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$Q-$CLASSICAL ORTHOGONAL POLYNOMIALS: A VERY CLASSICAL APPROACH*

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Abstract. The $q-$classical orthogonal polynomials defined by Hahn satisfy a Sturm-Liouville type equation in geometric differences. Working with this, we classify the $q-$classical polynomials in twelve families according to the zeros of the polynomial coefficients of the equation and the behavior concerning to q^{-1}. We determine a $q-$analogue of the weight function for the twelve families, and we give a representation of its orthogonality relation and its $q-$integral. We describe this representation in some normal and special cases (indeterminate moment problem and finite orthogonal sequences). Finally, the Sturm-Liouville type equation allows us to establish the correspondence between this classification and the Askey Scheme.

Key words. orthogonal $q-$polynomials, classical polynomials.

AMS subject classifications. 33D25.

1. Hahn's generalization of the classical orthogonal polynomials. The $q-$ classical orthogonal polynomials were introduced by Wolfgang Hahn in connection with the $q-$derivative [7]:

a) They are orthogonal in widespread sense, that is, in the three-term recurrence relation (TTRR) for the monic polynomials

$$xP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}\ ,\ n \geq 0 \quad , \quad P_{-1} = 0\ ,\ P_0 = 1\ , \tag{1.1}$$

it is required that $\beta_n \neq 0\,,\ n \geq 1$ or, equivalently, in terms of the corresponding functional, it must be regular, that is, the principal submatrices of the Hankel matrix for the moment sequence are nonsingular.

b) Since the classical polynomials are characterized as the only ones whose sequence of derivatives is also orthogonal, Hahn considers the $\mathrm{L}-$derivative and studies the orthogonal polynomials (OPS) whose sequence of $\mathrm{L}-$derivatives is also orthogonal.

The $\mathrm{L}-$derivative with parameters q and ω includes as particular cases the difference operator with step ω and the $q-$derivative (ϑ in the work by Hahn):

$$\begin{gathered} \mathrm{L}_{q,\omega} f(x) = \tfrac{f(qx+\omega)-f(x)}{(q-1)x+\omega}\ ,\ \mathrm{L}_{1,\omega} = \triangle_\omega\ , \\ \mathrm{L}_{q,0} = \Theta\ ,\ |q| \neq 1\ ,\ \Theta f(x) = \tfrac{f(qx)-f(x)}{(q-1)x}\ . \end{gathered} \tag{1.2}$$

We get the normal derivative when $q \to 1\,,\ \omega \to 0$. In this way, Hahn considers the $\mathrm{L}-$classical polynomials as a generalization of the classical polynomials ($\mathrm{D}-$classical polynomials) and discrete classical polynomials ($\triangle_\omega-$classical polynomials).

Traditionally two OPS are considered equal whenever we can pass from one to another by means of an affine transformation of the variable. The affine transformation of the variable, $\mathrm{A}_{a,b} f(x) = f(ax+b)$, modifies the parameters of the $\mathrm{L}-$derivative. Taking into account

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the effect of the dilation, $\mathrm{H}_a f(x) = f(ax)$, and the translation, $\mathrm{T}_b f(x) = f(x+b)$, we get:

$$
\begin{array}{ll}
|q| \neq 1 \ : \quad \mathrm{T}_b \mathrm{L}_{q,\omega} = \mathrm{L}_{q,\omega+(q-1)b} \mathrm{T}_b & \overset{b=\frac{\omega}{1-q}}{\Longrightarrow} \quad \mathrm{T}_b \mathrm{L} = \Theta \mathrm{T}_b \ , \\
|q| = 1 \ : \quad \mathrm{H}_a \mathrm{L}_{q,\omega} = a^{-1} \mathrm{L}_{q,a^{-1}\omega} \mathrm{H}_a & \overset{a=\omega^{-1}}{\Longrightarrow} \mathrm{H}_a \mathrm{L} = a^{-1} \triangle_q \mathrm{H}_a \ .
\end{array} \tag{1.3}
$$

In another way the $\mathrm{L}-$classical polynomials with respect to $\mathrm{L}_{q,w}$, $|q| \neq 1$ could be transformed by means of an appropriate translation in the $\Theta-$classical polynomials ($q-$classical polynomials). If $|q| = 1$ a dilation could transform them into the $\triangle-$classical polynomials (discrete classical polynomials); see [5], [11], [12] and references contained therein. The study of the classical and classical discrete polynomials was very complete, so actually it is only necessary to study the $q-$classical polynomials.

Starting from the Sturm-Liouville type equation in geometric differences with polynomial coefficients ϕ and ψ, $\deg \phi \leq 2$ and $\deg \psi = 1$, from now on denoted $q-\mathbb{SL}$, Hahn obtained the first results for the solutions as $q-$hypergeometric series. Unfortunately, there was no later publication, where the details were all filled in, according to Tom Koornwinder. Thirty six years later, G. Andrews and R. Askey [1] continued Hahn's work. Since then, a large literature on classical polynomials from the $q-$hypergeometric point of view has been generated. So, the $q-$classical polynomials are presented as a cascade of $q-$hypergeometric functions. Starting from two polynomials ${}_4\phi_3$, that are not classical in the sense proposed by Hahn, the rest are obtained by means of special choices and changes of parameters for variables, confluent limits, etc. [9, part 4]. A consequence of this procedure is that there does not exist a general theory for this scheme but a lot of particular cases. Moreover, in this hypergeometric approach is not evident how the manipulations have an influence on the characteristic elements of each family. A. Nikiforov and V. Uvarov represented another standpoint in the hypergeometric approach [11], [12]. They developed a theory based on the $q - \mathbb{SL}$ equation, but the Nikiforov-Uvarov approach leads in the end to the hypergeometric representation of the OPS. In [2], the authors try to unify both, the $q-$Askey's scheme and Nikiforov et al. one. In fact they give a more general framework for the $q-$Askey's scheme based on a $q - \mathbb{SL}$ equation.

Our approach and classification leads from ϕ and ψ to the $q-$weight functions and to the possible intervals of integration so as to represent the orthogonality relation. The zeros of ϕ and $\phi^\star$ $[\phi^\star(x) = q^{-1}\phi(x) + (q^{-1}-1)x\psi(x)]$ give the main information about the orthogonality. Our classification is designed to illustrate how alterations of ϕ and ψ (or ϕ and $\phi^\star$) have an effect on the orthogonality relation. The class of polynomials defined by Hahn are very varied but not a labyrinth. Our approach follows the standard analytic procedure in the $\mathrm{D}-$classical case. Starting from the Sturm-Liouville equation, $\phi \mathrm{D}^2 P_n + \psi \mathrm{D} P_n = \lambda_n P_n$, we write it in the self-adjoint form $\mathrm{D}(\phi w \mathrm{D} P_n) = \lambda_n w P_n$. This self-adjoint form, together with the integration by parts and the determination of two different points of the completed real line $a, b \in \overline{\mathbb{R}}$ such that $(\phi w)(a) = 0 = (\phi w)(b)$ make it easy to get the integral representation of the orthogonality

$$
(\lambda_n - \lambda_m) \int_a^b P_n P_m w = \int_a^b \mathrm{D}(\phi w \mathrm{D} P_n) \cdot P_m - \int_a^b \mathrm{D}(\phi w \mathrm{D} P_m) \cdot P_n = \tag{1.4}
$$

$$
= \phi w \mathrm{D} P_n \cdot P_m |_a^b - \int_a^b \phi w \mathrm{D} P_n \mathrm{D} P_m - \phi w \mathrm{D} P_m \cdot P_n |_a^b + \int_a^b \phi w \mathrm{D} P_n \mathrm{D} P_m = 0 \ ,
$$

$$
n \neq m \quad \Longrightarrow \quad \lambda_n \neq \lambda_m \quad \Longrightarrow \quad \int_a^b P_n P_m w = 0 \ .
$$

Finally, to prove $\int_a^b P_n^2\omega \neq 0$, $n \geq 0$, we only have to check that w is continuous in $[a,b]$ and nonzero in (a,b) . Thus we have to determine the weight function w , characterized as a solution of the Pearson equation $\mathrm{D}(\phi w) = \psi w$ $[\Longleftrightarrow \frac{\mathrm{D}w}{w} = \frac{\psi - \mathrm{D}\phi}{\phi}]$. It is evident that the degree of ϕ and the fact that it has a double zero or simple zeros when the degree is two determines the solutions. In conclusion, the classification of the $\mathrm{D}-$classical orthogonal polynomials is based on these aspects of the polynomial ϕ .

The development of a $q-$analogue of this procedure, where $q-$hypergeometric functions are not needed, was started with the contribution by M. Frank [4]. Later, S. Häcker, [6], applied it to the little $q-$Jacobi case, and in [10] all the cases for $0 < q < 1$ were considered. Our classical approach to the $q-$classical polynomials is presented as follows. In Section 2, a classification of the $q-$classical polynomials in 12 families with respect to the $q-$analogue of the weight function is developed. In Section 3, the determination of the $q-$weight functions by means of a $q-$analogue of the Pearson equation is given. In Section 4, the foundations of the orthogonality relationship represented with $q-$integrals and $q-$weights and an overview of the determination of the positive definite cases are considered. In Section 5, some cases which yield indeterminate moment problems and finite OPS are analyzed. In Section 6, the equivalences with the Askey Scheme are presented.

2. $q-$classical polynomials: classification. The $q-$classical polynomials are orthogonal with respect to linear functionals which satisfy a $q-$difference equation of first order with polynomial coefficients [10]

$$\Theta(\phi\mathbf{u}) = \psi\mathbf{u} \quad , \quad \deg\phi \leq 2 \ , \ \deg\psi = 1 \ . \tag{2.1}$$

The operations and action of the operators in the dual space of the polynomials is defined by transposition, except the derivative where there is also a change of sign, i.e., $\langle\Theta\mathbf{u}, x^n\rangle = -\langle\mathbf{u}, \Theta x^n\rangle$. Thus, (2.1), is equivalent to [10]

$$\phi\Theta\Theta^\star P_n + \psi\Theta^\star P_n = \lambda_n P_n \quad , \quad n \geq 1 \ , \tag{2.2}$$

where $\Theta^\star$ is the $q^{-1}-$derivative operator, (1.2), $\Theta^\star f(x) = \frac{f(q^{-1}x) - f(x)}{(q^{-1}-1)x}$. Another formulation equivalent to (2.2) is

$$\phi^\star\Theta^\star\Theta P_n + \psi\Theta P_n = \lambda_n^\star P_n \quad , \quad \phi^\star(x) = q^{-1}\phi(x) + (q^{-1}-1)x\psi(x) \ , \tag{2.3}$$

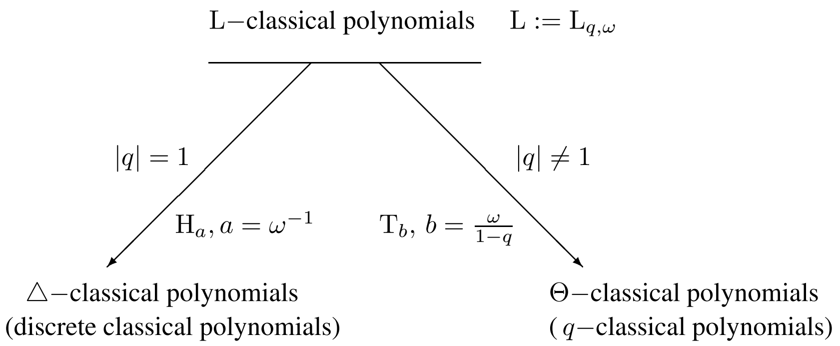
This is a well-known fact that has a special significance for us since

$$\Theta(\phi\mathbf{u}) = \psi\mathbf{u} \iff (2.2) \iff (2.3) \iff \Theta^\star(\phi^\star\mathbf{u}) = \psi\mathbf{u} \ , \tag{2.4}$$

that is, every $q-$classical functional/OPS is also $q^{-1}-$classical and vice versa.

Maybe this fact has gone unnoticed because when working in an analytical way if $0 < q < 1$ we have convergence in many expressions whereas with $q^{-1} > 1$ we have divergence. To see what comes next it is very important to keep (2.4) in mind. In fact we will see the $q-$classical OPS with a *stereoscopic* vision as q, $q^{-1}-$classical. We will refer to everything concerning the inverse basis as symmetric and we will mark it with * , for example: $\psi^\star = \psi$.

Let's recall the Hahn's scheme (1.3)



When $|q| \neq 1$, in order to normalize ϕ, we only need a dilation and the nonzero constant. The dilation H_a acting on the distributional equation of the functional $\mathbf{u}$, $\Theta(\phi\mathbf{u}) = \psi\mathbf{u}$, with the corresponding MOPS, (P_n), leads us to the normalized equation

$$\Theta(\widetilde{\phi}\widetilde{\mathbf{u}}) = \widetilde{\psi}\widetilde{\mathbf{u}} \quad , \quad \widetilde{\phi} = \mathrm{H}_a\phi \ , \ \widetilde{\psi} = a\mathrm{H}_a\psi \ , \ \widetilde{\mathbf{u}} = \mathrm{H}_{1/a}\mathbf{u} \ , \tag{2.5}$$

and the MOPS corresponding to $\widetilde{\mathbf{u}}$, $(\widetilde{P}_n)$, becomes $\widetilde{P}_n = a^{-n}\mathrm{H}_a P_n$. The factor c allows us to take ϕ monic. A straightforward consequence is that if the origin is a zero of ϕ, $\phi(0) = 0$, the origin will continue to be a zero in the normalized polynomial and those $c \neq 0$, $\phi(c) \neq 0$, will continue also to be a zero distinct of the origin after the dilation. Therefore, in the group of Laguerre and Jacobi polynomials, we will now distinguish among those that have a zero at the origin ($0-$zero) and those that do not vanish at the origin ($\emptyset-$zero). In general we will distinguish between:

$\emptyset-$zero families: $q-$Hermite, $\emptyset-$Laguerre, $\emptyset-$Jacobi,
and $0-$zero families: $0-$Laguerre, $0-$Jacobi, $q-$Bessel.

This is the vision from q. What happens for q^{-1}? If $\phi(x) = \widehat{a}x^2 + \bar{a}x + \dot{a}$ and $\psi(x) = \widehat{b}x + \bar{b}$, from (2.3), we get

$$\begin{aligned} \phi^\star(x) &= q^{-1}\phi(x) + (q^{-1}-1)x\psi(x) = \\ &= \underbrace{(q^{-1}\widehat{a} + (q^{-1}-1)\widehat{b})}_{\widehat{a}^\star} x^2 + \underbrace{(q^{-1}\bar{a} + (q^{-1}-1)\bar{b})}_{\bar{a}^\star} x + \underbrace{q^{-1}\dot{a}}_{\dot{a}^\star} \ . \end{aligned} \tag{2.6}$$

The immediate consequence is that every $q-\emptyset-$zero family is a $q^{-1}-\emptyset-$zero family and vice versa. The same is true for the $0-$zero families.

Notice that, from (2.6), if

$$\widehat{a}^\star = 0 \iff \widehat{b} = \tfrac{-\widehat{a}}{1-q} \quad \text{(main singularity)} \ , \tag{2.7}$$

the $\emptyset-$families are the $q^{-1}-$Laguerre ones, providing that $\deg\phi^\star = 1$, otherwise, if $\deg\phi^\star = 0$, that is,

$$\bar{a}^\star = 0 \iff \bar{b} = \tfrac{-\bar{a}}{1-q} \quad \text{(secondary singularity)} \ , \tag{2.8}$$

then they become in a q^{-1}Hermite family.

In the $0-$families, the framework is different. First, the two singularities cannot appear simultaneously. In fact, $\widehat{a}^\star = 0 = \bar{a}^\star$ implies $\phi^\star \equiv 0$, and so $\mathbf{u}$ is not regular. On the other hand, first, the $0-$Laguerre cannot have a main singularity, (2.7), since then

$$\widehat{a}^\star = q^{-1}\cdot 0 + (q^{-1}-1)\widehat{b} = 0 \implies \widehat{b} = 0 \implies \deg\psi < 1 \implies \mathbf{u}\,\text{is not regular} \ ,$$

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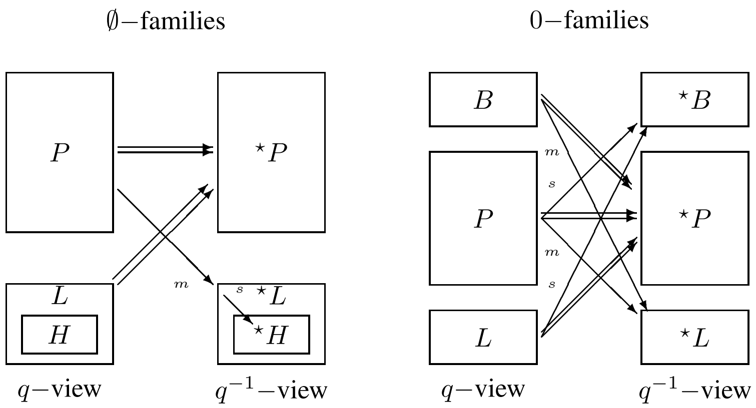
and, second, the $q-$Bessel cannot have a secondary singularity (2.8)

$$\bar{a}^{\star} = q^{-1} \cdot 0 + (q^{-1} - 1)\bar{b} = 0 \implies \bar{b} = 0 \implies \psi \,\text{divides}\, \phi \implies \mathbf{u}\,\text{is not regular} .$$

The following chart shows the situation (double arrow := no singularity, m := main singularity, s := secondary singularity)



Looking at the $q-$classical polynomials from q and q^{-1} we have 12 different families

$\emptyset-$Jacobi/*Jacobi	$q-$Bessel/*Jacobi
" /*Laguerre	" /*Laguerre
" /*Hermite	$0-$Jacobi/*Jacobi
$\emptyset-$Laguerre/*Jacobi	" /*Laguerre
$q-$Hermite/*Jacobi	" /*Bessel
	$0-$Laguerre/*Jacobi
	/*Bessel

3. $q-$classical polynomials: $q-$weight functions.

In this part, it will be justified that the zeros of ϕ and $\phi^{\star}$ determine the poles and zeros of the $q-$weight function. The weight function in the $\mathrm{D}-$cases satisfies the equation $\mathrm{D}(\phi\omega) = \psi\omega$. For our $q-$polynomials there is a $q-$analogue of the Pearson equation

$$\Theta^{\star}(\phi w) = q\psi w ,$$

which leads to the $q-$Sturm-Liouville equation in a self-adjoint form

$$\phi\Theta\Theta^{\star}P_n + \psi\Theta^{\star}P_n = \lambda_n P_n \iff \Theta\Big(\mathrm{H}^{-1}(\phi w)\Theta^{\star}P_n\Big) = \lambda_n w P_n .$$

We call w a $q-$weight function, and we get it as the solution of the $q-$Pearson equation. The equations in q and q^{-1} derivatives are reduced to an equation in q dilations $\mathrm{H} := \mathrm{H}_q$, $[\mathrm{H}f(x) = f(qx)]$

$$\Theta^{\star}(\phi w) = q\psi w \iff \phi w = q\mathrm{H}\phi^{\star}\mathrm{H}w \iff \phi(x)w(x) = \phi^{\star}(qx)w(qx) .$$

We solve these equations by a recurrent procedure

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$$\begin{tikzcd}
\phi w = q\mathrm{H}\phi^\star \mathrm{H} w \arrow[r] \arrow[d, "\mathrm{H}"'] & w = \underline{\mathrm{H}w}\,\frac{q\mathrm{H}\phi^\star}{\phi} \arrow[dr] & \\
\mathrm{H}\phi\underline{\mathrm{H}w} = \mathrm{H}(q\mathrm{H}\phi^\star)\mathrm{H}^2 w \arrow[ur] \arrow[d, "\mathrm{H}"'] & & w = \underline{\mathrm{H}^2 w}\,\frac{q\mathrm{H}\phi^\star}{\phi}\,\mathrm{H}\frac{q\mathrm{H}\phi^\star}{\phi} \arrow[d] \\
\mathrm{H}^2\phi\underline{\mathrm{H}^2 w} = \mathrm{H}^2(q\mathrm{H}\phi^\star)\mathrm{H}^3 w \arrow[urr] & \cdots\cdots & w = \mathrm{H}^n w \cdot \underbrace{\frac{q\mathrm{H}\phi^\star}{\phi}\cdot \mathrm{H}\frac{q\mathrm{H}\phi^\star}{\phi}\cdot \ldots \cdot \mathrm{H}^{n-1}\frac{q\mathrm{H}\phi^\star}{\phi}}_{\mathrm{H}^{(n)}\frac{q\mathrm{H}\phi^\star}{\phi}\ \left(=\prod_{k=0}^{n-1}\frac{q\phi^\star(q^{k+1}x)}{\phi(q^k x)}\right)}
\end{tikzcd}$$

Let us see what happens when n tends to infinity. If w is continuous at 0,

$$\lim_{n\to\infty} \mathrm{H}^n w = \lim_{n\to\infty} w(q^n x) = w(0)\ .$$

In order to deduce $\lim_{n\to\infty} \mathrm{H}^{(n)}\frac{q\mathrm{H}\phi^\star}{\phi}$ we need to consider infinite products: $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^n)$ and $(a,b;q)_\infty = (a;q)_\infty(b;q)_\infty$.

i) $\emptyset-$**cases**: Since the numerator polynomial and the denominator polynomial have the same nonzero independent term (see e.g. (2.6)), then the infinite product converges to

$$w(x) = w(0)\frac{(a_1^{\star-1}qx;q)_\infty(a_2^{\star-1}qx;q)_\infty}{(a_1^{-1}x;q)_\infty(a_2^{-1}x;q)_\infty}\ ,$$

where $a_1^\star$ and $a_2^\star$ are the zeros of $\phi^\star$ and a_1, a_2 those of ϕ. For any zero, for example a_1, it can be interpreted that

$$\deg\phi < 2 \implies a_1 = \infty \implies a_1^{-1} = 0 \implies (a_1^{-1}x;q)_\infty = 1\ .$$

The $q-$weights for the $\emptyset-$families are given in table 3.

These functions were already known by Hahn ([7], page 30), although he obtained them by another procedure. They are meromorphic functions in the complex plane with zeros in $a_i^\star q^{-n}$, $n \geq 1$ and poles in $a_i q^{-n}$, $n \geq 0$.

ii) $0-$**cases**: If the independent term is zero, several situations appear.

$\alpha)$ *No* $q^{\pm 1}-$*Bessel*. This is the simplest case also mentioned by Hahn. If both polynomials have nonzero $x-$term ($0-$Jacobi/*Jacobi, $0-$Jacobi/*Laguerre, $0-$Laguerre/*Jacobi) we eliminate a factor x of the numerator with another of the denominator, and we get a ratio of two polynomials with nonzero independent terms which do not coincide in general. To be able to introduce a factor that corrects this we assume the function w presents a zero or a pole in the origin introducing the factor $|x|^\alpha$. Then, the $q-$weights are

$$w(x) = |x|^\alpha\frac{(a_1^{\star-1}qx;q)_\infty}{(a_1^{-1}x;q)_\infty}\ ,$$

TABLE 3.1
The $q-$weights for the $\emptyset-$families

$\emptyset-$families	zeros of ϕ	zeros of $\phi^\star$	$q-$weight function
$\emptyset-$Jacobi/*Jacobi	$a_1 \neq \infty \neq a_2$	$a_1^\star \neq \infty \neq a_2^\star$	$w(x) = \dfrac{(a_1^{\star-1}qx, a_2^{\star-1}qx; q)_\infty}{(a_1^{-1}x, a_2^{-1}x; q)_\infty}$
/*Laguerre		$a_1^\star \neq \infty = a_2^\star$	$w(x) = \dfrac{(a_1^{\star-1}qx; q)_\infty}{(a_1^{-1}x, a_2^{-1}x; q)_\infty}$
/*Hermite		$a_1^\star = \infty = a_2^\star$	$w(x) = \dfrac{1}{(a_1^{-1}x, a_2^{-1}x; q)_\infty}$
$\emptyset-$Laguerre	$a_1 \neq \infty = a_2$	$a_1^\star \neq \infty \neq a_2^\star$	$w(x) = \dfrac{(a_1^{\star-1}qx, a_2^{\star-1}qx; q)_\infty}{(a_1^{-1}x; q)_\infty}$
$q-$Hermite	$a_1 = \infty = a_2$		$w(x) = (a_1^{\star-1}qx, a_2^{\star-1}qx; q)_\infty$

where once again $\deg\phi < 2$ implies $a_1 = \infty$.
$\beta)$ $q^{\pm1}-$*Bessel*. The $(\alpha)-$procedure can not be applied to the $q-$Bessel and $q^{-1}-$Bessel:

$(\beta 1)$ When the degree is different ($q-$Bessel/*Laguerre and $0-$Laguerre/*Bessel) we can use the function h: $h(x) = \sqrt{x^{\log_q x-1}}$. This function satisfies

$$\mathrm{H}h(x) = xh(x) \ .$$

In fact Häcker [6] uses it to solve the $q-$Bessel/*Laguerre case.
The following generalization of h, $h^{(\beta)}$ (we have not found any references to it in the literature) satisfies

$$\mathrm{H}h^{(\beta)}(x) = x^\beta h(x) \quad , \quad h^{(\beta)} = \sqrt{x^{\log_q x^\beta - \beta}} \ ,$$

and we have used $h^{(-1)}$ to solve the $0-$Laguerre/*Bessel case. In general the function h or its generalization can be used when the degrees of the polynomials are different. Hahn uses h in the $0-$Jacobi/*Laguerre case to prove that it corresponds to an indeterminate moment problem (generalizing the Stieltjes-Wigert polynomials). Notice that it was the only result developed with some detail in [7], but a mistake appears. It was corrected in a later article [8].

$(\beta 2)$ Finally, for the case when both polynomials have the same degree ($q-$Bessel/*Jacobi and $0-$Jacobi/*Bessel), the iterative solution using H leads to divergent expressions. So, we try to solve them using H^{-1}. Thus we get

$$w(x) = |x|^\alpha \frac{1}{(a_1^\star/x; q)_\infty} \quad \text{or} \quad w(x) = |x|^\alpha (a_1 q/x; q)_\infty \ .$$

TABLE 3.2
The $q-$weights for the $0-$families

$0-$families	zeros of ϕ	zeros of $\phi^\star$	$q-$weight function
$q-$Bessel/*Jacobi	$a_1 = 0\,,\, a_2 = 0$	$a_1^\star \neq \{^0_\infty \quad \alpha_2^\star = 0$	$w(x) = \|x\|^\alpha \dfrac{1}{(a_1^\star/x;q)_\infty}$ (b)
/*Laguerre		$a_1^\star = \infty\,,\, a_2^\star = 0$	$w(x) = \|x\|^\alpha \sqrt{x^{\log_q x - 1}}$ (c)
$0-$Jacobi/*Jacobi	$a_1 \neq \{^0_\infty\,,\, a_2 = 0$	$a_1^\star \neq \{^0_\infty\,,\, a_2^\star = 0$	$w(x) = \|x\|^\alpha \dfrac{(a_1^{\star -1} qx;q)_\infty}{(a_1^{-1}x;q)_\infty}$ (a)
/*Laguerre		$a_1^\star = \infty, a_2^\star = 0$	$w(x) = \|x\|^\alpha \dfrac{1}{(a_1^{-1}x;q)_\infty}$, (a)
/*Bessel		$a_1^\star = 0\,,\, a_2^\star = 0$	$w(x) = \|x\|^\alpha (a_1 q/x;q)_\infty$ (b)
$0-$Laguerre/*Jacobi	$a_1 = \infty\,,\, a_2 = 0$	$a_1^\star \neq 0\,,\, a_2^\star = 0$	$w(x) = \|x\|^\alpha (a_1^{\star -1} qx;q)_\infty$ (a)
/*Bessel		$a_1^\star = 0\,,\, a_2^\star = 0$	$w(x) = \|x\|^\alpha \sqrt{x^{\log_q \frac{1}{x} + 1}}$ (d)

We have not found any reference concerning these functions in the literature. In the first case, fixing $\widehat{a} = 1$ applying the standard normalization (non zero factor and dilation) over the distributional equation, the only free parameter is $\bar{b}$. Choosing it so that $\bar{b} = 2q^{2-\alpha}$ then [10]

$$\omega(x) = |x|^\alpha \tfrac{1}{(a_1^\star/x;q)_\infty} = |x|^\alpha e_q\left(\tfrac{a_1^\star}{x}\right) = |x|^\alpha e_q[-(1-q)2/x]\ ,$$

and $\lim_{q\to 1^-} \omega(x) = |x|^\alpha \exp(-2/x)$ is the Bessel weight function.

The $q-$weight functions for the $0-$families are shown in table 3

$$(a) \quad \alpha = -2 + \mathrm{Log}_q \frac{\bar{a}}{\bar{a}^\star}\ ,\ (b)\ \alpha = -3 + \mathrm{Log}_q \frac{\widehat{a}}{\widehat{a}^\star}\ ,\ (c)\ \alpha = -2 + \mathrm{Log}_q \frac{\widehat{a}}{\bar{a}^\star}\ ,\ (d)\ \alpha = 3 + \mathrm{Log}_q \frac{\widehat{a}^\star}{\bar{a}}$$

$$\phi = \widehat{a}x^2 + \bar{a}x + \dot{a} \quad , \quad \psi = \widehat{b}x + \bar{b} \quad , \quad \widehat{a}^\star = q^{-1}\widehat{a}$$

4. $q-$integral representation of the positive definite cases. The $q-$integral is a Riemann sum on an infinite partition $\{aq^n,\, n \geq 0\}$,

$$\textstyle\int_0^{a>0} f\, d_q := \sum_{n=0}^\infty f(aq^n)(aq^n - aq^{n+1}) = (1-q)a \sum_{n=0}^\infty f(aq^n) q^n\ ,$$

$$\textstyle\int_{a<0}^{0} f\, d_q := \sum_{n=0}^\infty f(aq^n)(aq^{n+1} - aq^n) = -(1-q)a \sum_{n=0}^\infty f(aq^n) q^n\ ,$$

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defined in such a way that we can apply the $q-$analogue of the Barrow rule,

$$\int_a^b \Theta F\, d_q = F(b) - F(a)\ .$$

This allows us to get the following integration by parts rules

$$\int_a^b f\Theta g\, d_q = \mathrm{H}^{-1}f\cdot g|_a^b - q\int_a^b g\Theta^\star f\, d_q \quad , \quad \int_a^b f\Theta g\, d_q = fg|_a^b - \int_a^b \mathrm{H}g\Theta f\, d_q\ . \tag{4.1}$$

On the other hand it can be generalized to unbounded intervals and to unbounded functions.

The Riemann-Stieltjes discrete integrals related with the $q-$classical polynomials can be represented as $q-$integrals. For example, for the $0-$Jacobi case (little $q-$Jacobi) we have

$$\sum_{k=0}^{\infty} \frac{(bq;q)_k}{(q;q)_k}(aq)^k p_m(q^k)p_n(q^k) = K\int_0^1 x^\alpha \frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty} p_m(x)p_n(x)\, d_qx\ ,\ a=q^\alpha\ ,\ b=q^\beta\ ,$$

$$w(x) = x^\alpha \frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty} (= x^\alpha[1-qx]_\beta\ ,\ \text{in the Hahn notation})\ .$$

Notice that the previous polynomials correspond to a positive definite case for $-1<\alpha$. For $-1<\alpha<0$ the $q-$integral converges.

The positive definite cases are deduced from the TTRR, (1.1), when $\beta_n>0,\ n\geq 1$. If $\phi(x) = \widehat{a}x^2 + \bar{a}x + \dot{a}$, $\psi(x) = \widehat{b}x + \bar{b}$, and $\mathrm{H}^n\phi(x) = \phi(q^n x)$ then

$$\beta_{n+1} = -\frac{q^n[n+1]\Big([n-1]\widehat{a}+\widehat{b}\Big)}{\Big([2n-1]\widehat{a}+\widehat{b}\Big)\Big([2n+1]\widehat{a}+\widehat{b}\Big)} \cdot \mathrm{H}^n\phi\left(-\frac{[n]\bar{a}+\bar{b}}{[2n]\widehat{a}+\widehat{b}}\right)\ ,\ n\geq 0\ . \tag{4.2}$$

This representation of β_n in terms of the coefficients of ϕ and ψ was obtained by N. Smaili [13] and S. Häcker [6] using different techniques. The determination of the positive definite cases has been done case by case for any real value of q, $|q|\neq 1$, by Häcker. A more global vision of the used procedures and, mainly, the positive definite cases which have not been considered by Häcker can be found in [10]. In all positive definite cases it is possible to represent the orthogonality relation using the $q-$integral and the $q-$weight function.

Thus, we have a self-adjoint form of the $q-$Sturm-Liouville equation, $q-$integration by parts. We only need two points $a,b\in\overline{\mathbb{R}}$, $a\neq b$, zeros of certain functions, so that $\int_a^b P_nP_mwd_q = 0$, $n\neq m$, see (1.4). If $n\neq m$, then $\lambda_n\neq\lambda_m$ and

$$(\lambda_n-\lambda_m)\int_a^b wP_nP_m\, d_q = \int_a^b (w\lambda_nP_n)P_m\, d_q - \int_a^b (w\lambda_mP_m)P_n\, d_q =$$

$$= \int_a^b \Theta\Big(\underbrace{\mathrm{H}^{-1}(\phi w)\Theta^\star P_n}_{g_1}\Big)\underbrace{P_m}_{f_1}\, d_q - \int_a^b \Theta\Big(\underbrace{\mathrm{H}^{-1}(\phi w)\Theta^\star P_n}_{g_2}\Big)\underbrace{P_n}_{f_2}\, d_q =$$

$$= \mathrm{H}^{-1}(\phi w)\Theta^\star P_n\cdot P_m|_a^b - \int_a^b \mathrm{H}\Big(\mathrm{H}^{-1}(\phi w)\Theta^\star P_n\Big)\Theta P_m\, d_q -$$

$$\underbrace{-\mathrm{H}^{-1}(\phi w)\Theta^\star P_m\cdot P_n|_a^b}_{\left(\mathrm{H}^{-1}(\phi w)\right)(a)=0=\left(\mathrm{H}^{-1}(\phi w)\right)(b)} + \int_a^b \underbrace{\mathrm{H}\Big(\mathrm{H}^{-1}(\phi w)\Theta^\star P_m\Big)\Theta P_n}_{\phi w\Theta P_m\Theta P_n}\, d_q = 0\ .$$

So a and b must cancel $\mathrm{H}^{-1}(\phi w)\Big[= \phi(q^{-1}x)w(q^{-1}x)\Big]$:

$$\Big(\mathrm{H}^{-1}(\phi w)\Big)(a) = 0 = \Big(\mathrm{H}^{-1}(\phi w)\Big)(b) \quad \Longleftrightarrow \quad \phi(aq^{-1})w(aq^{-1}) = 0 = \phi(bq^{-1})w(bq^{-1}) \ .$$

For instance, if a_1 and a_2 are zeros of ϕ we could take $a_1 q$ and $a_2 q$, $a_1, a_2 \in \overline{\mathbb{R}}$. There is a more interesting alternative, as we will see. Using the $q-$Pearson equation, we have

$$\phi w = q\mathrm{H}(\phi w) \quad \Longleftrightarrow \quad \mathrm{H}^{-1}(\phi w) = q^{-1}\phi^\star w \ ,$$

and so the zeros of $\phi^\star$, $a_1^\star$, $a_2^\star \in \overline{\mathbb{R}}$, constitute another choice. Also we can combine both possibilities, for example: $a_1 q$ and $a_2^\star$.

We will make some comments about the determination of the positive definite cases in order to facilitate the comprehension of what follows. First of all, we normalize the polynomial ϕ with $\widehat{a} = 1$ which does not alter either the functional or the orthogonal polynomial sequence. The study of the positive definite cases reduces to the study of the sign of the two factors of β_{n+1}, (4.2). The first factor is negative if the leading coefficient of ψ is positive, $\widehat{b} > 0$, and negative if $\widehat{b} < \frac{-1}{1-q}$ $\Big[[n] \overset{n\to\infty}{\longrightarrow} \frac{-1}{1-q},\, |q| < 1\Big]$. Notice that $\widehat{b} = \frac{-1}{1-q}$ represents the main singularity, (2.7). The other cases, $\frac{-1}{1-q} < \widehat{b} < 0$, have changes of sign and do not lead to positive definite cases. The second factor of (4.2) has more difficulties in the case $\deg\phi = 2$. If $\widehat{b} > 0$, then this second factor must be negative and $\frac{[n]\bar{a}+\bar{b}}{[2n]+\widehat{b}}$ must remain in the interval between the zeros of $\mathrm{H}^n\phi$ ($\mathrm{H}^n\phi$ with positive leading coefficient, $\widehat{a} = 1$). Equivalently, in this case, $\widehat{b} > 0$, the sequence $(\epsilon_n)_{n\geq 0}$,

$$\epsilon_n := -\frac{[n]\bar{a} + \bar{b}}{[2n] + \widehat{b}} \cdot q^n \ , \ n \geq 0 \ , \tag{4.3}$$

must remain between the zeros of ϕ, a_1 and a_2. Otherwise, if $\widehat{b} < \frac{-1}{1-q}$ then $\epsilon_n \notin [a_1, a_2]$, $n \geq 0$. In the case $\deg\phi = 1$, i.e., $\widehat{a} = 0$, for example, ϕ with positive leading coefficient, $\bar{a} > 0$, and a zero at a_0, we have positive definite cases iff $\epsilon_n < a_0$, $n \geq 0$, and so on.

The choice of the interval of integration is made to guarantee that $\int_a^b P_n^2 w d_q \neq 0$, $n \geq 0$, for which, it is enough that w be continuous and does not vanish inside the interval of integration. This has a difficulty since we have seen that even in the simplest cases, $\emptyset-$families, are infinite number of zeros, $a_i^\star q^{-n}$, $n \geq 1$, and infinite number of poles, $a_i q^{-n}$, $n \geq 0$. In the positive definite cases there is a situation that makes the problem simpler: $a_1^\star$ and $a_2^\star$ are real and

$$a_1^\star < 0 < a_2^\star \ , \tag{4.4}$$

or in the $0-$families, $a_1^\star = 0 < a_2^\star$, or, $a_1^\star < 0 = a_2^\star$. So in all cases we have the zeros out of $(a_1^\star, a_2^\star)$. For $a_1^\star < 0 < a_2^\star$ we have

$$\ldots < a_1^\star q^{-n} < \ldots < a_1^\star q^{-1} < a_1^\star < 0 < a_2^\star < a_2^\star q^{-1} < \ldots < a_2^\star q^{-n} < \ldots \ .$$

[Notice that, (2.6), $\dot{a} = \dot{a}^\star q$ and $\dot{a} = a_1 a_2$, $\dot{a}^\star = \widehat{a}^\star a_1^\star a_2^\star$, with, (2.6), $\widehat{a}^\star = q^{-1}\widehat{a} + (q^{-1} - 1)\widehat{b}$, $\widehat{a} = 1$ yields (4.4) with $\dot{a} \neq 0$.]

Now we come to the poles. The better case occurs when a_1 and a_2 are out of the interval $[a_1^\star, a_2^\star]$ but this does not usually happens. So we have the previous problem. We know the relative situation of the zeros of ϕ and ψ in the positive definite cases (we have the explicit expression β_n in terms of the coefficients of ϕ and ψ). The main question is to know the relative position of the zeros of ϕ and $\phi^\star$ in these cases.

5. An example: the $\emptyset$ and $0-$Jacobi cases. The $\emptyset-$Jacobi/*Jacobi, have three basic forms in order to be positive definite. With monic ϕ, $\phi(x) = (x-a_1)(x-a_2)$ and $\psi(x) = \widehat{b}(x-b_0)$, they are positive definite if

$$
\begin{array}{ll}
\text{a)}\ \widehat{b} > 0 & ,\quad a_1 < 0 < a_2 \quad , \quad a_1 < b_0 < a_2 \\
\text{b)}\ \widehat{b} < \dfrac{-1}{1-q} & \left\{ \begin{array}{lll} 0 < a_1 < a_2 & , & b_0 < a_1 \\ a_1 < a_2 < 0 & , & a_2 < b_0 \end{array} \right. \\
\text{c)}\ \widehat{b} < \dfrac{-1}{1-q} & \left\{ \begin{array}{lll} 0 < a_1 < a_2 & , & a_2 < b_0 \\ a_1 < a_2 < 0 & , & b_0 < a_1 \end{array} \right.
\end{array}
$$

In the (a)-cases, the sequence $(\epsilon_n)_{n\geq 0}$ with $\widehat{a} = 1$, (4.3), $\epsilon_0 = b_0$, which converges to 0, belongs to the interval (a_1, a_2), and this is guaranteed if $a_1 < 0 < a_2$, and $a_1 < b_0 < a_2$. In the (b)-cases, (ϵ_n) must be out of $[a_1, a_2]$ and so a_1 and a_2 are both positive or negative, and it is sufficient that b_0 is out of $[a_1, a_2]$ to yield this. In the (c)-cases, the condition is not sufficient. Another necessary condition is that a_1 and a_2 were close enough so that the sequence (ϵ_n), which converges to zero, jumps over the interval $[a_1, a_2]$. If $\epsilon_{n_0} \in (a_1, a_2)$ then we have $\beta_{n_0} < 0$ [$\epsilon_{n_0} = a_1$ or $\epsilon_{n_0} = a_2$ yields $\beta_{n_0} = 0$.] If $a_1 = a_2$, a discrete number of values of b_0 do not lead to positive definite cases, while $a_1, a_2 \in \mathbb{C} \setminus \mathbb{R}$ leads always to positive definite cases for all values of b_0 $(\widehat{b} < \frac{-1}{1-q})$.

We point out that a normalization procedure with a dilation applied to the corresponding OPS, (2.5), can put it into a quasidefinite class, that is, the normalized representant of the class could not be positive definite. This is the case of the Big $q-$Jacobi polynomials that represents the class with

$$\phi(x) = aq(x-1)(bx+c) = abq(x-1)(x+c/b).$$

The dilation that allows us to send a zero of ϕ to 1, $a_1 = 1$, is H_{a_1}:

$$\phi(x) = (x-a_1)(x-a_2) \overset{\mathrm{H}_{a_1}}{\longrightarrow} (a_1x-a_1)(a_1x-a_2) = a_1^2(x-1)(x-a_2/a_1)\,. \tag{5.1}$$

If (P_n) satisfies a TTRR, (1.1), with $\alpha_n \in \mathbb{R}$, $n \geq 0$, and $\beta_n > 0$, $n \geq 1$, a positive definite MOPS before the dilation, then $(\widetilde{P}_n)$, after the dilation (2.5), satisfies a TTRR with $\widetilde{\alpha}_n = a_1^{-1}\alpha_n$, $\widetilde{\beta}_n = a_1^{-2}\beta_n$. If $a_1 \in \mathbb{C}\setminus\mathbb{R}$ then $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$ are also complex$-$valued.

The relative position for the zeros of ϕ and $\phi^\star$ is different in each case:

$$
\begin{array}{l}
(a) \Longleftrightarrow [a_1, a_2] \supset [a_1^\star, a_2^\star]\,, \\
(b) \Longleftrightarrow [a_1, a_2] \cap [a_1^\star, a_2^\star] = \emptyset\,, \\
(c) \Longleftrightarrow [a_1, a_2] \subset [a_1^\star, a_2^\star]\,.
\end{array} \tag{5.2}
$$

We can consider three different subtypes of positive definite $\emptyset-$Jacobi/*Jacobi. In Figure (5) and Figure (5) the intervals of integration for all $\emptyset-$Jacobi/*Jacobi cases are represented. In the above discussed case, a_1 and a_2 are complex-valued, a meaningful normalization procedure is to send $a_1^\star$ to -1 or $a_2^\star$ to 1 with a dilation of ratio $-a_1^\star$ or $a_2^\star$, (4.4), acting on $\Theta^\star(\phi^\star\mathbf{u}) = \psi^\star\mathbf{u}$, (5.1). Further, OPS which are initially positive definite, $\beta_n > 0$, $n < n_0$, but $\beta_{n_0} \leq 0$ can appear in the (c)-cases. Then we have the so called finite OPS. In this case, $\emptyset-$Jacobi/*Jacobi, are the $q-$Hahn polynomials.

Notice that two possible intervals of integration appear in the (c)-cases ($a_1 \neq a_2$). For a positive definite case, a_1 and a_2 must be close enough $0 < a_1 < a_2 < a_1q^{-1}$ or $a_2q^{-1} < a_1 < a_2 < 0$ so that the poles are out of the interval of integration. So, we arrive

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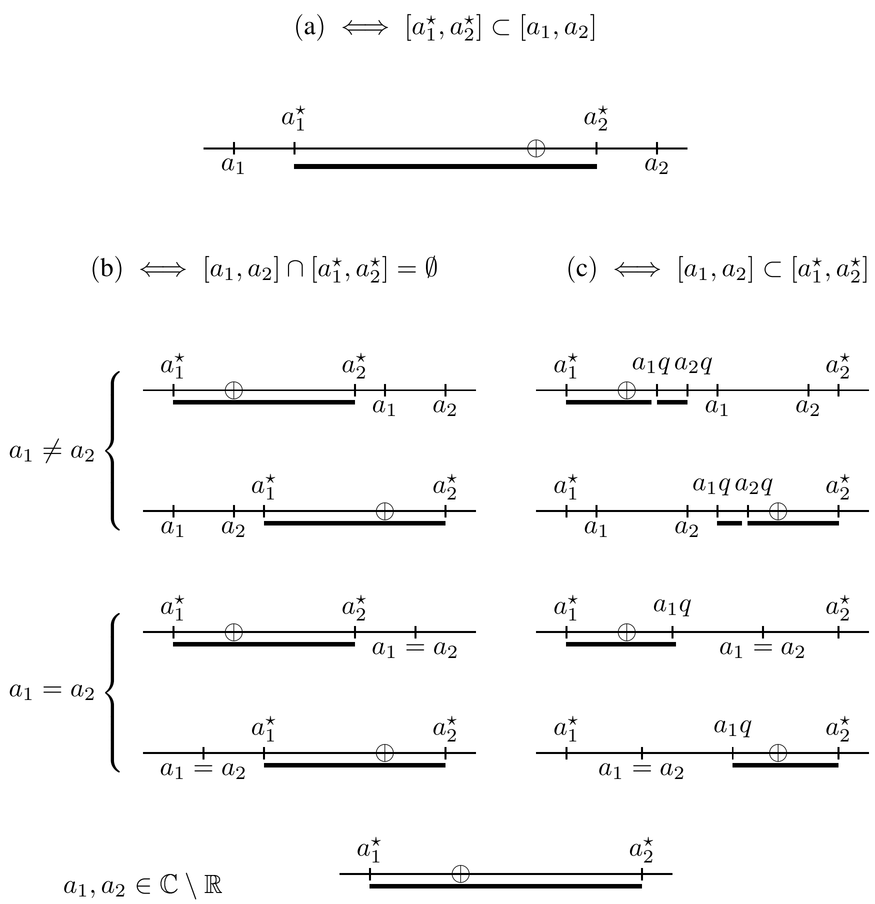
FIG. 5.1. *Intervals of integration for the $q-$weight function corresponding to the $\emptyset-$Jacobi/* Jacobi OPS*



to the same conclusion: a_1 and a_2 must be close enough. On the other hand, two different finite intervals do not lead necessarily to different orthogonality representations. The behavior of the $0-$Jacobi/*Jacobi cases $(a_1^\star = 0 < a_2^\star$ or $a_1^\star < 0 = a_2^\star)$ is the same in the (a) and (b)-cases. Notice that it is not possible that they were positive definite in the (c)-cases because $(\epsilon_n) \to 0$.

We now determine that the intervals of integration for the singular $\emptyset-$Jacobi ($\emptyset-$Jacobi /*Laguerre, (2.7), and $\emptyset-$Jacobi/*Hermite, (2.7) and (2.8)). Here, the discussion of the positive definite cases is different: (ϵ_n) must be also out of (a_1, a_2) but now (ϵ_n) diverges to $+\infty$ if $b_0 > a_1 + a_2$, or to $-\infty$ if $b_0 < a_1 + a_2$, or is constant if $b_0 = a_1 + a_2$, the case of a secondary singularity, (2.8). They are positive definite if

$$\text{a)} a_1 < 0 < a_2 \left\{ \begin{array}{lll} b_0 < a_1 & \underset{a_1 < a_1 + a_2 < a_2}{\overset{a_1 < a_1 + a_2 < a_2}{\Longrightarrow}} & (\epsilon_n) \searrow_{-\infty} \\ a_2 < b_0 & \Longrightarrow & (\epsilon_n) \nearrow^{\infty} \end{array} \right.$$

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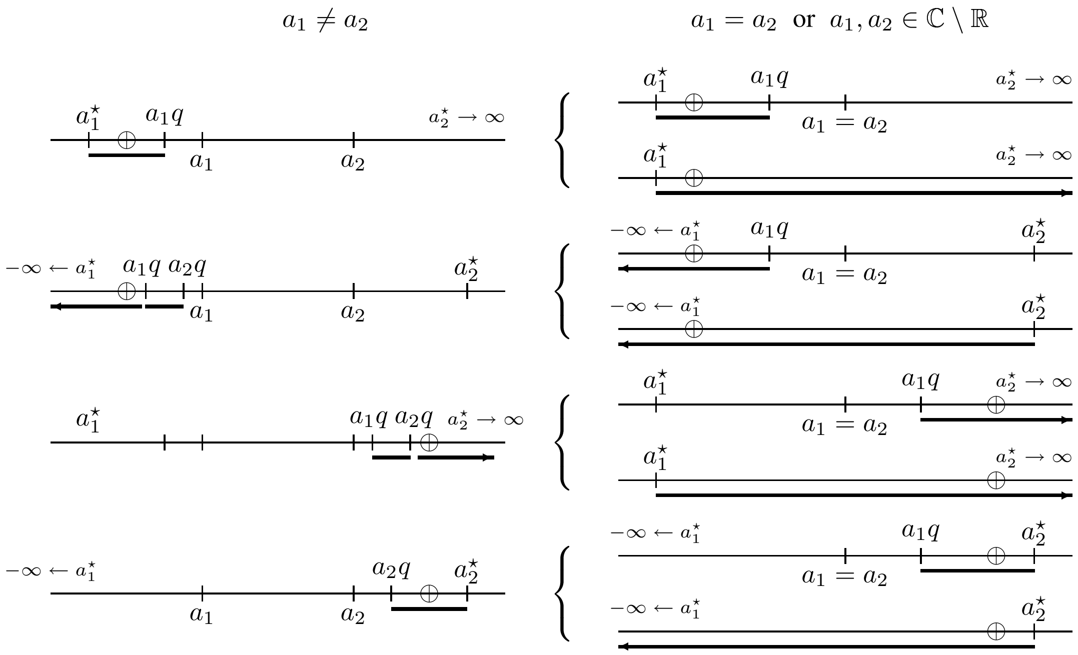
$$\text{b)}\left\{\begin{array}{lcll} 0<a_1<a_2 & , & b_0<a_1 \implies & (\epsilon_n)\searrow_{-\infty} \\ a_2<a_1<0 & , & a_1<b_0 \implies & (\epsilon_n)\nearrow^{\infty} \end{array}\right.$$

$$\text{c1)}\left\{\begin{array}{lcll} 0<a_1<a_2 & , & a_2<b_0<a_1+a_2 \implies & (\epsilon_n)\searrow_{-\infty} \\ a_1<a_2<0 & , & a_1+a_2<b_0<a_1 \implies & (\epsilon_n)\nearrow^{\infty} \end{array}\right.$$

$$\text{c2)}\left\{\begin{array}{lcll} 0<a_1<a_2 & , & a_1+a_2\le b_0 \implies & (\epsilon_n)\nearrow^{\infty} \text{ or constant} \\ a_1<a_2<0 & , & b_0\le a_1+a_2 \implies & (\epsilon_n)\searrow_{-\infty} \text{ or constant} \end{array}\right.$$

In the (a), (b), and (c2)-cases the condition is also sufficient. In the (c1)-cases, (ϵ_n) must also jump over the interval $[a_1, a_2]$. Finally, we have the same cases (4.2), with $a_1^\star$ or $a_2^\star$ equal to ∞. Now, when two intervals of integration appear in the (c)-cases, one is finite and the other is infinite. The finite OPS are also (c)-cases: the quantum $q-$Krawtchouk in the Askey's Scheme.

FIG. 5.2. *Intervals of integration in the (c)-cases for the $q-$weight function corresponding to the $\emptyset-$Jacobi/Laguerre and $\emptyset-$Jacobi/*Hermite ($a_1^\star \to -\infty$ and $a_2^\star \to \infty$) OPS.*



Now, the following question arises: What is the relationship with an indeterminate moment problem? In the case of the $q^{-1}-$Hermite ($\emptyset-$Jacobi /* Hermite), T. Chihara, [3, pp.197, 198], refers to the existence of one indeterminated moment problem. The indetermination of the moment problem is not only due to the unbounded integration interval, as Hahn pointed out. In this case the indetermination of the moment problem appears only when the two zeros of ϕ, $a_1 = 1$, $a_2 = a$, satisfies

$$1 < a < q^{-1} \quad \text{i.e.} \quad q < aq < 1$$

that allows us to consider two different intervals of integration: bounded, $[q, aq]$ and unbounded, $(-\infty, q]$.

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The $0-$Jacobi/*Laguerre follows the same scheme. They can be positive definite in the (c2) form, and we find the corresponding finite family: the $q-$Krawtchouk OPS.

In [10] different intervals of integration are described in all the cases. Even more, the integral$-$valued representation of $q-$Bessel/*Jacobi are obtained in an unusual case: w is complex valued for the positive definite cases and in any possible integration interval there are infinite poles of ω.

6. Our classification and the Askey Scheme. To conclude this work we point out the comparison of our classification scheme with the Askey's one just as R. Koekoek and R. Swarttouw present it. We emphasize the fact that a work with the hypergeometric feeling shows in a systematic way the $q-\mathbb{SL}$ equation. We have established the equivalences through it. By the way our $q-$Bessel/*Jacobi are alternative $q-$Charlier. There are very few references about them in the literature.

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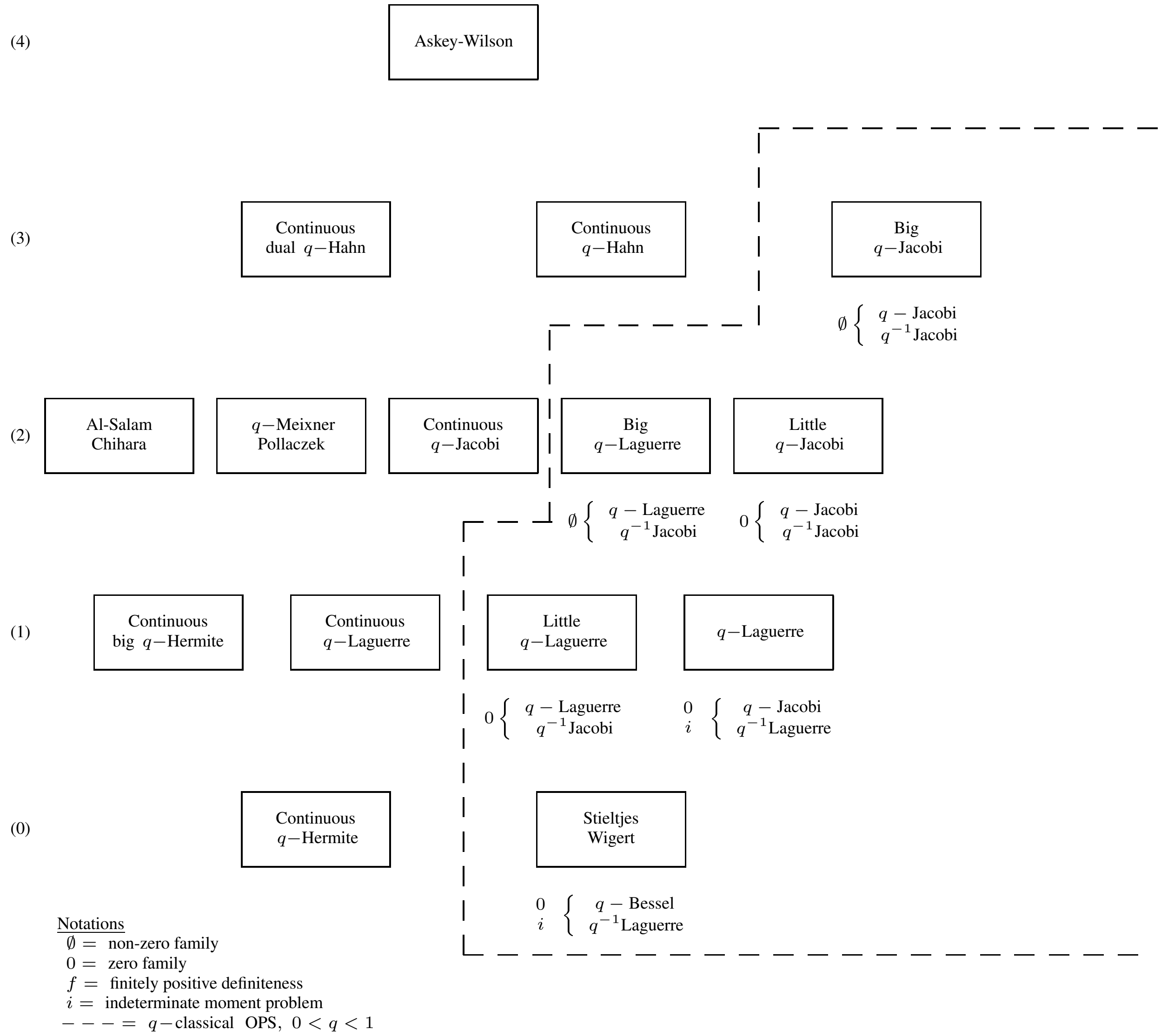
SCHEME OF BASIC HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS



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