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Inequalities for the Polygamma Functions with Application

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

We present some inequalities for the polygamma functions. As an application, we give the upper and lower bounds for the expression $\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma$, where $\gamma=0.57721...$ is the Euler's constant.

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1 Inequalities for the polygamma functions

The gamma function is usually defined for $Re\ z>0$ by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt.$$

The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{1+k} - \frac{1}{z+k}\right),$$

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}$$

for $Re\ z > 0$ and $n = 1, 2, ...$, where $\gamma = 0.57721...$ is the Euler's constant.

M. Merkle [2] established the inequality

$$\frac{1}{x} + \frac{1}{2x^2} + \sum_{k=1}^{2N} \frac{B_{2k}}{x^{2k+1}} < \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} <$$

$$< \frac{1}{x} + \frac{1}{2x^2} + \sum_{k=1}^{2N+1} \frac{B_{2k}}{x^{2k+1}}$$

for all real $x > 0$ and all integers $N \geq 1$, where B_k denotes Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} t^j.$$

The first five Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$$

The following Theorem establishes a more general result.

Theorem 1. *let $m \geq 0$ and $n \geq 1$ be integers, then we have for $x > 0$,*

(1) $$\ln x - \frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x) <$$

$$< \ln x - \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}}$$

and

$$\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}} < \tag{2}$$

$$< (-1)^{n+1}\psi^{(n)}(x) < \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}.$$

Proof. From Binet's formula [6, p. 103]

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}dt,$$

we conclude that

$$\psi(x) = \ln(x) - \frac{1}{2x} - \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t}dt \tag{3}$$

and therefore

$$(-1)^{n+1}\psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)t^{n-1}e^{-xt}dt. \tag{4}$$

It follows from Problem 154 in Part I, Chapter 4, of [3] that

$$\sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!}t^{2j} < \frac{t}{e^t - 1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \tag{5}$$

for all integers $m \geq 0$. the inequality (5) can be also found in [4].

From (3) and (5) we conclude (1), and we obtain (2) from (4) and (5). The proof of Theorem 1 is complete.

Note that $\psi(x+1) = \psi(x) + \frac{1}{x}$ (see [1, pag. 258]), (1) can be written as

$$\frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j}\frac{1}{x^{2j}} < \psi(x+1) - \ln x < \frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j}\frac{1}{x^{2j}}, \tag{6}$$

and (2) can be written as

$$(7)\qquad \frac{(n-1)!}{x^n}-\frac{n!}{2x^{n+1}}+\sum_{j=1}^{2m+1}\frac{B_{2j}}{2j}\frac{\Gamma(n+2j)}{x^{n+2j}}<$$

$$(-1)^{n+1}\psi^{(n)}(x+1)<\frac{(n-1)!}{x^n}-\frac{n!}{2x^{n+1}}+\sum_{j=1}^{2m+1}\frac{B_{2j}}{(2j)!}\frac{\Gamma(n+2j)}{x^{n+2j}}.$$

In particular, taking in (6) $m=0$ we obtain for $x>0$,

$$(8)\qquad \frac{1}{2x}-\frac{1}{12x^2}<\psi(x+1)-\ln x<\frac{1}{2x},$$

and taking in (7) $m=1$ and $n=1$ we obtain for $x>0$,

$$(9)\qquad \frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\frac{1}{42x^7}<\frac{1}{x}-\psi'(x+1)<$$

$$<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}.$$

The inequalities (8) and (9) play an important role in the proof of Theorem 2 in Section 2.

2 Inequalities for Euler's constant

Euler's constant $\gamma=0.57721...$ is defined by

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right).$$

It is of interest to investigate the bounds for the expression $\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma$. The inequality

$$\frac{1}{2n}-\frac{1}{8n^2}<\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma<\frac{1}{2n}$$

is called in literature Franel's inequality [3, Ex. 18].

It is given [1, p. 258] that $\psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma$, and then we get

$$\sum_{k=1}^{n-1} \frac{1}{k} - \gamma = \psi(n+1) - \ln n. \tag{10}$$

Taking in (6) $x = n$ we obtain that

$$\frac{1}{2n} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{n^{2j}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{n^{2j}}. \tag{11}$$

The inequality (11) provides closer bounds for $\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma$.

L. Toth [5, pag. 264] proposef the following problems:

(i) Prove that for every positive integers n we have

$$\frac{1}{2n + \frac{2}{5}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}.$$

(ii) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ cannot be replaced by a slightly larger number.

The following Theorem 2 answers the problem due to Tóth.

Theorem 2. *For every positive integers n,*

$$\frac{1}{2n + a} \le \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + b}, \tag{12}$$

with the posible constants

$$a = \frac{1}{1-\gamma} - 2 \text{ and } b = \frac{1}{3}.$$

Proof. By (10), the inequality (12) can be rearranged as

$$b < \frac{1}{\psi(n+1) - \ln n} - 2n \leq a.$$

Define for $x > 0$,

$$\phi(x) = \frac{1}{\psi(x+1) - \ln x} - 2x.$$

Differentiating ϕ and utilizing (8) and (9) reveals that for $x > \frac{12}{5}$,

$$(\phi(x+1) - \ln x)^2 \psi'(x) = \frac{1}{x} - \phi'(x+1) - 2(\psi(x+1) - \ln x)^2 <$$

$$< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2 = \frac{12 - 5x}{360x^5} < 0,$$

and then the function ϕ strictly decreases with $x > \frac{12}{5}$.

$$\phi(1) = \frac{1}{1-\gamma} - 2 = 0.3652721186544155...,$$

$$\phi(2) = \frac{1}{\frac{3}{2} - \gamma - \ln 2} - 4 = 0.35469600731465752...,$$

$$\phi(3) = \frac{1}{\frac{11}{6} - \gamma - \ln 3} - 6 = 0.34898948531361115... \,.$$

Therefore, the sequence

$$\phi(n) = \frac{1}{\psi(n+1) - \ln n} - 2n, \ n \in \mathbb{N}$$

is strictly decreasing. This leads to

$$\lim_{n \to \infty} \phi(n) < \phi(n) \leq \phi(1) = \frac{1}{1-\gamma} - 2.$$

Making use of asymptotic formula of ψ (see [1, pag. 259])

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4})(x \to \infty),$$

we conclude that

$$\lim_{n\to\infty} \phi(n) = \lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{\frac{1}{3} + O(x^{-2})}{1 + O(x^{-1})}.$$

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