β in (3.6), γ in (3.5) and c_0 in Lemma 3.1 we have*

$$\frac{1-\beta}{1+\gamma c_0} ||De|| \leq ||D \overset{o}{e}|| \leq ||De||.$$

Proof. We insert $v = \overset{o}{e}$ in (3.1) and we obtain

$$||D \overset{o}{e}||^2 = (De, D \overset{o}{e}) \leq ||De|| \, ||D \overset{o}{e}||,$$

which proves the bound from above. From the definition of $P_i u$ it follows that

$$(D(P_2 u - P_1 u), Dv) = 0 \quad \forall v \in S_1,$$

and hence

$$\begin{aligned} ||D(P_2 u - P_1 u)||^2 &= (D(P_2 u - P_1 u), D(P_2 u - P_1 u - I_1(P_2 u - P_1 u)) \\ &= (D \overset{o}{e}, D(P_2 u - P_1 u)) - (D \overset{o}{e}, DI_1(P_2 u - P_1 u)) \\ &\leq ||D \overset{o}{e}|| \, ||D(P_2 u - P_1 u)|| + \gamma ||D \overset{o}{e}|| \, ||DI_1(P_2 u - P_1 u)|| \\ &\leq (1 + \gamma c_0) ||D \overset{o}{e}|| \, ||D(P_2 u - P_1 u)||. \end{aligned}$$

From the resulting estimate

$$||D(P_2 u - P_1 u)|| \leq (1 + \gamma c_0) ||D \overset{o}{e}||$$

it follows that

$$\begin{aligned} ||De|| &\leq ||D(u - P_2 u)|| + ||D(P_2 u - P_1 u)|| \\ &\leq \beta(h) ||De|| + (1 + \gamma c_0) ||D \overset{o}{e}||. \end{aligned}$$

□

We conclude this section by presenting some numerical results for the anisotropic mesh in Fig. 1. We use a slight modification of $\overset{o}{e}$ which is well suited for use with multilevel methods. Instead of S_2 the space of continuous, piecewise linear functions on the refined mesh is used. The space S^0 then consists of all piecewise linear functions on the refined mesh vanishing on the nodal points of the actual mesh. Now $\overset{o}{e}$ is the first iterate of a hierarchical multilevel method. If the mesh is globally refined in the next step no computing time is wasted. The disadvantage of this method is that we can only expect $\beta \sim \frac{1}{2}$ in condition (3.6) in contrast to $\beta(h) \sim h$ when using the space S_2. On the other hand, the proof of Theorem 3.2 remains valid in this new setting.

Consider $-\Delta u = 10$ in $\Omega = (0,1)^2$. Denoting by h the length of the larger sides and denoting by a the aspect ratio of the triangles we obtain the following results:

	$h^{-1} = 8$			$h^{-1} = 16$		
	$a = 1$	4	8	$a = 1$	4	8
$\|\|D \overset{o}{e}\|\|$	0.2072	0.1769	0.1758	0.1234	0.0883	0.0877
$\|\|De\|\|$	0.4122	0.2876	0.2337	0.2072	0.1384	0.1144

From Theorem 3.2 we expect that

$$\frac{1-\beta}{1+\gamma c_0} \sim \frac{1}{2}.$$

Hence, the theory is exactly confirmed by the numerical results stated above. Furthermore, there is no dependence of the error indicator on the aspect ratio a.

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