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A FOOTNOTE ON QUATERNION BLOCK-TRIDIAGONAL SYSTEMS*

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Abstract. Two direct methods, for solving a system of linear equations having matrices of quaternion entries as coefficients and independent terms, are proposed. Second-kind Chebyshev polynomials, in a matrix argument, are used as a tool to do certain calculations.

Key words. block tridiagonal matrices, quaternions, Chebyshev polynomials.

AMS subject classifications. 65F05.

1. Commutative case. Consider the linear system $\mathbb{A}X = B$, written as follows:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & An \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

where $\mathbb{A}$ is a $n \times n$ matrix, partitioned into pairwise commuting blocks A_{ij}, $i, j = 1, ..., n$; $X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}^T$ and $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}^T$.

In general, we can solve the given system applying a generalized Cramer rule.

• When each B_j, for $j = 1, ..., n$, commutes with each A_{ij}, $i, j = 1, ..., n$, the solution of the system is given by X, with blocks $X_i = \Delta^{-1}\Delta_i$, $i = 1, ..., n$, where Δ is the formal determinant of the coefficient matrix and Δ_i are the formal determinants obtained by replacing the ith column in Δ, with the block–independent terms [10].

• When some B_j, $j = 1, ..., n$, do not commute with some A_{ij}, $i, j = 1, ..., n$, Δ_i must be replaced by a symbolic determinant — Δ_i^* — formally analogous to Δ_i; however, the B_j, $j = 1, ..., n$, should appear on the right in the computation of Δ_i^* [9].

If $\mathbb{A}$ is a full matrix, then Δ_i^* (and also Δ_i) is too difficult to compute. We have to choose another way of finding the solution. We may apply this to the inverse of $\mathbb{A}$, computing the solution by $X = \mathbb{A}^{-1}B$.

2. Noncommutative case. When passing from a commutative situation to a noncommutative, the use of the Cramer rule brings many difficulties. The references [7, 11] are just a sample of the extensive work done, mainly by Chinese mathematicians, on quaternion matrices. In this note, we use a very special matrix form, which is the reason why we could make the following extension.

In the special case of $\mathbb{A}$ being a block–tridiagonal block–symmetric matrix (of order n) of the form $\mathbb{A} = [(-I)\,(A)\,(-I)]$, with I denoting the identity matrix of order p and

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$A \in \mathcal{M}_{p\times p}(\mathbb{H})$ and $B_j \in \mathcal{M}_{p\times q}(\mathbb{H})$, $j = 1, ..., n$, it is possible to apply Chebyshev polynomials to compute Δ, Δ_i^* and each block of $\mathbb{A}^{-1}$ (denoted by $(\mathbb{A}^{-1})_{ij}$, $i, j = 1, ..., n$).

(a) Using Onana and Mbala's method [4], we have

$$\Delta^{-1} = J_n^{-1}$$
$$\Delta_i^* = J_{n-i}\left(\sum_{k=1}^{i-1} J_{k-1}B_k(-1)^{i+k}\right) + J_{i-1}\left(\sum_{k=1}^{n} J_{n-k}B_k(-1)^{i+k}\right)$$

where J_n, $n \in \mathbb{N}$, denotes the modified n^{th}–order second–kind Chebyshev polynomials, with matrix argument A.

(b) Using Kershaw's [2] or Rózsa and Romani's [6] method, we have

$$(\mathbb{A}^{-1})_{rs} = U_n^{-1}U_{r-1}U_{n-s}, \;\; 1 \le r \le s \le n, \;\; (\mathbb{A}^{-1})_{rs} = (\mathbb{A}^{-1})_{sr}$$

where U_n, $n \in \mathbb{N}$, denotes the n^{th}-order second-kind Chebyshev polynomials, with matrix argument $\frac{1}{2}A$.

3. Remarks. a) In the scalar case, the determinant of tridiagonal matrices is a certain kind of continuant [5]. To compute a continuant, a three term recurrence is usually used [5, 1, 8, 3]. Some special kind of continuants were first associated, as far as we know, with the expressions of Chebyshev Polynomials, in 1900, by Pascal [5], as referred to in [6].

We followed the generalization to the matrix case of such association, given in [2, 6].

b) We underline that the quaternionic skew–field $\mathbb{H}$ is noncommutative. The present extension is possible, because the only matrices involved are powers of A, and thus, in this particular case, we have commutativity between blocks. We have block symmetry, as well, but scalar symmetry is lost.

c) Another remark is due. In both methods, it is necessary to compute the inverse of quaternionic matrices (J_n^{-1} and U_n^{-1}).

Following [12], we can guarantee the unicity of the inverse of a quaternionic matrix, when it exists, and we can compute it.

The process to obtain the inverse is, briefly, the following: we transform the quaternionic matrix W into a complex one — denoted by W_c —; then, we invert W_c (in classical sense), obtaining the complex matrix W_c^{-1}; finally, transforming W_c^{-1} into a quaternionic matrix, we obtain W^{-1}. The main result involved in this process is the existence of the isomorphism φ, defined [12] as follows:

$$\varphi \;:\; M_{n\times n}(\mathbb{H}) \longrightarrow M_{2n\times 2n}(\mathbb{C})$$
$$M = M_1 + M_2 j \longmapsto \varphi(M) = \begin{bmatrix} M_1 & M_2 \\ -\overline{M_2} & \overline{M_1} \end{bmatrix}.$$

As an illustrative example of this process, let us compute the inverse of the matrix

$$W = \begin{bmatrix} j & -3k \\ -k & 1+j+k \end{bmatrix} \in M_{2\times 2}(\mathbb{H}).$$

We decompose W as follows

$$W = W_1 + W_2 j = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -3i \\ -i & 1+i \end{bmatrix} j.$$

Applying φ to W, we obtain

$$\varphi(W) \equiv W_c = \begin{bmatrix} 0 & 0 & 1 & -3i \\ 0 & 1 & -i & 1+i \\ -1 & -3i & 0 & 0 \\ -i & -1+i & 0 & 1 \end{bmatrix} \in M_{4\times 4}(\mathbb{C}).$$

The inverse of W_c is the complex matrix $W_c^{-1} = \begin{bmatrix} T_1 & T_2 \\ -\overline{T_2} & \overline{T_1} \end{bmatrix}$, with

$$T_1 = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6}i \\ \frac{1}{18}i & \frac{1}{18} \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} -\frac{1}{3}+\frac{1}{6}i & -\frac{1}{6}+\frac{2}{3}i \\ -\frac{1}{18}+\frac{2}{9}i & -\frac{2}{9}-\frac{1}{18}i \end{bmatrix}.$$

Finally, the inverse of W is $\varphi^{-1}(W_c^{-1})$, which is the quaternion matrix

$$W^{-1} = \begin{bmatrix} \frac{1}{6}-\frac{1}{3}j+\frac{1}{6}k & -\frac{1}{6}i-\frac{1}{6}j+\frac{2}{3}k \\ \frac{1}{18}i-\frac{1}{18}j+\frac{2}{9}k & \frac{1}{18}-\frac{2}{9}j-\frac{1}{18}k \end{bmatrix} \in M_{2\times 2}(\mathbb{H}).$$

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