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A Logarithmically Completely monotonic Function Involving the Gamma Functions [1]

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In memoriam of Associate Professor Ph. D. Luciana Lupa's

Abstract

We show that the function $x \to \frac{[\Gamma(x+1)]^{1/x}}{x[\Gamma(x+2)]^{1/(x+1)}}$ is logarithmically completely monotonic on $(0,\infty)$. This answers a question by A.Vernescu.

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1 Introduction

The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \quad (x>0)$$

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is one of the most important functions in analysis and its applications. The history and the development of this function are described in detail in [12]. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [16, p. 16] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\,\mathrm{d}t,$$
$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt}\,\mathrm{d}t$$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286\dots$ is the Euler-Mascheroni constant.

There exists a very extensive literature on these functions. In particular, inequalities, monotonicity and complete monotonicity properties for these functions have been published. Please refer to the papers [1, 2, 3] and the references therein.Recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \ge 0 \tag{1}$$

for $x \in I$ and $n \ge 0$. Let $\mathcal{C}$ denote the set of completely monotonic functions.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \ge 0 \tag{2}$$

for $k \in \mathbb{N}$ on I. Let $\mathcal{L}$ on $(0, \infty)$ stand for the set of logarithmically completely monotonic functions.

A function f on $(0, \infty)$ is called a Stieltjes transform if it can be written in the form

$$f(x) = a + \int_0^\infty \frac{\mathrm{d}\mu(s)}{s + x}, \tag{3}$$

where a is a nonnegative number and μ a nonnegative measure on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\mu(s) < \infty.$$

The set of Stieltjes transforms is denoted by $\mathcal{S}$.

The notion "logarithmically completely monotonic function" was posed explicitly in [19] and published formally in [18] and a much useful and meaningful relation $\mathcal{L} \subset \mathcal{C}$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in [18, 19]. Motivated by the papers [19, 20], among other things, it is proved in [8] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L} \subset \mathcal{C}$. The class of logarithmically completely monotonic functions can be characterized as the infinitely divisible completely monotonic functions which are established by Horn in [14, Theorem 4.4] and restated in [8, Theorem 1.1]. There have been a lot of literature about the (logarithmically) completely monotonic functions, for example, [4, 5, 7, 8, 9, 10, **?**, 13, 15, 18, 19, 20, 21] and the references therein.

When studying a problem on upper bound for permanents of $(0,1)$–matrices, in 1964 H. Minc and L. Sathre [17] discovered several noteworthy inequalities involving $(n!)^{1/n}$. Their main result states: If $\phi(n) = (n!)^{1/n}$, then

$$1 < n\frac{\phi(n+1)}{\phi(n)} - (n-1)\frac{\phi(n)}{\phi(n-1)} \tag{4}$$

holds for all integerers $n \geq 2$. To prove the inequality (4), they established the function

$$h(x) = x\frac{[\Gamma(x+2)]^{1/(x+1)}}{[\Gamma(x+1)]^{1/x}} \tag{5}$$

is strictly concave on $[6,\infty)$. In [6] A.Vernescu note that that h is logarithmically concave, but did not give its proof. We here consider logarithmically complete monotonicity of the function $1/h$.

Theorem 1.1. *Let the function h defined by (5), then $1/h$ is logarithmically completely monotonic in $(0,\infty)$.*

2 Lemma

Lema 2.1. *The function $f(x) = \frac{1}{[\Gamma(x+1)]^{1/x}}$ is logarithmically completely monotonic in $(0,\infty)$.*

Proof. Using Leibniz' rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$\begin{aligned}(\ln f(x))^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{x}\right)^{(k)} (-\ln\Gamma(x+1))^{(n-k)} \\ &= -\frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \psi^{(n-k-1)}(x+1) \qquad (6) \\ &\triangleq -\frac{1}{x^{n+1}} g(x).\end{aligned}$$

$$g'(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k!(n-k) x^{n-k-1} \psi^{(n-k-1)}(x+1) +$$

$$+ \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \psi^{(n-k)}(x+1) =$$

$$= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k!(n-k) x^{n-k-1} \psi^{(n-k-1)}(x+1) +$$

$$+ x^n \psi^{(n)}(x+1) + \sum_{k=1}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \psi^{(n-k)}(x+1) =$$

$$= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k!(n-k) x^{n-k-1} \psi^{(n-k-1)}(x+1) +$$

$$+ x^n \psi^{(n)}(x+1) + \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1} (k+1)! x^{n-k-1} \psi^{(n-k-1)}(x+1) =$$

$$= \sum_{k=0}^{n-1} \left[\binom{n}{k}(n-k) - \binom{n}{k+1}(k+1) \right] (-1)^k k! x^{n-k-1} \psi^{(n-k-1)}(x+1) +$$

$$+ x^n \psi^{(n)}(x+1) = x^n \psi^{(n)}(x+1) =$$

$$= x^n (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-(x+1)t} \, \mathrm{d}t.$$

If n is odd, then for $x > 0$,

$$g'(x) > 0 \Longrightarrow g(x) > g(0) = 0 \Longrightarrow (\ln f(x))^{(n)} < 0 \Longrightarrow$$

$$\Longrightarrow (-1)^n (\ln f(x))^{(n)}(x) > 0.$$

If n is even, then for $x > 0$,

$$g'(x) < 0 \Longrightarrow g(x) < g(0) = 0 \Longrightarrow (\ln f(x))^{(n)} > 0 \Longrightarrow$$

$$\Longrightarrow (-1)^n (\ln f(x))^{(n)}(x) > 0.$$

Hence,

$$(-1)^n (\ln f(x))^{(n)}(x) > 0 \tag{7}$$

for all real $x \in (0, \infty)$ and all integers $n \geq 1$. The proof is complete.

3 Proofs of theorems

It has been shown [18] that the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ is logarithmically completely monotonic on $(0, \infty)$. By Lemma 2.1, the function $\frac{1}{[\Gamma(x+1)]^{1/x}}$ is logarithmically completely monotonic in $(-1, \infty)$. From Leibniz' rule

$$(-1)^n [u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k u^{(k)}(x) (-1)^{n-k} v^{(n-k)}(x),$$

it is easy to see that the product of logarithmically completely monotonic functions is also logarithmically completely monotonic. Hence, the function

$$\frac{1}{h(x)} = \frac{[\Gamma(x+1)]^{1/x}}{x} \frac{1}{[\Gamma(x+2)]^{1/(x+1)}} \tag{8}$$

is logarithmically completely monotonic on $(0,\infty)$. The proof of Theorem 1.1 is complete.

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