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CREATION AND ANNIHILATION OPERATORS FOR ORTHOGONAL POLYNOMIALS OF CONTINUOUS AND DISCRETE VARIABLES*

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Abstract. We develop general expressions for the raising and lowering operators that belong to the orthogonal polynomials of hypergeometric type with discrete and continuous variable. We construct the creation and annihilation operators that correspond to the normalized polynomials and study their algebraic properties in the case of the Kravchuk/Hermite Meixner/Laguerre polynomials.

Key words. orthogonal polynomials, difference equations, creation and annihilation operators.

AMS subject classifications. 33C45, 33C80, 81R50.

1. Introduction. In a previous paper [5] we developed a method to construct raising and lowering operators for the Kravchuk polynomials of a discrete variable, using the properties of Wigner functions, and to calculate the continuous limit to the creation and annihilation operators for the solutions of the quantum harmonic oscillator.

In this contribution we apply the same method to other orthogonal polynomials of discrete and continuous variable. We give general formulas for all orthogonal polynomials of hypergeometric type [7]: difference/differential equations, recurrence relations, raising and lowering operators.

With the help of standard values we calculate these equations for the normalized functions of Kravchuk-Wigner and Meixner-Laguerre polynomials, and we construct the corresponding creation and annihilation operators.

The motivation for this work is the comparison of the study of the Sturm-Liouville problem in continuous case with the discrete case, in particular, the connection between the eigenfunctions and the creation and annihilation operators [1] [8]. This approach is becoming very important in the lattice formulation of field theories, where the physical properties of the model are analyzed in the lattice before the continuous limit is taken [6].

2. Basic relations between orthogonal polynomials of continuous and discrete variable. A polynomial of hypergeometric type of continuous variable satisfies the following fundamentals equations:

1. Differential equation:

$$\text{C1}: \sigma\left(s\right)\ y''_n\left(s\right)+\tau\left(s\right)\ y'_n\left(s\right)+\lambda_n\, y_n\left(s\right)=0, \tag{2.1}$$

where $\sigma\left(s\right)$ and $\tau\left(s\right)$ are polynomials of at most second and first degree, respectively and λ_n is a constant. This differential equation can be written in the form of an eigenvalue equation

$$\left(\sigma\left(s\right)\rho\left(s\right)y'_n\left(s\right)\right)'+\lambda_n\rho\left(s\right)y_n\left(s\right)=0,$$

where $\rho\left(s\right)$ is the weight function satisfying $\left(\sigma\left(s\right)\rho\left(s\right)\right)'=\tau\left(s\right)\rho\left(s\right)$ and $\lambda_n=-n\left(\tau'+\frac{n-1}{2}\sigma''\right)$.

2. Orthogonality relations:

$$\int_a^b y_n\left(s\right)y_m\left(s\right)\rho\left(s\right)ds=\delta_{nm}d_n^2$$

with d_n some normalization constant,

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3. Three term recurrence relations:

$$\text{C2}: sy_n(s) = \alpha_n y_{n+1}(s) + \beta_n y_n(s) + \gamma_n y_{n-1}(s) \tag{2.2}$$

where α_n, β_n, γ_n are constants.

4. Raising and lowering operators:

$$\text{C3}: \sigma(s)\, y_n^{'}(s) = \frac{\lambda_n}{n\tau_n^{'}}\left[\tau_n(s)\, y_n(s) - \frac{B_n}{B_{n+1}} y_{n+1}(s)\right], \tag{2.3}$$

where

$$\tau_n(s) = \tau(s) + n\sigma'(s)$$
$$\tau'_n = \tau' + n\sigma'' = -\frac{\lambda_{2n+1}}{2n+1}.$$

We can modify formula (C3) to a more suitable form. From

$$a_n = B_n \prod_{k\,=\,0}^{n-1}\left(\tau' + \frac{1}{2}(n+k-1)\,\sigma''\right) \quad , \quad a_0 = B_0,$$

we can prove the identity

$$\alpha_n = \frac{a_n}{a_{n+1}} = \frac{B_n}{B_{n+1}} \frac{\left(\tau' + \frac{n-1}{2}\sigma''\right)}{\left(\tau' + \frac{2n-1}{2}\sigma''\right)(\tau' + n\sigma'')} = -\frac{2n\,(2n+1)\,\lambda_n}{\lambda_{2n}\lambda_{2n+1}} \frac{B_n}{n\, B_{n+1}}$$

that when used in (C3) gives the simplified version

$$\text{C3}: \sigma(s)\, y'_n(s) = -\frac{\lambda_n}{n}\frac{2n+1}{\lambda_{2n+1}}\tau_n(s)\, y_n(s) - \frac{\lambda_{2n}}{2n}\alpha_n y_{n+1}(s). \tag{2.4}$$

Then using the recurrence relations (C2) gives

$$\text{C4}: \sigma(s)\, y'_n(s) = \left[-\frac{\lambda_n}{n}\frac{2n+1}{\lambda_{2n+1}}\tau_n(s) - \frac{\lambda_{2n}}{2n}(s-\beta_n)\right] y_n(s) + \frac{\lambda_{2n}}{2n}\gamma_n y_{n-1}(s). \tag{2.5}$$

Formulas (C3) and (C4) can be used to calculate solutions of the differential equations. In fact, if we put $n = 0$ in (C4) we get a differential equation whose solution is $y_0(s)$. Taking this value in (C3) we obtain by iteration all the polynomials satisfying (C1).

We can implement these formulas in the discrete case. A polynomial of hypergeometric type of discrete variable satisfies the following fundamental equations.

1. Difference equation:

$$\text{D1}: \sigma(x)\,\Delta\nabla y_n(x) + \tau(x)\,\Delta y_{\text{n}}(x) + \lambda_n y_n(x) = 0, \tag{2.6}$$

where $\sigma(x)$ and $\tau(x)$ are polynomial of at most second and first degree, respectively, and the forward (backward) difference operators are

$$\Delta f(x) = f(x+1) - f(s) \quad , \quad \nabla f(x) = f(x) - f(x-1).$$

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This difference equation can be written in the form of an eigenvalue equation

$$\Delta\left[\sigma(x)\rho(x)\nabla y_n(x)\right]+\lambda\rho(x)y_n(x)=0,$$

where $\rho(x)$ is the weight function satisfying

$$\Delta\left[\sigma(x)\rho(x)\right]=\tau(x)\rho(\mathrm{x}),$$

and

$$\lambda_n=-n\Delta\tau(x)-\frac{n(n-1)}{2}\Delta^2\sigma(\mathrm{x})=-n\left(\tau'+\frac{n-1}{2}\sigma''\right)$$

is the eigenvalue corresponding to the function $y_n(x)$.

2. Orthogonality relations:

The polynomial $P_n(x)$ of hypergeometric type satisfy the following orthogonal relations:

$$\sum_{2\varepsilon\,=\,a}^{b-1}P_n(x)P_m(x)\rho(x)=d_n^2\delta_{mn},$$

where δ_{mn} is the Kronecker symbol and d_n is some normalization constant.

3. Three term recurrence relation:

$$\text{D2}:xP_n(x)=\alpha_nP_{n+1}(x)+\beta_nP_n(x)+\gamma_nP_{n-1}(x)\tag{2.7}$$

with α_n, β_n, γ_n some constants.

4. Raising and lowering operators:

$$\text{D3}:\sigma(x)\nabla P_n(x)=\frac{\lambda_n}{n\tau'_n}\left[\tau_n(s)P_n(x)-\frac{B_n}{B_{n+1}}P_{n+1}(x)\right],\tag{2.8}$$

where

$$\begin{aligned}\tau_n(s)&=\tau(x+n)+\sigma(x+n)-\sigma(x)\\ \Delta\tau_n(s)&=\Delta\tau(x)+n\Delta^2\sigma(x)\end{aligned}$$

or

$$\tau'_n\;=\;\tau'+n\sigma''(x)=-\frac{\lambda_{2n+1}}{2n+1}$$

because $\sigma(x)$ and $\tau(x)$ are functions of at most second and first degree, respectively.

We can modify formula (D3) as we did in the continuous case with the help of the identities

$$a_n=B_n\prod_{k\,=\,0}^{n-1}\left(\tau'+\frac{1}{2}(n+k-1)\sigma''\right),a_0=B_0,\ \text{and}$$

$$\alpha_n=\frac{a_n}{a_{n+1}}=-\frac{2n}{\lambda_{2n}}\frac{(2n+1)}{\lambda_{2n+1}}\frac{n}{\lambda_n}\frac{B_n}{B_{n+1}}$$

Introducing these identities in (D3) gives a more simplified version

$$\text{D3}:\sigma(x)\nabla P_n(x)=-\frac{\lambda_n}{n}\frac{(2n+1)}{\lambda_{2n+1}}\tau_n(x)P_n(x)-\frac{\lambda_{2n}}{2n}\alpha_nP_{n+1}(x).\tag{2.9}$$

This expression defines the raising operator $P_{n+1}(x)$ in terms of the backward difference operator.

From this expression we can derive another lowering operator in terms of the forward difference operator. We substitute the operator ∇ in (D3) by its equivalent difference operator $\nabla = \Delta - \Delta\nabla$, and then we introduce the difference equation (D1) and the recurrence relation (D2) obtaining

$$\begin{aligned} &\mathrm{D4}: (\sigma(x) + \tau(x))\, \Delta P_n(x) = \\ &\quad \left[-\tfrac{\lambda_n}{n} \tfrac{2n+1}{\lambda_{2n+1}} \tau_n(x) - \lambda_n - \tfrac{\lambda_{2n}}{2n} x + \tfrac{\lambda_{2n}}{2n} \beta_n \right] P_n(x) + \tfrac{\lambda_{2n}}{2n} \gamma_n P_{n-1}. \end{aligned} \tag{2.10}$$

The advantage of expressions (D3) and (D4) is that all the coefficients are tabulated.

As in the continuous case, from (D4), putting $n = 0$ gives $P_0(x)$, and inserting this value in (D3) gives the solutions of (D1). Using the orthogonal polynomials of hypergeometric type we can construct the corresponding normalized functions (up to a phase factor)

$$\psi_n(x) = d_n^{-1} \sqrt{\rho(x)} P_n(x)$$

that satisfy equivalent relations and we denote them (ND1), (ND2), (ND3), (ND4) in the discrete case and (NC1), (NC2), (NC3), (NC4) in the continuous case. For instance, (NC1) becomes

$$\sigma(s)\psi_n''(s) + \sigma'(s)\psi_n'(s) - \rho(s)^{-\frac{1}{2}} \left(\sigma(s) \left(\rho(s)^{\frac{1}{2}} \right)' \right)' \psi_n(s) + \lambda_n \psi_n(s) = 0,$$

which corresponds to a self-adjoint operator of Sturm-Liouville type.

3. The Hermite and Kravchuk polynomials. We apply the results of section 2 to some orthogonal polynomials of continuous and discrete variables and to their corresponding normalized functions.

For the Hermite polynomials $H_n(s)$ we have

$$\mathrm{C1}: H''_n(s) - 2sH'_n(s) + 2nH(s) = 0, \tag{3.1}$$

$$\mathrm{C2}: sH_n(s) = \frac{1}{2} H_{n+1}(s) + nH_{n-1}(s), \tag{3.2}$$

$$\mathrm{C3}: H_{n+1}(s) = 2sH_n(s) - H'_n(s), \text{ and} \tag{3.3}$$

$$\mathrm{C4}: H_{n-1}(s) = \frac{1}{2n} H'_n(s). \tag{3.4}$$

Introducing the orthonormalized functions

$$\psi_n(s) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} e^{-s^2/2} H_n(s)$$

we get

$$\mathrm{NC1}: \psi''_n(s) + \left(2n + 1 - s^2\right) \psi_n(s) = 0, \tag{3.5}$$

$$\mathrm{NC2}: 2s\psi_n(s) = \sqrt{2(n+1)}\psi_{n+1}(s) + \sqrt{2n}\psi_{n-1}(s), \tag{3.6}$$

$$\mathrm{NC3}: \sqrt{n+1}\psi_{n+1}(s) = \frac{1}{\sqrt{2}} \left(s - \frac{d}{ds} \right) \psi_n(s), \text{ and} \tag{3.7}$$

$$\mathrm{NC4}: \sqrt{n}\psi_{n-1}(s) = \frac{1}{\sqrt{2}} \left(s + \frac{d}{ds} \right) \psi_n(s). \tag{3.8}$$

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(NC1) describes the quantum harmonic oscillator, and equations (NC3) and (NC4) result from the introduction of the familiar creation and annihilation operator

$$a^{+} \equiv \frac{1}{\sqrt{2}}\left(s - \frac{d}{ds}\right),\ a \equiv \frac{1}{\sqrt{2}}\left(s + \frac{d}{ds}\right).$$

From (NC4) with $n = 0$ we obtain $\psi_0\left(s\right)$ and inserting this value in (NC3) we obtain by iteration the solutions of the harmonic oscillator

$$\psi_n\left(s\right) = \frac{1}{\sqrt{n!}}\left(a^{+}\right)^{n}\psi_0\left(s\right).$$

In the discrete case we develop the Kravchuk polynomials $k_n^{(p)}\left(x, N\right)$ with $x = 0, 1, 2, \cdots N-1$ using

$$\begin{aligned}
&\sigma\left(x\right) = x;\\
&\tau\left(x\right) = \left(Np - x\right)/q;\ \tau_n\left(x\right) = \left(Np - x - n\right)/q \quad +n;\\
&\lambda_n = n/q;\\
&\alpha_n = n + 1/q;\ \beta_n = n + p\left(N - 2n\right);\ \gamma_n = pq\left(N - n + 1\right).
\end{aligned}$$

Inserting these values in the fundamental formulas we get

(3.9) $\mathrm{D1}: p\left(N-x\right)k_n\left(x+1\right) + \left[p\left(n+x-N\right) + q\left(n-x\right)\right]k_n\left(x\right) + qxk_n\left(x-1\right) = 0,$

(3.10) $\mathrm{D2}: xk_n\left(x\right) = \left(n+1\right)k_{n+1}\left(x\right) + \left[n + p\left(N-2n\right)\right]k_n\left(x\right) + pq\left(N-n+1\right)k_{n-1}\left(x\right) = 0,$

(3.11) $\mathrm{D3}: \left(n+1\right)k_{n+1}\left(x\right) = p\left(x+n-N\right)k_n\left(x\right) + qxk_n\left(x-1\right),$ and

(3.12) $\mathrm{D4}: q\left(N-n+1\right)k_{n-1}\left(x\right) = \left(x+n-N\right)k_n\left(x\right) + \left(N-x\right)k_n\left(x+1\right).$

For the normalized functions we use the Wigner functions, that appear in the representation of the rotation group, $d^{j}_{mm'}\left(\beta\right)$

$$\left(-1\right)^{m-m'}d^{j}_{mm'}\left(\beta\right) = d_n^{-1}\sqrt{\rho\left(x\right)}k_n^{(p)}\left(x, N\right)$$

with $N = 2j, m = j - n, m' = j - x, p = \sin^2\dfrac{\beta}{2},\ q = \cos^2\dfrac{\beta}{2}$

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After substitution we get

$$\text{(3.13)} \qquad \begin{aligned} \text{ND1}: &\sqrt{pq\,(N-x)\,(x+1)}d^j_{j-n,\ j-x-1}\,(\beta)+\\ &+\left(p\,(N-x-n)+q\,(x-n)\right)d^j_{j-n,\ j-x}\,(\beta)+\\ &+\sqrt{pqx\,(N-x+1)}d^j_{j-n,\ j-x+1}\,(\beta)=0, \end{aligned}$$

$$\text{(3.14)} \qquad \begin{aligned} \text{ND2}: &\left[-p(N-x-n)-q(n-x)\right]d^j_{j-n,\ j-x}+\\ &+\sqrt{pq\,(n+1)\,(N-n)}d^j_{j-n-1,\ j-x}\,(\beta)+\\ &+\sqrt{pqn\,(N-n+1)}d^j_{j-n+1,\ j-x}\,(\beta)=0, \end{aligned}$$

$$\text{(3.15)} \qquad \begin{aligned} \text{ND3}: &\sqrt{pq\,(n+1)\,(N-n)}d^j_{j-n-1,\ j-x}\,(\beta)=\\ &=p\,(N-x-n)\,d^j_{j-n,\ j-x}\,(\beta)+\\ &\sqrt{pqx\,(N-x+1)}d^j_{j-n,\ j-x+1},\ \text{and} \end{aligned}$$

$$\text{(3.16)} \qquad \begin{aligned} \text{ND4}: &\sqrt{pqn\,(N-n+1)}d^j_{j-n+1,\ j-x}\,(\beta)=\\ &=p\,(N-x-n)\,d^j_{j-n,\ j-x}\,(\beta)+\\ &\sqrt{pq\,(x+1)\,(N-x)}d^j_{j-n,\ j-x-1}\,(\beta)\,. \end{aligned}$$

The last four equations can be written down in terms of the new parameters $j=N/2\,,\ \ m=j-n\,,\ \ m'=j-x\,,\ \ p=\sin^2\beta/2\,,\ \ q=\cos^2\beta/2\,,\ \ \sqrt{pq}=\frac{1}{2}\sin\,\beta$:

$$\text{(3.17)} \qquad \begin{aligned} \text{ND1}: &\sqrt{(j+m')\,(j-m'+1)}d^j_{m,m'-1}\,(\beta)+\\ &+\frac{2}{\sin\,\beta}\left[m-m'\cos\,\beta\right]d^j_{m,m'}\,(\beta)+\\ &+\sqrt{(j-m')\,(j+m'+1)}d^j_{m,m'+1}\,(\beta)=0, \end{aligned}$$

$$\text{(3.18)} \qquad \begin{aligned} \text{ND2}: &\sqrt{(j+m)\,(j-m+1)}d^j_{m-1,m'}\,(\beta)-\\ &-\frac{2}{\sin\,\beta}\left[m'-m\,\cos\,\beta\right]d^j_{m,m'}\,(\beta)+\\ &+\sqrt{(j-m)\,(j+m+1)}d^j_{m+1,m}\,(\beta)=0, \end{aligned}$$

$$\text{(3.19)} \qquad \begin{aligned} \text{ND3}: &\frac{1}{2}\sin\beta\sqrt{(j+m)\,(j-m+1)}d^j_{m-1,m'}\,(\beta)=\\ &=\sin^2\frac{\beta}{2}\,(m+m')\,d^j_{m,m'}\,(\beta)+\\ &+\frac{1}{2}\sin\,\beta\sqrt{(j-m')\,(j+m'+1)}d^j_{m,m'+1}\,(\beta)\,,\ \text{and} \end{aligned}$$

$$\text{(3.20)} \qquad \begin{aligned} \text{ND4}: &\frac{1}{2}\sin\,\beta\sqrt{(j-m)\,(j+m+1)}d^j_{m+1,m'}\,(\beta)=\\ &=\sin^2\frac{\beta}{2}\,(m+m')\,d^j_{m,m'}\,(\beta)+\\ &+\frac{1}{2}\sin\,\beta\sqrt{(j+m')\,(j-m'+1)}d^j_{m,m'-1}\,(\beta)\,. \end{aligned}$$

Note that (ND1) and (ND2) are equivalent if we interchange $m\leftrightarrow m'$ and take in account the general property of Wigner functions

$$d^j_{m,m'}\,(\beta)=(-1)^{m-m'}d^j_{m',m}\,(\beta)\,.$$

The same property of duality applies to (ND3) and (ND4).

In [5] we constructed creation and annihilation operators with the help of (ND3) and (ND4)

$$A^{+} \equiv \frac{1}{\sqrt{2j}} \sqrt{(j+m)(j-m+1)} \quad , \quad A^{-} \equiv \frac{1}{\sqrt{2j}} \sqrt{(j-m)(j+m+1)}$$

that together with $A^0 \equiv \frac{1}{2j} m$, when apply to the spherical functions Y_{jm}, satisfy the $SO(3)$ algebra[1]

$$\left[A^{-}, A^{+}\right] = 2A^0 \quad , \quad \left[A^{\pm}, A^0\right] = \pm A^{\pm}.$$

Using the connection between the Wigner functions and the solutions of the quantum harmonic oscillator we proved [5] the limit relations

$$\text{(ND3)} \ \rightarrow \ \text{(NC3)} \quad , \quad \text{(ND4)} \ \rightarrow \ \text{(NC4)}.$$

Similar results have been obtained by Bijker, et al. [4] for the connection between the $su(2)$ algebra and the one dimensional anharmonic (Morse) oscillator, and by Atakishiev [1] for the lattice implementation of the quantum harmonic oscillator.

3.1. The Laguerre and Meixner polynomials. Using the general formulas of section 2, we get for the Laguerre polynomials $L_n^{\alpha}(s)$ of continuous variable

(3.21) $\quad \text{C1} : sL''_n(s) + (1+\alpha-s)\,L'_n(s) + nL_n(s) = 0,$

(3.22) $\quad \text{C2} : (n+1)\,L_{n+1}(s) + (n+s)\,L_{n-1}(s) = (2n+\alpha+1-s)\,L_n(s)\,,$

(3.23) $\quad \text{C3} : sL'_n(s) = -(1+\alpha-s)\,L_n(s) + (n+1)\,L_{n+1}(s)\,,$ and

(3.24) $\quad \text{C4} : sL'_n(s) = 2nL_n(s) - (n+\alpha)\,L_{n-1}(s)\,.$

For the normalized Laguerre functions

$$\psi_n(s) = \sqrt{\frac{n!}{\Gamma(\alpha+n+1)}} e^{-s/2} s^{\alpha/2} L_n^{\alpha}(s)$$

we obtain the following differential equation, recurrence relations and expressions for the raising and lowering operators:

(3.25) $$\text{NC1} : s^2\psi''_n(s) + s\psi'_n(s) + \frac{1}{2}\left[-s^2 - \alpha^2 + \alpha s + s\right]\psi_n(s) + sn\psi_n(s) = 0,$$

(3.26) $$\text{NC2} : \sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s) + \sqrt{n(n+\alpha)}\psi_{n-1}(s) = (2n+\alpha+1-s)\,\psi_n^{\alpha}(s)\,,$$

(3.27) $$\text{NC3} : \sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s) = \frac{1}{2}(2n+\alpha+2-s)\,\psi_n(s) + s\psi'_n(s)\,,$$ and

(3.28) $$\text{NC4} : \sqrt{n(n+\alpha)}\psi_{n-1}(s) = \frac{1}{2}(2n+\alpha-s) - s\psi'_n(s)\,.$$

The first equation, divided by s, corresponds to the self-adjoint operator of the Sturm-Liouville

[1]where we identify $A^{+} = \frac{1}{/sqrt2j} J_{-}$, $A^{-} = \frac{1}{/sqrt2j} J_{+}$, $A_0 = \frac{1}{j} J_z$.

problem for the normalized Laguerre functions.

The last two equations can be considered the creation and annihilation operators for the normalized Laguerre function. In fact, from (NC4) with n=0 we obtain ψ_0^α. And from (NC3) we easily obtain

$$\psi_n^\alpha(s) = \frac{1}{\sqrt{n!(\alpha+1)_n}}\left(A^+\right)^n \psi_0^\alpha,$$

where

$$A^+\psi_n(s) = \sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s)$$
$$A^-\psi_n(s) = \sqrt{n(n+\alpha)}\psi_{n-1}(s).$$

The Laguerre creation operator was given by Szafraniec [9]. His formula is equivalent to ours if we substitute (NC1) in (NC3).

Now we take the Meixner polynomials $m_n^\gamma(x)$ of discrete variable x, and apply the general expressions of section 2:

$$\text{D1}: \mu(x+\gamma)m_n(x+1) + (x-1)m_n(x-1) + [-\mu(x+n+\gamma)+n-x]m_n(x) = 0, \tag{3.29}$$

$$\text{D2}: \mu m_{n+1}(x) + n(n+\gamma-1)m_{n-1}(x) = [x(\mu-1)+n+\mu(n+\gamma)]m_n(x), \tag{3.30}$$

$$\text{D3}: \mu m_{n+1}(x) = \mu(x+n+\gamma)m_n(x) - xm_n(x-1), \text{ and} \tag{3.31}$$

$$\text{D4}: n(n+\gamma-1)m_{n-1}(x) = \mu(x+n+\gamma)m_n(x) - \mu(\gamma+x)m_n(x+1). \tag{3.32}$$

For the normalized Meixner polynomials

$$M_n^{(\gamma)}(x) = \sqrt{\frac{\mu^n(1-\mu)^\gamma}{n!(\gamma)_n}}\sqrt{\frac{\mu^x\Gamma(\gamma+x)}{\Gamma(x+1)\Gamma(\gamma)}}m_n^\gamma(x),$$

we have the following difference, recurrence equations and raising/lowering operators:

$$\text{ND1}: \sqrt{\mu(\gamma+x)(x+1)}M_n(x+1) + \sqrt{\mu x(x+\gamma-1)}M_n(x-1) - [\mu(x+n+\gamma)-n+x]M_n(x) = 0, \tag{3.33}$$

$$\text{ND2}: \sqrt{\mu(\gamma+n)(n+1)}M_{n+1}(x) + \sqrt{\mu n(n+\gamma-1)}M_{n-1}(x) - [\mu(x+n+\gamma)-x+n]M_n(x) = 0, \tag{3.34}$$

$$\text{ND3}: \sqrt{\mu(\gamma+n)(n+1)}M_{n+1}(x) = \mu(x+n+\gamma)M_n(x) - \sqrt{\mu x(x+\gamma-1)}M_n(x-1), \text{ and} \tag{3.35}$$

$$\text{ND4}: \sqrt{\mu n(n+\gamma-1)}M_{n-1}(x) = \mu(x+n+\gamma)M_n(x) - \sqrt{\mu(\gamma+x)(x+1)}M_n(x+1). \tag{3.36}$$

As in the case of Wigner functions there exists for the Meixner normalized functions a duality between (D1) $\leftrightarrow$ (D2) and (D3) $\leftrightarrow$ (D4) after interchanging $x \leftrightarrow n$.

The last two equations can be consider the creation and annihilation operators for the normalized Meixner functions:

$$\text{ND3}: A^+M_n(x) \equiv \sqrt{\mu(\gamma+n)(n+1)}M_{n+1}(x), \tag{3.37}$$

$$\text{ND4}: A^-M_n(x) \equiv \sqrt{\mu n(n+\gamma-1)}M_{n-1}(x). \tag{3.38}$$

From $A^{-}M_0\left(x\right)=0$ and (ND4) we obtain $M_0\left(x\right)$ and using this value in (ND3) we get

$$M_n\left(x\right)=\frac{1}{\sqrt{n!(\gamma)_n\mu^n}}\left(A^{+}\right)^n M_0\left(x\right).$$

The operators A^{+},A^{-} are adjoint conjugate of each other, and together with

$$A^0=\mu\left(2n+\gamma\right)$$

they close the $su(1,1)$ algebra

$$\left[A^{+},A^{-}\right]=A^0 \quad , \quad \left[A^{\pm},A^0\right]=\pm 2\mu A^{\pm}.$$

Finally, from the limit relation between the Meixner and Laguerre polynomials

$$\frac{1}{n!}{m_n}^{(\alpha+1,1-h)}\left(\frac{s}{h}\right)\xrightarrow[h\to 0]{} L_n^\alpha\left(s\right)$$

the limit relation between the corresponding normalized functions

$$M_n^{(\alpha+1,1-h)}\left(\frac{s}{h}\right)\xrightarrow[h\to 0]{}\ \psi_n^\alpha\left(s\right)$$

can be established along with the limit of the recurrence relations

$$\text{(ND2)}\ \xrightarrow[h\to 0]{}\ \text{(NC2)}.$$

The Meixner polynomials of discrete variable can be used in the solutions of the three dimensional harmonic oscillator in connection with the energy eigenvalues [3, formula (5.10)].

The Laguerre-Meixner creation and annihilation operators were presented by F.H. Szafraniec at Workshop on Orthogonal Polynomials in Mathematical Physics, June 24-26, 1996, Universidad Carlos III de Madrid, Leganes, Spain, (but not published).

Atakishiyev has given recently [2] the Hamiltonian and the creation and annihilation operators for the Meixner oscillator. In fact, his formula (43) and (60) are equivalent to our (ND1), (ND3), (ND4) for the normalized Meixner functions. They satisfy the same $su(1,1)$ algebra if we identify $A^{+}\leftrightarrow K_{+}$, $A^{-}\leftrightarrow K_{-}$, $A^0\leftrightarrow -K_0$.

The Laguerre polynomials of continuous variable are used in the models for the hydrogen atom, but with different weight function in the orthogonality relations [3, formula (5.10)]. The wave equation for the hydrogen atom can be considered as the differential equation of the Sturm-Liouville problem for the Laguerre functions

$$\psi_n(s)\equiv\psi_n^{2l+1}(s)=\sqrt{\rho_1(s)}L_{n-l-1}^{2l+1}(s),$$

where

$$\rho_1(s)=\sigma(s)\rho(s)=s^{2l+2}e^{-s}.$$

In fact, after substitution of $L_n(s)$ in (C1) for the Laguerre polynomials we get

$$\psi_n''(s)-[\rho_1(s)]^{-\frac{1}{2}}\left([\rho_1(s)]^{\frac{1}{2}}\right)''\psi_n(s)+(n-l-1)s^{-1}\psi_n(s)=0$$

which corresponds to a self-adjoint operator, from which orthogonality relations for $\psi_n(s)$ with weight function s^{-1} can be obtained.

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