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Integral Means of Certain Analytic Functions

Tadayuki Sekine, Shigeyoshi Owa and Rikuo Yamakawa

Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

We introduce the analytic functions $f(z) = z + \sum\limits_{k=n+1}^{\infty} a_k z^k$ ($n \in \mathbb{N}$) and $p(z) = z + \sum\limits_{s=1}^{m} b_{sj-s+1} z^{sj-s+1}$ ($j \geq n+1; n \in \mathbb{N}$) in the open unit disk $\mathbb{U}$. By means of the subordination theorem of J.E. Littlewood, we shall investigate the integral means with coefficients inequalities of analytic functions $f(z)$ and $p(z)$. Some applications of the integral mean of $f(z)$ are considered.

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1 Introduction

Let $\mathcal{A}_n$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (\, n \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $p(z)$ the analytic function in $\mathbb{U}$ defined by

$$p(z) = z + \sum_{s=1}^{m} b_{sj-s+1} z^{sj-s+1} \quad (j \geq n+1; n \in \mathbb{N}).$$

We recall the concept of subordination between analyric functions. Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$.

The following subordination theorem will be required in our present investigation.

Theorem A.(Littledood[1]) *If* $f(z)$ *and* $g(z)$ *are analytic in* $\mathbb{U}$ *with* $f(z) \prec g(z)$, *then, for* $\mu > 0$ *and* $z = re^{i\theta}(0 < r < 1)$

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \leq \int_0^{2\pi} |g(z)|^{\mu} \, d\theta.$$

Applying the theorem of Littlewood above, H. Silverman[3] has considered the integral means of $f(z)$ and $p(z)$, which are analytic functions with negative coefficients of the case of $n = 1, m = 1$ and $j = 2$. Recently, S. Owa and T. Sekine[2] have shown the integral means of $f(z) \in \mathcal{A}_n$ and $p(z)$ for the case of $m = 2$ and 3. In the present paper, we aim at investigating some conditions of coefficients for integral means of $f(z) \in \mathcal{A}_n$ and $p(z)(m \geq 2)$.

2 Integral means for $f(z)$ and $p(z)$

We begin by proving the following Lemma.

Lemma 1. Let $P_m(t)$ denote the polynominal of degree $m(m \geq 2)$ of the form

$$P_m(t) = c_1 t^m - c_2 t^{m-1} - \cdots - c_{m-1} t^2 - c_m t - d \quad (t \geq 0)$$

where $c_i (i = 1, 2, \cdots, m)$ are arbitrary positive constant and $d \geq 0$. Then $P_m(t)$ has unique solution for $t > 0$. If we denote the solution by t_0, $P_m(t) < 0$ for $0 < t < t_0$ and $P_m(t) > 0$ for $t > t_0$.

Proof. We shall prove Lemma 1 by mathematical induction. In the case where $m = 2$, it is clear that $P_2(t)$ has unique solution t_0 for $t > 0$ and $P_2(t) < 0$ for $0 < t < t_0$ and $P_2(t) > 0$ for $t > t_0$. Next, assuming that it is valid for $P_m(t)$, we prove that it is valid for $P_{m+1}(t)$. If we put $t = t_0$ as the solution of $P'_{m+1}(t)$, by assumption of mathematical induction $P_{m+1}(t)$ is monotone increasing for $t > t_0$ and monotone decreasing for $0 < t < t_0$. Thus, since $P_{m+1}(0) = -d \leq 0$, $P_{m+1}(t_0) < 0$. On the other hand, there exists some $t_1 (t_1 > t_0)$ such that $P_{m+1}(t_1) > 0$, because of $P_{m+1}(t) \uparrow \infty$ as $t \uparrow \infty$. Therefore there exists unique solution $t_2 (t_0 < t_2 < t_1)$ of $P_{m+1}(t)$ by intermediate value theorem and monotone increasing state of $P_{m+1}(t)$ for $t > t_0$. It is cerear that $P_{m+1}(t) < 0$ for $0 < t < t_0$ and $P_{m+1}(t) > 0$ for $t > t_0$. This completes the proof of Lemma 1.

Our first result for integral means is contained in the following theorem.

Theorem 2.1. *Let the functions $f(z) \in \mathcal{A}_n$ and $p(z)(m \geq 2)$ satisfy*

$$\sum_{k=n+1}^{\infty} |a_k| \leq |b_{mj-m+1}| - \sum_{s=1}^{m-1} |b_{sj-s+1}|$$

with

$$\sum_{s=1}^{m} |b_{sj-s+1}| < |b_{mj-m+1}|.$$

If there exists an analytic function $w(z)$ in U defined by

$$\sum_{s=1}^{m} b_{sj-s+1} w(z)^{s(j-1)} - \sum_{k=n+1}^{\infty} a_k z^{k-1} = 0, \tag{2.1}$$

then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \leq \int_0^{2\pi} |p(z)|^{\mu} \, d\theta \quad (\mu > 0).$$

Proof. Putting $z = re^{i\theta}(0 < r < 1)$, it follows that

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta = r^{\mu} \int_0^{2\pi} \left| 1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} \right|^{\mu} d\theta$$

and

$$\int_0^{2\pi} |p(z)|^{\mu} d\theta = r^{\mu} \int_0^{2\pi} \left| 1 + \sum_{s=1}^{m} b_{sj-s+1} z^{sj-s} \right|^{\mu} d\theta.$$

Applying Theorem A, it would suffice to show that

$$1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} \prec 1 + \sum_{s=1}^{m} b_{sj-s+1} z^{sj-s}. \tag{2.2}$$

Let us define the function $w(z)$ by

$$
\begin{aligned}
1+\sum_{s=1}^{m} b_{sj-s+1}\left\{w(z)\right\}^{s(j-1)}
&= b_{mj-m+1}w(z)^{m(j-1)} + b_{(m-1)j-m+2}w(z)^{(m-1)(j-1)} \\
&\quad + b_{(m-2)j-m+3}w(z)^{(m-2)(j-1)} + \cdots + b_{2j-1}w(z)^{2(j-1)} \\
&\quad + b_j w(z)^{j-1} = 1+\sum_{k=n+1}^{\infty} a_k z^{k-1}.
\end{aligned}
$$

or, by

$$
\begin{aligned}
\sum_{s=1}^{m} b_{sj-s+1}\left\{w(z)\right\}^{s(j-1)}
&= b_{mj-m+1}w(z)^{m(j-1)} + b_{(m-1)j-m+2}w(z)^{(m-1)(j-1)} \\
&\quad + b_{(m-2)j-m+3}w(z)^{(m-2)(j-1)} + \cdots + b_{2j-1}w(z)^{2(j-1)} \\
&\quad + b_j w(z)^{j-1} = \sum_{k=n+1}^{\infty} a_k z^{k-1}.
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
w(0)^{j-1}\Big\{& b_{mj-m+1}w(0)^{(m-1)(j-1)} + b_{(m-1)j-m+2}w(0)^{(m-2)(j-2)} \\
&+ b_{(m-2)(j-m+3}w(0)^{(m-3)(j-3)} + \cdots + b_{2j-1}w(0)^{j-1} + b_j\Big\} = 0
\end{aligned}
$$

Therefore, if there exists an analytic functions $w(z)$ which satisfies the equality (2.1), we have an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0)=0$.

Further, we prove that the analytic function $w(z)$ satisfies $|w(z)|<1(z\in\mathbb{U})$ for

$$
\sum_{k=n+1}^{\infty}|a_k| \leq |b_{mj-m+1}| - \sum_{s=1}^{m}|b_{sj-s+1}| \quad \left(\sum_{s=1}^{m}|b_{sj-s+1}| < |b_{mj-m+1}|\right).
$$

From the equality (2.1), we know that

$$\begin{aligned}&\left|b_{mj-m+1}w(z)^{mj-m}+b_{(m-1)j-m+2}w(z)^{(m-1)j-(m-1)}\right.\\&\quad\left.+b_{(m-2)j-m+3}w(z)^{(m-2)j-(m-2)}+\cdots+b_{2j-1}w(z)^{2j-2}+b_jw(z)^{j-1}\right|\\&\qquad\leq\sum_{k=n+1}^{\infty}\left|a_kz^{k-1}\right|<\sum_{k=n+1}^{\infty}|a_k|\end{aligned}$$

for $z\in\mathbb{U}$, so that

$$\begin{aligned}&|b_{mj-m+1}|\left|w(z)^{mj-m}\right|-\left|b_{(m-1)j-m+2}\right|\left|w(z)^{(m-1)j-(m-1)}\right|\\&\quad-\left|b_{(m-2)j-m+3}\right|\left|w(z)^{(m-2)j-(m-2)}\right|+\cdots-|b_{2j-1}|\left|w(z)^{2j-2}\right|\\&\qquad-|b_j|\left|w(z)^{j-1}\right|-\sum_{k=n+1}^{\infty}|a_k|<0\end{aligned}$$

for $z\in\mathbb{U}$.

Putting $t=|w(z)|^{j-1}$ $(t\geq 0)$, we define the polynominal $P(t)$ of degree m by

$$\begin{aligned}P(t)=|b_{mj-m+1}|\,t^m-\left|b_{(m-1)j-m+2}\right|t^{m-1}-\left|b_{(m-2)j-m+3}\right|t^{m-2}\\-\cdots-|b_{2j-1}|\,t^2-|b_j|\,t-\sum_{k=n+1}^{\infty}|a_k|\end{aligned}$$

By means of Lemma 1, if $P(1)\geq 0$, we have $t<1$ for $P(t)<0$. Hence for $|w(z)|<1$ $(z\in\mathbb{U})$, we need the following inequality

$$\begin{aligned}P(1)=|b_{mj-m+1}|-\left|b_{(m-1)j-m+2}\right|-\left|b_{(m-2)j-m+3}\right|\\-\cdots-|b_{2j-1}|-|b_j|-\sum_{k=n+1}^{\infty}|a_k|\geq 0\end{aligned}$$

so that,

$$\sum_{k=n+1}^{\infty} |a_k| \leq |b_{mj-m+1}| - \sum_{s=1}^{m-1} |b_{sj-s+1}|.$$

Therefore the subordination in (2.2) holds true, and this evidently completes the proof of Theorem 1.

Corollary 2.1. *Let the functions* $f(z) \in \mathcal{A}_n$ *and* $p(z)(m \geq 2)$ *satisfy the conditions in* Theorem 1 *then , for* $0 < \mu \leq 2$ *and* $z = re^{i\theta}$ $(0 < r < 1)$,

$$\begin{aligned} \int_0^{2\pi} |f(z)|^\mu d\theta &\leq 2\pi r^\mu \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2 \, r^{2s(j-1)}\right)^{\frac{\mu}{2}} \\ &< 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2\right)^{\frac{\mu}{2}} . \end{aligned}$$

Proof. Since,

$$\int_0^{2\pi} |p(z)|^\mu d\theta = \int_0^{2\pi} |z|^\mu \left|1 + \sum_{s=1}^{m} b_{sj-s+1} \, z^{sj-s}\right|^\mu d\theta,$$

applying the inequality of Hölder for $0 < \mu < 2$, we obtain that

$$
\begin{aligned}
&\int_0^{2\pi} |p(z)|^{\mu} d\theta \\
&\leq \left(\int_0^{2\pi} (|z|^{\mu})^{\frac{2}{2-\mu}} d\theta \right)^{\frac{2-\mu}{2}} \left\{ \int_0^{2\pi} \left(\left| 1 + \sum_{s=1}^{m} b_{sj-s+1}\, z^{sj-s} \right|^{\mu} \right)^{\frac{2}{\mu}} d\theta \right\}^{\frac{\mu}{2}} \\
&= \left(r^{\frac{2\mu}{2-\mu}} \int_0^{2\pi} d\theta \right)^{\frac{2-\mu}{2}} \left(\int_0^{2\pi} \left| 1 + \sum_{s=1}^{m} b_{sj-s+1}\, z^{sj-s} \right|^{2} d\theta \right)^{\frac{\mu}{2}} \\
&= \left(2\pi r^{\frac{2\mu}{2-\mu}} \right)^{\frac{2-\mu}{2}} \left\{ 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2\, r^{2s(j-1)} \right) \right\}^{\frac{\mu}{2}} \\
&= 2\pi r^{\mu} \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2\, r^{2s(j-1)} \right)^{\frac{\mu}{2}} \\
&< 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2 \right)^{\frac{\mu}{2}}.
\end{aligned}
$$

In case of $\mu = 2$, it is easy to see that

$$
\begin{aligned}
\int_0^{2\pi} |f(z)|^2 d\theta &\leq 2\pi r^2 \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2\, r^{2s(j-1)} \right) \\
&< 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2 \right).
\end{aligned}
$$

This comples the proof of Corollary 2.1.

3 Integral means for $f'(z)$ and $p'(z)$

Using the same techniques in Theorem 2.1, we obtain the following theorem.

Theorem 2.2 *Let the functions $f(z) \in \mathcal{A}_n$ and $p(z)(m \geq 2)$ satisfy*

$$\sum_{k=n+1}^{\infty} k|a_k| \leq (mj-m+1)|b_{mj-m+1}| - \sum_{s=1}^{m-1}(sj-s+1)|b_{sj-s+1}|$$

with

$$(mj-m+1)|b_{mj-m+1}| > \sum_{s=1}^{m-1}(sj-s+1)|b_{sj-s+1}|.$$

If there exists an analytic function $w(z)$ in $\mathbb{U}$ defined by

$$\sum_{s=1}^{m}(sj-s+1)b_{sj-s+1}\{w(z)\}^{s(j-1)} - \sum_{k=n+1}^{\infty} ka_k z^{k-1} = 0,$$

then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |p'(z)|^\mu d\theta.$$

Futher, with the help of the inequality of Hölder, we obtain

Corollary 2. *If the functions $f(z) \in \mathcal{A}_n$ and $p(z)(m \geq 2)$ satisfy the conditions in* Theorem 2, *then for $0 < \mu \leq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\begin{aligned}\int_0^{2\pi} |f'(z)|^\mu d\theta &\leq 2\pi \left(1 + \sum_{s=1}^{m}(sj-s+1)^2|b_{sj-s+1}|^2 r^{2s(j-1)}\right)^{\frac{\mu}{2}} \\ &< 2\pi \left(1 + \sum_{s=1}^{m}(sj-s+1)^2|b_{sj-s+1}|^2\right)^{\frac{\mu}{2}}.\end{aligned}$$

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Tadayuki Sekine
Office of Mathematics, College of Pharmacy, Nihon University,
7-1 Narashinodai 7chome, Funabashi-shi,
Chiba 274-8555, Japan
E-mail:*tsekine@pha.nihon-u.ac.jp*

Shigeyoshi Owa
Department of Mathematics, Kinki University,
Higashi-Osaka, Osaka 577-8520, Japan
E-mail:*owa@math.kindai.ac.jp*

Rikuo Yamakawa
Department of Natural Sciences, Shibaura Institute of Technology,
Fukusaku(Omia), Saitama 330-8570, Japan
E-mail:*yamakawa@sic.shibaura-it.ac.jp*