

Dirichlet problems for the biharmonic equation

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Dedicated to Prof. Dr. Dumitru Acu on the occasion of his 60th birthday

Abstract

Two kind of Dirichlet problems are solved explicitly for the inhomogenous biharmonic equation in the unit disc of the complex plane.

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1 Introduction

There are several possibilities to pose boundary conditions of Dirichlet type for the inhomogeneous biharmonic equation

$$(\partial_z \partial_{\bar{z}})^2 w = f \text{ in } D$$

for a regular domain D of the complex plane \mathbb{C} . Of course this is neither restricted to the complex nor to the two-dimensional case. One possibility is to prescribe

$$w = \varphi_0, \quad \partial_z \partial_{\bar{z}} w = \varphi_1 \text{ on } \partial D$$

an other

$$w = \varphi_0, \quad \partial_{\bar{z}} w = \varphi_1 \text{ on } \partial D.$$

Obviously the second condition in the last problem can be replaced by $\partial_z w = \varphi_1$. This will lead to a dual form of the solution. Other Dirichlet-type conditions are available, e.g.

$$\partial_{\bar{z}} w = \varphi_0, \quad \partial_z \partial_{\bar{z}} w = \varphi_1 \quad \text{on } \partial D$$

or

$$\partial_z w = \varphi_0, \quad \partial_z^2 \partial_{\bar{z}} w = \varphi_1 \quad \text{on } \partial D$$

etc., see [3]. The above first problem obviously is well-posed. This can be seen by reformulating the problem as the system

$$\begin{aligned} \partial_z \partial_{\bar{z}} w &= \omega \quad \text{in } \partial D, \quad w = \varphi_0 \quad \text{on } \partial D \\ \partial_z \partial_{\bar{z}} \omega &= f \quad \text{in } D, \quad \omega = \varphi_1 \quad \text{on } \partial D. \end{aligned}$$

It will turn out that the second problem is also well-posed.

Using the biharmonic Green function the solution can be given for arbitrary regular domains in analogy to e.g. [5]. In order to get explicit solutions the particular case of the unit disc is considered here.

2 First Dirichlet problem

Rewriting the problem

$$(1) \quad (\partial_z \partial_{\bar{z}})^2 w = f \quad \text{in } \mathbb{D}$$

$$w = \gamma_0, \quad \partial_z \partial_{\bar{z}} w = \gamma_1 \quad \text{on } \partial \mathbb{D}$$

as the system

$$(2) \quad \partial_z \partial_{\bar{z}} w = \omega \quad \text{in } \mathbb{D}, \quad w = \gamma_0 \quad \text{on } \partial \mathbb{D},$$

$$(2') \quad \partial_z \partial_{\bar{z}} \omega = f \quad \text{in } \mathbb{D}, \quad \omega = \gamma_1 \quad \text{on } \partial \mathbb{D}$$

and using the solution of (2) in the form

$$(3) \quad w(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} g_1(z, \zeta) \gamma_0(\zeta) \frac{d\zeta}{\zeta} + \frac{1}{\pi} \int_{\mathbb{D}} G_1(z, \tilde{\zeta}) \omega(\tilde{\zeta}) d\tilde{\xi} d\tilde{\eta}$$

where

$$(4) \quad g_1(z, \zeta) = \frac{1}{1 - z\tilde{\zeta}} + \frac{1}{1 - \bar{z}\zeta} - 1$$

is the Poisson kernel of the unit disc \mathbb{D} and

$$(5) \quad G_1(z, \zeta) = \log \left| \frac{1 - z\tilde{\zeta}}{\zeta - z} \right|^2$$

is twice the harmonic Green function an iteration process is providing the unique solution to (1). The solution (3) of (2) ist well known, see e.g. [3].

Applying (3) to problem (2') and eliminating ω gives

$$(6) \quad w(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} g_1(z, \zeta) \gamma_0(\zeta) \frac{d\zeta}{\zeta} + \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \hat{g}_2(z, \zeta) \gamma_1(\zeta) \frac{d\zeta}{\zeta} + \\ + \frac{1}{\pi} \int_{\mathbb{D}} \hat{G}_2(z, \zeta) f(\zeta) d\xi d\eta$$

with

$$(7) \quad \hat{g}_2(z\zeta) = \frac{1}{\pi} \int_{\mathbb{D}} G_1(z, \tilde{\zeta}) g_1(\tilde{\zeta}, \zeta) d\tilde{\xi} d\tilde{\eta}$$

and

$$(8) \quad \hat{G}_2(z, \zeta) = \frac{1}{\pi} \int_{\mathbb{D}} G_1(z, \tilde{\zeta}) G_1(\tilde{\zeta}, \zeta) d\tilde{\xi} d\tilde{\eta}.$$

Evaluating the right-hand side of (7) shows

$$(9) \quad \hat{g}_2(z, \zeta) = (|z|^2 - 1) \left[\frac{1}{z\tilde{\zeta}} \log(1 - z\tilde{\zeta}) + \frac{1}{\bar{z}\zeta} \log(1 - \bar{z}\zeta) + 1 \right].$$

This can be verified by applying

$$w(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w(\zeta) \left(\frac{1}{1 - z\tilde{\zeta}} + \frac{1}{1 - \bar{z}\zeta} - 1 \right) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int_{\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta) G_1(z, \zeta) d\xi d\eta,$$

see [3], to $\hat{g}_2(z, \zeta)$. In the same way

$$(10) \quad \begin{aligned} \hat{G}_2(z, \zeta) &= |\zeta - z|^2 \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2 - \\ &- (1 - |z|^2)(1 - |\zeta|^2) \left[\frac{1}{z\bar{\zeta}} \log(1 - z\zeta) + \frac{1}{\bar{z}\zeta} \log(1 - \bar{z}\zeta) \right] \end{aligned}$$

follows using

$$G_1(z, \zeta) = \partial_\zeta \partial_{\bar{\zeta}} |\zeta - z|^2 \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right| + \frac{1 - |z|^2}{1 - z\bar{\zeta}} + \frac{1 - |z|^2}{1 - \bar{z}\zeta}.$$

The function (6) is easily to be verified as a solution to the first Dirichlet problem (1). Using the properties of the Poisson kernel $w = \gamma_0$ on $\partial\mathbb{D}$ is seen. Differentiating (6) leads to

$$w_{z\bar{z}}(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} g_1(z, \zeta) \gamma_1 \frac{d\zeta}{\zeta} + \frac{1}{\pi} \int_{\mathbb{D}} G_1(z, \zeta) f(\zeta) d\xi d\eta$$

from which $w_{z\bar{z}} = \gamma_1$ on $\partial\mathbb{D}$ and $w_{z\bar{z}z\bar{z}} = f$ in \mathbb{D} are seen.

Theorem 1. *The first Dirichlet problem for the inhomogeneous biharmonic equation $w_{z\bar{z}z\bar{z}} = f$ in the unit disc \mathbb{D} with*

$$w = \gamma_0, \quad w_{z\bar{z}} = \gamma_1 \quad \text{on } \partial\mathbb{D}$$

is uniquely solvable (in distributional sense) for $f \in L_1(\mathbb{D}; \mathbb{C})$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D}; \mathbb{C})$. The solution is given by (6) with the kernel functions (9) and (10).

3 Second Dirichlet problem

For the second Dirichlet problem the biharmonic Green function given by Almansi [1] is proper. For the unit disc it is

$$G_2(z, \zeta) = |\zeta - z|^2 \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2 - (1 - |z|^2)(1 - |\zeta|^2).$$

Applying the Gauß theorems in complex form

$$\begin{aligned}\frac{1}{\pi} \int_D w_z(z) dx dy &= -\frac{1}{2\pi i} \int_{\partial D} w(z) d\bar{z}, \\ \frac{1}{\pi} \int_D w_{\bar{z}}(z) dx dy &= \frac{1}{2\pi i} \int_{\partial D} w(z) dz\end{aligned}$$

for regular domains D and continuously differentiable function w repeatedly to

$$\frac{1}{\pi} \int_{\mathbb{D}} w_{\zeta\bar{\zeta}\zeta\bar{\zeta}}(\zeta) G_2(z, \zeta) d\xi d\eta$$

and observing

$$\begin{aligned}\partial_{\zeta} G_2(z, \zeta) &= (\overline{\zeta - z}) \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2 - (1 - |z|^2) \left[\frac{|\zeta - z|^2}{(\zeta - z)(1 - \bar{z}\zeta)} - \bar{\zeta} \right], \\ \partial_{\zeta} \partial_{\bar{\zeta}} G_2(z, \zeta) &= \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2 - g_1(z, \zeta)(1 - |z|^2), \\ \partial_{\zeta}^2 \partial_{\bar{\zeta}} G_2(z, \zeta) &= -\frac{1}{\zeta - z} - \frac{\bar{z}}{1 - \bar{z}\zeta} - \frac{\bar{z}}{(1 - \bar{z}\zeta)^2} (1 - |z|^2)\end{aligned}$$

such that on $\partial\mathbb{D}$ for any $z \in \mathbb{D}$

$$G_2(z, \zeta) = 0, \quad \partial_{\zeta} G_2(z, \zeta) = 0, \quad \partial_{\bar{\zeta}} G_2(z, \zeta) = 0$$

gives

$$\begin{aligned}& \frac{1}{\pi} \int_{\mathbb{D}} w_{\zeta\bar{\zeta}\zeta\bar{\zeta}}(\zeta) G_2(z, \zeta) d\xi d\eta = \\ &= \frac{1}{\pi} \int_{\mathbb{D}} \left\{ \partial_{\bar{\zeta}} [w_{\zeta\bar{\zeta}\zeta}(\zeta) G_2(z, \zeta)] - \partial_{\zeta} w_{\zeta\bar{\zeta}}(\zeta) G_{2\bar{\zeta}}(z, \zeta) + w_{\zeta\bar{\zeta}}(\zeta) G_{2\zeta\bar{\zeta}}(z, \zeta) \right\} d\xi d\eta = \\ &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \{ w_{\zeta\bar{\zeta}\zeta}(\zeta) G_2(z, \zeta) d\zeta + w_{\zeta\bar{\zeta}}(\zeta) G_{2\bar{\zeta}}(z, \zeta) d\bar{\zeta} \} +\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\pi} \int_{\mathbb{D}} \left\{ \partial_{\zeta} [w_{\bar{\zeta}}(\zeta) G_{2\zeta\bar{\zeta}}(z, \zeta)] - w_{\bar{\zeta}}(\zeta) G_{2\zeta\bar{\zeta}}(z, \zeta) \right\} d\xi d\eta = \\
& = -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} w_{\bar{\zeta}}(\zeta) G_{2\zeta\bar{\zeta}}(z, \zeta) d\bar{\zeta} + \\
& + \frac{1}{\pi} \int_{\mathbb{D}} w_{\bar{\zeta}}(\zeta) \left[\frac{1}{\zeta - z} + \frac{\bar{z}}{1 - \bar{z}\zeta} + \frac{\bar{z}}{(1 - \bar{z}\zeta)^2} (1 - |z|^2) \right] d\xi d\eta = \\
& = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w_{\bar{\zeta}}(\zeta) g_1(z, \zeta) (1 - |z|^2) d\bar{\zeta} + \frac{1}{\pi} \int_{\mathbb{D}} w_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z} + \\
& + \frac{1}{\pi} \int_{\mathbb{D}} \partial_{\bar{\zeta}} \left[w(\zeta) \left(\frac{\bar{z}}{1 - \bar{z}\zeta} + \frac{\bar{z}}{(1 - \bar{z}\zeta)^2} (1 - |z|^2) \right) \right] d\xi d\eta = \\
& = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w(\zeta) \left(\frac{\bar{z}}{1 - \bar{z}\zeta} + \frac{\bar{z}}{(1 - \bar{z}\zeta)^2} (1 - |z|^2) \right) d\zeta - \\
& - \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w_{\bar{\zeta}}(\zeta) g_1(z, \zeta) (1 - |z|^2) \frac{d\zeta}{\zeta^2} + \frac{1}{\pi} \int_{\mathbb{D}} w_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z}.
\end{aligned}$$

Using the Cauchy-Pompeiu formula, see e.g. [2],

$$w(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_{\mathbb{D}} w_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z}$$

results in a representation formula.

Lemma. Any $w \in C^4(\mathbb{D}; \mathbb{C}) \cap C^3(\bar{\mathbb{D}}; \mathbb{C})$ is representable by

$$\begin{aligned}
w(z) & = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w(\zeta) \left[g_1(z, \zeta) + \frac{\bar{z}\zeta}{(1 - \bar{z}\zeta)^2} (1 - |z|^2) \right] \frac{d\zeta}{\zeta} - \\
& - \frac{1}{\pi} \int_{\mathbb{D}} w_{\zeta\bar{\zeta}\zeta}(\zeta) G_2(z, \zeta) d\xi d\eta.
\end{aligned}$$

This representation provides the solution to the second Dirichlet problem.

Theorem 2. *The second Dirichlet problem for the inhomogeneous biharmonic equation*

$$\begin{aligned} w_{z\bar{z}z\bar{z}} &= f \text{ in } \mathbb{D}, \\ w &= \gamma_0, w_{\bar{z}} = \gamma_1 \text{ on } \partial\mathbb{D} \end{aligned}$$

is uniquely solvable. The solution is

$$\begin{aligned} (11) \quad w(z) &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_0(\zeta) \left[g_1(z, \zeta) + \frac{\bar{z}\zeta}{(1-\bar{z}\zeta)^2} (1-|z|^2) \right] \frac{d\zeta}{\zeta} - \\ &\quad - \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_1(\zeta) g_1(z, \zeta) (1-|z|^2) \frac{d\zeta}{\zeta^2} - \\ &\quad - \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_2(z, \zeta) d\xi d\eta. \end{aligned}$$

Proof. Uniqueness is obvious. If there is a solution it has the representation (11). That (12) in fact provides a solution is shown by verification. At once $w = \gamma_0$ on $\partial\mathbb{D}$ is seen by the properties of the Poisson kernel and the second Green function. Differentiating (12) shows

$$\begin{aligned} w_{\bar{z}}(z) &= \frac{2}{2\pi i} \int_{\partial\mathbb{D}} \gamma_0(\zeta) \frac{1}{(1-\bar{z}\zeta)^3} (1-|z|^2) d\zeta + \\ &\quad + \frac{z}{2\pi i} \int_{\partial\mathbb{D}} \gamma_1(\zeta) \left[g_1(z, \zeta) \frac{1}{\zeta} - \frac{1-|z|^2}{(1-\bar{z}\zeta)^2} \right] \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_{2z}(z, \zeta) d\xi d\eta. \end{aligned}$$

This verifies the second boundary condition. While the boundary integrals in (12) are biharmonic functions the area integral provides a particular solution to the differential equation. This is seen from

$$\begin{aligned} &\partial_z^2 \partial_{\bar{z}} \left[-\frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_2(z, \zeta) d\xi d\eta \right] = \\ &= -\frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) \left[\frac{1}{\zeta-z} - \frac{\bar{\zeta}}{1-z\bar{\zeta}} - \frac{\bar{\zeta}}{(1-z\bar{\zeta})^2} (1-|\zeta|^2) \right] d\xi d\eta. \end{aligned}$$

As this is the Pompeiu operator, see e.g. [2], and an additional analytic function, thus

$$\partial_z^2 \partial_{\bar{z}}^2 \left[-\frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_2(z, \zeta) d\xi d\eta \right] = f(z).$$

Both kind of Dirichlet problems can be similarly solved for the inhomogeneous polyharmonic equation

$$(\partial_z \partial_{\bar{z}})^n w = f.$$

For the second kind problem this is done in [4]. The explicit form of the solution to the first Dirichlet problem is not yet worked out.

References

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