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On a subclass of functions with negative coefficients

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

We determine conditions for a function to be n-close to convex of order α, $\alpha \in [0,1)$, $n \in \mathbb{N}$, with negative coefficients.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In ([4]) the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0, j = 2, 3, ...,\ z \in U \tag{1}$$

was introduced.

The purpose of this paper is to give a condition for $f \in T$ to be n-close to convex of order α, $\alpha \in [0,1)$, $n \in \mathbb{N}$, and to determine some properties of this class.

2 Preliminary results

Let D^n be the Sălăgean differential operator (see [2]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

Remark 2.1. *If* $f \in T, f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$ *then* $D^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j$.

Theorem 2.1.[2]. *If* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$ *then the next assertions are equivalent:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* S^* *is the well-known class of starlike functions.*

Definition 2.1.[2]. *Let* $\alpha \in [0,1)$ *and* $n \in \mathbb{N}$, *then*

$$S_n(\alpha) = \left\{ f \in A : Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, z \in U \right\}$$

is the set of n-starlike functions of order α.

Remark 2.2. *If* $f \in S_n(\alpha)$ *according to the definition of the Sălăgean differential operator we can write that*

$$Re\frac{z(D^n f(z))'}{D^n f(z)} > \alpha$$

and thus the function $F(z) = D^n f(z) \in S(\alpha)$, $\alpha \in [0, 1)$, *where*

$$S(\alpha) = \left\{ h \in A : Re\frac{zh'(z)}{h(z)} > \alpha,\ z \in U \right\}.$$

Definition 2.2.[2]. $T_n(\alpha) = T \bigcap S_n(\alpha)$.

Definition 2.3.[3]. *Let* $\alpha \in [0, 1), \beta \in (0, 1]$ *and let* $n \in \mathbb{N}$*; we define the class* $T_n(\alpha, \beta)$ *of n-starlike functions of order* α *and type* β *with negative coefficients by*

$$T_n(\alpha, \beta) = \{f \in A : |J_n(f, \alpha; z)| < \beta, z \in U\},$$

where

$$J_n(f, \alpha; z) = \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{\dfrac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha},\ z \in U$$

Remark 2.3. *The class* $T_0(\alpha, 1)$ *is the class of starlike functions of order* α *with negative coefficients;* $T_1(\alpha, 1)$ *is the well-known class of convex functions of order* α *with negative coefficients;* $T_n(\alpha, 1)$ *is the class of n-starlike functions of order* α *with negative coefficients i.e.* $T_n(\alpha, 1) = T \bigcap S_n(\alpha)$. *We also note that the functions in* $T_n(\alpha, \beta)$ *are univalent because* $T_n(\alpha, \beta) \subset T_n(\alpha, 1)$, $\beta \in (0, 1)$ *and* $T_n(\alpha_1, \beta) \subset T_n(\alpha, \beta)$ *with* $1 > \alpha_1 > \alpha \geq 0$, $\beta \in (0, 1]$.

Theorem 2.2.[3]. *Let $\alpha \in [0,1), \beta \in (0,1]$ and $n \in \mathbb{N}$. The function f of the form (1) is in $T_n(\alpha,\beta)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[j-1+\beta(j+1-2\alpha)]a_j \leq 2\beta(1-\alpha)$$

The result is sharp and the extremal functions are:

$$f_j(z) = z - \frac{2\beta(1-\alpha)}{j^n[j-1+\beta(j+1-2\alpha)]} z^j, j = 2, 3, ...$$

From this result we have $T_{n+1}(\alpha,\beta) \subset T_n(\alpha,\beta)$, $n \in \mathbb{N}$.

Definition 2.4.[3]. *Let $I_c : A \to A$ be the integral operator defined by $f = I_c(F)$, where $c \in (-1,\infty)$, $F \in A$ and*

(2) $$f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} F(t) dt.$$

We note if $F \in A$ is a function of the form (1), then

(3) $$f(z) = I_c F(z) = z - \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j.$$

Remark 2.4. *In [3] is showed that if $F \in T_n(\alpha,\beta)$ then $f = I_c(F) \in T_n(\alpha,\beta)$.*

Definition 2.5.[1]. *Let $f \in A$. We say that f is n-close to convex of order α with respect to a half-plane, and denote by $CC_n(\alpha)$ the set of these functions, if there exists $g \in S_n(0) = S_n$ so that*

$$Re \frac{D^{n+1} f(z)}{D^n g(z)} > \alpha, \ z \in U,$$

where $n \in \mathbb{N}, \alpha \in [0,1)$.

Remark 2.5. *$CC_0(\alpha) = CC(\alpha)$, where $CC(\alpha)$ is the well-known class of close to convex functions of order α.*

Remark 2.6. *In [1] the author show that if $n \in \mathbb{N}$ and $\alpha \in [0,1)$ then $CC_{n+1}(\alpha) \subset CC_n(\alpha)$ and thus the functions from $CC_n(\alpha)$ are univalent.*

Remark 2.7. *From Remark 2.3 and Theorem 2.2 we have for f of the form (1) with $f \in T_n(\alpha, 1) = T_n(\alpha)$:*

$$\sum_{j=2}^{\infty} j^n (j-\alpha) a_j \leq 1-\alpha, \text{ where } \alpha \in [0,1)$$

3 Main results

Definition 3.1. *Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \dots$, $z \in U$. We say that f is in the class $CCT_n(\alpha), \alpha \in [0,1)$, $n \in \mathbb{N}$, with respect to the function $g \in T_n(0)$, if:*

$$Re \frac{D^{n+1} f}{D^n g} > \alpha, \; z \in U.$$

Theorem 3.1. *Let $\alpha \in [0,1)$ and $n \in \mathbb{N}$. The function $f \in T$ of the form (1) is in $CCT_n(\alpha)$, with respect to the function $g \in T_n(0)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, \dots$, if and only if*

$$\sum_{j=2}^{\infty} j^n [j a_j + (2-\alpha) b_j] < 1-\alpha \tag{4}$$

Proof. Let $f \in CCT_n(\alpha)$, with $\alpha \in [0,1)$. We have

$$Re \frac{D^{n+1} f(z)}{D^n g(z)} > \alpha.$$

If we take $z \in [0,1)$, we have (see Remark 2.1):

$$\frac{1-\sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1}}{1-\sum_{j=2}^{\infty} j^{n} b_j z^{j-1}} > \alpha \tag{5}$$

From $g \in T_n(0) = T_n(0,1)$, $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j, b_j \geq 0, j = 2, 3, ...,$ we have (see Remark 2.7):

$$\sum_{j=2}^{\infty} j^{n+1} b_j \leq 1. \tag{6}$$

We have: $\sum\limits_{j=2}^{\infty} j^n b_j z^{j-1} \leq \sum\limits_{j=2}^{\infty} j^{n+1} b_j z^{j-1} < \sum\limits_{j=2}^{\infty} j^{n+1} b_j$.

From (6) we obtain: $\sum\limits_{j=2}^{\infty} j^n b_j z^{j-1} < 1$ and thus $1 - \sum\limits_{j=2}^{\infty} j^n b_j z^{j-1} > 0$.

In this condition from (5) we obtain:

$$1-\sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1} > \alpha \left[1-\sum_{j=2}^{\infty} j^{n} b_j z^{j-1}\right]$$

Letting $z \to 1^-$ along the real axis we have:

$$1-\sum_{j=2}^{\infty} j^{n+1} a_j > \alpha - \sum_{j=2}^{\infty} j^{n} \alpha b_j,$$

and thus:

$$\sum_{j=2}^{\infty} j^n [j a_j - \alpha b_j] < 1-\alpha.$$

From $\sum\limits_{j=2}^{\infty} j^n [j a_j + (2-\alpha) b_j] > \sum\limits_{j=2}^{\infty} j^n [j a_j - \alpha b_j]$ we have that from

$$\sum_{j=2}^{\infty} j^n [j a_j + (2-\alpha) b_j] < 1-\alpha \tag{7}$$

we obtain $Re\dfrac{D^{n+1}f(z)}{D^n g(z)} > \alpha$.

Now let take $f \in T$ and $g \in T_n(0)$ for which the relation (4) hold.

The condition $Re\dfrac{D^{n+1}f(z)}{D^n f(z)} > \alpha$ is equivalent with

$$\alpha - Re\left(\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right) < 1 \tag{8}$$

We have

$$\alpha - Re\left(\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right) \leq \alpha + \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right| =$$

$$= \alpha + \left|\frac{1 - \sum\limits_{j=2}^{\infty} j^{n+1} a_j z^{j-1}}{1 - \sum\limits_{j=2}^{\infty} j^n b_j z^{j-1}} - 1\right| \leq \alpha + \frac{\sum\limits_{j=2}^{\infty} j^n |b_j - ja_j| \cdot |z|^{j-1}}{1 - \sum\limits_{j=2}^{\infty} j^n b_j |z|^{j-1}} \leq$$

$$\leq \alpha + \frac{\sum\limits_{j=2}^{\infty} j^n |b_j - ja_j|}{1 - \sum\limits_{j=2}^{\infty} j^n b_j} \leq \alpha + \frac{\sum\limits_{j=2}^{\infty} j^n (b_j + ja_j)}{1 - \sum\limits_{j=2}^{\infty} j^n b_j} =$$

$$= \frac{\alpha + \sum\limits_{j=2}^{\infty} j^n [ja_j + (1-\alpha) b_j]}{1 - \sum\limits_{j=2}^{\infty} j^n b_j}$$

Using (8) we obtain:

$$\alpha + \sum_{j=2}^{\infty} j^n [ja_j + (2-\alpha) b_j] < 1$$

that is the condition (4).

Remark 3.1. *If we take* $f \equiv g$ *we obtain from Theorem 3.1*

$$\sum_{j=2}^{\infty} j^n a_j[ja_j + 2 - \alpha] < 1 - \alpha$$

From $\sum\limits_{j=2}^{\infty} j^n a_j[j + 2 - \alpha] > \sum\limits_{j=2}^{\infty} ja_j(j - \alpha)$ *we obtain:*

$$\sum_{j=2}^{\infty} ja_j(j - \alpha) < 1 - \alpha$$

Thus we obtain the result from Remark 2.7.

Remark 3.2. *From the proof of the Theorem 3.1 we obtain a necessary condition for a function* $f \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$ *to be in the class* $CCT_n(\alpha)$, $\alpha \in [0, 1)$, $n \in \mathbb{N}$, *with respect to the function* $g \in T_n(0)$, $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$*:*

$$\sum_{j=2}^{\infty} j^n[ja_j - \alpha b_j] < 1 - \alpha.$$

Theorem 3.2. *If* $F \in CCT_n(\alpha), \alpha \in [0, 1), n \in \mathbb{N}$, *with respect to the function* $G \in T_n(0)$ *and* $f = I_c(F)$, $g = I_c(F)$ *where* I_c *is defined by (2), then* $f \in CCT_n(\alpha)$ *with respect to the function* $g \in T_n(0)$ *(see Remark 2.4)*

Proof. From $F(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j, a_j \geq 0, j = 2, 3, ...$ and $f(z) = I_c(F)(z)$ we have (see (3)):

$$f(z) = z - \sum_{j=2}^{\infty} \alpha_j z^j, \text{ where } \alpha_j = \frac{c+1}{c+j} a_j, \ j = 2, 3, ...$$

From $G(z)=z-\sum\limits_{j=2}^{\infty}b_jz^j, b_j\geq 0, j=2,3,...$ and $g(z)=I_c(G)(z)$ we have:

$$g(z)=z-\sum_{j=2}^{\infty}\beta_jz^j,\ \text{where}\ \beta_j=\frac{c+1}{c+j}b_j,\ j=2,3,...$$

From $F\in CCT_n(\alpha)$ with respect to the function $G\in T_n(0)$ we have (see Theorem 3.1):

$$\sum_{j=2}^{\infty}j^n[ja_j+(2-\alpha)b_j]<1-\alpha. \tag{9}$$

From Theorem 3.1 we need only to show that:

$$\sum_{j=2}^{\infty}j^n[j\alpha_j+(2-\alpha)\beta_j]<1-\alpha.$$

We have for $c\in(-1,\infty)$ and $j=2,3,...$:

$$\sum_{j=2}^{\infty}j^n[j\alpha_j+(2-\alpha)\beta_j]=$$

$$=\sum_{j=2}^{\infty}\frac{c+1}{c+j}j^n[ja_j+(2-\alpha)b_j]<\sum_{j=2}^{\infty}j^n[ja_j+(2-\alpha)b_j]$$

From (9) we have:

$$\sum_{j=2}^{\infty}j^n[j\alpha_j+(2-\alpha)\beta_j]<1-\alpha.$$

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