General Mathematics Vol. 10, No. 3–4 (2002), 17–22

A note on the equation $(x + y + z)^2 = xyz$

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Abstract

Generally, integer solutions to equations of three or more variables are given in various parametric forms (see [3]). In [2] it is proved that the diophantine equation $x + y + z = xyz$ has solutions in the units of the quadratic field $Q(\sqrt{d})$ if and only if $d = -1, 2$, or 5 and in these cases all solutions are also given. The problem of finding all of its solutions remains open. In this paper we will construct different families of infinite positive integer solutions to the equation:

(1) $$(x + y + z)^2 = xyz$$

We will indicate a general method of generating such families of solutions by using the theory of Pell's equations. It seems that the problem of finding all solutions to equation (1) is a difficult one and it is still open.

2000 Mathematical Subject Classification: 11D72, 11A99

1 Solutions of the equation $Ax^2 - By^2 = C$

We will present a general method for solving the equation

$$Ax^2 - By^2 = C \tag{2}$$

In the special case $C = 1$, the equation (2) was studied in [1].

Theorem 1. *Let A, B be positive integers such that AB is not a perfect square and let C be a nonzero integer. If the equations $u^2 - ABv^2 = C$ and $Aq^2 - Bt^2 = 1$ are solvable, then (2) is also solvable and all its solutions (x, y) are given by*

$$x = q_0u + Bt_0v, \quad y = t_0u + Aq_0v \tag{3}$$

where (u, v) is any solution to the above general Pell's equation and (s_0, t_0) is the minimal solution to $Aq^2 - Bt^2 = 1$.

Proof. We have

$$Ax^2 - By^2 = A(q_0u + Bt_0v)^2 - B(t_0u + Aq_0v)^2 =$$

$$= (Aq_0^2 - Bt_0)^2(u^2 - ABv^2) = 1 \cdot C = C.$$

It follows that (x, y), given in (3), is a solution to the equation (2).

Conversely, let (x, y) be a solution to (2), and let (q_0, t_0) be the minimal solution to the equation $Aq^2 - Bt^2 = 1$. Then (u, v), where $u = Ax_0x - Bt_0y$ and $v = -t_0x + q_0y$ is a solution to the general Pell's equation $u^2 - ABv^2 = C$. Solving the above system of linear equations with unknowns x and y yields $x = q_0u + Bt_0v$ and $y = t_0u + Aq_0v$, i.e. (x, y) has the form (3).

Remark 1. *Consider the three diophantine equations in the Theorem:*

(I) $Ax^2 - By^2 = C$

(II) $u^2 - ABv^2 = C$

(III) $Aq^2 - Bt^2 = 1.$

The following implications are true:

(II) and (III) are solvable then (I) is solvable

(I) and (III) are solvable then (II) is solvable

(I) and (II) are solvable and there exist solutions (x, y) *and* (u, v) *such that*

$$\frac{ux - Bvy}{C} \text{ and } \frac{-Avx + uy}{C}$$

are both integers then (III) is solvable.

The first implication was proved in the above Theorem.

For the second implication, if (x, y) and (q, t) are solutions to (I) and (II), respectively, then (u, v), with $u = Aqx - Bty$ and $v = -tx + qy$ is a solution to (II). Moreover, each solution to (II) is of the above form. Indeed, if (u, v) is an arbitrary solution to (II), then (x, y), where $x = qu + Btv$ and $y = tu + Aqv$ is a solution to (I). Thus, solving the above system of linear equation in u, v, it follows that $u = Aqx - Bt$ and $v = -tx + qy$.

In order to prove the third implication, let (x, y) and (u, v) be a solution to (I) and (II), respectively, for which

$$\frac{ux - Bvy}{c} \quad \text{and} \quad \frac{-Avx + uy}{C} \in \mathbb{Z}.$$

Then (q, t) is a solution to (II).

In what follows, we are not interested in finding all solutions to the equation (2) that will arise. A family of solutions to such equations can be generated in the following way: if (x_0, y_0) is a solution to the equation (2) and $(r_m, s_m)m \geq 1$ is the general solution to its Pell's resolvent $r^2 - ABs^2 = 1$, then $(x_m, y_m)m \geq 1$, where

$$x_m = x_0 u_m, \quad y_m = y_0 u_m + Ax_0 v_m \tag{4}$$

are also solutions to the equations (2). The proof is similar to the one in the first part of the Theorem's proof.

2 Four families of solutions to the equation (1)

Recall that $D > 0$ is not a square, then the Pell's equation $r^2 - Ds^2 = 1$ is solvable and all of its solutions are given by

$$r_m + s_m\sqrt{D} = (r_1 + s_1\sqrt{D})^m, \ m \geq 1, \tag{5}$$

where (r_1, s_1) is its minimal nontrivial solution.

We start by performing the transformations

$$x = \frac{u+v}{2} + a, \ y = \frac{u-v}{2} + a, \ z = b \tag{6}$$

where a and b are nonzero integer parameters that will be determined in a convenient manner. The equation becomes

$$(u + 2a + b)^2 = \frac{b}{4}(u^2 - v^2) + abu + a^2 b.$$

Imposing the conditions $2(2a+b) = ab$ and $b(b-4) > 0$ yield the general Pell's equation

$$(b-4)u^2 - bv^2 = 4[(2a+b)^2 - a^2 b]. \tag{7}$$

The imposed conditions are equivalent to $(a-2)(b-4) = 8$, $b < 0$ or $b > 4$. A simple case analysis shows that the only pairs of positive integers (a, b) satisfying them are: $(3, 12)$, $(4, 8)$, $(6, 6)$, $(10, 5)$.

The following table contains the general Pell's equations (7) corresponding to the above pairs (a, b), their Pell's resolvents, both equations with their fundamental solutions.

By using the formula (4) we obtain the following sequences of solutions to the e equations (7):

$$u_m^{(1)} = 18r_m^{(1)} + 36s_m^{(1)}, \quad v_m^{(1)} = 12r_m^{(1)} + 36s_m^{(1)}$$

where $r_m^{(1)} + s_m^{(1)}\sqrt{6} = (5 + 2\sqrt{6})^m$, $m \geq 1$;

$$u_m^{(2)} = 16r_m^{(2)} + 16s_m^{(2)}, \quad v_m^{(2)} = 8r_m^{(2)} + 16s_m^{(2)}$$

where $r_m^{(2)} + s_m^{(2)}\sqrt{2} = (3 + 2\sqrt{2})^m$, $m \geq 1$;

$$u_m^{(3)} = 18r_m^{(3)} + 18s_m^{(3)}, \quad v_m^{(3)} = 6r_m^{(3)} + 18s_m^{(3)}$$

where $r_m^{(3)} + s_m^{(3)}\sqrt{3} = (2 + \sqrt{3})^m$, $m \geq 1$;

$$u_m^{(4)} = 25r_m^{(4)} + 25s_m^{(4)}, \quad v_m^{(4)} = 5r_m^{(4)} + 25s_m^{(4)}$$

where $r_m^{(4)} + s_m^{(4)}\sqrt{5} = (9+4\sqrt{5})^m, \ m \geq 1$.

Formulas (6) yield following four families of nonzero integer solutions to equation (1):

$$x_m^{(1)} = 15r_m^{(1)} + 36s_m^{(1)} + 3, \quad y_m^{(1)} = 3r_m^{(1)} + 3, \quad z_m^{(1)} = 12, \quad m \geq 1$$
$$x_m^{(2)} = 12r_m^{(2)} + 16s_m^{(2)} + 4, \quad y_m^{(2)} = 4r_m^{(2)} + 4, \quad z_m^{(2)} = 8, \quad m \geq 1$$
$$x_m^{(3)} = 12r_m^{(3)} + 18s_m^{(3)} + 6, \quad y_m^{(3)} = 6r_m^{(3)} + 6, \quad z_m^{(3)} = 6, \quad m \geq 1$$
$$x_m^{(4)} = 15r_m^{(4)} + 25s_m^{(4)} + 10, \ \ y_m^{(4)} = 10r_m^{(4)} + 10, \ \ z_m^{(4)} = 5, \ \ m \geq 1.$$

References

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