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Some Remarks about Mastrogiacomo Cohomolgy

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

We prove the isomorphism between the Mastrogiacomo cohomology group and the 1-dimensional Čech cohomology group with real coefficients of a differential manifold, using a fine resolution of the constant sheaf R. Some interesting results are obtained if the manifold is foliated.

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1 Mastrogiacomo Cohomology

Following [2], we present the Mastrogiacomo cohomology on a differentiable manifold.

Let M be a n-dimensional paracompact manifold and denote by $\Omega^0(M)$ the set of differentiable functions on M.

We recall that two maps $f, g \in \Omega^0(M)$ determine the same *2-jet at* $x \in M$ if $f(x) = g(x) = 0$ and if for every curve $\gamma : R \to M$ with $\gamma(0) = x$, the curves $f \circ \gamma$, $g \circ \gamma$ have a second order contact at zero. If, moreover, the maps f and g have the first derivative in x equal to zero, then they determine the same *homogeneous 2-jet at* x.

The *2-jet of* f *at* x is denoted by $j_x^2 f$. Obviously, $j_x^2 f$ depends on the germ of f at x only.

Let $(U, (x^i)), (\tilde{U}, (\tilde{x}^{i_1}))$ be two local charts with $U \cap \tilde{U} \neq \emptyset$. More generally, a *2-jet at* $x \in M$ is a combination locally given by

$$\omega_x = \omega_i(x) \cdot j_x^2 x^i + \frac{1}{2}\omega_{ij}(x) \cdot j_x^2 x^i \cdot j_x^2 x^j, \tag{1}$$

with $\omega_{ij} = \omega_{ji}$ and where the coefficient functions are satisfying the following conditions in $U \cap \tilde{U}$

$$\tilde{\omega}_i = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}; \quad \tilde{\omega}_{i_1 i_2} = \omega_{j_1 j_2} \frac{\partial x^{j_1}}{\partial \tilde{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \tilde{x}^{i_2}} + \omega_j \frac{\partial^2 x^j}{\partial \tilde{x}^{i_1} \partial \tilde{x}^{i_2}}. \tag{2}$$

The sets of all 2-jets, homogeneous 2-jets at x are denoted by $J_x^2 M$ and $\Theta_x^2(M)$, respectively. The space $J^2 M = \cup_{x \in M} J_x^2(M)$ of 2-jets on M is a fiber bundle over M with the fiber of dimension $\frac{n(n+3)}{2}$ and we denote by $J^2(M)$ its sections, namely the *space of fields of* 2*-jets* on M. The space $\Theta^2 M = \cup_{x \in M} \Theta_x^2(M)$ of homogeneous 2-jets on M is also a fiber bundle over M with the fiber of dimension $\frac{n(n+1)}{2}$ and we denote by $\Theta^2(M)$ its sections, namely the *space of homogeneous fields of* 2*-jets* on M.

Let $j^2 : \Omega^0(M) \longrightarrow J^2(M)$ be the map who assigns to a function f the field $j^2 f$ of 2-jets of f on M. This map is called the *second differential* on M. In local coordinates we have

$$j^2 f = \frac{\partial f}{\partial x^i} j^2 x^i + \frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j} j^2 x^i \cdot j^2 x^j. \tag{3}$$

The field of 2-jets $\omega \in J^2(M)$ is called, (see [2]):

(i) *exact* if $\omega = j^2 f$ for some $f \in \Omega^0(M)$;

(ii) *closed* if it is locally exact.

We denote by $E^2(M)$, $C^2(M)$ the space of exact and closed fields of 2-jets, respectively. We have $E^2(M) \subset C^2(M)$ and the *Mastrogiacomo cohomology group of* M is the following quotient group:

$$H^2_j(M) = C^2(M)/E^2(M). \tag{4}$$

It is isomorphic with the 1-dimensional cohomology group with real coefficients of M ([2], theorem II).

2 A new definition of Mastrogiacomo cohomology

In this section we define an operator μ on $J^2(M)$ which allows us to define the cohomology group (4) in a natural way.

Let be the following map

$$\mu : J^2(M) \to \Theta^2(M),$$

locally given in a local chart $(U,(x^i))$ by

$$\mu(\omega)=\frac{1}{2}\left[\omega_{ij}-\frac{1}{2}\left(\frac{\partial\omega_i}{\partial x^j}+\frac{\partial\omega_j}{\partial x^i}\right)\right]j^2x^i\cdot j^2x^j, \tag{5}$$

where ω is given by (1). A simple calculation shows that the functions

$$\lambda_{ij}=\left[\omega_{ij}-\frac{1}{2}\left(\frac{\partial\omega_i}{\partial x^j}+\frac{\partial\omega_j}{\partial x^i}\right)\right],$$

satisfy (2), so $\mu(\omega)$ is an homogeneous field of 2-jets on M. This map is also a surjective morphism. Indeed, for every homogeneous field of 2-jets on M, let it be ϖ, we have $\varpi=\mu(\varpi)$. From here, we can see that the restriction of μ at the subspace $\Theta^2(M)$ of the domain $J^2(M)$, is just the identity map on $\Theta^2(M)$.

Moreover, the morphism μ has the following important property:

$$Imj^2\subset Ker\mu, \tag{6}$$

which follows easy from the relations (3) and (5). Hence, the sequence

$$O\to\Omega^0(M)\xrightarrow{j^2}J^2(M)\xrightarrow{\mu}\Theta^2(M)\to O, \tag{7}$$

is a semiexact one. We can remark that in sequence (7) we have

$$Imj^2=E^2(M). \tag{8}$$

We wish to define the Mastrogiacomo cohomology group as a cohomology group of a certain semiexact sequence. For this reason, we give the following definition:

Definition 1. *A field $\omega\in J^2(M)$ is called admissible if in a local chart $(U,(x^i))$ its coefficient functions ω_i from (1) verify*

$$\frac{\partial\omega_i}{\partial x^j}=\frac{\partial\omega_j}{\partial x^i}, \tag{9}$$

for every $i, j = \overline{1, n}$.

Taking into account the relations (2), the conditions from the previous definition have geometrical meaning. We shall denote by $J_a^2(M)$ the space of all admissible fields of 2-jets on M. We have the following sequence of subspaces:

$$\Theta^2(M) \subset J_a^2(M) \subset J^2(M),$$

and the inclusion

$$Imj^2 \subset J_a^2(M). \tag{10}$$

If we consider the canonical projection of fields of 2-jets on 1-forms

$$\pi : J^2(M) \to \Omega^1(M),$$

locally given by

$$\pi\left(\omega_i \cdot j^2x^i + \frac{1}{2}\omega_{ij} \cdot j^2x^i \cdot j^2x^j\right) = \omega_i dx^i,$$

it is easy to verify the following assertion:

Proposition 1. *We have the equality*

$$J_a^2(M) = \pi^{-1}(Kerd), \tag{11}$$

where d *is the exterior derivative on* M.

Let μ_a be the restriction of μ at $J_a^2(M)$. From the relations (6) and (10) results that the following sequence is also a semiexact one:

$$O \to \Omega^0(M) \xrightarrow{j^2} J_a^2(M) \xrightarrow{\mu_a} \Theta^2(M) \to O. \tag{12}$$

The following assertion is a *lema Poincaré* type proposition.

Proposition 2. *If ω is an admissible field of 2-jets on M such that $\mu_a(\omega) = 0$, then, for every open subset U of M, there exists a differentiable function f_U which satisfies $\omega = j^2 f_U$ in U. In other words, we could say that we have the equality*

$$Ker\mu_a = C^2(M). \tag{13}$$

Proof. Let $\omega \in J_a^2(M)$, $\mu_a(\omega) = 0$. Taking into account the locally forms (1), (5), and the conditions (9), in a local chart $(U, (x^i))$ we have the following system:

$$\begin{aligned} \omega &= \omega_i \cdot j^2 x^i + \frac{1}{2}\omega_{ij} \cdot j^2 x^i \cdot j^2 x^j, \\ \frac{\partial \omega_i}{\partial x^j} &= \frac{\partial \omega_j}{\partial x^i}, \\ \omega_{ij} &= \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i}\right), i, j = \overline{1, n}. \end{aligned}$$

which has a locally solution, $\omega = j^2 f_U$ in U, with $f_U \in \Omega^0(U)$. So, $Ker\mu_a \subset C^2(M)$. For the reverse inclusion, let ω be a locally exact field of 2-jets on M. Then, in every open domain U there exists $f_U \in \Omega^0(U)$ such that $\omega = j^2 f_U$ in U. Hence ω is admissible and in every U we have $\mu(\omega) = 0$. It results that $\omega \in Ker\mu_a$.

From the previous proposition and the relation (8) we obtain that the Mastrogiacomo cohomology group defined in (4) is just the first cohomology group of the semiexact sequence (12). So, we could consider this one to be the definition of the Mastrogiacomo cohomology.

3 A fine resolution for the constant real sheaf

We shall prove the isomorphism between $H_j^2(M)$ and the 1-dimensional Čech cohomology group with real coefficients of M, using the new definition of the Mastrogiacomo cohomology from the previous section. For this reason, we see that the semiexact sequence (12) determine a semiexact sequence of sheaves

$$O \to \tilde{R} \xrightarrow{i} \Omega^0 \xrightarrow{j^2} J_a^2 \xrightarrow{\mu_a} \Theta^2, \tag{14}$$

where $\tilde{R}$ is the sheaf associated to the presheaf of germs of locally constant functions on M, Ω^0 is the sheaf of germs of differentiable functions, Θ^2 is the sheaf of germs of sections of the fiber bundle $\Theta^2(M)$, and J_a^2 is the sheaf which assigns to every open subset V of M the germs of the admissible fields of 2-jets on V. Taking into account that M is paracompact and the proposition 2.2, there exists a base of topology B_M on M such that

$$O \to \tilde{R}(U) \xrightarrow{i} \Omega^0(U) \xrightarrow{j^2} J_a^2(U) \xrightarrow{\mu_a} \Theta^2(U),$$

is an exact sequence for every $U \in \mathrm{B}_M$. Hence, the sequence of shaves (14) is an exact one. Unfortunately, the sheaf J_a^2 is not fine, so this sequence is not a fine resolution for $\tilde{R}$.

We shall construct another semiexact sequence of groups which has the property that one of its cohomology group is just $H_j^2(M)$.

The sections of the fiber bundles product $\Theta^2 M \times A_2(TM;R)$ are the elements of the direct product $\Theta^2(M) \times \Omega^2(M)$, where $\Omega^2(M)$ is the space of 2- differential forms on M. We have the following morphism of groups

$$\tau : J^2(M) \to \Theta^2(M) \times \Omega^2(M), \tau(\omega) = (\mu(\omega), (d \circ \pi)(\omega)), \tag{15}$$

where $\omega \in J^2(M)$, μ is the map defined in (5), d is the exterior derivative and π is the projection of fields of 2-jets on 1-forms. For every differentiable function f on M we have

$$\tau\left(j^2 f\right) = \left(\mu\left(j^2 f\right), (d \circ \pi)\left(j^2 f\right)\right) = \left(\left(\mu \circ j^2\right) f, d(df)\right) = (0,0),$$

because $d^2 = 0$ and $\mu \circ j^2 = 0$ (see relation (6)). It follows that the following sequence is semiexact:

$$O \to \Omega^0(M) \xrightarrow{j^2} J^2(M) \xrightarrow{\tau} \Theta^2(M) \times \Omega^2(M). \tag{16}$$

Proposition 3. *There is the following equality:*

$$Ker\tau = Ker\mu_a.$$

Proof. We have

$$\begin{aligned} Ker\tau &= \left\{\omega \in J^2(M), \mu(\omega) = 0, (d \circ \pi)(\omega) = 0\right\} = \\ &= \left\{\omega \in J^2(M), \omega \in Ker\mu, \pi(\omega) \in Kerd\right\} = Ker\mu \cap \pi^{-1}(Kerd). \end{aligned}$$

Taking into account the proposition 2.1, results $Ker\ \tau = Ker\ \mu \cap J_a^2(M)$, so we obtained $Ker\ \tau = Ker\ \mu_a$.

From the previous result and the proposition 2.1, it follows that

$$Ker\tau = C^2(M). \tag{17}$$

The above equality has two consequences. The first one is that the Mastrogiacomo cohomology group is the first cohomology group of the semiexact sequence (16)

$$H_j^2(M) = Ker\tau/Imj^2. \tag{18}$$

The second one is that locally we have $Ker\tau = Imj^2$, which means that the following sequence of sheaves is an exact one:

$$O \to \tilde{R} \xrightarrow{i} \Omega^0 \xrightarrow{j^2} J^2 \xrightarrow{\tau} \Theta^2 \times \Omega^2, \tag{19}$$

where J^2 is the sheaf of germs of sections of the fiber bundle J^2M, and Ω^2 is the sheaf of germs of differential 2-forms on M. It is well-known that the sheaf of germs of sections of a bundle is a fine sheaf, so J^2, Θ^2 and Ω^2 are fine sheaves. Then $\Theta^2 \times \Omega^2$ is also a fine sheaf. The sheaf Ω^0 is fine, too. So we have the main theorem of this paper:

Theorem 1. *The sequence (19) is a fine resolution for the sheaf $\tilde{R}$.*

From the theorem 3.1 results, (see [6]).

Proposition 4. *The first cohomology group of the sequence (16) is isomorphic to the 1-dimensional Čech cohomology group with coefficients in the sheaf $\tilde{R}$.*

From the above proposition and the relations (8) and (17), we obtained the isomorphism between $H_j^2(M)$ and the 1-dimensional Čech cohomology group with real coefficients.

4 The case of a foliated manifold

We introduced in [5], the basic and leafwise 2-jets on a foliated manifold M, and the basic and leafwise Mastrogiacomo cohomology, respectively.

Let $\mathcal{F}$ be a m-dimensional foliation on the n-dimensional manifold M. The number $p = n - m$ is called the codimension of the foliation. On the foliated manifold $(M, \mathcal{F})$ there exists an atlas which charts are called adapted to foliation and has the following form:

$$\left(U, (x^a, x^u)_{a=\overline{1,p};u=\overline{p+1,m}}\right). \tag{20}$$

The leaves of the foliation are locally given by the equations $x^a = const.$ A differentiable function $f \in \Omega^0(M)$ is basic if it is constant on the leaves, which means that in every adapted chart it depends only by the first p coordinates. The ring of basic functions is denoted by $\Phi(M)$.

Definition 2.(see [5]) *The field $\omega \in J^2(M)$ is called basic if in an adapted chart it has the form*

$$\omega = \omega_a \cdot j^2x^a + \frac{1}{2}\omega_{ab} \cdot j^2x^a \cdot j^2x^b, \tag{21}$$

and $\omega_a, \omega_{ab} \in \Phi(M)$.

It is proved that the above conditions have geometrical meaning. We denoted the space of fields of basic 2-jets on M by $J^{b,2}(M/\mathcal{F})$. The space of fields of homogeneous basic 2-jets is denoted by $\Theta^{b,2}(M/\mathcal{F})$.

It is interesting now to restrict the sequence (16) to the space of basic functions. For every $f \in \Phi(M)$, j^2f is locally given in an adapted chart

(20) by

$$j^2 f = \frac{\partial f}{\partial x^a} j^2 x^a + \frac{1}{2} \frac{\partial^2 f}{\partial x^a \partial x^b} j^2 x^a \cdot j^2 x^b,$$

and $\frac{\partial f}{\partial x^a}$, $\frac{\partial^2 f}{\partial x^a \partial x^b}$ are also basic functions, so $j^2 f \in J^{b,2}(M/\mathcal{F})$. For every field $\omega \in J^{b,2}(M/\mathcal{F})$, we have in an adapted chart (20)

$$\begin{aligned} \tau(\omega) &= (\mu(\omega), (d \circ \pi)(\omega)) = \\ &= \left(\frac{1}{2} \left(\omega_{ab} - \frac{1}{2} \left(\frac{\partial \omega_b}{\partial x^a} + \frac{\partial \omega_a}{\partial x^b} \right) \right) \cdot j^2 x^a \cdot j^2 x^b, d(\omega_a dx^a) \right), \end{aligned}$$

so $\tau(\omega) \in \Theta^{b,2}(M/\mathcal{F}) \times \Omega^2(M/\mathcal{F})$, where $\Omega^2(M/\mathcal{F})$ is the space of basic 2-forms on M. Hence, we obtained the following semiexact sequence:

$$O \to \Phi(M) \xrightarrow{j^2} J^{b,2}(M/\mathcal{F}) \xrightarrow{\tau} \Theta^{b,2}(M/\mathcal{F}) \times \Omega^2(M/\mathcal{F}),$$

which first cohomology group is exactly the basic Mastrogiacomo cohomology defined in [5]. Indeed, $\mathrm{Im} j^2$ is the group of exact fields of basic 2-jets, and a lema Poincaré type proposition could be proved: $Ker\tau$ is the group of the locally exact fields of basic 2-jets on M.

The leafwise 2-jets of a foliated manifold are defined as the equivalence classes of differentiable functions which determine the same 2-jet on leaves. We denoted by $J^{l,2}(M)$ the bundle of leafwise 2-jets on M, and by $J^2(\mathcal{F})$ the space of sections of this bundle. The locally expression on a generally field of leafwise 2-jet in an adapted chart (20) is

$$\omega^l = \omega_u \cdot j^{l,2} x^u + \frac{1}{2} \omega_{uv} \cdot j^2 x^u \cdot j^2 x^v, \tag{22}$$

where $j^{l,2} : \Omega^0(M) \to J^2(\mathcal{F})$ is locally defined by

$$j^{l,2} f = \frac{\partial f}{\partial x^u} j^{l,2} x^u + \frac{1}{2} \frac{\partial^2 f}{\partial x^u \partial x^v} j^{l,2} x^u \cdot j^{l,2} x^v.$$

It easy to see that the restriction of the sequence (16) at leaves determine the following semiexact sequence:

$$O \to \Omega^0(M) \overset{j^{l,2}}{\to} J^2(\mathcal{F}) \overset{\tau_1}{\to} \Theta^2(\mathcal{F}) \times \Omega^{0,2}(M), \tag{23}$$

where $\Theta^2(\mathcal{F})$ is the space of fields of homogeneous leafwise 2-jets, $\Omega^{0,2}(M)$ is the space of (0,2)-forms on M (see for instance [9]), and the map τ_1 is locally defined as it follows:

$$\tau_1\left(\omega^l\right) = \left(\frac{1}{2}\left(\omega_{uv} - \frac{1}{2}\left(\frac{\partial \omega_u}{\partial x^v} + \frac{\partial \omega_v}{\partial x^u}\right)\right) \cdot j^{l,2}x^u \cdot j^{l,2}x^v, d_{01}\left(\omega_u \theta^u\right)\right). \tag{24}$$

We remind that d_{01} is the foliated derivative and it acts on $(0,q)$-forms, with result in the space of $(0,q+1)$-forms. In (24), the 1-forms $\{\theta^u\}$ are the components of an adapted local cobasis $\{dx^a, \theta^u\}$ in the adapted chart $(U, (x^a, x^u))$.

Following an analogous argument as in the previous section, we obtain a fine resolution of the sheaf Φ of germs of basic functions on M:

$$O \to \Phi \to \Omega^0 \overset{j^{l,2}}{\to} J^2 \overset{\tau_1}{\to} \Theta^2 \times \Omega^{0,2}.$$

Then results the isomorphism between the 1-dimensional Čech cohomology group with coefficients in the sheaf Φ, and the group $Ker\tau_1 \;/\; Imj^{l,2}$. Moreover the first cohomology group of the sequence (23) is just the leafwise Mastrogiacomo cohomology from [5]. Hence we have:

Theorem 2. *The leafwise Mastrogiacomo cohomology group is isomorphic with the 1-dimensional Čech cohomology group with coefficients in Φ.*

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