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On some integral inequalities with modified argument and applications

Olaru Ion-Marian

Dedicated to Professor Emil C. Popa on his 60th anniversary

Abstract

In this paper we study the following inequalities

$$x(t) \le A + B\int_a^t x(g(s))ds, t \in [a,b], A, B \in \mathbb{R}_+$$

and hears applications to study of data dependence for functional differential equations.

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1 Introduction

A study about integral- inequalities with modified argument was made in [1].We will study below the integral-inequalities:

$$(1) \qquad x(t) \le A + B\int_a^t x(g(s))ds, t \in [a,b], A, B \in \mathbb{R}_+$$

where:

$(H1)$ $g : [a,b] \to [a_1, b], g \in C^1([a,b])$ with the derivate that satisfies the following condition:

$$-1 \leq g\prime(t) \leq m < 0, \text{ for all } t \in [a,b].$$

An example of that function is the follows:

Example 1.1.

$g : [a,b] \longrightarrow [a_1, b],\ a_1 < a, g(t) = -\frac{a}{b}t - \frac{c}{b}$ *with:*

(i) $a < 0, b < 0, -1 \leq -\frac{a}{b} \leq m < 0.$

(ii) $c = -a^2 - b^2.$

Next we consider the following set:

$$S_g = \{x \in C([a,b], \mathbb{R}_+) \mid x(s) + g'(s)x(g(s)) \geq 0 \text{ for all } s \in [a,b],$$

$$C[a,b], \mathbb{R}_+ = \{x : [a,b] \to \mathbb{R}_+, x \text{ continuous}\}$$

Remark 1.1. *We observe that S_g is the closed set with respect to the topology generated by the uniform norm and $0 \in S_g$.*

Remark 1.2. *From the condition*

$$x(s) + g'(s)x(g(s)) \geq 0, \text{ for all } s \in [a,b]$$

by integrating on $[a,t]$ we obtain

$$-\int_a^t g'(s)x(g(s))ds \leq \int_a^t x(s)ds$$

which implies

$$\int_{g(t)}^{g(a)} x(u)du \leq \int_a^t x(s)ds.$$

Next ,we define the Picard operator and weakly Picard operator notions on a metric space X by (see [5]):

Definition 1.1. *(i)An operator $A : X \to X$ is weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n \in N}$$

converges ,for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

(ii)If the operator A is WPO and $F_A = \{x^*\}$then by definition A is a Picard operator(PO).

Here F_A is the fixed points set of A and A^n is the n order iteration of the operator A.

Next we use the following theorem see [5]:

Theorem 1.1. *Let (X, d) and (Y, ρ) be two metric space and*

$$A : X \times Y \to X \times Y, A(x, y) = (B(x), C(x, y)).$$

We suppose that

(i)(Y, ρ) is a complete metric space .

(ii) The operator $B : X \to X$ is weakly Picard operator.

(iii)There exists $a \in [0, 1)$ such that $C(x, \cdot)$ is an a-contraction,for all $x \in X$.

(iv)If $(x^*, y^*) \in F_A$ then $C(\cdot, y^*)$ is continuous in x^*.

Then A is weakly Picard operator.If B is Picard operator then A is a Picard operator.

Data dependence with respect to initial conditions was study in [2],[3],[4]

2 Main results

Proposition 2.1. *If $x_0 \in S_g$ is an solution of inequalities (1) then :*

$$x_0(t) \leq Ae^{\frac{B}{-m}(t-a)}.$$

Proof. Because g is strictly decreasing wel make the variable change $g(s) = u$ and we obtain:

$$x_0(t) \leq A + B \int\limits_{g(a)}^{g(t)} x_0(u)(g^{-1}(u))'du \leq A + \frac{B}{-m} \int\limits_{g(t)}^{g(a)} x_0(u)du \leq$$

$$\leq A + \frac{B}{-m} \int\limits_{a}^{t} x_0(s)ds.$$

From Gronwall Lema we have:

$$x_0(t) \leq Ae^{\frac{B}{-m}(t-a)}.$$

Next we consider the following Cauchy problem:

$$x'(t) = f(t, x(g(t))), t \in [a, b] \tag{2}$$

$$x(t) = \varphi(t) \qquad , t \in [a_1, a]. \tag{3}$$

where:

(H_2) $f \in C^1([a, b] \times \mathbb{R}^n), f(t, 0) = 0,$ for all $t \in [a, b]$, $\varphi \in C([a_1, a], [-b, b]^n)$.

(H_3) There exists $L_f \leq \frac{-m}{(b-a_1)e}$ such that $\|\frac{\partial f}{\partial u_i}(t, u)\|_{\mathbb{R}^n} \leq L_f$ for all $t \in [a, b], u \in \mathbb{R}^n, i = \overline{1, n}$.

The problem (2)+(3)is equivalent with

(4) $$x(t) = \begin{cases} \varphi(a) + \int\limits_a^t f(s, x(g(s)))ds, & t \in [a,b] \\ \varphi(t) & , \ t \in [a_1, a] \end{cases}$$

Further wel apply the Banach principle to the restriction of operator L at closed ball from $C([a_1,b])$ conveniently chosen , where $L : C([a_1,b]) \to C([a_1,b])$ is defined by :

(5) $$Lx(t) = \begin{cases} \varphi(a) + \int\limits_a^t f(s, x(g(s)))ds, & t \in [a,b] \\ \varphi(t) & , \ t \in [a_1, a] \end{cases}.$$

On $C([a_1,b])$ we define the Bielecki norm :

$$\|x\|_B = \max_{t\in[a_1,b]} \|x(t)\|_{R^n} e^{-\tau(t-a_1)}.$$

Because $\dfrac{e^{\tau(b-a_1)}}{\tau} \geq (b-a_1)e$ for all $\tau \geq 0$ and using that $\frac{-m}{L_f} \geq (b-a_1)e$ we choose $\tau_0 > 0$ such that $\dfrac{e^{\tau_0(b-a_1)}}{\tau_0} < \dfrac{-m}{L_f}$.

Next,we choose in the definition of Bieleski norm this τ_0.

For g which satisfied the hypothesis (H_1), we define the following set:

$$S_{gn} = \{x \in C([a,b],\mathbb{R}^n) \mid \|x(s)\|_{R^n} + g'(s)\|x(g(s))\|_{\mathbb{R}^n} \geq 0, \text{ for all } s \in [a,b]\}$$

(H_4) We suppose that there exists a set $A \subseteq S_{gn}$ such that for all $x, y \in A$ we have $x + y \in A$.

Remark 2.1. *For g having the property* (H_1) *the set* $A = \{ce^{\alpha t} \mid c, \alpha \in \mathbb{R}_+\} \subset S_{g1}$ *verifies* (H_4).

Proposition 2.2. *We suppose that:*

(a)The hypothesis $(H_1)-(H_4)$ *are satisfied.*

(b)There exists $R>\frac{b}{1-\dfrac{L_f}{-m\tau_0}e^{\tau_0(b-a_1))}}$ *such that* $\overline{B}(0,R)\subset A$

Then :

(i)The problem (2)+(3) are a unique solutions $x(\cdot,\varphi)$.

(ii)The solution $x(t,\varphi)$ *is continuous with respect the* φ.

Proof. First we show that $\overline{B}(0,R)$ is a invariant set for the operator L. Let be $x\in\overline{B}(0,R)$.Then

$$\|Lx(t)\|_{\mathbb{R}^n}\leq b+\int_a^t L_f\|x(g(s))\|_{\mathbb{R}^n}ds\leq b+L_fR\int_a^t e^{\tau_0(g(s)-a_1)}ds\leq$$

$$\leq b+\frac{L_fR}{-m}\int_{g(t)}^{g(a)}e^{\tau_0(u-a_1)}du\overset{e^{\tau_0(t-a)}\in S_{g1}}{\leq}b+\frac{L_fR}{-m}\int_a^t e^{\tau_0(s-a_1)}ds\leq$$

$$\leq b+\frac{L_fR}{-m\tau}e^{\tau_0(b-a_1)}.$$

It follow that $\|Ax\|_B\leq\|Ax\|_C\leq b+\frac{L_fR}{-m\tau}e^{\tau(b-a_1)}$. Here $\|\cdot\|_C$ is Chebyshev norm.

We obtain that $L(\overline{B}(0,R))\subseteq\overline{B}(0,R)$.

Let be $x,y\in\overline{B}(0,R)$.Then

$$\|Ax(t)-Ay(t)\|_{\mathbb{R}^n}\leq L_f\int_a^t\|x(g(s))-y(g(s))\|_{\mathbb{R}^n}ds\leq$$

$$\leq\frac{L_f}{-m}\int_{g(t)}^{g(a)}\|x(u))-y(u))\|_{\mathbb{R}^n}du\leq\frac{L_f}{-m}\|x-y\|_B\int_{g(t)}^{g(a)}e^{\tau_0(u-a_1)}du\leq$$

$$\leq\frac{L_f}{-m}\|x-y\|_B\int_a^t e^{\tau_0(s-a_1)}ds\leq\frac{L_f}{-m\tau_0}e^{\tau_0(b-a_1)}\|x-y\|_B.$$

We obtain that L is a contraction map on $\overline{B}(0,R)$. In consequence there exists a unique solution $x(\cdot,\varphi)$ in $\overline{B}(0,R)$ of (2)+(3)

(ii)We suppose that there exists $\eta>0$ such that $\|\varphi_1(t)-\varphi_2(t)\|_{R^n}\le\eta$ for all $t\in[a_1,a]$.Let be $x(\cdot,\varphi_1),x(\cdot,\varphi_2)$ the solutions for the Cauchy problem (2)+(3)with initial conditions φ_1,φ_2.Then

$$\|x(t,\varphi_1)-x(t,\varphi_2)\|_{\mathbb{R}^n}\le\eta+\int_a^t\|x(g(s),\varphi_1)-x(g(s),\varphi_2)\|_{\mathbb{R}^n}ds\le$$

$$\le\eta+\frac{L_f}{-m}\int_{g(t)}^{g(a)}\|x(u,\varphi_1)-x(u,\varphi_2)\|_{\mathbb{R}^n}du\le\eta+\frac{L_f}{-m}\int_a^t\|x(s,\varphi_1)-x(s,\varphi_2)\|_{\mathbb{R}^n}ds$$

Result that $\|x(t,\varphi_1)-x(t,\varphi_2)\|_{\mathbb{R}^n}\le\eta e^{\frac{L_f}{-m}(b-a)}$.

Next we consider the following Cauchy problem:

$$x'(t)=f(t,x(g(t),\lambda))\qquad,t\in[a,b],\lambda\in J \tag{6}$$

$$x(t)=\varphi(t)\qquad,t\in[a_1,a] \tag{7}$$

where $J\subset R$ a compact interval,and

(H_5) $f\in C^1([a,b]\times\mathbb{R}^n\times J),f(t,0)=0,$ for all $t\in[a,b]$, $\varphi\in C([a_1,a],[-b,b]^n)$.

(H_6)There exists $L_f,M\in\mathbb{R}_+,L_f\le\frac{-m}{(b-a_1)e}$ such that $\|\frac{\partial f}{\partial u_i}(t,u,\lambda)\|_{R^n}\le L_f,\frac{\partial f}{\partial\lambda}(t,u,\lambda)\le M$,for all $t\in[a,b],u\in\mathbb{R}^n,\lambda\in J$.

The problem (6)+(7) is equivalent with:

$$x(t)=\begin{cases}\varphi(a)+\int_a^t f(s,x(g(s),\lambda)ds & ,t\in[a,b],\lambda\in J\subset\mathbb{R}\\ \varphi(t) & ,t\in[a_1,a]\end{cases} \tag{8}$$

Proposition 2.3. *We suppose that:*

(a)The hypothesis $(H_1), (H_4), (H_5), (H_6)$ *are satisfied.*

(b)There exists $R > \max \left\{ \frac{M(b-a}{1-\frac{L_f}{-m\tau_0}e^{\tau_0(b-a_1))}}, \frac{b}{1-\frac{L_f}{-m\tau_0}e^{\tau_0(b-a_1))}} \right\}$ *such that* $\overline{B}(0,R) \subset A$

Then :

(i)The problem (6)+(7) are a unique solutions $x(\cdot, \varphi, \lambda) \in \overline{B}(0,R)$.

(ii)The solution $x(t,\varphi,\lambda)$ *is derivable with respect the* λ.

Proof. Using the Proposition 2.2 ,we have that there exists a unique solutions $\overline{x}(t,\varphi,\lambda)$ which verify

$$(9) \qquad \overline{x}(t,\varphi,\lambda) = \begin{cases} \varphi(a) + \int\limits_a^t f(s, \overline{x}((g(s),\varphi,\lambda),\lambda)ds, t \in [a,b] \\ \varphi(t) \qquad\qquad\qquad\qquad\qquad\qquad , t \in [a_1, a] \end{cases}$$

We consider C defined on $\overline{B}(0,R) \times \overline{B}(0,R)$,by

$$(10) \qquad C(x,y)(t) =$$

$$= \begin{cases} \int\limits_a^t \frac{\partial f}{\partial u}(s, x(g(s),\lambda))y(g(s),\lambda)ds + \int\limits_a^t \frac{\partial f}{\partial \lambda}(s, x(g(s),\lambda),\lambda), t \in [a,b] \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , t \in [a_1, a] \end{cases}$$

From

$$\|C(x,y)(t)\|_{\mathbb{R}^n} \leq \int\limits_a^t \text{Ł}_f \|y(g(s))\|_{\mathbb{R}^n} ds + M(b-a) \leq \int\limits_{g(t)}^{g(a)} \|y(u)\|_{\mathbb{R}^n} du + M(b-a)$$

$$\leq \frac{L_f}{-m} \frac{e^{\tau_0}(b-a_1)}{\tau_0} R + M(b-a) \leq R$$

we have $C(\overline{B}(0,R) \times \overline{B}(0,R)) \subseteq \overline{B}(0,R)$ Let be $x \in \overline{B}(0,R)$ From

$$\|C(x,y)(t) - C(x,z)(t)\|_{\mathbb{R}^n} \leq L_f \int_a^t \|y(g(s) - z(g(s)))\|_{\mathbb{R}^n} ds \leq$$

$$\leq \frac{L_f}{-m} \frac{e^{\tau_0}(b-a_1)}{\tau_0} \|y-z\|_B$$

we have that $C(x,\cdot)$ is a contraction map.It follow that

$$x_{n+1} = L(x_n), n \geq 0$$

$$y_{n+1} = C(x_n, y_n), n \geq 0$$

converges uniformly (with respect to $t \in [a,b]$) to $(\overline{x}, \overline{y}) \in F_A$, for all $x_0, y_0 \in C([a_1,b])$.

If we take $x_0 = 0, y_0 = 0$,then $y_1 = \dfrac{\partial x_1}{\partial \varphi}$.

By induction we prove that

$$y_n = \frac{\partial x_n}{\partial \varphi}, n \in N.$$

Thus

$$x_n \longrightarrow \overline{x} \text{ as } n \to \infty$$
$$\frac{\partial x_n}{\partial \lambda} \longrightarrow \overline{y} \text{ as } n \to \infty.$$

These imply that there exists $\frac{\partial \overline{x}}{\partial \lambda}$ and $\frac{\partial \overline{x}}{\partial \lambda} = \overline{y}$.

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"Lucian Blaga" University of Sibiu
Department of Mathematics
Str. Dr. I. Rațiu, No. 5-7
550012 - Sibiu, Romania
E-mail address: olaruim@yahoo.com