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Certain Subclasses of Analytic Functions with Negative Coefficients Defined by Generalized Sălăgean Operator[1]

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Abstract

We introduce the subclass $T_j(n, m, \gamma, \alpha, \lambda)$ of analytic functions with negative coefficients defined by generalized Sălăgean operator D^n_λ. Coefficient estimates, some important properties of the class $T_j(n, m, \gamma, \alpha, \lambda)$ and distortion theorems are determined. Further, extremal properties and radii of close-to-convexity, starlikeness and convexity of the class $T_j(n, m, \gamma, \alpha, \lambda)$ are obtained.

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1 Introduction

Let A_j denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k (j \in N = \{1, 2, ...\}) \tag{1.1}$$

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which are analytic in the unit disc $U=\{z:|z|<1\}$. In [1] AL-Oboudi defined the generalized Sălăgean operator as following :

$$D_{\lambda}^{0}f(z)=f(z), \tag{1.2}$$

$$\begin{aligned} D_{\lambda}^{1}f(z) &= (1-\lambda)f(z)+\lambda zf'(z) \\ &= D_{\lambda}f(z) \qquad ,\lambda\geq 0 \end{aligned} \tag{1.3}$$

and

$$D_{\lambda}^{n}f(z)=D_{\lambda}(D^{n-1}f(z)). \tag{1.4}$$

If $f(z)$ is given by (1.1), then from (1.3) and (1.4) we see that

$$D_{\lambda}^{n}f(z)=z+\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^{n}a_{k}z^{k}. \tag{1.5}$$

When $\lambda=1,\ j=1$ we get Sălăgean's differential operator [9].

Let T_j denote the subclass of A_j consisting of functions of the form

$$f(z)=z-\sum_{k=j+1}^{\infty}a_{k}z^{k}(a_{k}\geq 0\ ,j\in N). \tag{1.6}$$

With the above operator D_{λ}^{n}, we say that a function $f(z)$ belonging to A_j is in the class $A_j(n,m,\gamma,\alpha,\lambda)$ if and only if

$$Re\left\{\frac{D_{\lambda}^{n+m}f(z)/D_{\lambda}^{n}f(z)}{\gamma\ (D_{\lambda}^{n+m}f(z)/D_{\lambda}^{n}f(z))+1-\gamma}\right\}>\alpha, z\in U, \tag{1.7}$$

for some $\alpha,\gamma\in[0,1),\ \lambda\geq 0,\ \ j\in N, n,m\in N_{0}^{*}=N\cup\{0\}$.

Furthermore, by specializing the parameters n,m,γ,α and λ, we obtain the following subclasses studied by various other authors:

(i) $T_1(0,1,0,\alpha,1)=T^{*}(\alpha)$ and $T_1(1,1,0,\alpha,1)=C(\alpha)$(Silverman [11]);

(ii) $T_j(0,1,0,\alpha,1) = T_\alpha(j)$ and $T_j(1,1,0,\alpha,1) = C_\alpha(j)$(Chatterjea [5] and Srivastava et al. [12]);

(iii) $T_1(n,1,0,\alpha,1) = T(n,\alpha)$ (Hur and Oh [8]);

(iv)$T_1(0,1,\gamma,\alpha,1) = T(\gamma,\alpha)$ and $T_1(1,1,\gamma,\alpha,1) = C(\gamma,\alpha)$
(Altintas and Owa [2]);

(iv) $T_1(n,1,\gamma,\alpha,1) = T_n(\gamma,\alpha)$ (Aouf and Cho [3] and Cho and Aouf [6]);

(v) $T_j(n,m,0,\alpha,1) = T_j(n,m,\alpha)$ (Sekine [10] and Hossen, Sălăgean and Aouf [7]);

(vi) $T_j(n,m,\gamma,\alpha,1) = T_j(n,m,\gamma,\alpha)$(Aouf, Darwish and Attiya[4]).

2 Coefficient estimates

Lemma 2.1. *Let the function* $f(z)$ *be defined by* (1.6) *with* $j = 1$.*Then* $f(z) \in T_j(n,m,\gamma,\alpha,\lambda)$ *if and only if*

(2.1) $$\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}a_k \le 1-\alpha,$$

for $n \in N_0$, $m \in N_0$, $0 \le \alpha < 1$, $\lambda \ge 0$ *and* $0 \le \gamma < 1$.*The result is sharp .*

Assume the inequality (2.1) holds and let $|z| = 1$.Then we have

(2.2) $$\left|\frac{\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}}{\gamma\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}+1-\gamma}-1\right| \le$$

$$\le \frac{(1-\gamma)\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k|z|^{k-1}-(1-\gamma)\sum_{k=2}^{\infty}[1+(k-1)]^n a_k|z|^{k-1}}{1-\gamma\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k|z|^{k-1}-(1-\gamma)\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n a_k|z|^{k-1}}$$

$$\leq \frac{(1-\gamma)\left\{\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k - \sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n}a_k\right\}}{1-\gamma\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k-(1-\gamma)\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n}a_k} \leq 1-\alpha$$

which implies (1.7). Thus it follows from this fact that $f(z) \in T_1(n,m,\gamma,\alpha,\lambda)$. Conversely, assume that the function $f(z)$is in the class $T_1(n,m,\gamma,\alpha,\lambda)$.Then

$$Re\left\{\frac{\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}}{\gamma\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}+1-\gamma}\right\} = \tag{2.3}$$

$$= Re\left\{\frac{1-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k z^{k-1}}{1-\gamma\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k z^{k-1}-(1-\gamma)\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n}a_k z^{k-1}}\right\}$$

$$> \alpha (z \in U)$$

for $z \in U$.Choose values of z on the real axis so that $D_\lambda^{n+m}f(z)/D_\lambda^{n+m}f(z)$ $/\left\{\gamma\, D_\lambda^{n+m}f(z)/D_\lambda^n f(z)+1-\gamma\right\}$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we obtain

$$\begin{aligned} 1-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k \quad &\geq \quad \alpha\left\{1-\gamma\sum_{k=2}^{\infty}[1+(k-1)\lambda]^{n+m}a_k\right. \\ &\left. -(1-\gamma)\sum_{k=2}^{\infty}[1+(k-1)]^{n}a_k\right\} \end{aligned} \tag{2.4}$$

which gives (2.1). Finally the result is sharp with the extremal function given by

$$f(z) = z - \frac{1-\alpha}{[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}z^k \tag{2.5}$$

$(k \geq 2)$.

With the aid of Lemma 2.1, we prove

Theorem 2.1. *Let the function* $f(z)$ *be defined by* (1.6). *Then* $f(z) \in T_j(n, m, \gamma, \alpha, \lambda)$*if and only if*

$$\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \{[1+(k-1)\lambda]^m (1-\gamma\alpha) - \alpha(1-\gamma)\} a_k \leq 1-\alpha \tag{2.6}$$

for $n \in N_0, m \in N_0, 0 \leq \alpha < 1, 0 \leq \gamma < 1$ *and* $\lambda \geq 0$. *The result is sharp for the function*

$$f(z) = z - \frac{(1-\alpha)}{[1+(k-1)\lambda]^n \{[1+(k-1)\lambda]^m (1-\gamma\alpha) - \alpha(1-\gamma)\}} z^k \tag{2.7}$$

$(k \geq j+1)$.

Putting $a_k = 0 (k = 2, 3, ..., j)$ in Lemma 2.1, we can prove the assertion of Theorem1.

Corollary 2.1. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $T_j(n, m, \gamma, \alpha, \lambda)$. *Then*

$$a_k \leq \frac{1-\alpha}{[1+(k-1)\lambda]^n \{[1+(k-1)\lambda]^m (1-\alpha\gamma) - \alpha(1-\gamma)\}} (k \geq j+1). \tag{2.8}$$

The equality in (2.8) is attained for the function $f(z)$ given by (2.7).

3 Some Properties of the class $T_j(n, m, \gamma, \alpha, \lambda)$

Theorem 3.1. *Let* $0 \leq \alpha < 1,\ 0 \leq \gamma_1 \leq \gamma_2 < 1,\ \lambda \geq 0,\ n \in N_0,$ *and* $m \in N_0$.*Then*

$$T_j(n, m, \gamma_1, \alpha, \lambda) \subset T_j(n, m, \gamma_2, \alpha, \lambda).$$

It follows from Theorem 2.1., that

$$\begin{aligned}
&\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^{n}\{[1+(k-1)\lambda]^{m}(1-\alpha\gamma_2)-\alpha(1-\gamma_2)\}a_k\\
\leq\ &\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^{n}\{[1+(k-1)\lambda]^{m}(1-\alpha\gamma_1)-\alpha(1-\gamma_1)\}a_k\\
\leq\ &1-\alpha
\end{aligned}$$

for $f(z) \in T_j(n, m, \gamma_1, \alpha, \lambda)$. Hence $f(z) \in T_j(n, m, \gamma_2, \alpha, \lambda)$.

Theorem 3.2. *Let* $0 \leq \alpha < 1, 0 \leq \gamma < 1, \lambda \geq 0,$ *and* $n, m \in N_0$. *Then*

(i) $T_j(n+1, m, \gamma, \alpha, \lambda) \subset T_j(n, m, \gamma, \alpha, \lambda)$,

(ii) $T_j(n, m+1, \gamma, \alpha, \lambda) \subset T_j(n, m, \gamma, \alpha, \lambda)$,

and

(iii) $T_j(n+1, m+1, \gamma, \alpha, \lambda) \subset T_j(n, m, \gamma, \alpha, \lambda)$.

The proof follows immediatly from Theorem 2.1.

4 Distortion Theorems

Theorem 4.1. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $T_j(n, m, \gamma, \alpha, \lambda)$. *Then we have*

$$\left|D_{\lambda}^{i}f(z)\right| \geq |z| - \frac{(1-\alpha)}{[1+j\lambda]^{n-i}\{[1+j\lambda]^{m}(1-\alpha\gamma)-\alpha(1-\gamma)\}}\,|z|^{j+1} \tag{4.1}$$

and

$$\left|D_{\lambda}^{i}f(z)\right| \leq |z| + \frac{(1-\alpha)}{[1+\lambda j]^{n-i}\{[1+j\lambda]^{m}(1-\alpha\gamma)-\alpha(1-\gamma)\}}\,|z|^{j+1} \tag{4.2}$$

for $z \in U$, *where* $0 \leq i \leq n$. *Then equalities in* (4.1) *and* (4.2) *are attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{(1-\alpha)}{[1+j\lambda]^{n}\{[1+\lambda j]^{m}(1-\alpha\gamma)-\alpha(1-\gamma)\}}z^{j+1}. \tag{4.3}$$

Proof. Note that $f(z) \in T_j(n, m, \gamma, \alpha, \lambda)$ if and only if $D^i f(z) \in T_j(n-i, m, \gamma, \alpha, \lambda)$ and that

$$D^i_\lambda f(z) = z - \sum_{k=j+1}^{\infty} [1 + (k-1)\lambda]^i a_k z^k. \tag{4.4}$$

Using Theorem 1, we note that
(4.5)

$$[1 + \lambda j]^{n-i} \left\{ [1 + \lambda j]^m (1 - \gamma\alpha) - \alpha(1 - \gamma) \right\} \sum_{k=j+1}^{\infty} [1 + (k-1)\lambda]^i a_k \leq 1 - \alpha),$$

that is, that

$$\sum_{k=j+1}^{\infty} [1 + (k-1)\lambda]^i a_k \leq \frac{(1-\alpha)}{[1 + \lambda j]^{n-i} \{[1 + \lambda_j]^m (1 - \alpha\gamma) - \alpha(1-\gamma)\}}. \tag{4.6}$$

It follows from (4.4) and (4.6) that

$$\left| D^i_\lambda f(z) \right| \geq |z| - \frac{(1-\alpha)}{[1 + \lambda j]^{n-i} \{[1 + \lambda j]^m (1 - \alpha\gamma) - \alpha(1-\gamma)\}} |z|^{j+1} \tag{4.7}$$

and

$$\left| D^i_\lambda f(z) \right| \leq |z| + \frac{(1-\alpha)}{[1 + \lambda j]^{n-i} \{[1 + \lambda j]^m (1 - \alpha\gamma) - \alpha(1-\gamma)\}} |z|^{j+1}. \tag{4.8}$$

Finally, we note that the equalities in (4.1) and (4.2) are attained for the function $f(z)$ defined by

$$D^i_\lambda f(z) = z - \frac{(1-\alpha)}{[1 + \lambda_j]^{n-i} \{[1 + \lambda_j]^m (1 - \alpha\gamma) - \alpha(1-\gamma)\}} |z|^{j+1}. \tag{4.9}$$

This complete the proof of Theorem 4.1.

Corollary 4.1. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $T_j(n, m, \gamma, \alpha, \lambda)$*. Then we have*

$$|f(z)| \geq |z| - \frac{(1-\alpha)}{[1 + j\lambda]^n \{[1 + j\lambda]^m (1 - \alpha\gamma) - \alpha(1-\gamma)\}} |z|^{j+1} \tag{4.10}$$

and

$$|f(z)| \leq |z| + \frac{(1-\alpha)}{[1+\lambda j]^n\{[1+\lambda j]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}\,|z|^{j+1} \tag{4.11}$$

for $z \in U$. *The equalities in* (4.10) *and* (4.11) *are attained for the function* $f(z)$ *given by* (4.3).

Proof. Taking $i = 0$ in Theorem 4.1., we can easily show (4.10) and (4.11).

5 Extremal properties of the class T_j(n,m,γ,α,λ)

Theorem 5.1. *The class* $T_j(n, m, \gamma, \alpha, \lambda)$ *is convex.*

Proof. Let the functions

$$f_\nu(z) = z - \sum_{k=j+1}^{\infty} a_{\nu,k} z^k (a_{\nu.k} \geq 0, \nu = 1, 2) \tag{5.1}$$

be in the class $T_j(n, m, \gamma, \alpha, \lambda)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = t f_1(z) + (1-t)\ f_2(z) \quad (0 \leq t \leq 1) \tag{5.2}$$

is in the class $T_j(n, m, \gamma, \alpha, \lambda)$. Since

$$h(z) = z - \sum_{k=j+1}^{\infty} [t a_{1,k} + (1-t) a_{2,k}]\, z^k, \tag{5.3}$$

with the aid of Theorem 2.1. , we have

$$\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}.$$

$$\{t a_{1,k} + (1-t) a_{2,k}\ \}\ \leq 1-\alpha \tag{5.4}$$

which implies that $h(z) \in T_j(n, m, \gamma, \alpha, \lambda)$. Hence $T_j(n, m, \gamma, \alpha, \lambda)$ is convex.

As a consequence of Theorem 5.1. we can obtain the extreme points of the class $T_j(n, m, \gamma, \alpha, \lambda)$.

Theorem 5.2. *Let* $f_j(z) = z$ *and*

$$f_k(z) = z - \frac{1-\alpha}{[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}} z^k \tag{5.5}$$

$$(n, m \in No, k \geq j+1)$$

for $0 \leq \alpha < 1$, *and* $0 \leq \gamma < 1, \lambda \geq 0$. *Then* $f(z)$ *is in the class* $T_j(n, m, \gamma, \alpha, \lambda)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z), \tag{5.6}$$

where $\mu_k \geq 0$ *for* $k \geq j$ *and* $\sum_{k=j}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) \tag{5.7}$$

$$= \mu_j f_j(z) + \sum_{k=j+1}^{\infty} \mu_k f_k(z)$$

$$= (1 - \sum_{k=j+1}^{\infty} \mu_k) +$$

$$+ \sum_{k=j+1}^{\infty} \mu_k \left\{ z - \frac{1-\alpha}{[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}} \right\}$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}} z^k.$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{[1+(k-1)\lambda]^n \left\{[1+(k-1)\lambda]^m (1-\alpha\gamma)-\alpha(1-\gamma)\right\}(1-\alpha)\mu_k}{(1-\alpha)[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}$$

$$= \sum_{k=j+1}^{\infty} \mu_k = 1-\mu_j \leq 1. \tag{5.8}$$

So by Theorem 2.1., $f(z) \in T_j(n,m,\gamma,\alpha,\lambda)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_j(n,m,\gamma,\alpha,\lambda)$.Then

$$a_k \leq \frac{(1-\alpha)}{[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}(k \geq j+1). \tag{5.9}$$

Setting

$$\mu_k = \frac{[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}{(1-\alpha)} a_k (k \geq j+1) \tag{5.10}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k$$

we notice that $f(z)$ can be expressed in the form (5.6). This completes the proof of Theorem 5.2.

Corollary 5.1. *The extreme points of the class $T_j(n,m,\gamma,\alpha,\lambda)$ are the functions $f_k(z)(k \geq j)$ given by Theorem 5.2.*

6 Radii of close-to-convexity, starlikeness and convexity

Theorem 6.1. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $T_j(n,m,\gamma,\alpha,\lambda)$.*Then* $f(z)$ *is close - to- convex of order* $\rho(0 \le \rho < 1)$ *in* $|z| < r_1(n,m,\gamma,\alpha,\lambda,\rho)$, *where*

$$r_1(n,m,\gamma,\alpha,\lambda,\rho) = \tag{6.1}$$

$$= \inf_k \left\{ \frac{(1-\rho)[1+(k-1)\lambda]^{n+1}\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}{k(1-\alpha)} \right\}^{\frac{1}{k-1}}$$

$(k \ge j+1)$.

The result is sharp, with the extremal function $f(z)$ *given by* (2.7).

It is sufficient to show that

$$|f^{'}(z)-1| = 1-\rho(0 \le \rho < 1)$$

for $|z| < r_1(n,m,\gamma,\alpha,\lambda,\rho)$.

We have

$$|f^{'}(z)-1| = \left| -\sum_{k=j+1}^{\infty} ka_k z^{k-1} \right| \le \sum_{k=j+1}^{\infty} ka_k |z|^{k-1}.$$

Thus $|\ f^{'}(z)-1| \le 1-\rho$ if

$$\sum_{k=j+1}^{\infty} (\frac{k}{1-\rho})\, a_k\, |z|^{k-1} \le 1. \tag{6.2}$$

Hence, by Theorem 2.1., (6.2) will be true if

$$\frac{k\,|z|^{k-1}}{(1-\rho)} \le \frac{[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}{(1-\alpha)} \tag{6.3}$$

or if

$$|z| \le \left[\frac{(1-\rho)[1+(k+1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\alpha\gamma)-\alpha(1-\gamma)\}}{k(1-\alpha)} \right]^{\frac{1}{k-1}}$$

$(k \geq j+1)$.

The theorem follows easily from (6.3).

Theorem 6.2. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $T_j(n, m, \gamma, \alpha, \lambda)$*; then* $f(z)$ *is starlike of order* $\rho (0 \leq \rho < 1)$ *in* $|z| < r_2(n, m, \gamma, \alpha, \lambda, \rho)$*,where*

$$r_2(n, m, \gamma, \alpha, \lambda, \rho) = \tag{6.4}$$

$$= \inf_k \left\{ \frac{(1-\rho)[1+(k-1)\lambda]^n \{[1+(k-1)\lambda]^m (1-\gamma\alpha) - \alpha(1-\gamma)\}}{(k-\rho)(1-\alpha)} \right\}^{\frac{1}{k-1}}$$

$(k \geq j+1)$.

The result is sharp, with extremal function $f(z)$ *given by* (2.7).

Proof. We must show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$ for $|z| < r_2(n, m, \gamma, \alpha, \lambda, \rho)$. We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{k=j+1}^{\infty} (k-1)\, a_k \, |z|^{k-1}}{1 - \sum_{k=j+1}^{\infty} a_k \, |z|^{k-1}}.$$

Thus $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$ if

$$\sum_{k=j+1}^{\infty} \frac{(k-\rho)\, a_k \, |z|^{k-1}}{(1-\rho)} \leq 1. \tag{6.5}$$

Hence, by Theorem 2.1., (6.5) will be true if

$$\frac{(k-\rho) a_k \, |z|^{k-1}}{(1-\rho)} \leq \frac{[1+(k-1)\lambda]^n \{[1+(k-1)\lambda]^m (1-\gamma\alpha) - \alpha(1-\gamma)\}}{(1-\alpha)}$$

or if

$$|z| \leq \tag{6.6}$$

$$\leq \left\{ \frac{(1-\rho)[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\gamma\alpha)-\alpha(1-\gamma)\}}{(k-\rho)(1-\alpha)} \right\}^{\frac{1}{k-1}},$$

$$(k \geq j+1).$$

The theorem follows easily from (6.6).

Corollary 6.1. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $T_j(n, m, \gamma, \alpha, \lambda)$; *then* $f(z)$ *is convex of order* $\rho (0 \leq \rho < 1)$ *in* $|z| < r_3(n, m, \gamma, \alpha, \lambda, \rho)$,*where*

$$r_3(n, m, \gamma, \alpha, \lambda, \rho)$$

(6.7)
$$= \inf_k \left\{ \frac{(1-\rho)[1+(k-1)\lambda]^n\{[1+(k-1)\lambda]^m(1-\gamma\alpha)-\alpha(1-\gamma)\}}{k(k-\rho)(1-\alpha)} \right\}^{\frac{1}{k-1}}$$

$$(k \geq j+1).$$

The result is sharp, with extremal function $f(z)$ *given by* (2.7).

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