General Mathematics Vol. 15, No. 1 (2007), 35–40

On Some Limits and Series Arising From Semigroup Theory [1]

Sorin G. Gal

Dedicated to Professor Alexandru Lupaş on the occasion of his 65th birthday anniversary and to the memory of Professor Luciana Lupaş

Abstract

In this note we consider some interesting limits and series arising from the theory of semigroups of linear operators on non-locally convex spaces (p-Fréchet spaces, $0 < p < 1$).

2000 Mathematics Subject Classification: 40A05, 40A30.

Key words and phrases: Limits, p-elementary functions, $0 < p < 1$.

1 Introduction

In the proof of the well-known Cernoff's product formula in semigroup theory on Banach spaces, a key result is the following inequality (see [1])

$$\frac{\sum_{k=0}^{\infty} \frac{n^k}{k!}|k-n|}{ne^n} \leq \frac{1}{\sqrt{n}},$$

[1] *Received 19 December, 2006*

Accepted for publication (in revised form) 9 January, 2007

which obviously implies

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{\infty} \frac{n^k}{k!}|k-n|}{ne^n} = 0.$$

In order to obtain a Cernoff-type formula in the theory of semigroups on p-Fréchet spaces, $0 < p < 1$, in the very recent paper [2], we had to prove that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{\infty}[n^k/k!]^p|k-n|}{n^p e^{np}} = 0,$$

for every $0 < p < 1$.

In Section 2 we reproduce the elegant proof in [2] of this limit and we consider an open question concerning more general type of limits suggested by this one.

Suggested by the same paper [2], Section 3 contains simple considerations on some p-series with $0 < p \leq 1$, which for $p = 1$ define well-known elementary real functions of real variable.

2 Limits

We present

Theorem 2.1. ([2]) *For every* $0 < p < 1$ *it follows*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{\infty}[n^k/k!]^p|k-n|}{n^p e^{np}} = 0.$$

Proof. Since the proof is elegant and might be useful in the proofs of more general limits, we reproduce it below.

Let $r \geq 2$ be an arbitrary even number. We will prove that the above limit is equal to 0, for any $\frac{1}{r} < p < 1$, which obviously implies that the above limit is equal to 0 for any $0 < p < 1$. Denote by s the conjugate of r, i.e. $\frac{1}{r} + \frac{1}{s} = 1$,$(s = \frac{r}{r-1})$,

$$\gamma(n) = \sum_{k=0}^{+\infty} \left(\frac{n^k}{k!}\right)^{\frac{pr-1}{r-1}},$$

and

$$F(n) = \sum_{k=0}^{\infty} [n^k/k!]^p |k-n| =$$

$$\sum_{k=0}^{\infty} [n^k/k!]^{p-\frac{1}{r}} (n^k/k!)^{1/r} [(n-k)^r]^{1/r}.$$

Applying now the Hölder's inequality to $F(n)$, we obtain

$$F(n) \leq \left(\sum_{k=0}^{\infty} [n^k/k!]^{(p-\frac{1}{r})s}\right)^{1/s} \left(\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^r\right)^{1/r} =$$

$$(\gamma(n))^{\frac{r-1}{r}} \left(\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^r\right)^{1/r}.$$

It is obvious that $\gamma(0) = 1$. Then, considering n as a real variable and differentiating with respect to n, by simple calculations we get

$$\gamma'(n) = \frac{pr-1}{r-1} \sum_{k=1}^{+\infty} \left(\frac{n^k}{k!}\right)^{\frac{pr-1}{r-1}-1} \frac{kn^{k-1}}{k!} \leq$$

$$\frac{pr-1}{r-1} n^{\frac{pr-r}{r-1}} \gamma(n).$$

Integrating this differential inequality with respect to n (from 0 to n), we easily arrive at the inequality

$$\gamma(n) \leq e^{n^{(pr-1)/(r-1)}},$$

for all $n \in \mathbb{N}$.

Therefore,

$$0 < \frac{F(n)}{(ne^n)^p} \leq \frac{[\gamma(n)]^{(r-1)/r}\left(\sum_{k=0}^{+\infty}\frac{n^k}{k!}(n-k)^r\right)^{1/r}}{(ne^n)^p}.$$

But it is easy to show that

$$\sum_{k=0}^{+\infty}\frac{n^k}{k!}(n-k)^r = e^n P_r(n),$$

where $P_r(n)$ is a polynomial in n of degree at most r, which implies

$$0 < \frac{F(n)}{(ne^n)^p} \leq \frac{[\gamma(n)]^{(r-1)/r}[P_r(n)]^{1/r}e^{n/r}}{(ne^n)^p} \leq$$

$$e^{\frac{r-1}{r}n^{(pr-1)/(r-1)}}\frac{[P_r(n)]^{1/r}e^{n/r}}{(ne^n)^p}.$$

But for sufficiently large n we have

$$\frac{r-1}{r}n^{\frac{pr-1}{r-1}} + \frac{n}{r} - np < 0,$$

(actually the left-hand side tends to $-\infty$ with $n \to +\infty$), which immediately implies

$$\lim_{n\to+\infty}\frac{F(n)}{(ne^n)^p} = 0,$$

and the theorem is proved.

Remark 2.1. *Would be interesting to find for every* $0 < p < 1$, *a concrete sequence (the best if it is possible)* $(A_n(p))_{n\in\mathbb{N}}$, *with* $lim_{n\to\infty}A_n(p) = 0$, *such that to have*

$$\frac{\sum_{k=0}^{\infty}[n^k/k!]^p|k-n|}{n^pe^{np}} \leq A_n(p), \textit{for all } n \in \mathbb{N}.$$

Note that for $p = 1$, *by [1] we have* $A_n(1) = \frac{1}{\sqrt{n}}$, $n \in \mathbb{N}$.

Remark 2.2. *Theorem 2.1 suggests to define the more general expressions*

$$E_n(p,q,\beta,\gamma) = \frac{\sum_{k=0}^{\infty}[n^k/k!]^p|k-n|^q}{n^\beta e^{n\gamma}},$$

with $0 < p, q, \beta, \gamma$. *It is an open question to consider and calculate (if exist) the limits* $lim_{n\to\infty} E_n(p,q,\beta,\gamma)$, *for all the possible situations between* p, q, β *and* γ. *Note that Theorem 2.1 (together with [1] for* $p = 1$*) states that* $lim_{n\to\infty} E_n(p,1,p,p) = 0$, *for all* $0 < p \le 1$.

3 p-Series, $0 < p < 1$

Suggested by the considerations in [2], we can introduce the following functions.

Definition 3.1. *For any fixed* $0 < p \le 1$, *the p-functions*

$$exp_p(x) = \sum_{k=0}^{\infty}\left(\frac{x^k}{k!}\right)^p,$$

$$cos_p(x) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{x^{2k}}{(2k)!}\right)^p,$$

$$sin_p(x) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{x^{2k+1}}{(2k+1)!}\right)^p,$$

will be called p-exponential, p-cosine and p-sine function, respectively. For $p = 1$, *the above series define the classical exponential, cosine and sine, respectively.*

Remark 3.1. *Of course that in a similar way, we can define p-logarithm, p-hyperbolic cosine, p-hyperbolic sine, p-tangent, so on.*

Remark 3.2. *Applying the ratio test, it is very easy to see that* $exp_p(x)$, $cos_p(x)$ *and* $sin_p(x)$ *are well defined for any* $x \in \mathbb{R}$, $0 < p \le 1$.

In our opinion, would be of interest to solve the following

Open Questions. 1) Find elementary properties of the above mentioned p-functions for $0 < p < 1$. Also, would be of interest to find some known (classical) lower and upper functions (the best if it is possible) for each p-function. For example, in the case of $exp_p(x)$, the inequality $(\sum_{k=0}^{\infty} a_k)^p \leq \sum_{k=1}^{\infty} a_k^p$ valid for $a_k \geq 0, k = 0, 1, ...,$, implies that $exp_p(x) \geq [exp(x)]^p$, for all $x \geq 0$, where $exp(x)$ denotes the classical exponential. The finding of the (best) upper function for $exp_p(x)$ seems to be more complicated.

2) It is known that the classical $exp(x)$ can be expressed as the limit (when $n \to \infty$) of the sequence $(1 + \frac{x}{n})^n, n \in \mathbb{N}$.

The question is what sequence would have as limit the value $exp_p(x)$, for a fixed $0 < p < 1$?

References

[1] P.R. Chernoff, *Note on product formulas for operator semigroups*, J. Funct. Analysis, **2**(1968), 238-242.

[2] S.G. Gal and J.A. Goldstein, *Semigroups of linear operators on p-Fréchet spaces,* $0 < p < 1$, Acta Math. Hungar., 2006, under press.

Department of Mathematics and Computer Science
University of Oradea
Str. Universitatii No. 1
410087 Oradea, Romania
E-mail address: *galso@uoradea.ro*