General Mathematics Vol. 16, No. 1 (2008), 73-81

Best simultaneous approximation in linear 2-normed spaces[1]

S. Elumalai and R. Vijayaragavan

Abstract

In this paper we established some of the results of the best simultaneous approximation in the context of linear 2-normed space.

2000 Mathematics Subject Classification: 41A50, 41A52, 41A99, 41A28.

Key words: Linear 2-normed space, strictly covex, uniformly convex, 2-functional and best simultaneous approximation.

1 Introduction

The problem of simultaneous approximation has been studied by several authors. Diaz and McLaughlin [1,2] and Dunham [4] have considered the simultaneous approximation of two real-valued functions defined on $[a, b]$. Several results of best simultaneous approximation in the context of normed linear space were obtained by Goel, et al. [8,9]. Subsequently, Elumalai S. and coworkers have developed best approximation theory with respect to 2-norm to a considerable extent [5,6,7]. The main aim of this paper is to

[1] *Received 1 July 2007*

Accepted for publication (in revised form) 10 December 2007

drive existence and uniqueness of the best simultaneous approximation in the context of linear 2-normed space. Section 2 provides some definitions that are used in the sequel. Some main results of the set of best simultaneous approximation are established in Section 3.

2 Preliminaries

Definition 2.1. *Let X be a linear space over $\mathbb{R}$ with dimension $X > 1$ and let $||\cdot,\cdot|| : X \times X \to \mathbb{R}$ be a mapping with the following properties:*
(i) $||x,y|| > 0$ and $||x,y|| = 0$ if and only if x and y are linearly dependent,
(ii) $||x,y|| = ||y,x||$,
(iii) $||\lambda x,y|| = |\lambda|||x,y||$,
(iv) $||x+y,z|| = ||x,z|| + ||y,z||$, for all $x,y,z \in X$ and $\lambda -$ a scalar.

Then the mapping $||\cdot,\cdot||$ is called a 2-norm and the pair $(X, ||\cdot,\cdot||)$ is called a linear 2-normed space.

Definition 2.2. *A sequence $\{x_n\}$ is a linear 2-normed space X is called a convergent sequence if there is an $x \in X$ such that $\lim\limits_{n\to\infty} ||x_n - x, z|| = 0$ for all $z \in X$.*

Definition 2.3. *A linear 2-normed space $(X, ||\cdot,\cdot||)$ is said to be strictly convex if $||a+b,c|| = ||a,c||+||b,c||$, $||a,c|| = ||b,c|| = 1$ and $c \in X \setminus V(a,b)$, where $V(a,b)$ is the subspace of X generated by a and b, which implies that $a = b$.*

A linear 2-normed space $(X, ||\cdot,\cdot||)$ is said to be strictly convex if and only if $||x,z|| = ||y,z|| = 1, x \neq y$ and $z \in X \setminus V(x,y)$ implies that $||\frac{x+y}{2}, z|| < 1$.

Definition 2.4. *A linear 2-normed space $(X, ||\cdot,\cdot||)$ is said to be uniformly convex if for any sequences $\{x_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ in X, $||x_n, z|| \leq 1, ||y_n, z|| \leq 1, n = 1,2,3,\ldots, \lim\limits_{n\to\infty} ||\dfrac{x_n+y_n}{2}, z|| = 1$ and $V(c) \cap \{\cap_{n=1}^{\infty} V(x_n,y_n)\} = \{0\}$ implies that $\lim\limits_{n\to\infty} ||x_n - y_n, z|| = 0$.*

Example 2.1. *Let* $X = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ *with 2-norm defined as* $x = (a_1, b_1, c_1)$,$y = (a_2, b_2, c_2)$

$$||x, y|| = \sqrt{(b_1c_2 - b_2c_1)^2 + (a_1c_2 - a_2c_1)^2 + (a_1b_2 - a_2b_1)^2}.$$

Then $(X, ||\cdot, \cdot||)$ *is both strictly convex and uniformly convex.*

Definition 2.5. *Let* $(X, ||\cdot, \cdot||)$ *be a linear 2-normed space. Let* F *be any bounded subset of* X *and* K *be a subset of* X. *An element* $k^* \in K$ *is said to be a best simultaneous approximation to the set* F, *if*

$$d(F, K)_z = \sup_{f\in F} ||f - k^*, z||, z \in X \setminus V(f, k^*).$$

Where

$$d(F, K)_z = \inf_{k\in K} \sup_{f\in F} ||f - k, z||, z \in X \setminus V(f, k).$$

Definition 2.6. *A 2-functional is a real-valued mapping defined on* $A \times M$, *where* A *and* M *are linear subspaces of a linear 2-normed space* $(X, ||\cdot, \cdot||)$.

Definition 2.7. *A 2-functional* f *is said to be continuous at* (x, y) *if for a given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*
$|f(x, y) - f(z, s)| < \varepsilon$ *whenever* $||x - z, y|| < \delta$ *and* $||z, y - s|| < \delta$ *or* $||x - -z, s|| < \delta$ *and* $||x, y - s||$. *Then* f *is said to be continuous at each point of this domain.*

3 Main Results

Lemma 3.1. *Let* $(X, ||\cdot, \cdot||)$ *be a linear 2-normed space, let* $K \subset X$ *and* F *be a bounded subset of* X. *Then* $\Phi(k, z) = \sup_{f\in F} ||f - k, z||, z \in X \setminus V(f, k)$ *is a continuous functional on* X.

Proof. For any $f \in F$ and $k, k' \in X$, we have

$$||f - k, z|| \leq ||f - k', z|| + ||k - k', z||, z \in X \setminus V(f, k, k').$$

Then

$$\sup_{f\in F}||f-k,z|| \le \sup_{f\in F}(||f-k',z|| + ||k-k',z||).$$

Now, if

$$||k-k',z|| < \epsilon, \text{ then } \Phi(k,z) \le \Phi(k',z) + \epsilon.$$

By interchanging k and k', we obtain

$$\Phi(k',z) \le \Phi(k,z) + \epsilon.$$

Thus

$$|\Phi(k,z) - \Phi(k',z)| < \epsilon,$$

which completes the proof.

Lemma 3.2. *Let $(X, ||\cdot,\cdot||)$ be a linear 2-normed space. Let K be a finite dimensional subspace of X. Then there exists a best simultaneous approximation $k^* \in K$ to any given compact subset $F \subset X$.*

Proof. Since F is compact, there exists a finite constant M such that $||f,b|| \le M$, for all $f \in F$ and $b \in X$

Now we define the subset S of K as $S \equiv S(0, 2M)$. Then

$$\inf_{k\in S}\sup_{f\in F}||f-k,b|| = \inf_{k\in K}\sup_{f\in F}||f-k,b||,\ b \in X \setminus V(f,k) \le M.$$

Since S is compact, the continuous functional $\Phi(k,b)$ attains its minimum over S for some $k^* \in K$. Which is the best simultaneous approximation to F.

Lemma 3.3. *Let $(X, ||\cdot,\cdot||)$ be a linear 2-normed space and let K be a convex subset of X and $F \subset X$. If $k_1, k_2 \in K$ are two best simultaneous approximations to F by elements of K. Then $\overline{k} = \lambda k_1 + (1-\lambda)k_2, (0 \le \lambda \le 1)$ is also a best simultaneous approximation to F.*

Proof. For $z \in X \setminus V(f, \overline{k})$,

$$\begin{aligned} \sup_{f\in F} \|f - \overline{k}, z\| = & \sup_{f\in F} \|f - (\lambda k_1 + (1-\lambda)k_2), z\| \\ & = \sup_{f\in F} \|\lambda(f - k_1) + (1-\lambda)(f - k_2), z\| \\ & \leq \sup_{f\in F}(\lambda\|(f - k_1, z\| + (1-\lambda)\|f - k_2, z\|) \\ & \leq \lambda \sup_{f\in F} \|f - k_1, z\| + (1-\lambda) \sup_{f\in F} \|f - k_2, z\| \\ & = \lambda d(F,K)_z + (1-\lambda) d(F,K)_z \\ & = d(F,K)_z. \end{aligned} \tag{1}$$

$$\begin{aligned} d(F,K)_z = & \inf_{k\in K} \sup_{f\in F} \|f - \overline{k}, z\| \\ & \leq \sup_{f\in F} \|f - \overline{k}, z\| \end{aligned} \tag{2}$$

$$d(F,K)_z = \sup_{f\in F} \|f - \overline{k}, z\| \tag{3}$$

Which proves the result.

Theorem 3.1. *Let $(X, \|\cdot,\cdot\|)$ be a strictly convex linear 2-normed space. Let K be a finite dimensional subspace of X. Then there exists one and only one best simultaneous approximation from the elements of K to any given compact subset $F \subset X$.*

Proof. The existence of a best simultaneous approximation follows from the Lemma 3.2.

Suppose k_1 and $k_2 (k_1 \neq k_2)$ are best simultaneous approximations to F. Then for $z \in X \setminus U(f, k_1, k_2)$,

$$\begin{aligned} \inf_{k\in K} \sup_{f\in F} \|f - k, z\| = & \sup_{f\in F} \|f - k_1, z\| \\ & = \sup_{f\in F} \|f - k_2, z\| \\ & = d. \end{aligned} \tag{4}$$

Then by Lemma 3.3, $\frac{k_1+k_2}{2}$ is also the best simultaneous approximation,i.e,

$$\sup_{f\in F}||f-\frac{k_1+k_2}{2},z||=d. \tag{5}$$

Since F is compact there exists an f_0 such that

$$\sup_{f\in F}||f-\frac{k_1+k_2}{2},z||=||f_0-\frac{k_1+k_2}{2},z||=d. \tag{6}$$

From (4), $||f_0-k_1,z||\leq d$ and $||f_0-k_2,z||\leq d$

Then by strict convexity, we have

$$||f_0-k_1+f_0-k_2,z||<2d.$$

That is

$$||f_0-\frac{k_1+k_2}{2},z||<d.$$

which is a contradiction to (6).

Theorem 3.2. *Let K be a closed and convex subset of a uniformly convex 2-Banach space X. Then for any compact subset $F\subset X$, there exists a unique best approximation to F form the elements of K.*

Proof. Let

$$d=\inf_{k\in F}\sup_{f\in F}||f-k,z||,z\in X\backslash V(f,k) \tag{7}$$

and $\{k_n\}$ be any sequence of elements in K such that

$$\lim_{n\to\infty}\sup_{f\in F}||f-k_n,z||=d.$$

Also, let

$$d_m=\sup_{f\in F}||f-k_m,z||,m\geq 1,\text{ and } z\in X\backslash V(f,k_m).$$

Then $d_m \geq d$, which implies that

$$\frac{||f - k_m, z||}{d_m} \leq 1, \text{ for } f \in F. \tag{8}$$

Now, we consider

$$\frac{1}{2}\left[\frac{k_m}{d_m} + \frac{k_n}{d_n}\right] = \frac{(d_n k_m + k_n d_m)(d_m + d_n)}{(d_m + d_n)2d_m d_n} \tag{9}$$

and let $y_{m,n} = \dfrac{d_n k_m + d_m k_n}{d_m + d_n}$. Then since K is a convex, $y_{m,n} \in K$. Hence

$$\sup_{f \in F} ||f - y_{m,n}, z|| \geq d$$

and

$$\sup_{f \in F} = ||\frac{d_m + d_n}{2d_m d_n} \cdot f - \frac{1}{2}\left\{\frac{k_m}{d_m} + \frac{k_n}{d_n}\right\}, z||$$
$$= \sup_{f \in F} ||f - y_{m,n}, z|| \cdot \left(\frac{d_m + d_n}{2d_m d_n}\right) \geq d \cdot \left(\frac{d_m + d_n}{2d_m d_n}\right).$$

Since F is a compact subset of X, there exists an $f \in F$ such that

$$\left|\left|\frac{f - k_m}{d_m} + \frac{f - k_n}{d_n}, z\right|\right| \geq d \cdot \frac{(d_m + d_n)}{d_m d_n}.$$

By (8) and the uniform convexity of the 2-norm it follows that for a given $\epsilon > 0$, there exists an N such that

$$\left|\left|\frac{f - k_m}{d_m} - \frac{f - k_n}{d_n}, z\right|\right| < \epsilon$$

for $m, n > N$ and $z \in X \backslash V(f, k_n)$.

Since $d_m \to d$ as $m \to \infty$ we can easily see that the sequence $\{k_n\}$ is a Cauchy sequence, hence if converges to some $k \in K \subset X$ as K is closed. This provides that K is a best simultaneous approximation.

Assume that there exists two best simultaneous approximations k_1 and k_2. Then there exists sequences $\{k_n\}$ and $\{k_m\}$ such that $k_n \to k_1$ as $n \to \infty$ and $k_m \to k_2$ as $m \to \infty$.

Again,

$$\lim_{n\to\infty} \sup_{f\in F} ||f - k_n, z|| = d = \lim_{n\to\infty} \sup_{f\in F} ||f - k_m, z||.$$

This implies that

$$\sup_{f\in F} ||f - k_1, z|| = \sup_{f\in F} ||f - k_2, z||$$

$$k_1 = k_2.$$

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S.ELUMALAI
Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Chennai - 600 005,
Tamilnadu,
India.

R. VIJAYARAGAVAN
School of Science and Humanities,
Vellore Institute of Technology University,
Vellore - 632 014,
Tamilnadu,
India.